Making sense of non-Hermitian Hamiltonians

Crab Lender
Practised Nymphets
Washing Nervy Tuitions

Making sense of non-Hermitian Hamiltonians

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Washington University

Quantum Mechanics

- "Anyone who thinks he can contemplate quantum mechanics without getting dizzy hasn't properly understood it." – Niels Bohr
- "Anyone who thinks they know quantum mechanics doesn't." – Richard Feynman
- "I don't like it, and I'm sorry I ever had anything to do with it." – Erwin Schrödinger

Axioms of Quantum Mechanics

- Causality
- Locality
- Stability of the vacuum
- Relativity
- Probabilistic interpretation

Dirac Hermiticity …

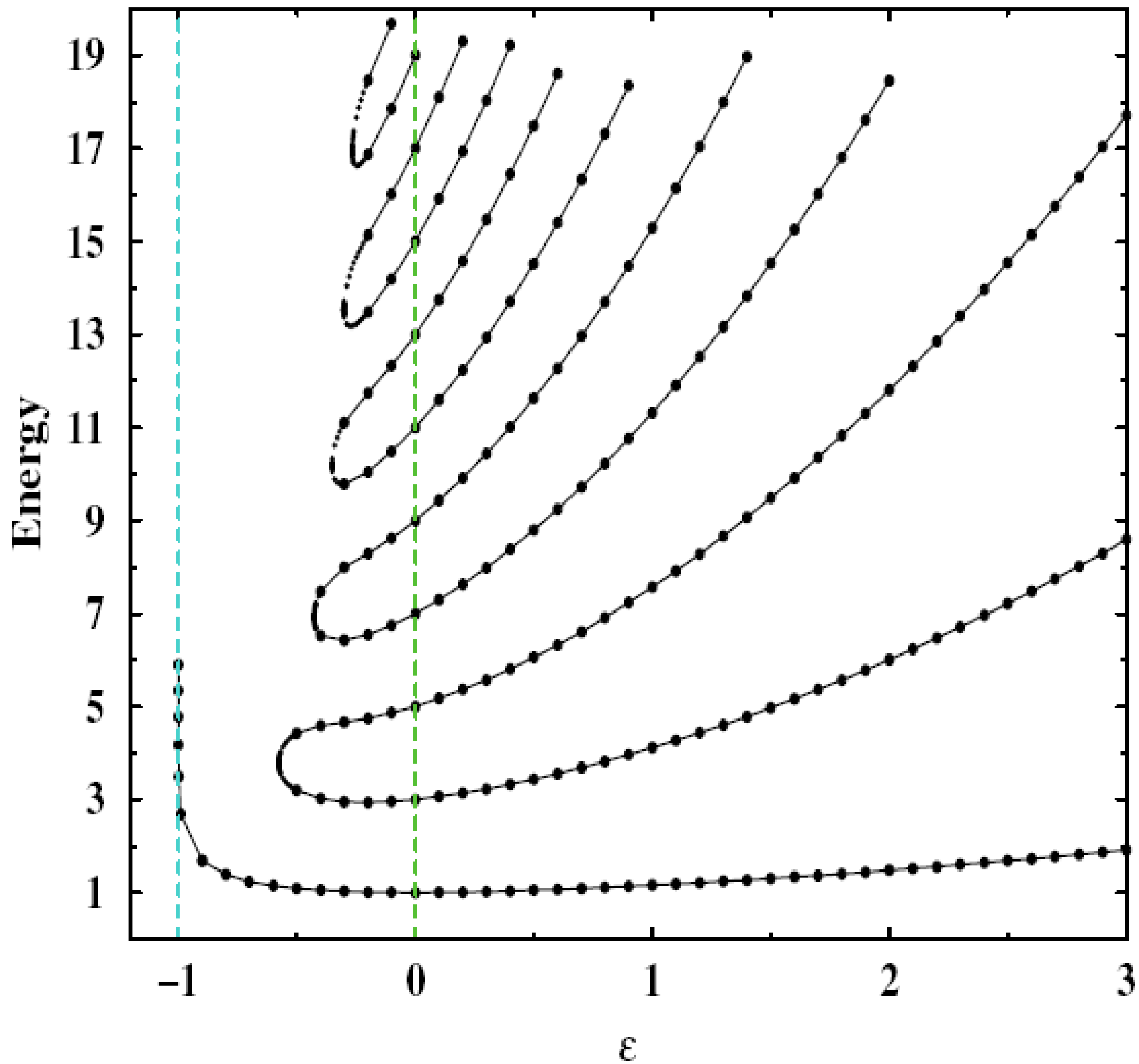
- guarantees real energy and conserved probability
- but … is mathematical and not physical

$$H = p^2 + ix^3$$

$$H = p^2 + ix^3$$

Wait a minute…
This model has
PT symmetry

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Some references …

- CMB and S. Boettcher, PRL **80**, 5243 (1998)
- CMB, D. Brody, H. Jones, PRL **89**, 270401 (2002)
- CMB, D. Brody, and H. Jones, PRL **93**, 251601 (2004)
- CMB, D. Brody, H. Jones, B. Meister, PRL **98**, 040403 (2007)
- CMB and P. Mannheim, PRL **100**, 110402 (2008)
- CMB, Reports on Progress in Physics **70**, 947 (2007)
- P. Dorey, C. Dunning, and R. Tateo, JPA **34**, 5679 (2001)
- P. Dorey, C. Dunning, and R. Tateo, JPA **40**, R205 (2007)

The original discoverers of PT symmetry:


JONES
BERRY
MEISINGER
MILTON
BOETTCHER
DUNNE
SAVAGE
WANG
The PT BUNCH


PT
Posteitaliane
POSTAMAT
SE VOCÊ
É PT DE CORAÇÃO,
QUERO FALAR
COM VOCÊ.
PT

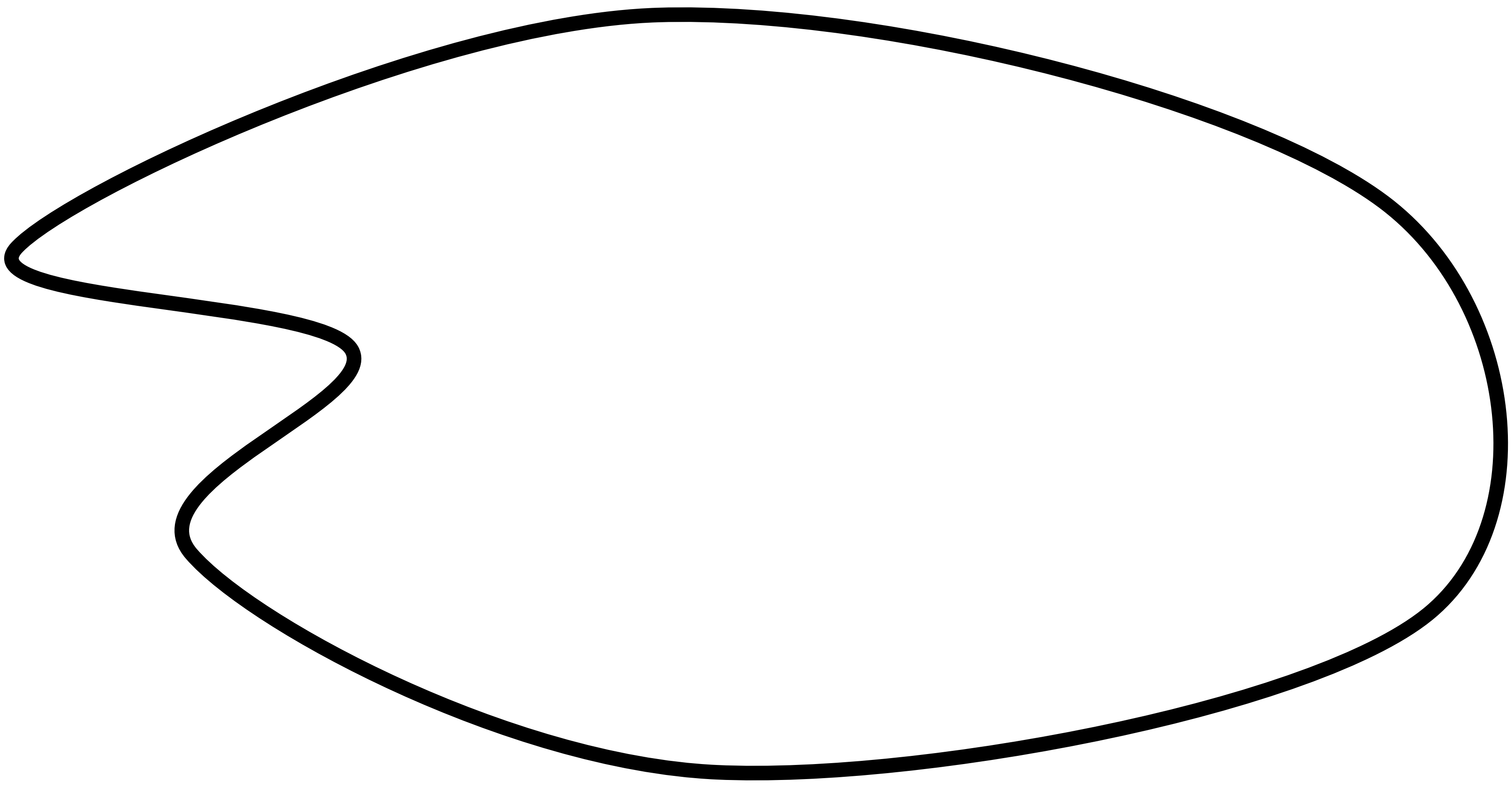

PALAZZO
PITTI
CORTILE
DELL'AMMANNATI
2
1
INGRESSO PRINCIPALE
MAIN ENTRANCE
1 BIGLIETTERIA
TICKETS
2 INGRESSO MUSEI
MUSEUMS ENTRANCE
PT
S-MAX. Your life.


Pt Lake
TRAILHEAD
SAN ISABEL
National Forest
FOREST SERVICE
U S
DEPARTMENT OF AGRICULTURE
U.S. DEPARTMENT·OF·AGRICULTURE


PT HIGHWAY
LITTER CONTROL
NEXT 2 MILES
SEROPTIMIST INT
OF SARATOGA


PT OF
INTEREST


PT CRUISER
PT
SE VOCÊ



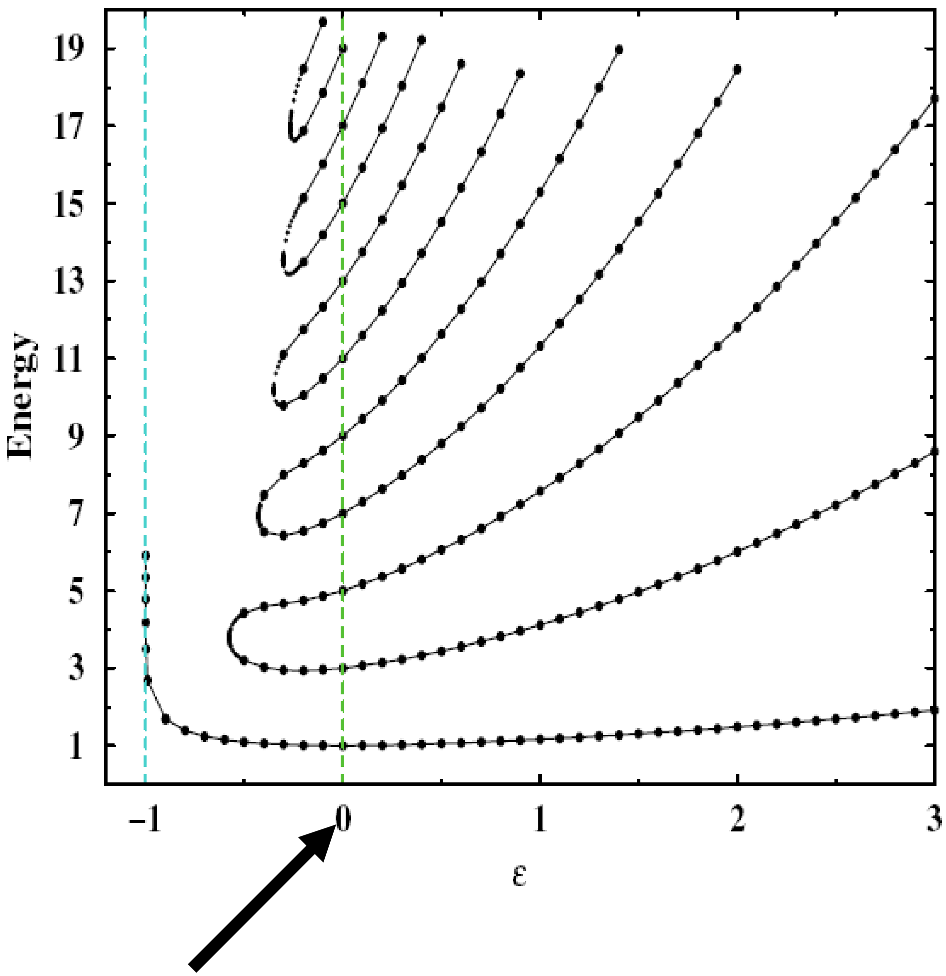
MY TALK

Outline of Talk

- Beginning
- Middle
- End
- (applause)

