Four-fermi theory at a non-Lorentz invariant fixed point

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References

Most of this talk is based on an ongoing project

- with G. Mandal and S. Wadia, arXiv:0905.2928
- with G. Mandal, P. Nag and S. Wadia, work in progress

References to other works appear in the appropriate places.

Motivation

- The four-fermi theory that we will discuss is an example of a theory at a fixed point characterized by anisotropic scaling symmetry, i.e. space and time directions scale differently:

$$x \longrightarrow \lambda x\,, \quad t \longrightarrow \lambda^z t$$

– a Lifshitz-like fixed point, labeled by the exponent z

- Example: Free scalar field theory at $z = 2$:

$$\int dt\, d^d x \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x^2 \phi)^2 \right)$$

Such theories violate Lorentz invariance

Motivation

Field Theory – High Energy Physics

- New examples of ultraviolet complete field theories; Present 4-fermi model provides an UV completion of the familiar Lorentz-invariant (low-energy effective) 4-fermi field theories, like the Nambu–Jona-Lasinio model.
- In this model, fermion mass is generated dynamically and a composite scalar field arises as a collective excitation of fermions around the broken symmetry vacuum. Such a scenario raises the possibility of eliminating the Higgs field and the associated hierarchy problem.
- Of course, the main issue then is to ensure that Lorentz invariance, consistent with observational constraints, emerges at low energies.......

Motivation

String Theory

- 2-dim string theory has a nonperturbative formulation in terms of a $z = 2$ system of fermions!
- In recent studies, examples of flows between fixed-points with different values of z have been constructed in string theory:
  - Kachru, Liu and Mulligan, arXiv:0808.1725 – example of $z = 2$ theory at the boundary flowing to $z = 1$ theory (in the dual geometry) in the IR bulk.
  - Azeyanagi, Li and Takayanagi, arXiv:0905.0688 – example of a $z = 1$ theory at the UV boundary flowing to an anisotropic scale invariant theory in the IR bulk. These are constructions in Type IIB theory.

Motivation

Condensed Matter Physics

Many strongly correlated fermion systems exhibit Lifshitz type multi-critical points.

Examples:

- space-like anisotropic fixed points appear in realistic magnetic substances, e.g. MnP. These substances are modeled by an axial next-to-nearest-neighbour Ising model. A competition between the ferromagnetic nearest-neighbour and antiferromagnetic next-to-nearest-neighbour interactions (along a single lattice axis) produces a modulated phase, in addition to the usual ferromagnetic and paramagnetic ones.

Motivation

- quantum dimer models, e.g. Rokhsar-Kivelson model, which is believed to be in the universality class of the $z = 2$ scalar field theory near the so-called RK critical point. A (euclidean) Lagrangian that reproduces properties of a class of quantum dimer models is

$$L = \frac{1}{2}(\partial_\tau h)^2 + \frac{1}{2}\rho_2(\nabla h)^2 + \frac{1}{2}\rho_4(\nabla^2 h)^2 + \lambda \cos(2\pi h)$$

These models may explain some features of many systems of interest, like high T_c superconductivity, polymer physics, ferro-electric liquid crystals, etc.

OUTLINE

- Lorentz violations – observational constraints
- 4-fermi model at z=3; relevant and marginal deformations
- exact solution in the limit of a large number of species
- relevant deformation and Lorentz-invariant low-energy theory
- anomalies
- application to Standard Model

Lorentz Violations

- A general framework to discuss Lorentz violations (LV) and explore their potentially observable ramifications exists
  - Colladay and Kostelecky, hep-ph/9809521
  - Coleman and Glashow, hep-ph/9812418
- Use Standard Model as the underlying field theory framework. Assume that in the preferred frame laws of physics have exact translation and rotation invariance and that CMB is isotropic in this frame – "the rest frame of the universe"
- Assuming exact gauge invariance and renormalizability, there are 46 CPT-even independent LV parameters which preserve anomaly cancellation

Lorentz Violations

- Most of the parameters can be subsumed in a changed dispersion relation involving a separate “maximum attainable velocity” (MAV) for each paticle:

$$E_a^2 = \vec{p_a}^2 c_a^2 + m_a^2 c_a^4$$

c_a is the MAV for particle ‘a’

- Having a different MAV for different particles can have dramatic consequences – some effects abruptly turn on or off. In such a scenario, high accuracy is obtained from high energy rather than high precision.

