Recursive Structures in String Theory

Anirban Basu

Institute of Physics
Bhubaneswar

9.6.10

- It has been long believed that there could be a unified description of all phenomena in nature. This line of thought has led to several concrete steps in this direction, each has provided remarkable insight into the laws of nature.
- At the level of our present understanding, there are three forces in nature, which govern all interactions: gravitation, electroweak and strong.
- Gravitation was the earliest force to be analyzed, and it unified phenomena as diverse as an apple falling on the ground, and the motion of planets around the sun.

- The theory of electroweak interactions unified electromagnetism (which in turn unified electricity and magnetism, thus putting charges at rest and in motion on an equal footing) with the weak interactions, providing explanations for many phenomena, such as beta decay.
- Strong interactions explain the stability of matter, and govern the internal dynamics of mesons and baryons, by giving the theory of interaction of quarks and antiquarks with gluons.

- It is plausible that these forces can be unified into a common framework, thus providing a unified theory for all observed phenomena. The challenge is to find this unified quantum theory.
- Apart from gravity, the other interactions can be well understood at the quantum level (apart from some issues like confinement, which can be understood at a qualitative level). This is because these interactions are renormalizable.
- Gravity, however, is not renormalizable. Thus a quantum theory of gravity must be a finite theory.

- String theory is a promising candidate for being the unified theory of all interactions. It essentially replaces point particles by strings (open and closed), and the vibrations of these strings produce the particle content of our universe. Among its massless modes, string theory has gravitons as well as gauge fields. It is also a perturbatively finite theory. So let us take it seriously.
- We shall be interested in these massless modes today. We shall try to understand how they encode the structure of scattering amplitudes in string theory, and hence provide us with a valuable tool to understand perturbative as well as non-perturbative aspects of the theory.

- The principal aim is to construct the effective action (EA) of string theory. This is the action whose degrees of freedom are the massless modes of the theory.
- Since in constructing the effective action, one has to integrate over the various modes of the theory propagating in loops when one goes beyond the classical regime, one has to specify what are the modes that are being integrated over.

- Since these are the modes that are running in the string loops, there is no natural way to decouple massless from massive modes when one is calculating a loop diagram in string perturbation theory. Hence in construcing the effective action, one naturally allows all modes to propagate in the loops, including the massless ones.

- Thus, to summarize, terms in the effective action represent (at the perturbative level) string amplitudes, where the external on-shell modes (represented by vertex operators on the world sheet) are massless, while all modes are allowed to propagate in the loops.
- Since this procedure of constructing the effective action does not separate massive from massless modes in any way, the whole procedure is invariant under duality symmetries of the theory. Hence, the equations of motion following from the effective action are duality covariant.

- There is a very nice and useful interplay between world sheet scattering amplitudes and the spacetime effective action.
- The spacetime effective action describes the dynamics of the low lying degrees of freedom at large distance scales. Thus at the two derivative level, it must reduce to (the supersymmetric completion of) General Relativity, and it does. The higher derivative corrections encode the stringy degrees of freedom, and thus the effective action is a perturbative expansion in α', the inverse string tension.

- Importantly, at a fixed order in the α' expansion, it is exact in g_s, the string coupling, and hence contains complete information about the perturbative as well as non–perturbative contributions to that scatterimg amplitude.
- To summarize, at a fixed order in the α' expansion, the effective action contains information to all orders in g_s.

- Now consider a string scattering amplitude at a fixed order in the genus expansion (i.e., at a fixed order in g_s).
- This gives the complete amplitude to all orders in α', the inverse string tension. This follows by taking the amplitude and performing a momentum expansion. These terms yield the higher derivative corrections to the two derivative action in the spacetime effective action, of course, at a fixed order in the genus expansion.

- To summarize, a string scattering amplitude at a fixed order in the genus expansion, provides information to all orders in the α' expansion.
- Thus an interplay between the spacetime effective action and world sheet amplitudes provides strong constraints on the structure of the effective action of string theory. This can be successfully implemented in certain cases. We shall consider one case in detail today.