How to “prove” that the eigenvalues are real

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



PT Boundary

How to "prove" that the eigenvalues are real

The proof is really hard!
You need to use
(3)Bethe ansatz
(4)Monodromy group
(5)Baxter T-Q relation
(6)Functional Determinants

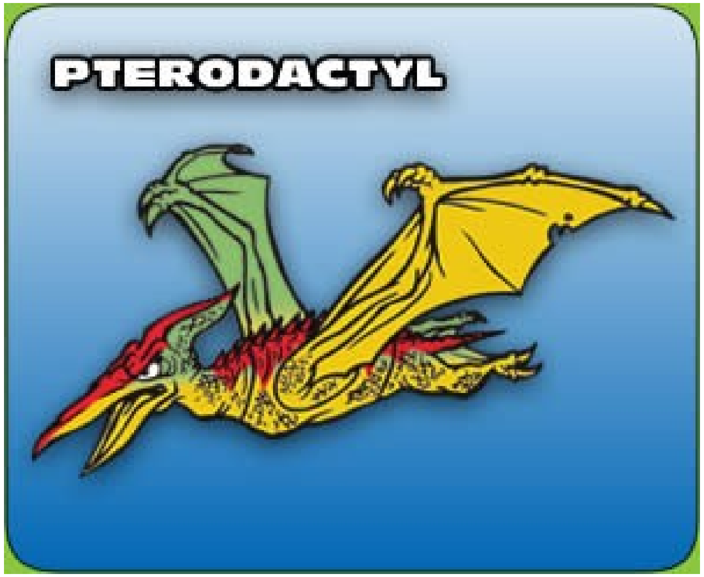
PT Boundary

Greatest murder mystery of all time…
Extinction of over 90% of species!

Permian era

Triassic era

PT Boundary



OK, so the eigenvalues are real … But is this quantum mechanics??

- Probabilistic interpretation??
- Hilbert space with a positive metric??
- Unitarity??

Dirac, Bakerian Lecture 1941, Proceedings of the Royal Society A

Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results. The physical interpretation of relativistic quantum mechanics that one gets by a natural development of the non-relativistic theory involves these things and is thus in contradiction with experiment. We therefore have to consider ways of modifying or supplementing this interpretation.

The Hamiltonian determines its own adjoint

$$[\mathcal{C}, \mathcal{PT}] = 0,$$

$$[\mathcal{C}^2 = 1],$$

$$[\mathcal{C}, H] = 0$$

Replace $\dagger$ by $\mathcal{CPT}$

Unitarity

With respect to the *CPT* adjoint the theory has UNITARY time evolution.

Norms are strictly positive!
Probability is conserved!

OK, we have unitarity…
But is PT quantum mechanics useful??

- It revives quantum theories that were thought to be dead
- It is beginning to be observed experimentally

The Lee Model

$$V \to N + \theta, \qquad N + \theta \to V.$$

$$H = H_0 + g_0 H_1,$$

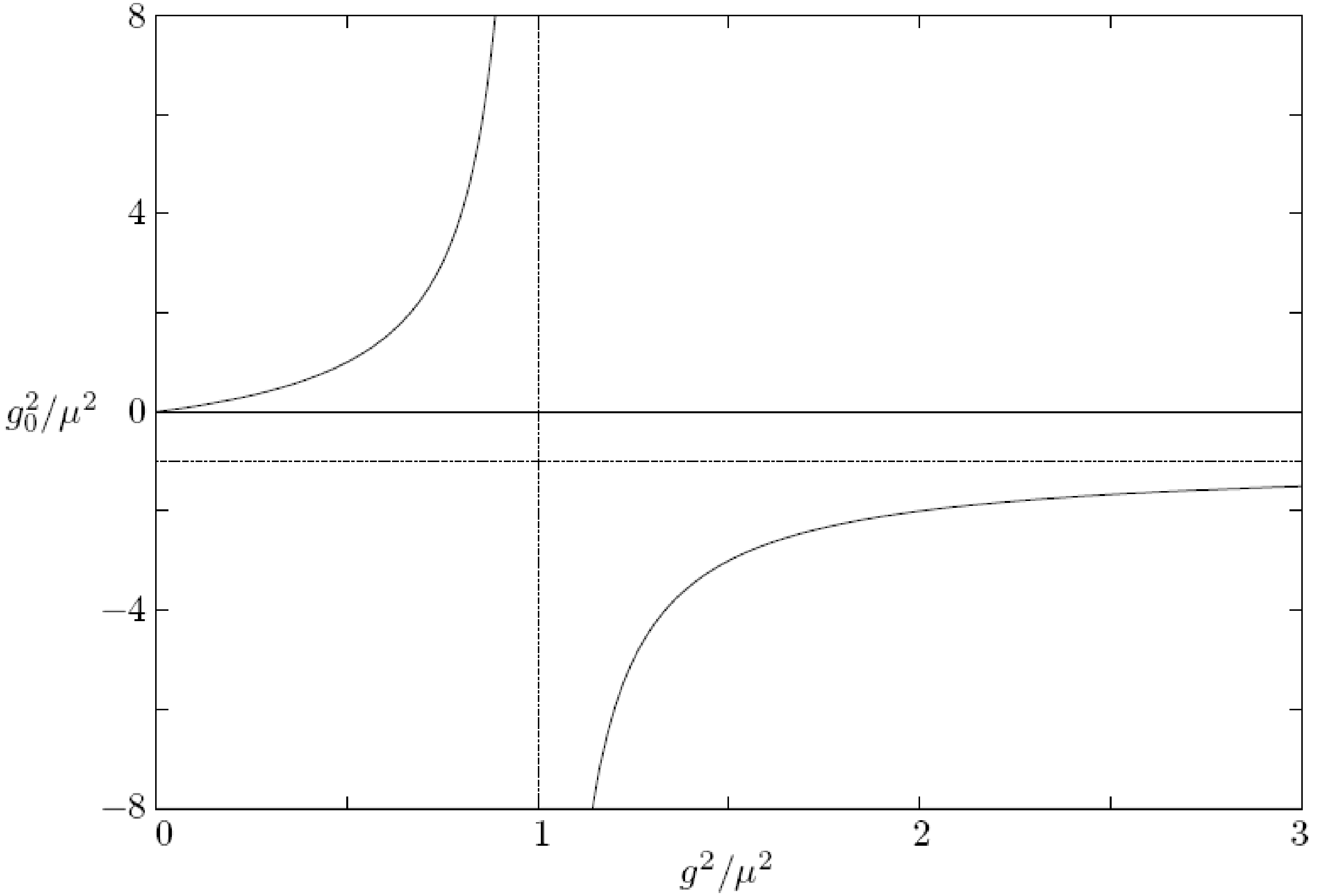
$$H_0 = m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a,$$

$$H_1 = V^\dagger N a + a^\dagger N^\dagger V.$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

The problem:



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

MR0076639 (17,927d) 81.0X
Källén, G.; **Pauli, W.**
On the mathematical structure of T. D. Lee's model of a renormalizable field theory.
Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), *no. 7, 23 pp.*

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR0064658 (16,317b)] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V \rightleftarrows N+\theta$ but forbidding $N \rightleftarrows V+\theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/{g_0}^2 = 1 - Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio $g^2/{g_0}^2$ should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR0046306 (13,713d); Källén, Helv. Phys. Acta **25** (1952), 417–434; MR0051156 (14,435l); Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR0072756 (17,332e); Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR0064652 (16,315e)]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value $g_{\rm crit}$. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

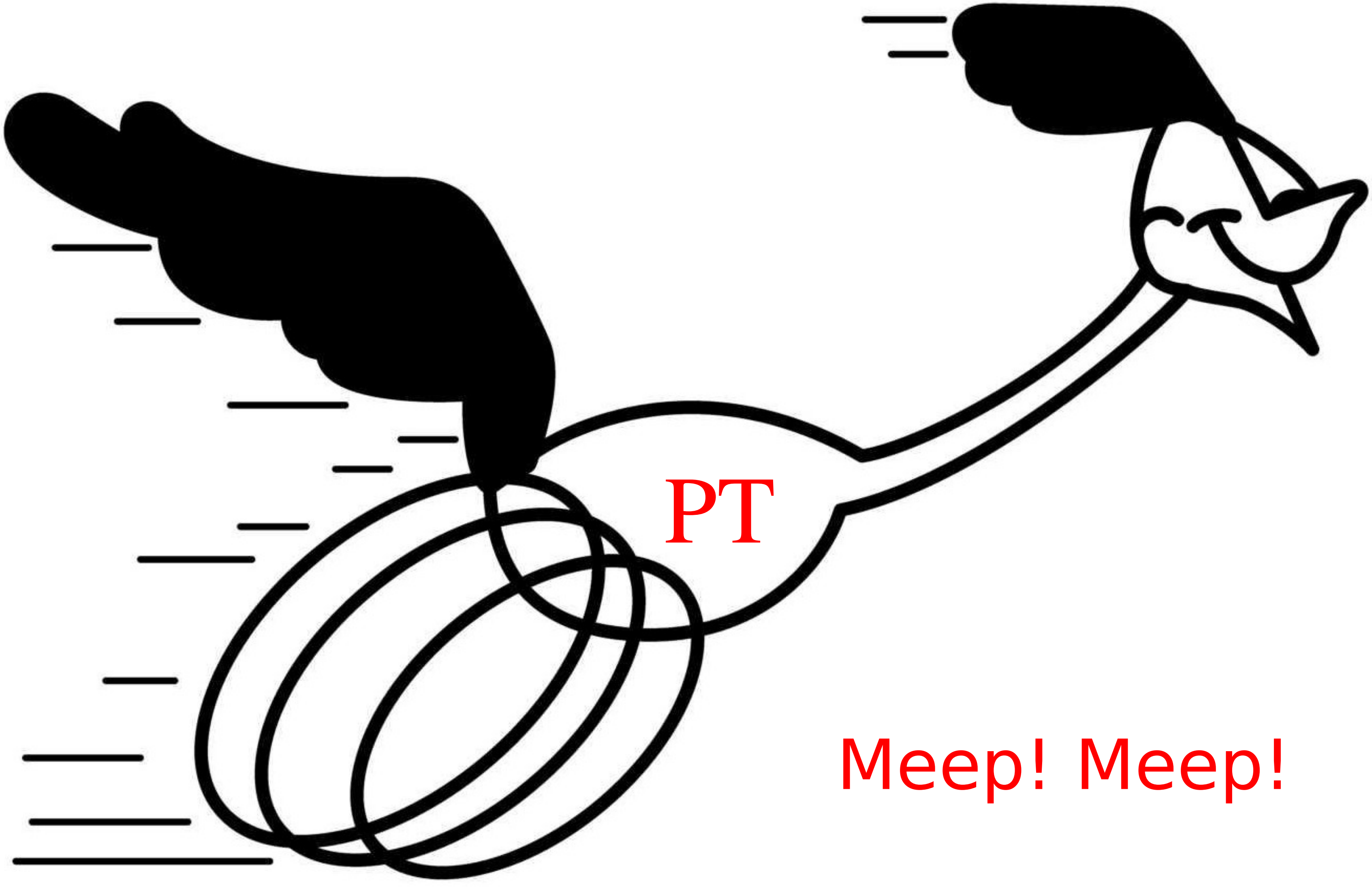
The authors discover the remarkable result that if $g > g_{\rm crit}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g > g_{\rm crit}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