Lorentz Violations

- Example 1: Suppose $c_\gamma > c_e$. In this case, e^+e^- pair creation is kinematically allowed for an isolated photon if

$$E_\gamma > 2m_e/\sqrt{\delta_{\gamma e}}, \qquad \delta_{\gamma e} = c_\gamma^2 - c_e^2$$

  Such photons rapidly pair create (mean free path $\sim$ few cms). Primary cosmic ray photons with energy upto 20 TeV have been detected $\implies \delta_{\gamma e} < 2 \times 10^{-15}$

- Example 2: Suppose $c_\gamma > c_{\pi^0}$. Then, the decay $\pi^0 \to 2\gamma$ of pions with energy $E_\pi > m_\pi/\sqrt{\delta_{\gamma\pi^0}}$ is kinematically forbidden! Suppose $\delta_{\gamma\pi^0} \sim 10^{-22}$. Then, primary cosmic rays with pions of energy greater than 10^{19} eV are stable and should be seen.

Lorentz Violations

- Evading the Greisen-Zatsepin-Kuzmin cut-off? Ultra high energy cosmic ray protons lose energy by inelastic scattering off the CMB photons:

$$p + \gamma(CMB) \to \Delta(1232) \to p + \pi, \qquad E_p > \frac{m_\Delta^2 - m_p^2}{2\omega}$$

For 2.73° K photons, $\omega \approx 2.35 \times 10^{-4}$ eV $\Longrightarrow$ a proton threshold energy of about 10^{20} eV.

If $c_\pi > c_p$, the threshold condition gets altered. In fact, photopion production is impossible unless $\delta_{\pi p} \leqslant \omega^2/m_\pi^2$. Using recent HiRes and Auger data, this implies (Scully and Stecker, arXiv:0811.2230) the limit $\delta_{\pi p} < 6 \times 10^{-23}$

Lorentz Violations

- Going beyond renormalizable LV terms – higher dimensional operators in an effective low energy theory approach – main effect is to change dispersion relations with terms that are higher than second power in momentum. This gives an effective MAV that changes with energy:

$$c_a(E) = c_a\left(1 + \eta_1(\frac{E}{M}) + \eta_2(\frac{E}{M})^2 + \cdots\right)$$

  H.E.S.S. collaboration has measured the scale M for photons using the linear form. They report the bound $M > 7 \times 10^{17}$ Gev

The 4-Fermi Model

- We will assume invariance under space-time translations and spatial rotations. The basic dof are then space-time fields which transform as rpns. of the rotation group. In particular, fermions are 2-component $SU(2)$ spinors.
- The dof of our model are:

  2N species of fermions $\psi_{ai}(t, \vec{x})$, $a = 1, 2; i = 1, ..., N$,

  which belong to the fundamental representation of $SU(N)$ and transform under the flavour group $U(1)_1 \times U(1)_2$:

$$\psi_{ai} \to e^{i\alpha_a}\psi_{ai}, \; a = 1, 2$$

The Model

An action which is consistent with the above symmetries is:

$$S = \int d^3\vec{x}\, dt \left[{\psi^\dagger}_{1i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma}\, \partial^2 \right) \psi_{1i} + {\psi^\dagger}_{2i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma}\, \partial^2 \right) \psi_{2i} \right.$$
$$\left. + g^2\, {\psi^\dagger}_{1i}\psi_{2i}{\psi^\dagger}_{2j}\psi_{1j} \right]$$

- Note the sign flip of the spatial derivative term between the two flavours $a = 1$ and $a = 2$; this ensures that the Lagrangian is invariant under the parity operation $\psi_{1i}(t, \vec{x}) \rightarrow \psi_{2i}(t, -\vec{x})$.
- We will study the dynamics of this action in the large N limit, holding the 'tHooft coupling $\lambda = g^2 N$ fixed.