- We shall consider some terms in the effective action of type IIB string theory in ten spacetime dimensions. These are a very special set of terms in the effective action, in the sense that they receive only a finite number of perturbative contributions. Thus, they are called "protected" interactions.
- This case turns out to be tractable because of the high amount of supersymmetry as well as duality. So, let us recall some facts about type IIB string theory, which are relevant for our purposes.

- Type IIB string theory is maximally supersymmetric, i.e., it preserves 32 real supercharges. It is a chiral theory, and has $(2,0)$ spacetime supersymmetry. Thus it has 2 gravitini, both of which have the same chirality. However, this theory is free of anomalies, and hence is a consistent quantum theory.

- Since we are dealing with the effective action, let us consider the massless modes of the theory. Among the bosonic degrees of freedom, they are the graviton $g_{\mu\nu}$, a complex scalar field τ, a complex two form potential $B_{\mu\nu}$, and a four form potential $C_{\mu\nu\rho\sigma}$, with a self–dual five form field strength. The fermionic degrees of freedom are given by the Weyl gravitini ψ_μ and the dilatini λ. The gravitini and the dilatini have opposite chiralities.

- The theory has a conjectured $SL(2,\mathbb{Z})$ duality symmetry, under which

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

where

$$a, b, c, d \in \mathbb{Z}, \quad \text{and} \quad ad - bc = 1.$$

- The complex scalar field (modulus) τ is actually given by

$$\tau = C_0 + ie^{-\phi},$$

where C_0 is a RR scalar, and $\langle e^{\phi}\rangle = g_s$.

- Thus, when $C_0 = 0$, the transformation $\tau \to -1/\tau$ implies that

$$g_s \to \frac{1}{g_s}.$$

Thus the theory has strong–weak coupling duality.

- The various other fields also transform non–trivially under $SL(2,\mathbb{Z})$.

- We shall exploit the maximal supersymmetry and the $SL(2,\mathbb{Z})$ invariance of the theory a great deal in this talk. So let us write down some formulae which will be relevant for our purposes.
- After transforming to the Einstein frame, such that the Einstein–Hilbert term is canonically normalized (as opposed to the string frame), the various massless fields transform non–trivially under $SL(2,\mathbb{Z})$, which we shall discuss shortly.

- Supersymmetry imposes strong constraints on the possible interactions in the effective action. The first correction to the two derivative terms in the effective action (i.e., supergravity) are eight derivative terms. (Contrast this to theories with 16 supersymmetries, where the first corrections have 4 derivatives.)

- For the sake of simplicity, we shall only look at a subset of interactions in the effective action which are composed of the graviton, the gravitini and the dilatini. Thus a generic interaction we shall consider is of the form

$$\int d^{10}x\sqrt{-g}f(\tau,\bar{\tau})\mathcal{O}(x),$$

where $\mathcal{O}$ is a composite operator constructed out of $g_{\mu\nu},\psi_\mu$ and λ. The aim is to determine $f(\tau,\bar{\tau})$ exactly.
- It turns out that $f(\tau,\bar{\tau})$ is a "modular form" of $SL(2,\mathbb{Z})$ which we shall soon talk about. Its "weight" is fixed uniquely given the $U(1)$ charges of the various massless modes. (This $U(1)$ symmetry is a local symmetry of type IIB supergravity, but not of type IIB string theory.)

- In the Einstein frame, the $U(1)$ charges of the relevant fields are given by

$$U(1)_g = 0, \quad U(1)_\lambda = 3/2, \quad U(1)_\psi = 1/2.$$

- If the operator $\mathcal{O}$ has $U(1)$ charge q, then $f(\tau,\bar{\tau})$ is a modular form of weight $(q/2, -q/2)$.

- We shall look at certain operators in the effective action which are of the form described above.
- These are operators of the form $D^{2k}\mathcal{R}^4$ for $k = 0, 2, 3$. Here $\mathcal{R}^4$ is a $(\text{curvature})^4$ coupling which involves only the Weyl tensor (which is the traceless part of the Riemann tensor), while D^{2k} stands for derivatives with appropriate contractions. Thus, at the linearized level, this involves the four graviton amplitude. These are interactions at the 8, 12 and 14 derivative level in the effective action.
- At the 8 derivative level, we shall also look at interactions of the form λ^{16} and $\lambda^{15}\gamma^\mu\psi^*_\mu$, which are the maximally fermionic interactions.