Reviewed by *A. Salam*

“A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

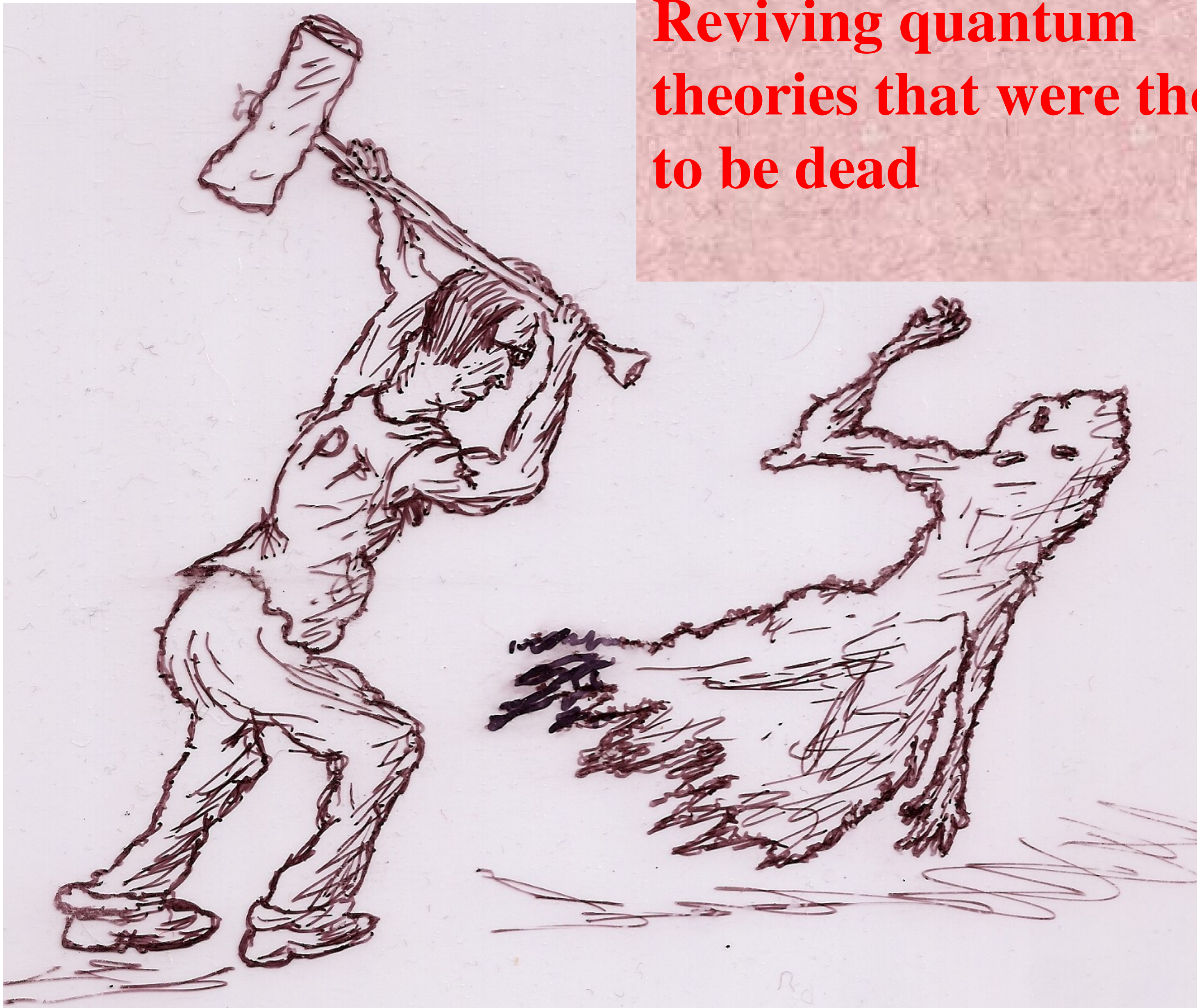
G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)

PT quantum mechanics to the rescue…



Meep! Meep!

GHOSTBUSTING: Reviving quantum theories that were thought to be dead



Pais-Uhlenbeck action

$$I = \frac{\gamma}{2}\int dt\left[\ddot{z}^2 - \left(\omega_1^2+\omega_2^2\right)\dot{z}^2 + \omega_1^2\omega_2^2 z^2\right]$$

Gives a fourth-order field equation:

$$z''''(t) + (\omega_1^2+\omega_2^2)z''(t) + \omega_1^2\omega_2^2 z(t) = 0$$

CMB and P. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008)
CMB and P. Mannheim, Phys. Rev. D **78**, 025002 (2008)

The problem: A fourth-order field equation gives a propagator like

$$G(E) = \frac{1}{(E^2 + m_1^2)(E^2 + m_2^2)}$$

$$G(E) = \frac{1}{m_2^2 - m_1^2}\left(\frac{1}{E^2 + m_1^2} - \frac{1}{E^2 + m_2^2}\right)$$

GHOST!

There are now two possible realizations…

(I) If a_1 and a_2 annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2|\Omega\rangle = 0,$$

then the energy spectrum is real and bounded below. The state $|\Omega\rangle$ is the ground state of the theory and it has zero-point energy $\frac{1}{2}(\omega_1 + \omega_2)$. The problem with this realization is that the excited state $a_2^\dagger|\Omega\rangle$, whose energy is ω_2 above ground state, has a *negative Dirac norm* given by $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$.

(II) If a_1 and $a_2^\dagger$ annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2^\dagger|\Omega\rangle = 0,$$

then the theory is free of negative-norm states. However, this realization has a different and equally serious problem; namely, that the energy spectrum is unbounded below.

There can be many realizations!

$$H = p^2 - x^4$$

$$-\psi''(x) - x^4\psi(x) = E\psi(x)$$

Equivalent Dirac Hermitian Hamiltonian:

$$\mathcal{C} = e^{Q}\mathcal{P} \qquad \tilde{H} = e^{-Q/2} H e^{Q/2}$$

$$\tilde{H} = p^2 + 4x^4 - 2\hbar x$$

$$Q = \alpha pq + \beta xy$$

$$\beta = \gamma^2 \omega_1^2 \omega_2^2 \alpha \quad \text{and} \quad \sinh(\sqrt{\alpha\beta}) = \frac{2\omega_1\omega_2}{\omega_1^2 - \omega_2^2}$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2}$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2$$

No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck model, CMB and P. Mannheim, PRL **100**, 110402 (2008)

TOTALITARIAN PRINCIPLE

"Everything which is not forbidden is compulsory."

---M. Gell-Mann

And there are now observations in table-top optics experiments!

Observing $\mathcal{PT}$ symmetry using wave guides:

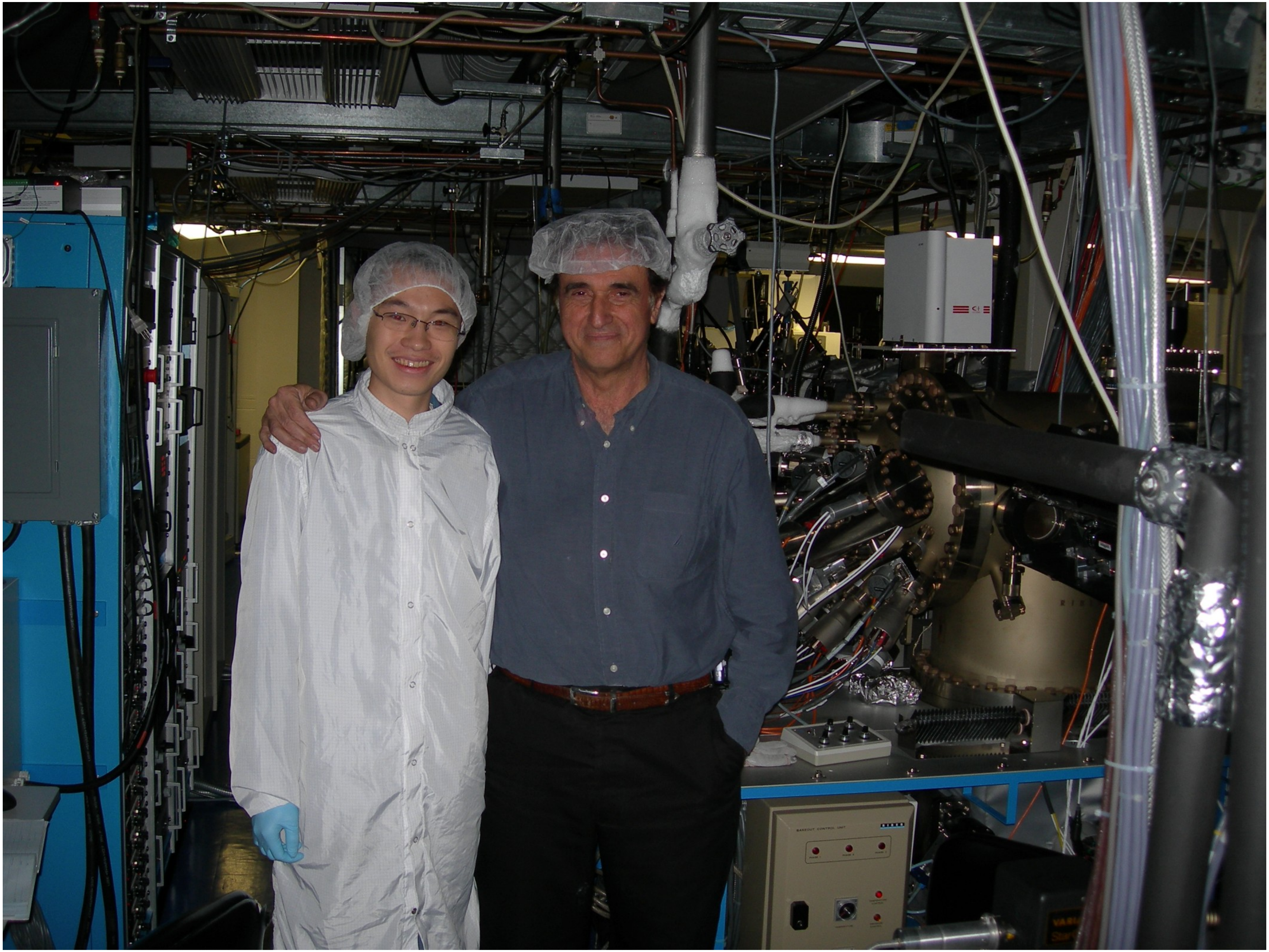
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, PRL **100**, 030402 (2008)
- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, PRL **100**, 103904 (2008)

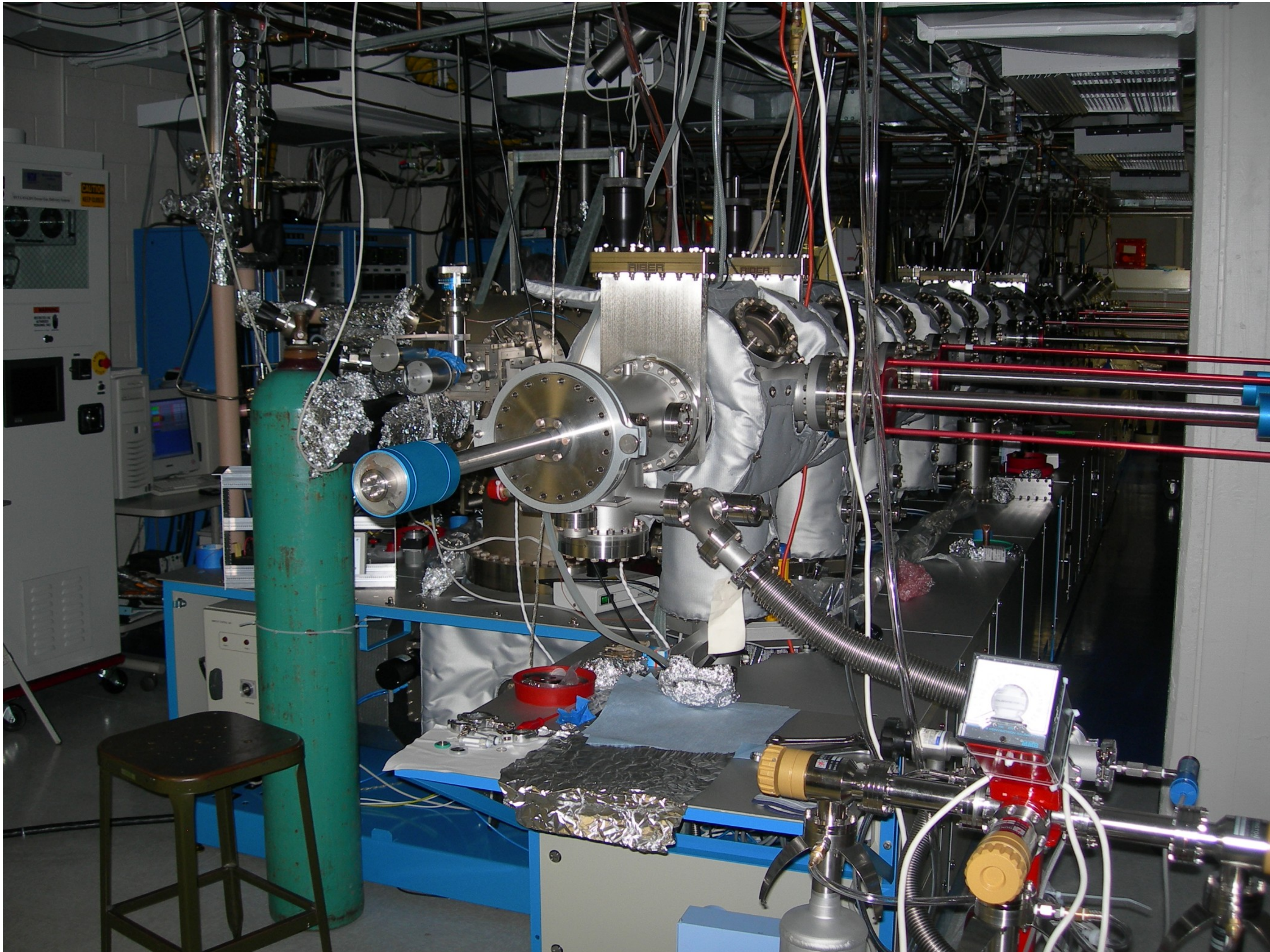
Date: Thu, 13 Mar 2008 23:04:45 -0400
From: Demetrios Christodoulides <demetri@creol.ucf.edu>
To: Carl M. Bender <cmb@wuphys.wustl.edu>
Subject: Re: Benasque workshop on non-Hermitian Hamiltonians