The Model

- According to $z = 3$ scaling dimensions, $[x] = -1, [t] = -3$. It follows that $[\psi] = 3/2$. In this case, all the three terms appearing in the above action are of dimension 6 and hence marginal.
- Recall that in the usual context of a $3 + 1$ dimensional Lorentz invariant theory, any interaction involving four fermions represents an irrelevant operator and so must be understood as a low energy effective interaction.
- Here the marginality of the interaction leads one to hope that the theory is perhaps uv-complete. This is indeed the case since the four-fermi coupling turns out to be asymptotically free.

The Model

A more general $z = 3$ action with all relevant and marginal couplings, which is consistent with all the symmetries, is:

$$\begin{aligned} S \;=\; & \int d^3\vec{x}\, dt \Big[\psi^\dagger{}_{1i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma} \; \left((-i\partial)^2 + g_1 \right) + g_2(-i\partial)^2 \right) \psi_{1i} \\ & + \psi^\dagger{}_{2i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma} \; \left((-i\partial)^2 + g_1 \right) + g_2(-i\partial)^2 \right) \psi_{2i} \\ & + g_3 \left(\psi^\dagger{}_{1i}\psi_{1i} + \psi^\dagger{}_{2i}\psi_{2i} \right) + g_4^2 \left(\left(\psi^\dagger{}_{1i}\psi_{1i} \right)^2 + \left(\psi^\dagger{}_{2i}\psi_{2i} \right)^2 \right) \\ & + g_5^2 \left(\psi^\dagger{}_{1i}\psi_{1i}\psi^\dagger{}_{2j}\psi_{2j} \right) + g_6^2 \left(\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j} \right) \Big] , \end{aligned}$$

The earlier action corresponds to putting all the couplings $g_1, ..., g_5 = 0$ and setting $g_6 = g$.

Large-N Solution

- One can eliminate the 4-fermi interaction using a standard Gaussian trick. This introduces a complex scalar field and gives the following action, which is equivalent to the 4-fermi action:

$$S = \int d^3\vec{x}\, dt \left[{\psi^\dagger}_{1i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma}\partial^2 \right) \psi_{1i} + {\psi^\dagger}_{2i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma}\partial^2 \right) \psi_{2i} \right.$$
$$\left. + \phi^* {\psi^\dagger}_{1i}\psi_{2i} + \phi {\psi^\dagger}_{2i}\psi_{1i} - \frac{1}{g^2}|\phi|^2 \right]$$

- The scalar field ϕ is an $SU(N)$-singlet and is charged under the axial $U(1)$ parametrized by $\exp[i(\alpha_1 - \alpha_2)]$.

Large-N Solution

- Action is now quadratic in fermions $\Longrightarrow$ one can integrate them out $\Longrightarrow$ Effective action for the boson:

$$S_{\text{eff}}[\phi] = -iN \,\mathrm{Tr} \ln \tilde{D} - \frac{1}{g^2} \int |\phi|^2$$

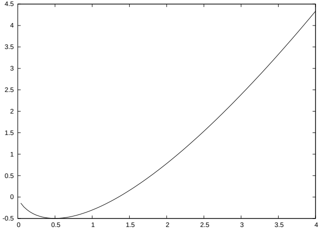
- Vacuum solutions minimize the effective potential (assuming ϕ to be real and space-time independent in the vacuum)

$$V_{\text{eff}}(\phi) = \frac{1}{g^2} \int |\phi|^2 - 2N \int \frac{dk_0 d^3\vec{k}}{(2\pi)^4} \ln(k_0^2 + |\vec{k}|^6 + |\phi|^2)$$

Large-N Solution

- With a cut-off Λ in place, we get

$$V_{\text{eff}}(\phi) = N|\phi|^2 \left(\frac{1}{\lambda} - \frac{1}{12\pi^2} \ln\left(\frac{\Lambda^6}{|\phi|^2}\right) - \frac{1}{12\pi^2} \right) + \text{constant}$$



At the minimum, $|\phi| = m^3 = \Lambda^3 \exp[-6\pi^2/\lambda]$

- The treatment of the effective potential and the RG flow presented above is exact in the strict $N = \infty$ limit.

