Neutrino Physics: Lecture 9

Three flavour mixing

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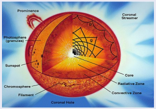
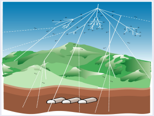
Two disconnected two-flavour mixings

Solar neutrino parameters



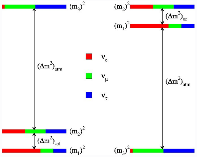
$\Delta m^2_\odot \approx 8 \times 10^{-5}\ \text{eV}^2$, $\theta_\odot \approx 32^\circ$ ✓

Atmospheric neutrino parameters



$\Delta m^2_{\text{atm}} \approx 2.5 \times 10^{-3}\ \text{eV}^2$, $\theta_{\text{atm}} \approx 45^\circ$ ✓

Normal vs. inverted mass ordering / mass hierarchy



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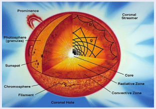

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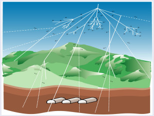

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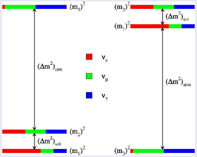

Outline

1. General formalism for three flavours

2. Matching with atmospheric and solar solution

Outline

Flavour eigenstates and mass eigenstates

$$\left.\begin{array}{l}\nu_e\\ \nu_\mu\\ \nu_\tau\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\nu_1\\ \nu_2\\ \nu_3\end{array}\right.$$

Definitions of flavour eigenstates

- ν_e: Interacts to give e^-
- ν_μ: Interacts to give μ^-
- ν_τ: Interacts to give τ^-

Definitions of mass eigenstates

- ν_3: The state that does not participate in solar oscillations (the state "further away")
- ν_2: The heavier state involved in solar oscillations
- ν_1: The lighter state involved in solar oscillations

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Assigning Δm^2 values from available data

Solar data

$$\Delta m^2_\odot = m_2^2 - m_1^2 \equiv \Delta m^2_{21}$$

- $\Delta m^2_{21} > 0$ by definition

Atmospheric data

$$\Delta m^2_{atm} \equiv m_3^2 - (m_2^2 + m_1^2)/2$$

- $\Delta m^2_{31} \equiv m_3^2 - m_1^2$
- $\Delta m^2_{32} \equiv m_3^2 - m_2^2$
- $\Delta m^2_{31} \approx \Delta m^2_{32} \approx \Delta m^2_{atm}$
- Sign of Δm^2_{atm} unknown:
  Measured quantity $P_{\mu\mu} = 1 - \sin^2 2\theta \sin^2(\Delta m^2_{atm} L/(4E))$

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- $\Delta m^2_{32} \equiv m_3^2 - m_2^2$
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- Sign of Δm^2_{atm} unknown:
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The mixing matrix

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

PMNS matrix: elements in general complex

$$\nu_e = U_{e1}\nu_1 + U_{e2}\nu_2 + U_{e3}\nu_3$$

Vacuum oscillations with three neutrinos

Starting with a flavour eigenstate ν_α:

$$|\nu_\alpha(0)\rangle = U_{\alpha 1}|\nu_1\rangle + U_{\alpha 2}|\nu_2\rangle + U_{\alpha 3}|\nu_3\rangle$$

After propagation:

$$|\nu_\alpha(t)\rangle = U_{\alpha 1}|\nu_1\rangle e^{-i\frac{m_1^2 L}{2E}} + U_{\alpha 2}|\nu_2\rangle e^{-i\frac{m_2^2 L}{2E}} + U_{\alpha 3}|\nu_3\rangle e^{-i\frac{m_3^2 L}{2E}}$$

Projection on flavour eigenstate $|\nu_\beta\rangle$:

$$\langle\nu_\beta|\nu_\alpha(t)\rangle = U^*_{\beta 1} U_{\alpha 1} e^{-i\frac{m_1^2 L}{2E}} + U^*_{\beta 2} U_{\alpha 2} e^{-i\frac{m_2^2 L}{2E}} + U^*_{\beta 3} U_{\alpha 3} e^{-i\frac{m_3^2 L}{2E}}$$