Dear Carl,
I have some good news from Greg Salamo (U. of Arkansas). His students (who are now visiting us here in Florida) have just observed a PT phase transition in a passive AlGaAs waveguide system. We will be submitting soon these results as a post-deadline paper to CLEO/QELS and subsequently to a regular journal. We are still fighting against the Kramers-Kronig relations, but the phase transition effect is definitely there. We expect even better results under TE polarization conditions. I will bring them over to Israel.

In close collaboration with us, more teams (also best friends!) are moving ahead in this direction. Moti Segev (from Technion) is planning an experiment in an active-passive dual core optical fiber -- fabricated in Southampton, England. More experiments will be carried later in Germany by Detlef Kip. Christian (his post doc) just left from here with a possible design. If everything goes well, with a bit of luck we may have an experimental explosion in the PT area. I wish the funding situation was a bit better. So far everything is done on a shoe-string budget (it is subsidized by other projects). Let us see...

All the best
Demetri

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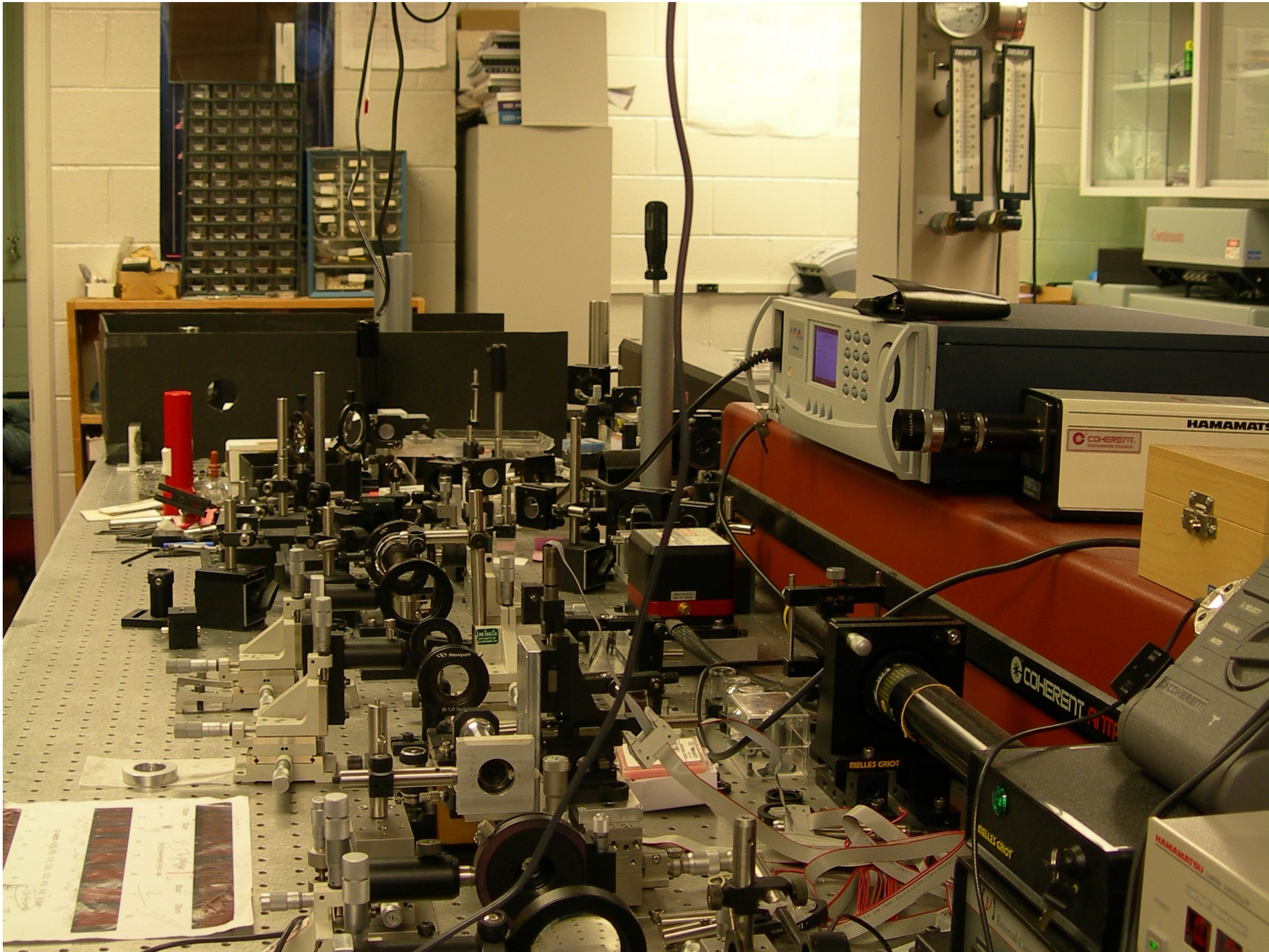
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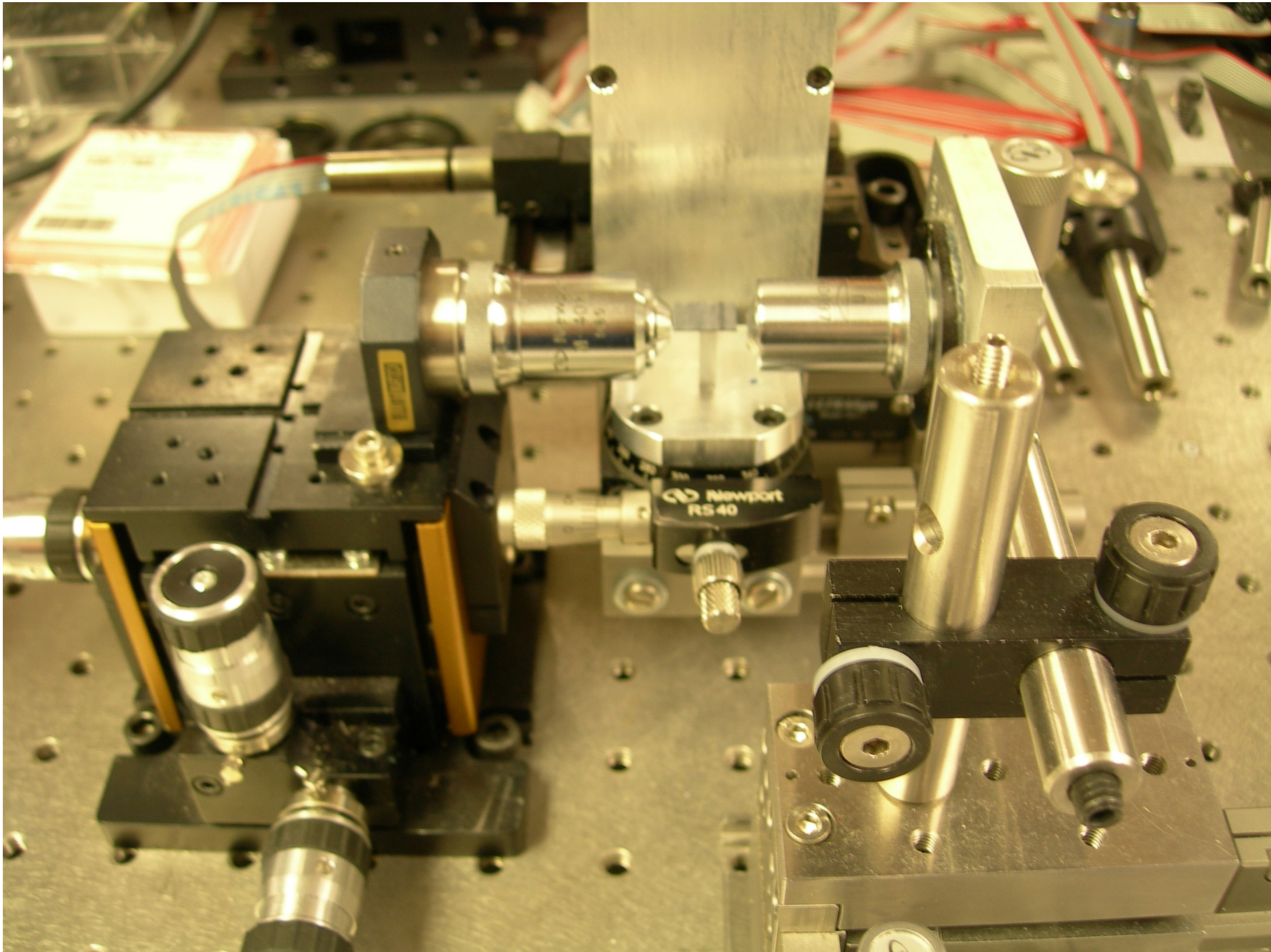
COHERENT
Instruments Division
COHERENT
MELLES GRIOT
MELLES GRIOT
HAMAMATSU