Large-N Solution

- $V_{\rm eff}(\phi)$ should be cut-off independent. Define a (cut-off independent) running coupling:

$$\frac{1}{\lambda(\mu)} = \frac{1}{\lambda(\Lambda)} - \frac{1}{2\pi^2}\ln(\frac{\Lambda}{\mu}),$$

with beta-function

$$\beta(\lambda(\mu)) = \mu\frac{d\lambda(\mu)}{d\mu} = -\frac{\lambda(\mu)^2}{2\pi^2}$$

Here μ is an arbitrary scale. Then,

$$V_{\rm eff}(\phi) = N|\phi|^2\left(\frac{1}{\lambda(\mu)} - \frac{1}{12\pi^2}\ln(\frac{\mu^6}{|\phi|^2}) - \frac{1}{12\pi^2}\right)$$

Deformations

- Action with all possible terms consistent with symmetries:

$$
\begin{aligned}
S \;=\; & \int d^4x \left[\psi^\dagger{}_{1i}\left(i\partial_t - i\vec{\partial}.\vec{\sigma}\,(-i\partial)^2 - g_1 i\vec{\partial}.\vec{\sigma} \;+ g_2(-i\partial)^2\right)\psi_{1i}\right. \\
& +\psi^\dagger{}_{2i}\left(i\partial_t + i\vec{\partial}.\vec{\sigma}\,(-i\partial)^2 - g_1 i\vec{\partial}.\vec{\sigma} \;+ g_2(-i\partial)^2\right)\psi_{2i} \\
& +g_3\left(\psi^\dagger{}_{1i}\psi_{1i} + \psi^\dagger{}_{2i}\psi_{2i}\right) + g_4^2\left(\left(\psi^\dagger{}_{1i}\psi_{1i}\right)^2 + \left(\psi^\dagger{}_{2i}\psi_{2i}\right)^2\right) \\
& \left.+g_5^2\left(\psi^\dagger{}_{1i}\psi_{1i}\psi^\dagger{}_{2j}\psi_{2j}\right) + g_6^2\left(\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j}\right)\right],
\end{aligned}
$$

- Marginal couplings $\Longrightarrow g_4,\; g_5,\; g_6$
- Relevant couplings $\Longrightarrow g_1,\; g_2,\; g_3$

Deformations

- In the treatment so far we have set $g_1 = \cdots = g_5 = 0$, $g_6 = g$, but the full system can also be analysed exactly in the large-N limit.

- $g_1 = 0$ – new fermion condensates appear, $< \psi^\dagger{}_{ai} \psi_{ai} > \neq 0$, but all the new marginal couplings remain exactly marginal $\Longrightarrow$ vacuum solution remains unchanged.

- $g_1 \neq 0$ – this relevant deformation is

$$g_1 \left(\psi^\dagger{}_{1i} (-i\vec{\partial}.\vec{\sigma}) \psi_{1i} + \psi^\dagger{}_{2i} (i\vec{\partial}.\vec{\sigma}) \psi_{2i} \right)$$

- The operator multiplying g_1 is the usual spatial derivative term of the Dirac action. At $z = 3$, g_1 is a relevant coupling with dimension 2. This deformation causes flow to the Lorentz invariant $z = 1$ fixed point.

Restoration of Lorentz invariance

- A quick way of seeing this is the following. Assume $g_1 = M^2 > 0$. (g_1 can be negative – different phases for different signs in other examples.) Now look at the fermion dispersion relation, which is exact in large N:

$$k_0^2 - k^2(k^2 + M^2)^2 - |\phi|^2 = 0$$

For $k << M$, this is approximately $k_0^2 - k^2 M^4 - |\phi|^2 = 0$

- Now, rescale energy $k_0 = k_0' M^2$ (equivalently rescale time, $k_0/M^2 = \frac{i}{M^2}\frac{\partial}{\partial t} = i\frac{\partial}{\partial t'} = k_0'$). The new dispersion relation is the standard Lorentz-invariant mass-shell condition:

$$(k_0')^2 = (k^2 + {m_*}^2), \qquad m_* = |\phi|/M^2$$

RG flow to low energies

- An appropriate framework for systematically deriving the low energy theory is Wilson's RG.
- In our case, the natural cut-off scale for the low energy theory is the scale M that determines the lorentz violations. Thus, we will integrate out the modes of ψ_{ia} and ϕ fields with energies above M.
- In the large N limit this program can be explicitly carried out exactly