- At the 12 derivative level, the similar interactions we shall consider are $\hat{G}^4\lambda^{16}$ and $\hat{G}^4\lambda^{15}\gamma^\mu\psi^*_\mu$, where

$$\hat{G}_{\mu\nu\rho} \sim \bar{\psi}^*_{[\mu}\gamma_\nu\psi_{\rho]} + \bar{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda + \ldots,$$

is the supercovariant field strength.

- The coefficients of these interactions are modular forms. Their weights are fixed by the above discussion. They are

$$[\mathcal{R}^4] = (0,0), \quad [\lambda^{16}] = (12,-12), \quad [\lambda^{15}\gamma^\mu\psi^*_\mu] = (11,-11),$$

at the 8 derivative level,

$$[D^4\mathcal{R}^4] = (0,0), \quad [\hat{G}^4\lambda^{16}] = (14,-14),$$

$$[\hat{G}^4\lambda^{15}\gamma^\mu\psi^*_\mu] = (13,-13),$$

at the 12 derivative level, and

$$[D^6\mathcal{R}^4] = (0,0),$$

at the 14 derivative level.

- These coefficients can be exactly calculated. One way to calculate them is using the equivalence of M theory on T^2 when the volume of $T^2 \to 0$, and type IIB string theory in 10 dimensions. The four graviton amplitude can be calculated to low loop orders exactly in $d = 11$ supergravity, and one can use duality to regularize the divergences, which then lead to the couplings we need in type IIB string theory. The maximally fermionic couplings can then be deduced using linearized supersymmetry and $SL(2,\mathbb{Z})$ duality.

- Alternatively, they can be calculated using the Noether procedure order by order in the α' expansion, taking into account the corrections to the classical supersymmetry transformations as well.
- To analyze these coefficients, let us define the relevant modular forms.

- Under $SL(2,\mathbb{Z})$ transformations, when

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d},$$

a modular form $\Phi^{(m,n)}(\tau,\bar{\tau})$ of weight (m,n) transforms as

$$\Phi^{(m,n)}(\tau,\bar{\tau}) \rightarrow (c\tau + d)^m (c\bar{\tau} + d)^n \Phi^{(m,n)}(\tau,\bar{\tau}).$$

- Let us also define certain weight $(0,0)$ modular forms, called the non–holomorphic Eisenstein series, by

$$E_s(\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}.$$

They satisfy the Poisson equation on moduli space, since

$$4\tau_2^2 \frac{\partial^2 E_s}{\partial\tau\partial\bar{\tau}} = s(s-1)E_s.$$

- Another important property of the Eisenstein series is their large τ_2 (small g_s) expansion. Using Poisson resummation, one finds that there are only two perturbative contributions, given by

$$E_s(\tau,\bar{\tau}) = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(s)}\tau_2^{1-s} + \ldots,$$

while the non–perturbative contributions are given by

$$\ldots + \frac{2\pi^s\sqrt{\tau_2}}{\Gamma(s)}\sum_{m\neq 0,n\neq 0}\Big|\frac{m}{n}\Big|^{s-1/2}K_{s-1/2}(2\pi|mn|\tau_2)e^{2\pi i mn\tau_1}.$$

- To see that these terms are indeed non-perturbative in nature, we expand for weak coupling (use $K_s(x) \sim e^{-x}$), to get the (anti) D–instanton action

$$e^{2\pi ik\tau},$$

for $k \in \mathbb{Z}$.

- Thus the Eisenstein series has 2 perturbative contributions, as well as fluctuations in D–instanton backgrounds. What has all this got to do with the couplings in the effective action?

- The protected couplings, it turns out, are intimately related to the Eisenstein series. In fact, the $\mathcal{R}^4$ coupling is given by

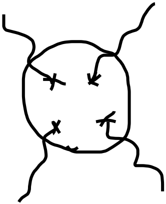
$$E_{3/2}(\tau,\bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\sqrt{\tau_2}} + \ldots,$$

while the $D^4\mathcal{R}^4$ coupling is given by

$$E_{5/2}(\tau,\bar{\tau}) = 2\zeta(5)\tau_2^{5/2} + \frac{4\pi^4}{135\tau_2^{3/2}} \ldots .$$

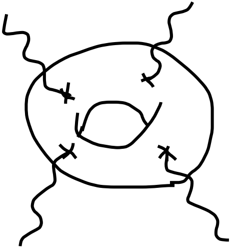
- Thus the $\mathcal{R}^4$ coupling receives perturbative contributions only at tree level and at one loop, while the $D^4\mathcal{R}^4$ coupling receives contributions only at tree level and at two loops.