Newport
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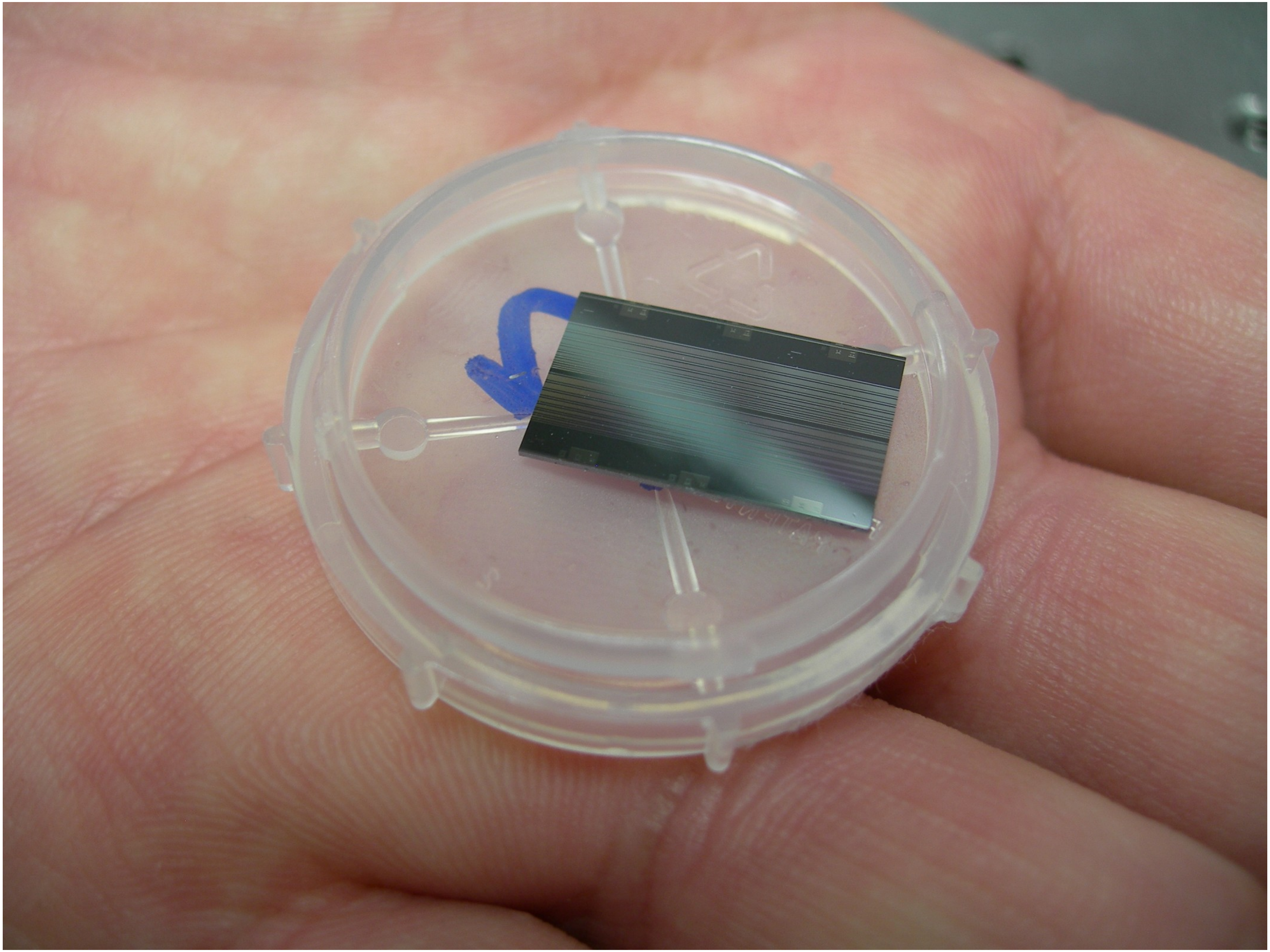


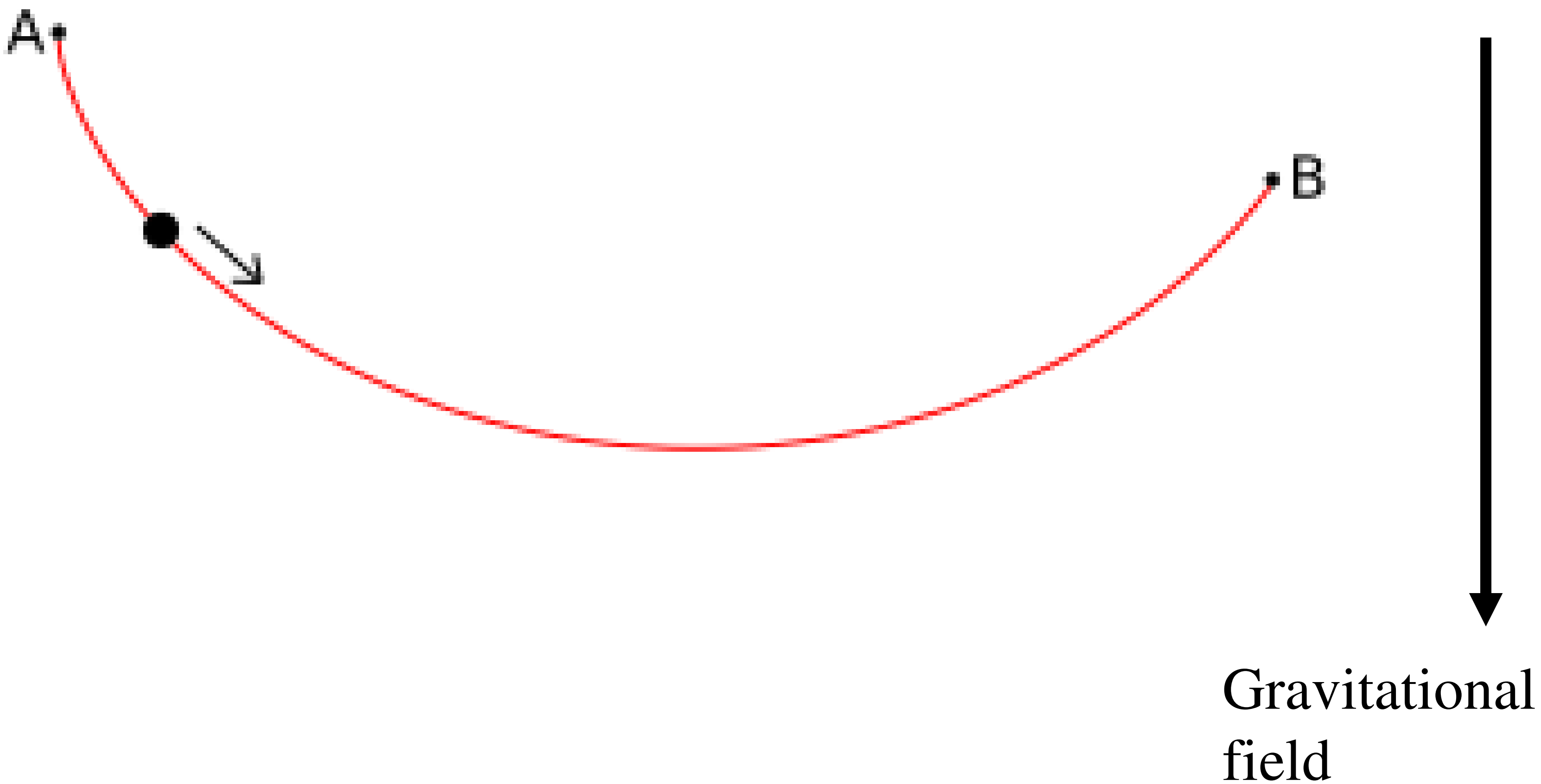
OK, but how do we interpret a non-Hermitian Hamiltonian??

Solve the quantum brachistochrone problem…

Classical Brachistochrone

- Newton
- Bernoulli
- Leibniz
- L'Hôpital

Classical Brachistochrone is a cycloid



Quantum Brachistochrone

$$|\psi_I\rangle \to |\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle$$

Constraint: $\omega = E_{\max} - E_{\min}$

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Hermitian case

$$H = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix} \qquad (r,\ s,\ u,\ \theta \text{ real})$$

$$H = \tfrac{1}{2}(s+u)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n}$$

$$\mathbf{n} = \frac{1}{\omega}(2r\cos\theta, 2r\sin\theta, s-u) \qquad \omega^2 = (s-u)^2 + 4r^2$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\exp(i\phi\,\sigma\cdot\mathbf{n}) = \cos\phi\,\mathbf{1} + i\sin\phi\,\sigma\cdot\mathbf{n}$$

$$|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$$

becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-\frac{1}{2}i(s+u)t/\hbar} \begin{pmatrix} \cos\frac{\omega t}{2\hbar} - i\frac{s-u}{\omega}\sin\frac{\omega t}{2\hbar} \\ \\ -i\frac{2r}{\omega}e^{i\theta}\sin\frac{\omega t}{2\hbar} \end{pmatrix}$$

$$t = \frac{2\hbar}{\omega}\arcsin\frac{\omega|b|}{2r}$$

Minimize t over all positive *r* while maintaining constraint

$$\omega^2 = (s-u)^2 + 4r^2.$$

Minimum evolution time:

$$\tau\omega = 2\hbar \arcsin |b|.$$

Looks like uncertainty principle but is merely rate times time = distance

Note that if $a = 0$ and $b = 1$, we have $\tau = \pi\hbar/\omega$ for the smallest time required to transform $\binom{1}{0}$ to the orthogonal state $\binom{0}{1}$. The time τ required to transform a vector into an orthogonal vector is called the *passage time*.

Non-Hermitian PT-symmetric Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (r,\ s,\ \theta \text{ real})$$

$\mathcal{T}$ is complex conjugation and $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$E_{\pm} = r\cos\theta \pm \sqrt{s^2 - r^2 \sin^2\theta} \qquad \text{real if } s^2 > r^2 \sin^2\theta$$

$$\mathcal{C} = \frac{1}{\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}$$

where $\sin\alpha = (r/s)\sin\theta.$

Exponentiate H

$$H = (r\cos\theta)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n},$$

where

$$\mathbf{n} = \frac{2}{\omega}(s, 0, ir\sin\theta)$$

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta.$$

$$e^{-iHt/\hbar}\begin{pmatrix}1\\0\end{pmatrix} = \frac{e^{-itr\cos\theta/\hbar}}{\cos\alpha}\begin{pmatrix}\cos(\frac{\omega t}{2\hbar} - \alpha)\\ -i\sin\left(\frac{\omega t}{2\hbar}\right)\end{pmatrix}$$

Consider the pair of vectors used in the Hermitian case: $|\psi_I\rangle = \binom{1}{0}$ and $|\psi_F\rangle = \binom{0}{1}$. (Note that these two vectors are not orthogonal with respect to the $\mathcal{CPT}$ inner product.) Observe that the evolution time needed to reach $|\psi_F\rangle$ from $|\psi_I\rangle$ is $t = (2\alpha - \pi)\hbar/\omega$. Optimizing this result over allowable values for α as α approaches $\frac{1}{2}\pi$, the optimal time τ tends to zero!

Interpretation…

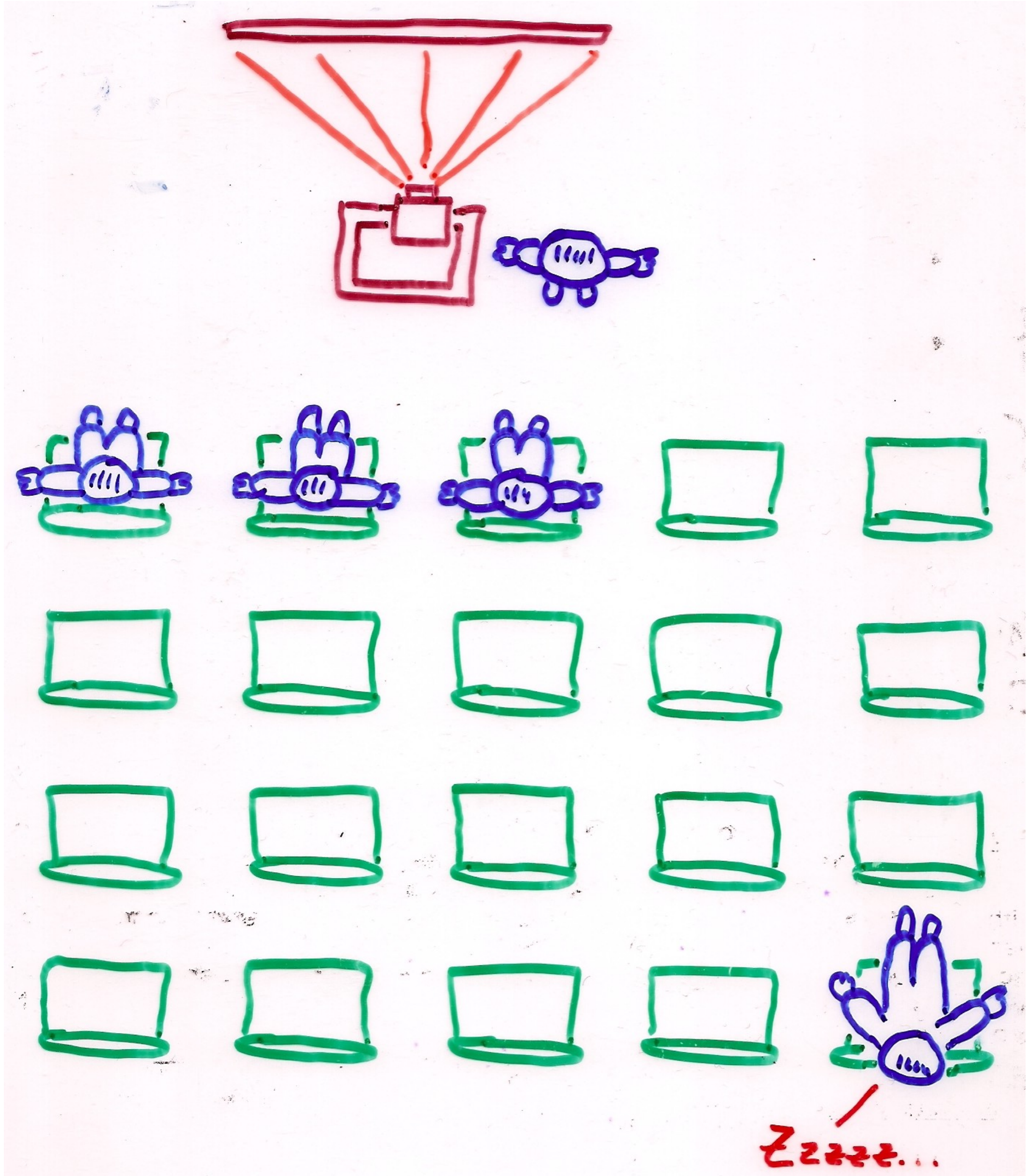
Finding the optimal PT-symmetric Hamiltonian amounts to constructing a wormhole in Hilbert space!

