Spin Precession Description of Neutrino Oscillations

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Outline

- Density matrix formalism
- Use density matrix to describe a neutrino beam
- 2 flavor case (ν_e-ν_μ)
- Evolution of **S**: analogous to spin precession in magnetic field
- Understanding oscillations with spin precession analogy

Density Matrix Formalism

- Pure Ensemble: a collection of physical states characterized by the same ket $|\alpha\rangle$
- Mixed Ensemble: a fraction of the members with relative population w_i are characterized by ket $|\alpha^{(i)}\rangle$
- The density matrix formalism helps in quantitative description of physical situations with mixed ensembles.
- Populations are constrained by $\sum_i w_i = 1$
- the kets $|\alpha^{(i)}\rangle$ need not be orthogonal.

- Measurement of some observable A gives the expectation value:

$$\langle A \rangle = \sum_i w_i \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle$$

$$= \sum_i \sum_m w_i \left| \langle m | \alpha^{(i)} \rangle \right|^2 m \qquad |m\rangle : \text{Eigen ket of } A$$

- Probabilistic concepts enters twice

- quantum mechanical probability $\left| \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle \right|^2$

- statistical w_i weight for finding $|\alpha^{(i)}\rangle$ in the ensemble

- We define a density operator as $\rho = \sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$

Some properties

- Matrix element looks like

$$\langle m|\rho|n\rangle = \sum_i w_i \langle m|\alpha^{(i)}\rangle\langle\alpha^{(i)}|n\rangle$$

- Expectation value of the observable A:

$$\langle A\rangle = \sum_m \sum_n \langle m|\rho|n\rangle\langle n|A|m\rangle$$
$$= \mathrm{tr}(\rho A)$$

- ρ is Hermitian.

- $$\mathrm{tr}(\rho) = \sum_i \sum_m w_i \langle m|\alpha^{(i)}\rangle\langle\alpha^{(i)}|m\rangle$$
$$= \sum_i w_i \langle\alpha^{(i)}|\alpha^{(i)}\rangle = 1$$

- $$\mathrm{tr}(\rho^2) = \sum_i \sum_m \sum_n w_i^2 \langle m|\alpha^{(i)}\rangle\langle\alpha^{(i)}|n\rangle\langle n|\alpha^{(i)}\rangle\langle\alpha^{(i)}|m\rangle$$
$$= \sum_i w_i^2$$
$$= 1 \quad \text{for} \quad \text{pure} \quad \text{ensemble}$$
$$< 1 \quad \text{for} \quad \text{mixed} \quad \text{ensemble}$$

- The time evolution of the density matrix is given by

$$\boxed{i\hbar \frac{\partial \rho}{\partial t} = [H, \rho]}$$

Use density matrix to describe a ν beam

- A neutrino beam can be described by this density matrix operator

$$\rho(x) = \sum_{\alpha} |\nu_\alpha(x)\rangle w_\alpha \langle \nu_\alpha(x)|$$

- Special case of one initial flavor β can be obtained by setting $w_\alpha = \delta_{\alpha\beta}$
- Probability of detecting a ν_β at a distance x is give by

$$\boxed{P_\beta(x) = \langle \nu_\beta | \rho(x) | \nu_\beta \rangle = \rho^F_{\beta\beta}(x)}$$

where $\rho^F(x)$: density matrix in flavor basis

Evolution of ρ

- Evolution equation of $\rho^F(x)$ is given by

$$i\frac{d\rho^F}{dx} = \left[H_F, \rho^F\right]$$

- Recall $U_M^+ H_F U_M = H_M = \begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix}$

  U_M: the effective mixing matrix in matter

- The density matrix in effective mass basis in matter is

$$\rho^M = U_M^+ \rho^F U_M$$

- The evolution equation of ρ^M is

$$i\frac{d\rho^M}{dx} = \left[H_M, \rho^M\right] - i\left[U_M^+ \frac{dU_M}{dx}, \rho^M\right]$$