$R^4 \sim$



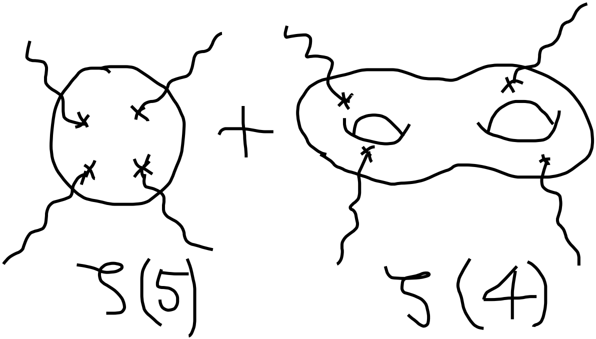
$\zeta(3)$

$+$



$\zeta(2)$

$D^4 R^4 \sim$



- Note that these protected interactions receive only a finite number of perturbative contributions.
- Their coefficients are modular forms that satisfy the Laplace equation on moduli space.

- What about the λ^{16} and the $\lambda^{15}\gamma^\mu\psi^*_\mu$ couplings that are in the same supermultiplet as the $\mathcal{R}^4$ coupling?
- They are simply given by appropriate number of modular covariant derivatives acting on $E_{3/2}$, which we now define.

- The modular covariant derivatives D_m and $\bar{D}_n$ are defined by

$$D_m = i\Big(\tau_2 \frac{\partial}{\partial\tau} - \frac{im}{2}\Big),$$

and

$$\bar{D}_n = -i\Big(\tau_2 \frac{\partial}{\partial\bar{\tau}} + \frac{in}{2}\Big),$$

respectively.

- When acting on modular forms, they change the modular weights, according to

$$D_m \Phi^{(m,n)} \longrightarrow \Phi^{(m+1,n-1)}, \quad \bar{D}_n \Phi^{(m,n)} \longrightarrow \Phi^{(m-1,n+1)}.$$

- Thus the λ^{16} and $\lambda^{15}\gamma^\mu\psi^*_\mu$ couplings are given by

$$D_{11}\ldots D_0 E_{3/2},$$

and

$$D_{10}\ldots D_0 E_{3/2},$$

respectively.
- Similarly, the $\hat{G}^4\lambda^{16}$ and $\hat{G}^4\lambda^{15}\gamma^\mu\psi^*_\mu$ couplings are given by

$$D_{13}\ldots D_0 E_{5/2},$$

and

$$D_{12}\ldots D_0 E_{5/2},$$

respectively.

- Let us consider the $D^6\mathcal{R}^4$ interaction given by

$$\int d^{10}x\sqrt{-g}f(\tau,\bar{\tau})D^6\mathcal{R}^4.$$

- $f(\tau,\bar{\tau})$ satisfies the equation

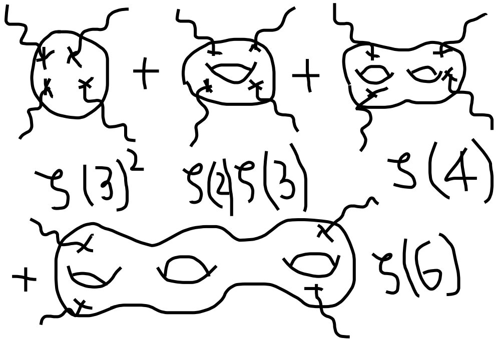
$$4\tau_2^2\frac{\partial^2 f}{\partial\tau\partial\bar{\tau}} = 12f - 6E_{3/2}^2.$$

- Thus, unlike the previous couplings, it satisfies Poisson equation on moduli space, with a source term that is given by the coefficient of the $\mathcal{R}^4$ interaction.