"The shortest path between two truths in the real domain passes through the complex domain."

-- Jacques Hadamard

[The Mathematical Intelligencer **13** (1991)]

Overview of talk:



Classical PT symmetry

Provides an intuitive explanation of what is going on…

Motion on the Real Axis

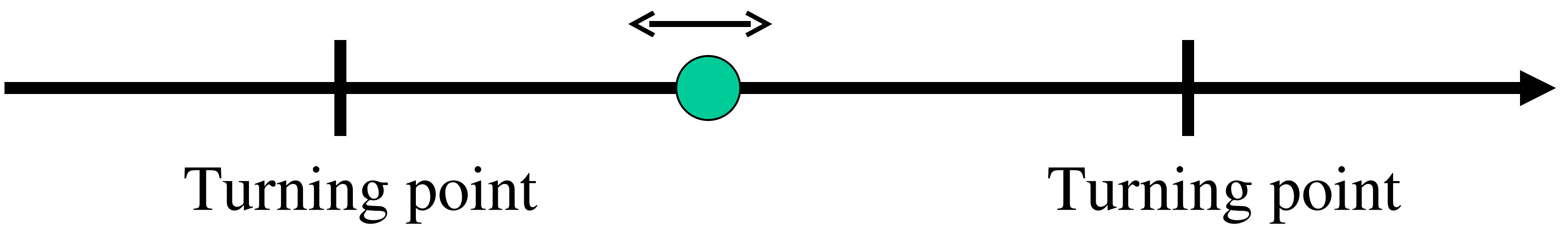
Motion of particles is governed by Newton's Law:

F=ma

In freshman physics this motion is restricted to the REAL AXIS.

Harmonic Oscillator: Particle on a Spring

Back and forth motion
on the real axis:



$$H = p^2 + x^2 \qquad (\epsilon = 0)$$

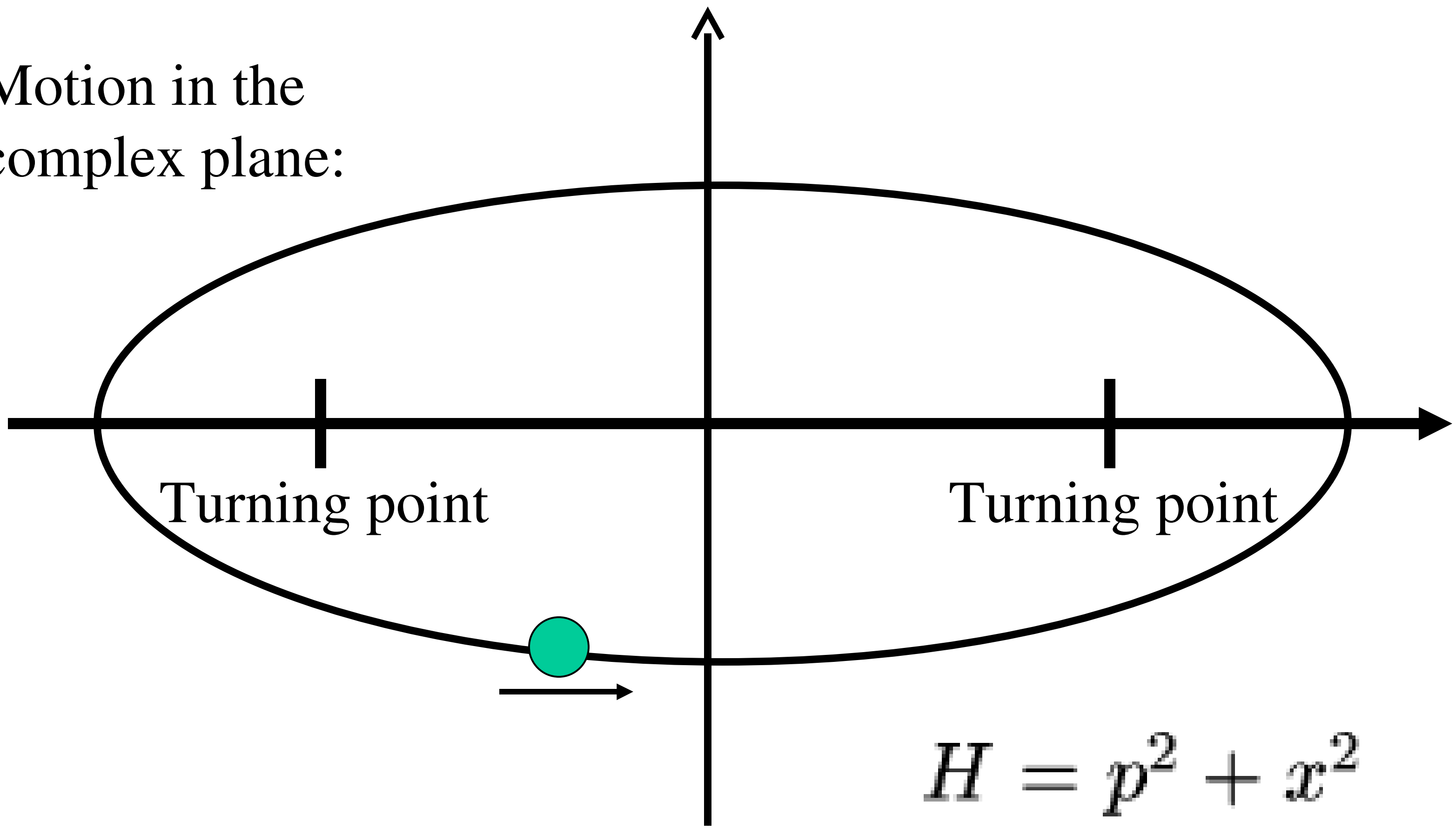
Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x}$$

Harmonic Oscillator:

Motion in the complex plane:

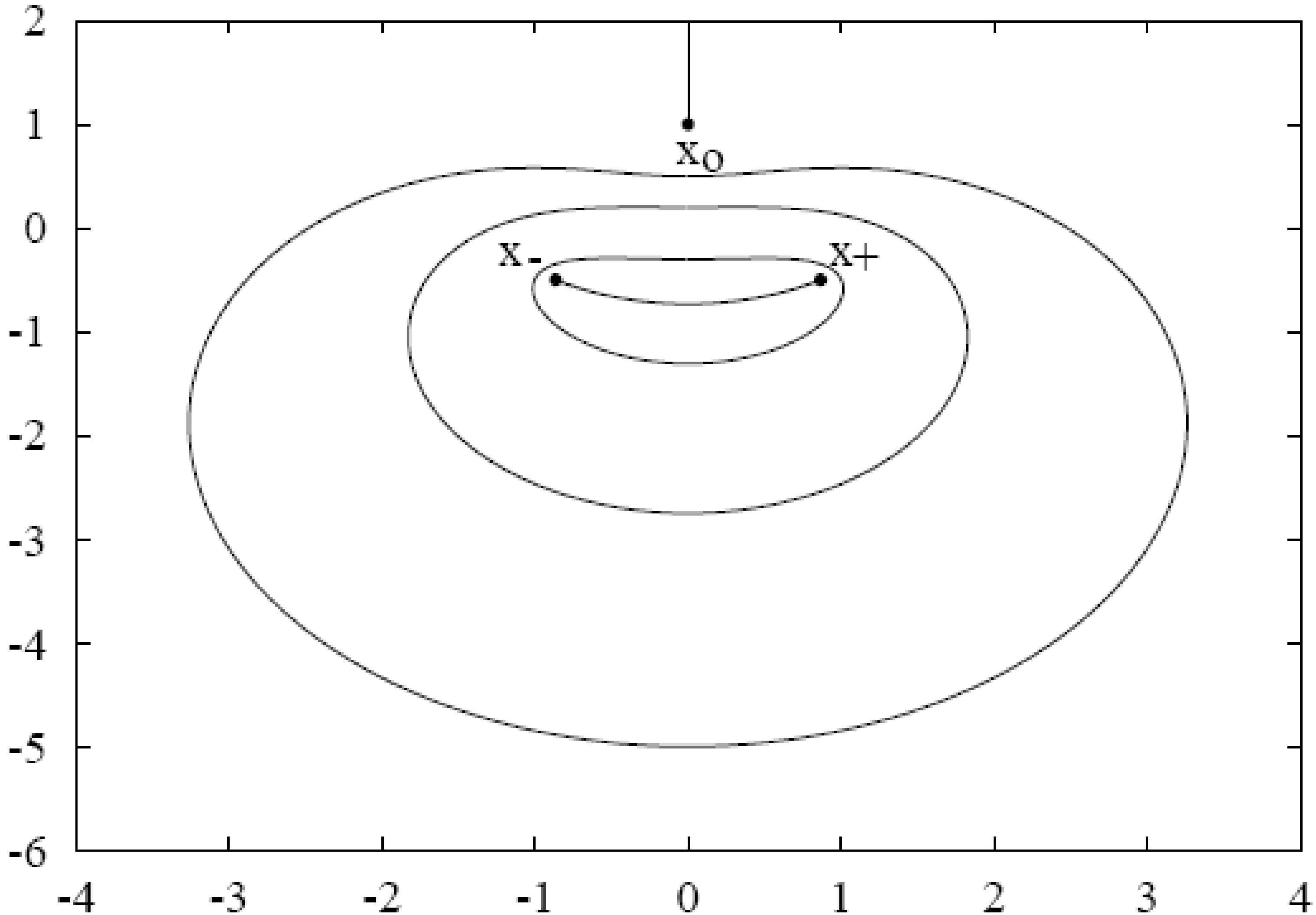


$$H = p^2 + x^2$$

($\epsilon = 0$)

$$H = p^2 + ix^3$$

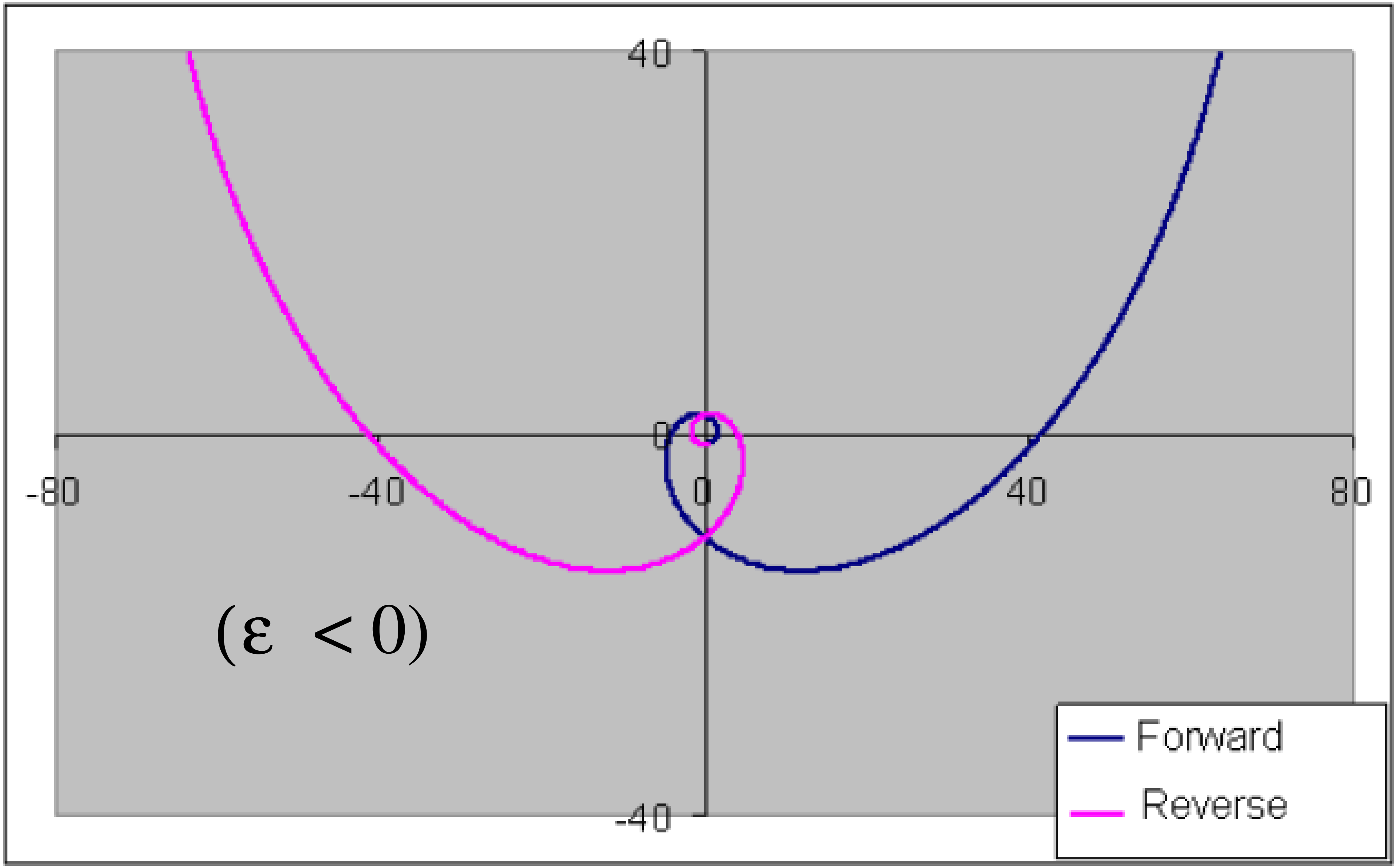
(ϵ = 1)