- For adiabatic case second term is negligible and can be ignored

$$i\frac{d\rho_{kj}^M}{dx} = \left[(H_M)_{kk} - (H_M)_{jj}\right]\rho_{kj}^M$$

- The diagonal elements of ρ^M remain constant and for $k \neq j$

$$\rho_{kj}^M(x) = \rho_{kj}^M(0)\left(-i\int_0^x \left[(H(x'))_{kk} - (H(x'))_{jj}\right]dx'\right)$$

2 flavor case (ν_e-ν_μ)

- A hermitian matrix X can be decomposed in terms of Pauli Spin matrices as

$$X = \frac{I}{2}\left[\mathrm{Tr}(X) + \sum_k \mathrm{Tr}(X\sigma^k)\sigma^k\right]$$

- Both $\mathrm{H_F}$ and ρ^F are Hermitian. $\mathrm{Tr}(\mathrm{H_F})=0$, $\mathrm{Tr}(\rho^F)=1$.

$$\Rightarrow H_F = -\frac{1}{2}\overline{\sigma}\cdot\overline{B}, \qquad \rho^F = \frac{1}{2}\left(I + \overline{\sigma}\cdot\overline{S}\right)$$

with the vectors,

$$\overline{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$$

$$\overline{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\overline{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$$

- Components of **B** and **S** in flavor basis are:

$$B_x = -\frac{\Delta m^2 \sin 2\vartheta}{2E} = -\frac{\Delta m_M^2 \sin 2\vartheta_M}{2E}, \qquad B_y = 0, \qquad B_z = \frac{\Delta m^2 \cos 2\vartheta - A_C}{2E} = \frac{\Delta m_M^2 \cos 2\vartheta_M}{2E},$$

$$S_x = 2\,\mathrm{Re}\,\rho^F_{e\mu}, \qquad S_y = -2\,\mathrm{Im}\,\rho^F_{e\mu}, \qquad S_z = \rho^F_{ee} - \rho^F_{\mu\mu},$$

- From the initial conditions for the density matrix, we have

$$S_x(0) = 0, \qquad S_y(0) = 0, \qquad S_z(0) = w_e - w_\mu$$

- The probabilities of detection of a ν_e or a ν_μ at a distance x is

$$P_e(x) = \rho^F_{ee}(x) = \frac{1}{2}\left(1 + S_z\right) = \frac{1}{2}\left(1 + \overline{S}\cdot\hat{z}\right)$$

$$P_\mu(x) = \rho^F_{\mu\mu}(x) = \frac{1}{2}\left(1 - S_z\right) = \frac{1}{2}\left(1 - \overline{S}\cdot\hat{z}\right)$$

Evolution of the vector **S**

- The evolution equation of the vector *S* is

$$\frac{d\overline{S}}{dx} = \overline{S} \times \overline{B}$$

- Analogous to magnetic moment (with *g*=1), precessing around a magnetic field. The precession frequency is given by

$$\omega = -\left|\overline{B}\right| = -\frac{\Delta m_M^2}{2E}$$

- Angle between **S** and **B**

$$\frac{\overline{S}(0)\cdot\overline{B}}{\left|\overline{S}(0)\right|\left|\overline{B}\right|}=\cos 2\vartheta_M$$

- Precession becomes clear by rotating the reference frame by $2\theta_M$ in 1-3 plane