- This equation can be solved, and $f(\tau,\bar{\tau})$ receives perturbative contributions only upto three loops. More explicitly,

$$f(\tau,\bar{\tau}) = 4\zeta(3)^2\tau_2^3 + 8\zeta(2)\zeta(3)\tau_2 + 24\zeta(4)\tau_2^{-1} + \frac{8}{9}\zeta(6)\tau_2^{-3}.$$

$D^6\mathcal{R}^4 \sim$



- The various protected interactions we have discussed so far exhibit some definite patterns. They all receive only a finite number of perturbative contributions. Their coefficients satisfy Laplace or Poisson equation on moduli space. This leads to the natural question:
- Can something be said about a generic term in a certain class of such protected interactions? We now explain some aspects of this is some detail.

- In general, it is quite complicated to go to arbitrary orders in the derivative expansion, and find out special properties of interactions in the effective action. However, because of the large amount of supersymmetry the theory enjoys, it turns out that for certain classes of interactions, the analysis is under control.
- These are terms in the effective action with maximal number of fermions at each order in the derivative expansion. When all fermions have been exhausted, the relevant terms in the subleading order in the derivative expansion is obtained by multiplying bosonic fields to the maximally fermionic ones.

- Thus, starting with these interactions, one can show that only a few of them mix with each other under tree level supersymmetry transformations. This also receives contributions from corrected supersymmetry transformations acting on the supergravity action. Finally, there are also contributions coming from terms in the effective action at intermediate orders in the derivative expansion, by the action of corrected supersymmetry transformations at the appropriate order.

- This can be succintly described by the equation

$$\delta S = (\sum_m \delta^{(m)})(\sum_n S^{(n)}) = 0.$$

- Thus this procedure essentially amounts to implementing the Noether procedure order by order in the derivative expansion, coupled with the constraints obtained by using the closure of the supersymmetry algebra.

- This analysis tells us that interactions of the form

$$\hat{G}^{2k}\lambda^{16}, \quad \hat{G}^{2k}\lambda^{15}\gamma^{\mu}\psi^*_{\mu}, \quad \hat{G}^{2(k-1)}(\hat{G}\cdot\hat{G}^*)\lambda^{16}$$

  fall in this class.
- The coefficients of these interactions are modular forms of weights $(12+k, -12-k)$, $(11+k, -11-k)$ and $(11+k, -11-k)$ respectively. In fact the last two modular forms are the same.
- Each of these couplings satisfy Poisson equation on moduli space, with source terms that are given by the couplings of lower order interactions. Thus, we see that the origin of Poisson equations on moduli space is spacetime supersymmetry.

- Thus one can solve for these coefficients recursively starting from small values of k. One can also show that these interactions receive only a finite number of perturbative contributions.
- Thus there are an infinite number of protected interactions in the effective action of type IIB string theory.

- Much remains to be said and done regarding the construction of effective actions, given a string compactification. We only discussed the simplest case of uncompactified string theory with maximal supersymmetry with a very simple moduli space. We also did not discuss at all the complicated case of non–local interactions.
- Compactifications which preserve sufficient supersymmetry are also sometimes amenable to exact treatment, in the sense that there are an infinite set of ptotected interactions which can be determined exactly.

- Examples are provided by $N = 2$ string theories in 4 dimensions, which can be realized by compactifying type IIA (or type IIB) string theory on Calabi–Yau threefolds. Certain interactions involving the graviphoton and the graviton can be determined to all orders in the derivative expansion.
- Clearly a lot remains to be done.

- Because the string effective action describes the dynamics of the massless modes of the string, it can be used in various aspects of string theory.
- One particular application is in the direction of the gauge/gravity duality. These higher derivative corrections give the large (but finite) ’t Hooft coupling corrections in the large N limit. (Supergravity gives the result for infinite ’t Hooft coupling.)

- Another useful application is in the area of black holes. Higher derivative corrections change the classical geometry obained at the level of supergravity, and leads to a detailed understanding of the thermodynamic properties of black holes.
- These corrections are also relevant when one studies string phenomenology. This is simply because one has to go beyond the two derivative action to understand in detail the structure and consequences of any proposed model.