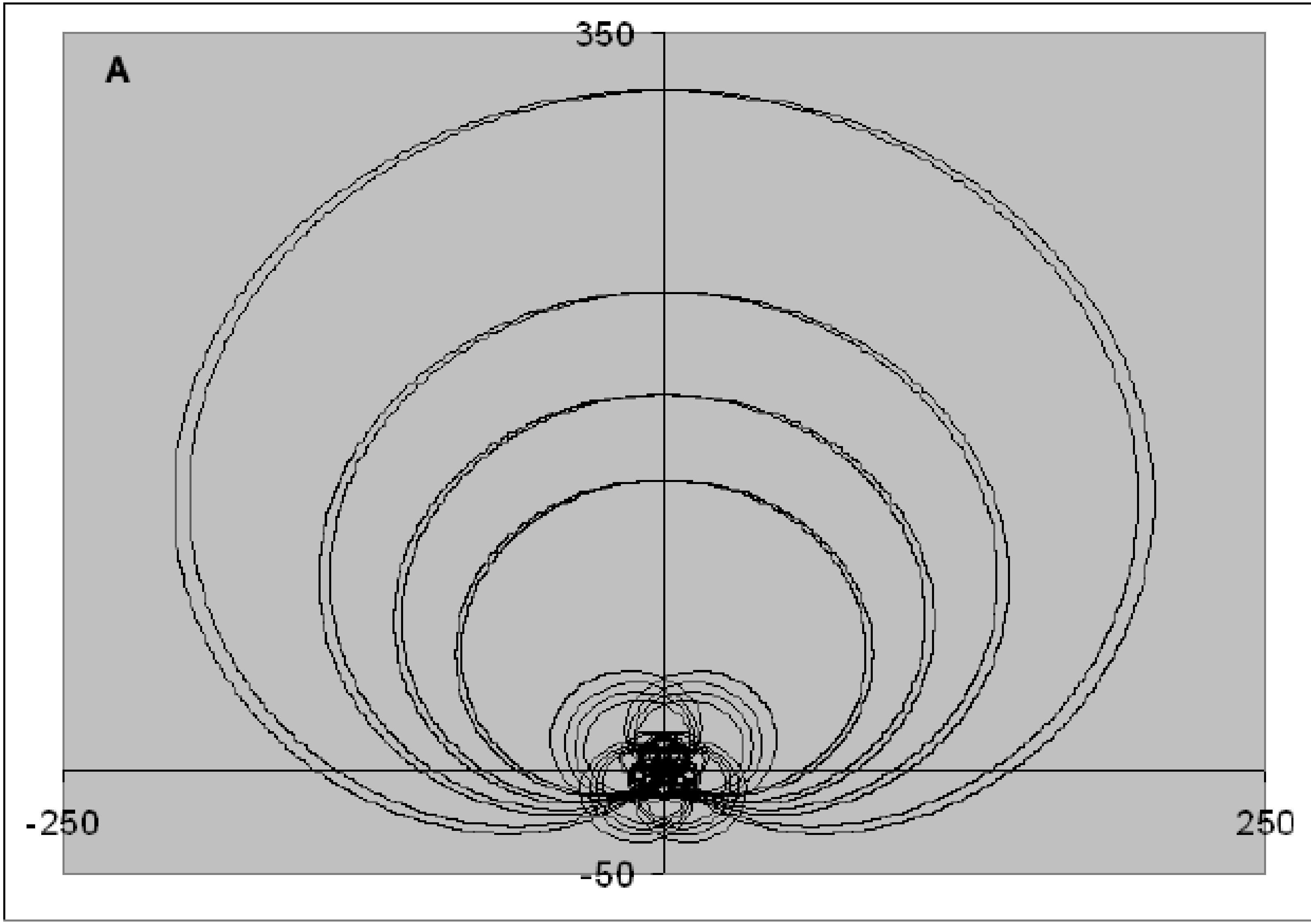


Bohr-Sommerfeld Quantization of a complex atom

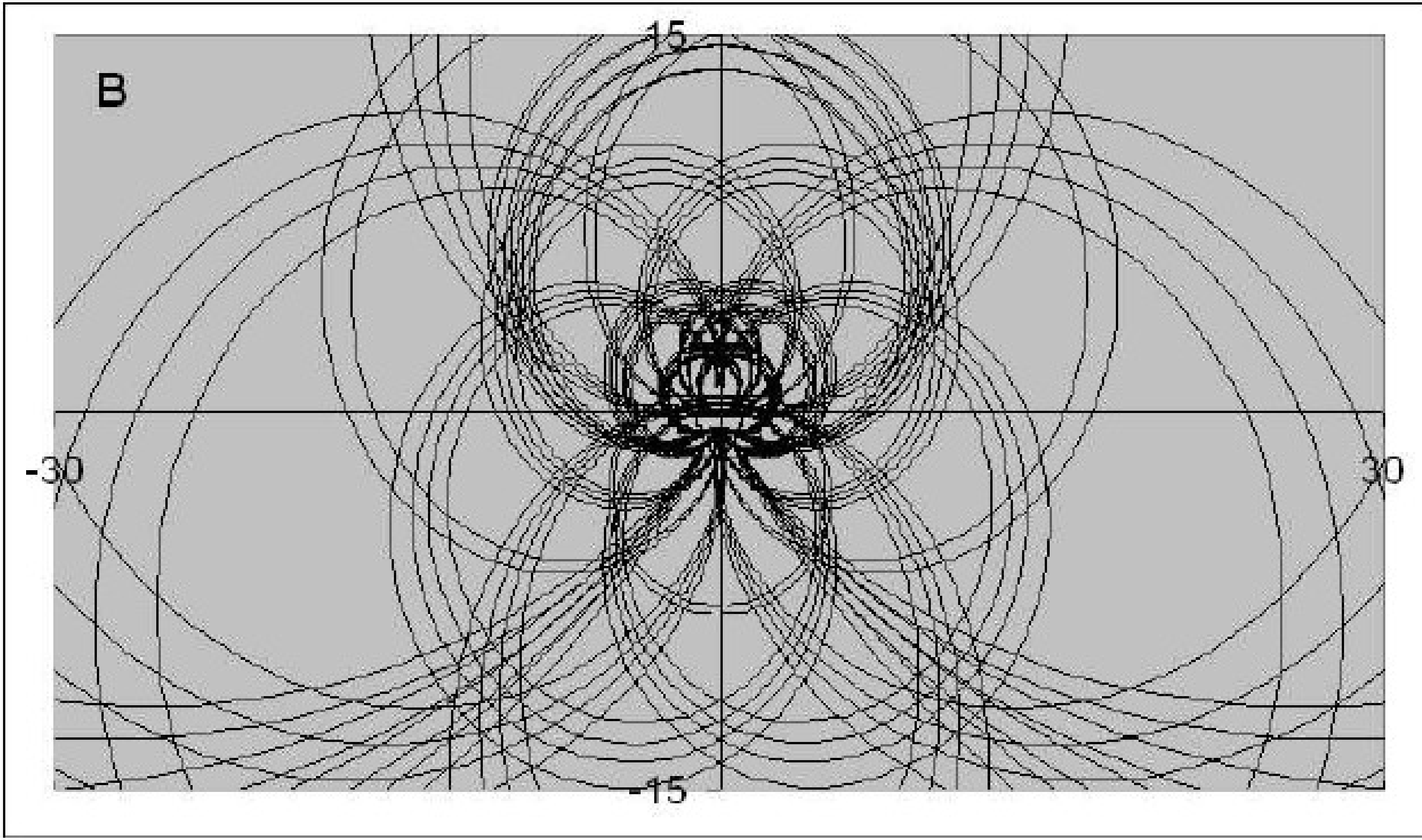
$$\oint dx\, p = \left(n + \frac{1}{2}\right)$$

Broken PT symmetry





$\epsilon = \pi - 2$ 11 sheets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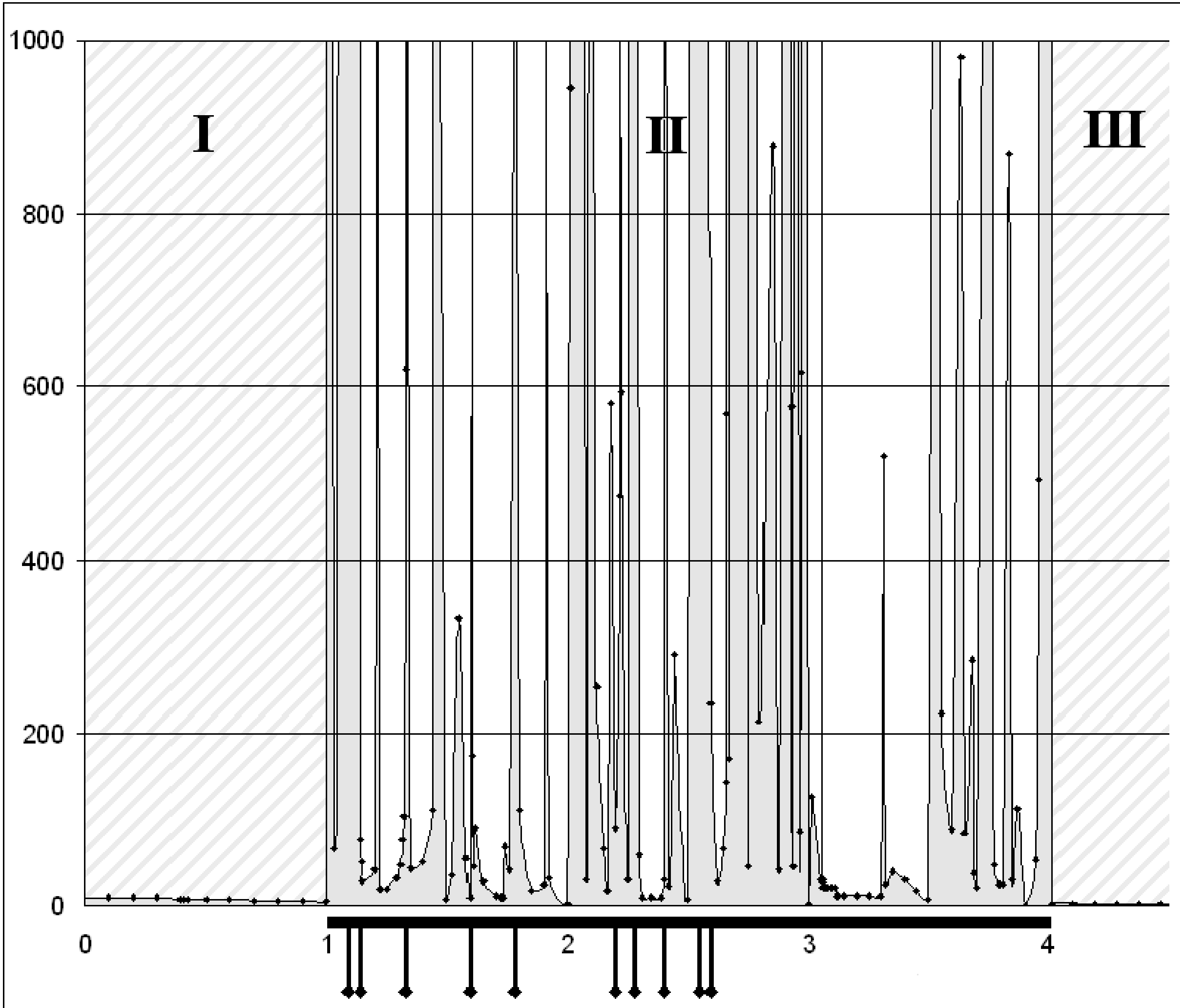