Low energy theory

- Step 1: Parametrize ϕ as

$$\phi(t,\vec{x}) = \left(\phi_0 + \frac{\sigma(t,\vec{x})}{\sqrt{N}}\right) e^{i\pi(t,\vec{x})/\sqrt{N}}$$

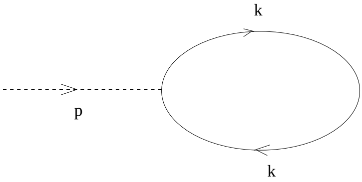
- The Nambu-Goldstone mode is the phase $\pi(t,\vec{x})$. It can be absorbed into fermions by an axial $U(1)$ rotation and disappers from the Yukawa coupling. But then it reappears through the fermion kinetic terms.

- The action is

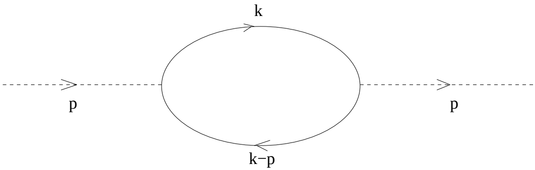
$$S = \int d^4x \left[\bar{\Psi}_i \left(i\gamma^0\partial_t + i(\vec{\gamma}.\vec{\partial})((i\vec{\partial})^2 + M^2) + \phi_0 + \frac{\sigma(t,\vec{x})}{\sqrt{N}} \right.\right.$$
$$\left.\left. +\gamma^0\gamma^5\frac{\partial_t\pi(t,\vec{x})}{2\sqrt{N}} + \cdots\right)\Psi_i - \frac{1}{g^2}\left(\phi_0 + \frac{\sigma(t,\vec{x})}{\sqrt{N}}\right)^2\right]$$

Low energy theory

- Step 2: The leading contributions in $1/N$ arise from
  - 1-point function of σ:



  - 2-point functions of σ and π:



Low energy theory

- Step 3 In the effective low energy action (the action with M as the cut-off) make the following rescalings

$$t \to t/M^2, \qquad \sigma(t,\vec{x}) \to M^2\sigma(t,\vec{x}).$$

Then, the effective action takes the schematic form

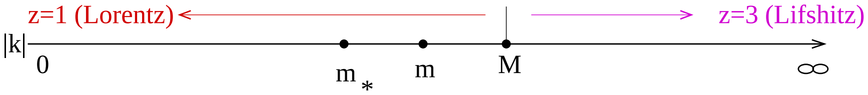
$$S = \int d^4x \left[\bar{\Psi}_i \left(i\gamma^\mu \partial_\mu + m_*\right) \Psi_i + \text{Yukawa terms} \right.$$
$$+c\partial_\mu\sigma(t,\vec{x})\partial^\mu\sigma(t,\vec{x}) - V(\sigma(t,\vec{x})) + \text{pion kinetic term}$$
$$\left. +\text{Lorentz violating terms} \sim (E^2/M^2, m_*^2/M^2)\right]$$

Low energy theory

- To leading order in $1/N$, the scale M does not get renormalized. Renormalization of M involves small $1/N$ corrections. A related point is that the renormalizations are differnt for ψ and σ fields.
- There are Lorentz violating terms present in the low-energy theory, given by dimension ≥ 4 operators. These lead to different and scale-dependent MAVs for ψ and σ fields. Thus, the structure of Lorentz violations that emerges is exactly like that described in the general scheme earlier.
- The Lorentz violations can be computed precisely and subjected to observational constraints (Iengo, Russo and Serone, arXiv:0906.3477)

Low energy theory

Observational constraints require the Lorentz-invariant mass scale m_* to be much smaller than M. This implies $|\phi_0| = m^3 << M^3$. This results in the hierarchy of scales



Application to particle physics

Anomalies

- Minimal coupling of fermions to a $U(1)$ gauge field

$$S = \int d^3x\, dt\, \left(\bar{\Psi}_i i \not{\mathbf{D}} \Psi_i + \cdots\right)$$

$$\not{\mathbf{D}} = \gamma^\mu \mathbf{D}_\mu,\ \mathbf{D}_t = D_t,\ \mathbf{D}_i = -D_i(\vec{D})^2,\ D_\mu = \partial_\mu + iA_\mu$$

- As in the Lorentz-invariant case, this theory has global axial $U(1)$ symmetry

$$\delta\Psi_i = i\alpha(x)\gamma^5\Psi_i,\ \delta\bar{\Psi}_i = \bar{\Psi}_i\, i\alpha(x)\gamma^5$$

Is the axial current conserved? Note that the current has a complicated expression in terms of fields.