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Conversion / survival probability

$$\langle \nu_\beta | \nu_\alpha(t) \rangle = U^*_{\beta 1} U_{\alpha 1} e^{-i \frac{m_1^2 L}{2E}} + U^*_{\beta 2} U_{\alpha 2} e^{-i \frac{m_2^2 L}{2E}} + U^*_{\beta 3} U_{\alpha 3} e^{-i \frac{m_3^2 L}{2E}}$$

$$\begin{aligned}
P_{\alpha\beta} &\equiv |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \\
&= |U^*_{\beta 1} U_{\alpha 1}|^2 + |U^*_{\beta 2} U_{\alpha 2}|^2 + |U^*_{\beta 3} U_{\alpha 3}|^2 \\
&\quad + U^*_{\beta 1} U_{\alpha 2} U_{\beta 2} U^*_{\alpha_1} e^{i \frac{m_2^2 - m_1^2}{2E} L} + c.c. \\
&\quad + U^*_{\beta 1} U_{\alpha 3} U_{\beta 3} U^*_{\alpha_1} e^{i \frac{m_3^2 - m_1^2}{2E} L} + c.c. \\
&\quad + U^*_{\beta 2} U_{\alpha 3} U_{\beta 3} U^*_{\alpha_2} e^{i \frac{m_3^2 - m_2^2}{2E} L} + c.c.
\end{aligned}$$

$P_{\alpha\beta}$: further simplification

Plaquette $\longrightarrow$ $\Box_{\alpha\beta ij} \equiv U_{\alpha_i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}$

$$
\begin{aligned}
P_{\alpha\beta} &= |U^*_{\beta 1} U_{\alpha 1}|^2 + |U^*_{\beta 2} U_{\alpha 2}|^2 + |U^*_{\beta 3} U_{\alpha 3}|^2 \\
&+ 2\mathrm{Re}(\Box_{\alpha\beta 12} e^{2i\Delta_{21}}) + 2\mathrm{Re}(\Box_{\alpha\beta 13} e^{2i\Delta_{31}}) + 2\mathrm{Re}(\Box_{\alpha\beta 23} e^{2i\Delta_{32}})
\end{aligned}
$$

$$
\begin{aligned}
P_{\alpha\beta} &= |U^*_{\beta 1} U_{\alpha 1}|^2 + |U^*_{\beta 2} U_{\alpha 2}|^2 + |U^*_{\beta 3} U_{\alpha 3}|^2 \\
&+ \sum_{i<j} 2\mathrm{Re}(\Box_{\alpha\beta ij}) \cos(2\Delta_{ji}) - \sum_{i<j} 2\mathrm{Im}(\Box_{\alpha\beta ij}) \sin(2\Delta_{ji})
\end{aligned}
$$

$$
P_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{i<j} 4\mathrm{Re}(\Box_{\alpha\beta ij}) \sin^2(\Delta_{ji}) - 2\sum_{i<j} \mathrm{Im}(\Box_{\alpha\beta ij}) \sin(2\Delta_{ji})
$$

$P_{\alpha\beta}$: further simplification

$$\square_{\alpha\beta ij} \equiv U_{\alpha_i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}$$

$$\begin{aligned} P_{\alpha\beta} &= |U^*_{\beta 1} U_{\alpha 1}|^2 + |U^*_{\beta 2} U_{\alpha 2}|^2 + |U^*_{\beta 3} U_{\alpha 3}|^2 \\ &+\mathrm{Re}(\square_{\alpha\beta 12} e^{2i\Delta_{21}}) + \mathrm{Re}(\square_{\alpha\beta 13} e^{2i\Delta_{31}}) + \mathrm{Re}(\square_{\alpha\beta 23} e^{2i\Delta_{32}}) \end{aligned}$$

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$$P_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{i<j} 4\mathrm{Re}(\square_{\alpha\beta ij}) \sin^2(\Delta_{ji}) - 2\sum_{i<j} \mathrm{Im}(\square_{\alpha\beta ij}) \sin(2\Delta_{ji})$$

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Features of $P_{\alpha\beta}$

$$P_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{i<j} 4\mathrm{Re}(\Box_{\alpha\beta ij})\sin^2(\Delta_{ji}) - 2\sum_{i<j}\mathrm{Im}(\Box_{\alpha\beta ij})\sin(2\Delta_{ji})$$

$$\Box_{\alpha\beta ij} \equiv U_{\alpha_i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}$$

$$\Delta_{ji} \equiv \frac{(m_j^2 - m_i^2)}{2E}$$

- No-oscillation term
- CP-conserving oscillation term
- CP-violating oscillation term

Parameterizing U_{PMNS} with three mixing angles

Neglect CP violation $\Rightarrow$ Real U_{PMNS} matrix

$$
\begin{aligned}
U &= R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12}) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}
\end{aligned}
$$