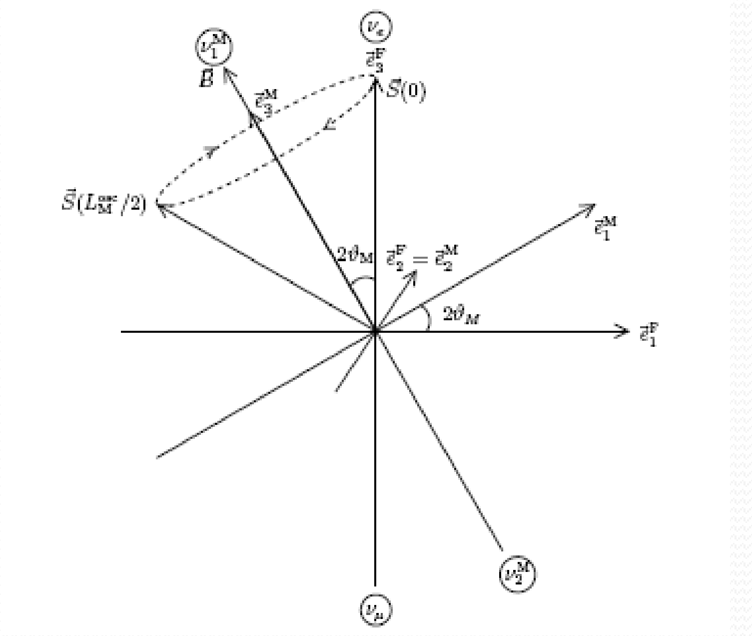
$$\hat{x}'=\cos 2\vartheta_M\,\hat{x}+\sin 2\vartheta_M\,\hat{z}$$
$$\hat{y}'=\hat{y}$$
$$\hat{z}'=-\sin 2\vartheta_M\,\hat{x}+\cos 2\vartheta_M\,\hat{z}$$

In this rotated frame **B** lies along z'

$$B'_x = \cos 2\vartheta_{\mathrm{M}} B_x + \sin 2\vartheta_M B_z = 0$$

$$B'_y = B_y = 0$$

$$B'_z = -\sin 2\vartheta_M B_x + \cos 2\vartheta_M B_z = \Delta m_M^2 / 2E$$



Fig(1):Precession of *S* around *B* with constant matter density $Ne<N_e^R$ in case of pure an intially pure ν_e beam

- The evolution equation of **S** thus becomes

$$\frac{dS'_x}{dx} = -\omega S_y, \qquad \frac{dS'_y}{dx} = \omega S_x \qquad , \frac{dS'_z}{dx} = 0$$

$$\frac{d^2 S'_{x,y}}{dx^2} = \omega^2 S'_{x,y} = 0$$

- Using the initial conditions on **S**, the solution is

$$S'_x(x) = \sin 2\vartheta_M (w_e - w_\mu) \cos(\omega x)$$
$$S'_y(x) = \sin 2\vartheta_M (w_e - w_\mu) \sin(\omega x)$$
$$S'_z(x) = \cos 2\vartheta_M (w_e - w_\mu)$$

- The third component of the **S** in flavor basis becomes

$$S_z(x) = \sin 2\vartheta_M S'_x(x) + \cos 2\vartheta_M S'_z(x) = (w_e - w_\mu)\left[1 - 2\sin^2 2\vartheta_M \sin^2\left(\frac{\omega x}{2}\right)\right]$$

- The probabilities of detecting ν_e and ν_μ are

$$P_e(x) = \frac{1}{2} + (w_e - w_\mu)\left[\frac{1}{2} - \sin^2 2\vartheta_M \sin^2\left(\frac{\omega x}{2}\right)\right]$$

$$P_\mu(x) = \frac{1}{2} + (w_\mu - w_e)\left[\frac{1}{2} - \sin^2 2\vartheta_M \sin^2\left(\frac{\omega x}{2}\right)\right]$$