Application to particle physics

- Use Fujikawa's argument and heat kernel method of regularization to compute the anomaly:

$$\partial_\mu J^{\mu 5} = 2\mathrm{Tr}\left(\gamma^5 \exp[i\not{D}^2/\Lambda^6]\right)$$

- The final answer is exactly as in the Lorentz-invariant case:

$$\partial_\mu J^{\mu 5} = -\frac{N}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

Application to particle physics

The Model

- We will now consider a simple extension of the system to describe Higgs mechanism in which the Higgs field is a composite object.

- The degrees of freedom of the extended model are:

$$\psi_{i\alpha}, \qquad \chi_i, \qquad \alpha = 1, 2$$

  $\psi_{i\alpha}$ transforms as the fundamental repn. of $SU(2)$ and χ_i as a singlet. The index α is gauged.

- The 4-fermi interaction

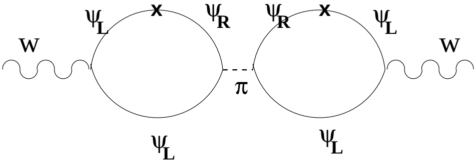
$$(\psi^{\dagger}{}_{i\alpha}\chi_i)(\chi_i^{\dagger}\psi_{i\alpha})$$

  leads to breaking of the $SU(2)$ symmetry.

Application to particle physics

- The bilinear order parameter $\chi_i^\dagger \psi_{i\alpha} \equiv \phi_\alpha$ acts as (a composite) Higgs field because its $SU(2)$ phase is eaten up by the gauge fields to which the $\psi_{i\alpha}$ couple.
- The key point is that gauge field masses arise from their gauge-invariant interactions with the ψ's because of the exchange of the would-be Nambu-Goldstone bosons, the "pions", $\pi(t,x)$, a well-known mechanism originally discovered in the context of Meissner effect.

Application to particle physics



- In addition, we have the usual quarks and leptons, appropriately coupled to the above fermions:

$$({\psi^\dagger}_{i\alpha}\chi_i)(q_2^\dagger q_{1\alpha}) + \mathrm{h.c.}, \qquad \epsilon^{\alpha\beta}({\psi^\dagger}_{i\alpha}\chi_i)(q_{1\beta}^\dagger q_2') + \mathrm{h.c.}$$

which is equivalent, after symmetry breaking, to the standard Yukawa couplings, $(\phi_\alpha^* q_2^\dagger q_{1\alpha} + \mathrm{h.c.})$ and $(\epsilon^{\alpha\beta}\phi_\alpha^* q_{1\beta}^\dagger q_2' + \mathrm{h.c.})$.

Application to particle physics

- After mass generation and Higgs mechanism, at low energies, the $\psi_{i\alpha}$ and χ_i combine to give massive fermions. It may be possible to arrange their masses to be sufficiently high, consistent with phenomenological constraints.
- Actually two χ_i's are needed to give mass to both components of $\psi_{i\alpha}$. Assuming that the pair $\chi_{ia},\ a = 1,\ 2$ transforms as a doublet of another $SU(2)$, one can arrange the four-fermi interactions to be invariant under this "custodial" $SU(2)$.
- A consistent application of these ideas to the entire Sandard Model will require extension of this work to the pure gauge sector at $z = 3$ and its RG flow down to low energies.

Summary

- Fermions at $z = 3$ in 4-dims have an asymptotically free four-fermi coupling. Hence, this theory provides an UV completion of relativistic effective four-fermi theories at low energies, without introducing new degrees of freedom.
- Our example has dynamical mass generation and a composite Higgs field. This could eliminates the need for a Higgs potential, thus avoiding the hierarchy problem.
- Phenomenologically, the key issue is that of acceptable Lorentz violations at low energies. This seems to require fine tuning.
- Applications to condensed matter physics may be more feasible at present. Combined with AdS/CFT, this could provide a powerful tool for solving problems in this area.