$c_i \equiv \cos\theta_i$, $s_i \equiv \sin\theta_i$

Outline

Reactor neutrino experiments

$\bar{\nu}_e \to \bar{\nu}_e$ with $\Delta_{32} \sim 1, \Delta_{21} \ll 1$

$\Delta m^2_{atm} L / 4E$

$\square_{ee12} = U_{e1} U_{e2} U^*_{e2} U^*_{e1}$ etc

$$P_{ee} = 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2(\Delta_{21})$$
$$-4|U_{e2}|^2|U_{e3}|^2 \sin^2(\Delta_{32}) - 4|U_{e1}|^2|U_{e3}|^2 \sin^2(\Delta_{31})$$

$$\approx 1 - 4|U_{e3}|^2 \sin^2(\Delta_{31})\left(|U_{e2}|^2 + |U_{e1}|^2\right)$$

$$P_{ee} \approx 1 - 4|U_{e3}|^2(1 - |U_{e3}|^2)\sin^2(\Delta_{31})$$

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2(\Delta_{31})$$

Reactor neutrino experiments

$\bar{\nu}_e \to \bar{\nu}_e$ with $\Delta_{32} \sim 1, \Delta_{21} \ll 1$

$$
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\end{aligned}
$$

$$P_{ee} \approx 1 - 4|U_{e3}|^2(1 - |U_{e3}|^2) \sin^2(\Delta_{31})$$

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2(\Delta_{31})$$

Reactor neutrino experiments

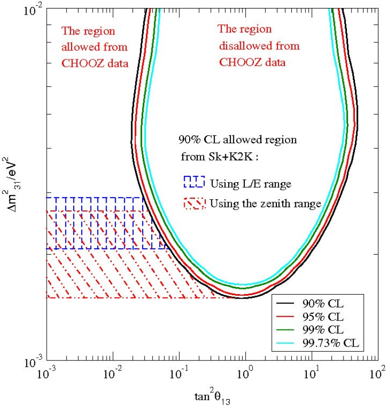
$\bar{\nu}_e \to \bar{\nu}_e$ with $\Delta_{32} \sim 1, \Delta_{21} \ll 1$

$$P_{ee} = 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2(\Delta_{21}) - 4|U_{e2}|^2|U_{e3}|^2 \sin^2(\Delta_{32}) - 4|U_{e1}|^2|U_{e3}|^2 \sin^2(\Delta_{31})$$

$$P_{ee} \approx 1 - \underbrace{4|U_{e3}|^2(1 - |U_{e3}|^2)}_{4 s_{13}^2 c_{13}^2} \sin^2(\Delta_{31})$$

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2(\Delta_{31})$$

Results from CHOOZ



- $E_\nu \approx 3$ MeV, $L \approx 1000$ m $\Rightarrow \Delta_{32} \sim 1, \Delta_{21} \ll 1$
- Disappearance experiment: null result (no disappearance)
- $|U_{e3}|^2(1 - |U_{e3}|^2) < 0.05$, consistent with zero
- $|U_{e3}|^2 < 0.05$ (3σ) or $|U_{e3}|^2 > 0.95$ (3σ)
- $\sin^2\theta_{13} < 0.05$ or $\sin^2\theta_{13} > 0.95$

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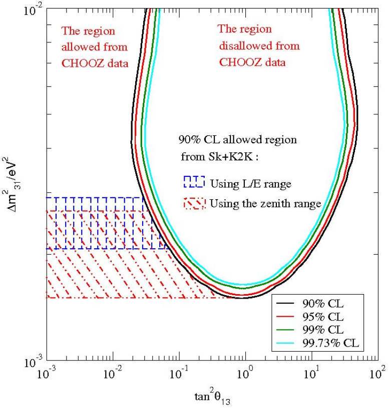

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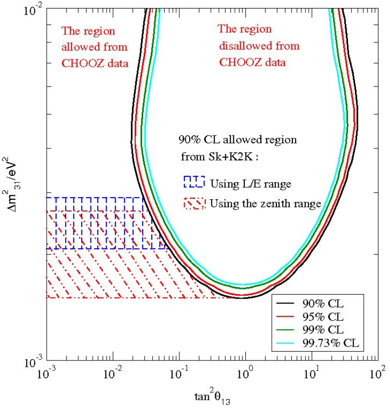

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Atmosphetic neutrino conversion probability

$U_{\mu\mu12} = U_{\mu1} U_{\mu2} U^*_{\mu2} U^*_{\mu1}$ etc

$$P_{\mu\mu} = 1 - 4|U_{\mu1}|^2|U_{\mu2}|^2\sin^2(\Delta_{21}) - 4|U_{\mu2}|^2|U_{\mu3}|^2\sin^2(\Delta_{32}) - 4|U_{\mu1}|^2|U_{\mu3}|^2\sin^2(\Delta_{31})$$

$$\approx 1 - 4|U_{\mu3}|^2\sin^2(\Delta_{31})\left(|U_{\mu2}|^2 + |U_{\mu1}|^2\right)$$

$$P_{\mu\mu} \approx 1 - 4|U_{\mu3}|^2(1 - |U_{\mu3}|^2)\sin^2(\Delta_{31})$$
$$\approx 1 - 4c_{13}^2 s_{23}^2(1 - c_{13}^2 s_{23}^2)\sin^2(\Delta_{31})$$