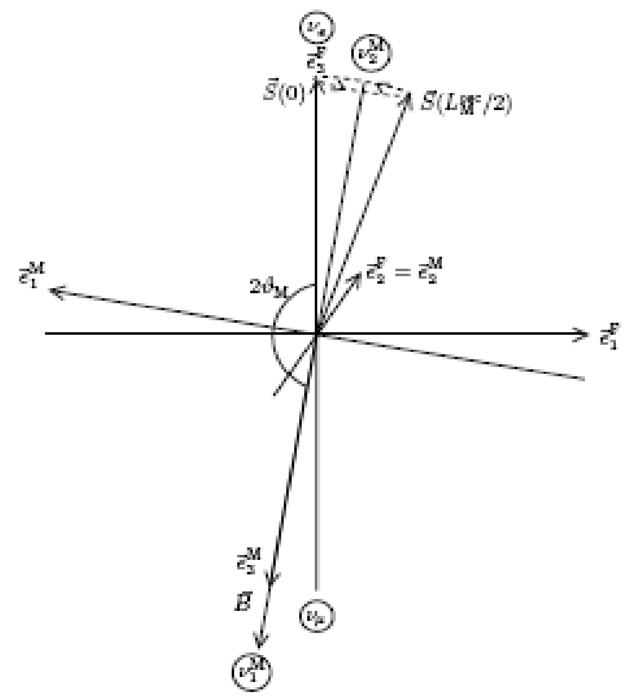
Solar case (variable matter density)

- Initially pure ν_e beam
- Very large matter density ($N_e >> N_e^R$)

→$\theta_M \approx \pi/2$

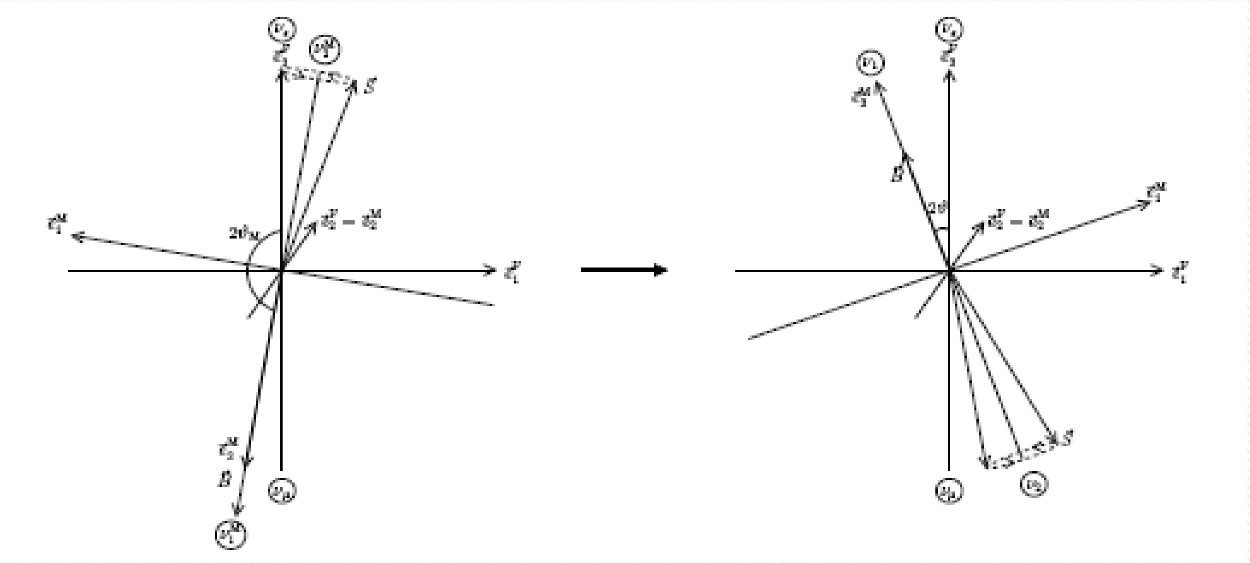
- ν_e almost coincides with ν_2^M
- **S** describes the surface on a narrow

cone around the negative z' axis



- When the ν passes through the resonance, B_x (and hence **B**) changes its value.

- If the resonance region is crossed adiabatically, the speed of rotation of **S** around **B** is much faster than the change in B.
- The cone swept by **S** is dragged by **B** and is finally rotated upside-down.
- Thus for small θ, the probability of $\nu_e \rightarrow \nu_\mu$ conversion is large



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Thank You