When $\sin\theta_{13} \approx 0$: $P_{\mu\mu} \approx 1 - \sin^2 2\theta_{23}\sin^2(\Delta m_{32}^2 L/4E)$
When $\sin\theta_{13} \approx 1$: $P_{\mu\mu} \approx 1 \quad \Rightarrow$ No oscillations

$\theta_{23} \approx \theta_{atm}, \quad \theta_{13} \approx 0$

Atmosphetic neutrino conversion probability

$$P_{\mu\mu} = 1 - 4|U_{\mu 1}|^2|U_{\mu 2}|^2 \sin^2(\Delta_{21}) - 4|U_{\mu 2}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{32}) - 4|U_{\mu 1}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{31})$$

$$\begin{aligned} P_{\mu\mu} &\approx 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) \sin^2(\Delta_{31}) \\ &\approx 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2(\Delta_{31}) \end{aligned}$$

When $\sin\theta_{13} \approx 0$: $P_{\mu\mu} \approx 1 - \sin^2 2\theta_{23} \sin^2(\Delta m_{32}^2 L/4E)$
When $\sin\theta_{13} \approx 1$: $P_{\mu\mu} \approx 1 \quad \Rightarrow$ No oscillations

$$\theta_{23} \approx \theta_{atm}, \quad \theta_{13} \approx 0$$

Atmosphetic neutrino conversion probability

$$P_{\mu\mu} = 1 - 4|U_{\mu 1}|^2|U_{\mu 2}|^2 \sin^2(\Delta_{21}) - 4|U_{\mu 2}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{32}) - 4|U_{\mu 1}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{31})$$

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$4 s_{23}^2 c_{23}^2$

When $\sin\theta_{13} \approx 0$: $P_{\mu\mu} \approx 1 - \sin^2 2\theta_{23} \sin^2(\Delta m_{32}^2 L/4E)$ ✓

When $\sin\theta_{13} \approx 1$: $P_{\mu\mu} \approx 1 \quad \Rightarrow$ No oscillations ✗

$\theta_{23} \approx \theta_{atm}, \quad \theta_{13} \approx 0$

Atmosphetic neutrino conversion probability

$$
\begin{aligned}
P_{\mu\mu} &= 1 - 4|U_{\mu 1}|^2|U_{\mu 2}|^2 \sin^2(\Delta_{21}) - \\
& \quad 4|U_{\mu 2}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{32}) - 4|U_{\mu 1}|^2|U_{\mu 3}|^2 \sin^2(\Delta_{31})
\end{aligned}
$$

$$
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P_{\mu\mu} &\approx 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) \sin^2(\Delta_{31}) \\
&\approx 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2(\Delta_{31})
\end{aligned}
$$

When $\sin\theta_{13} \approx 0$: $P_{\mu\mu} \approx 1 - \sin^2 2\theta_{23} \sin^2(\Delta m_{32}^2 L/4E)$
When $\sin\theta_{13} \approx 1$: $P_{\mu\mu} \approx 1 \quad \Rightarrow$ No oscillations

$\theta_{23} \approx \theta_{atm}, \quad \theta_{13} \approx 0$

Solar neutrino problem with three flavours

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = R_{23}(\theta_{23})\ R_{13}(0)\ R_{12}(\theta_{12}) \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

New basis:

$$\begin{pmatrix} \nu_e \\ \nu_x \\ \nu_y \end{pmatrix} = R^\dagger_{23}(\theta_{23}) \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

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Solar neutrino problem with three flavours

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = R_{23}(\theta_{23})\, R_{13}(0)\, R_{12}(\theta_{12}) \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

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Reducing solar neutrino problem to two flavours

$$\begin{pmatrix} \nu_e \\ \nu_x \\ \nu_y \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

- $\nu_3 = \nu_y$ decouples
- Two neutrino mixing:

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

- Mixing angle θ_{12}

$\theta_{12} = \theta_{\odot}$ (when $\theta_{13} = 0$)

Solar oscillations are $\nu_e \leftrightarrow \nu_x$

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- Mixing angle θ_{12}

✓ $\theta_{12} = \theta_\odot$ (when $\theta_{13} = 0$)

✓ Solar oscillations are $\nu_e \leftrightarrow \nu_x$

Net three-flavour picture (No CP violation)

Mass squared differences

- $\Delta m^2_{21} = \Delta m^2_{\odot} \approx 8 \times 10^{-5}$ eV^2 ✓
- $\Delta m^2_{31} \approx \Delta m^2_{32} \approx \Delta m^2_{atm} \approx \pm 2.5 \times 10^{-3}$ eV^2 ✓

Mixing angles

- $\nu_\alpha = R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta_{12})\ \nu_i$
- $\theta_{13} \approx 0$
- $\theta_{23} \approx \theta_{atm} \approx 45^\circ$
- $\theta_{12} \approx \theta_{\odot} \approx 32^\circ$

Neutrino mass-flavour spectrum