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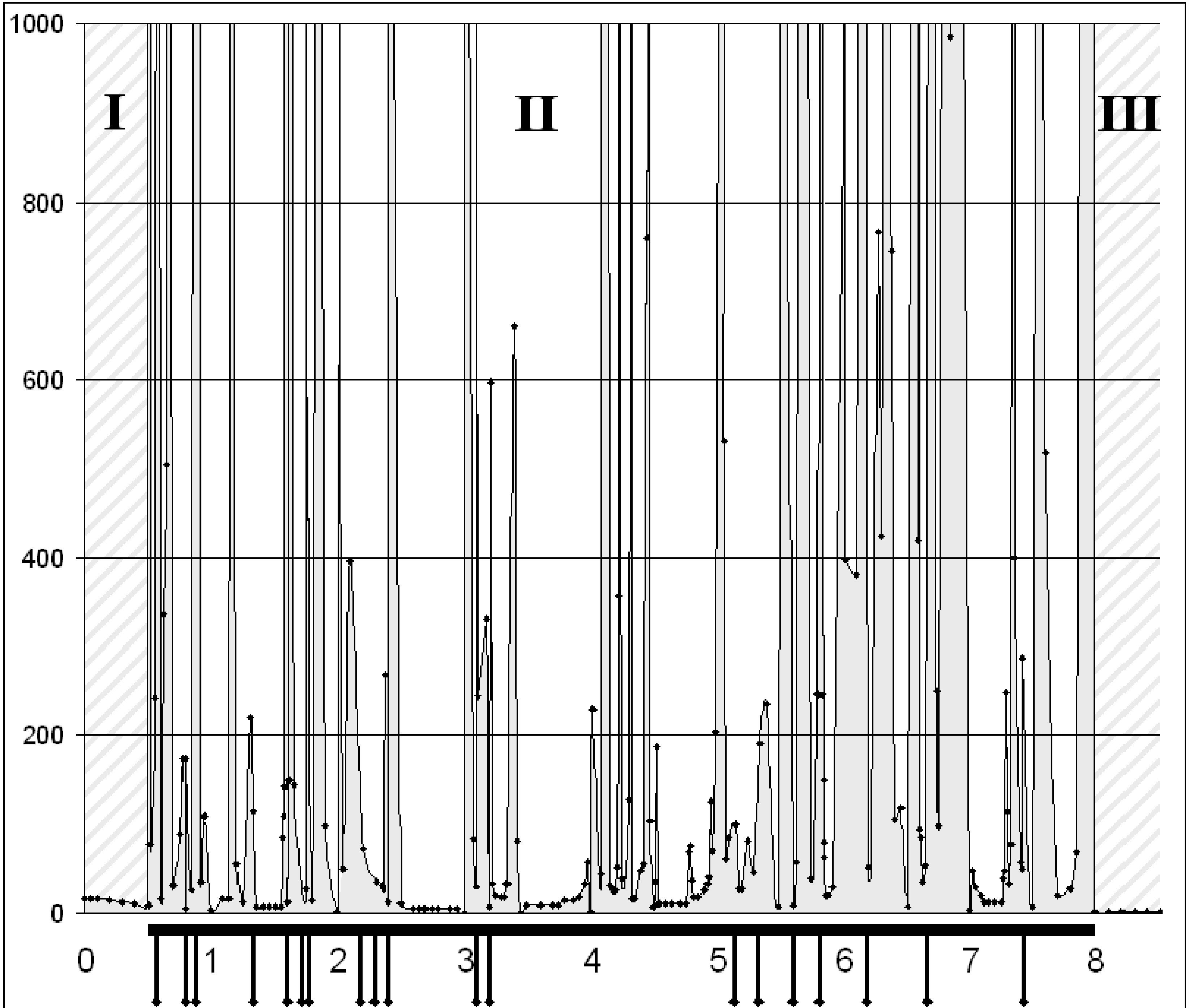

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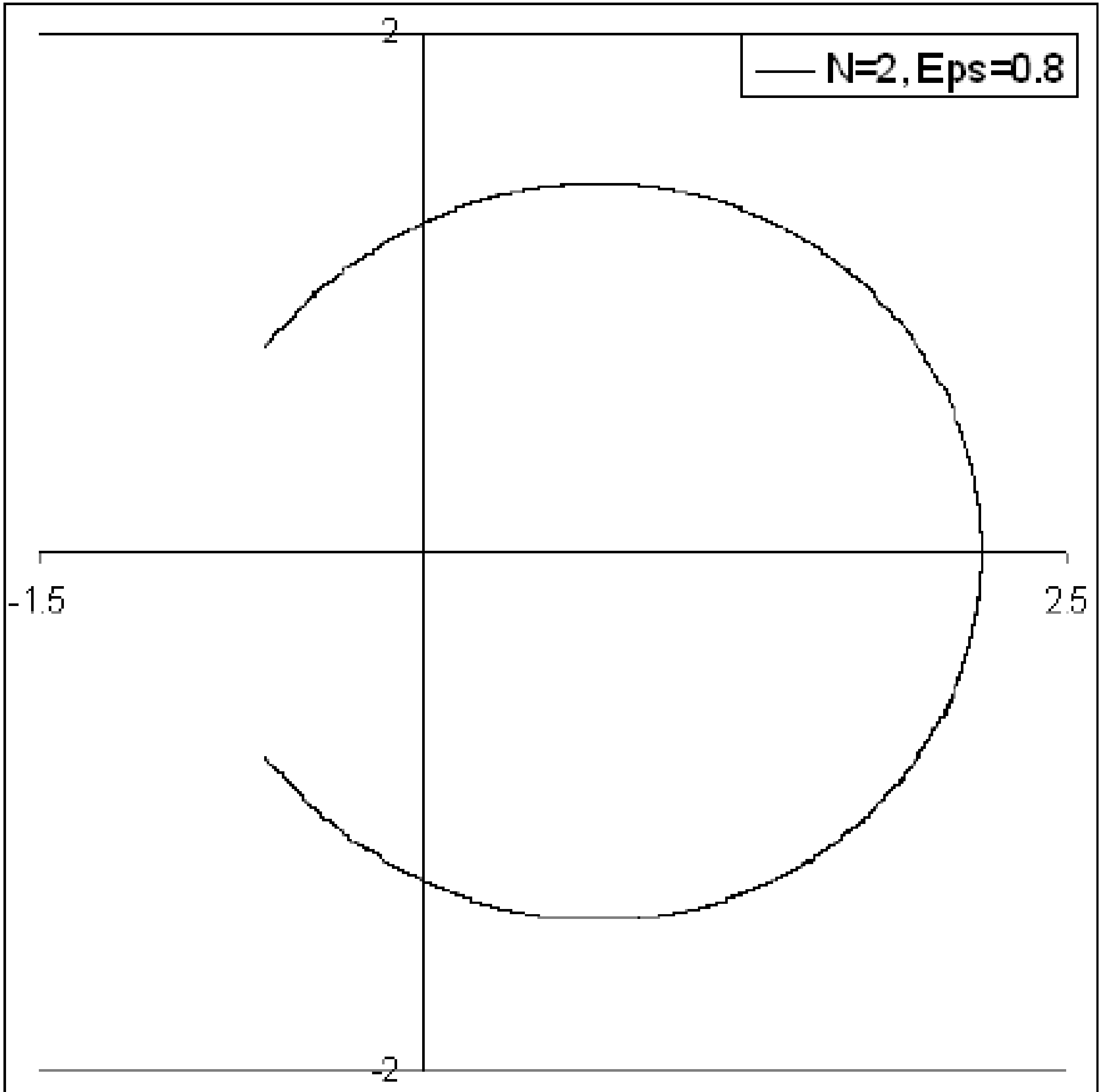
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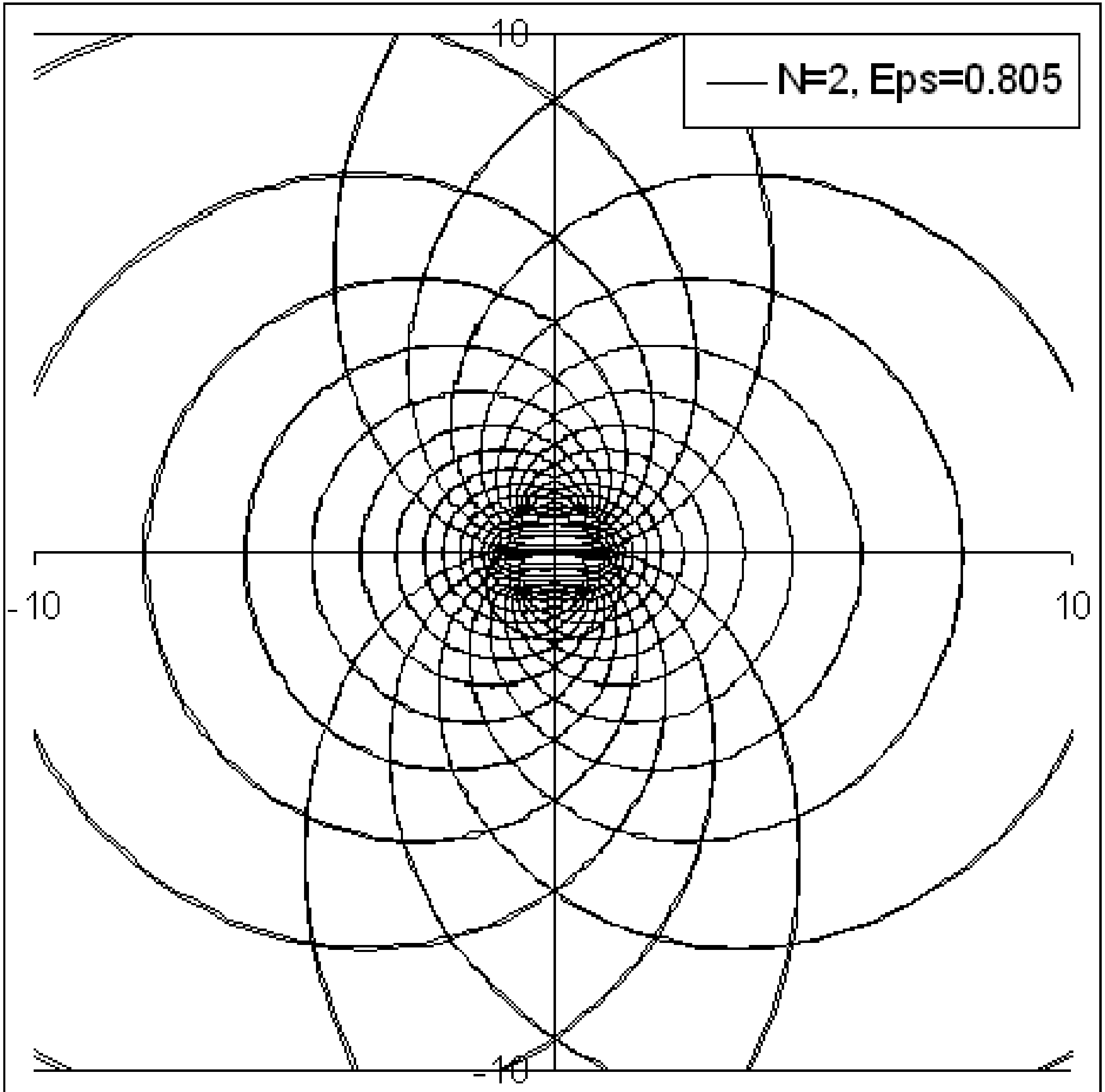

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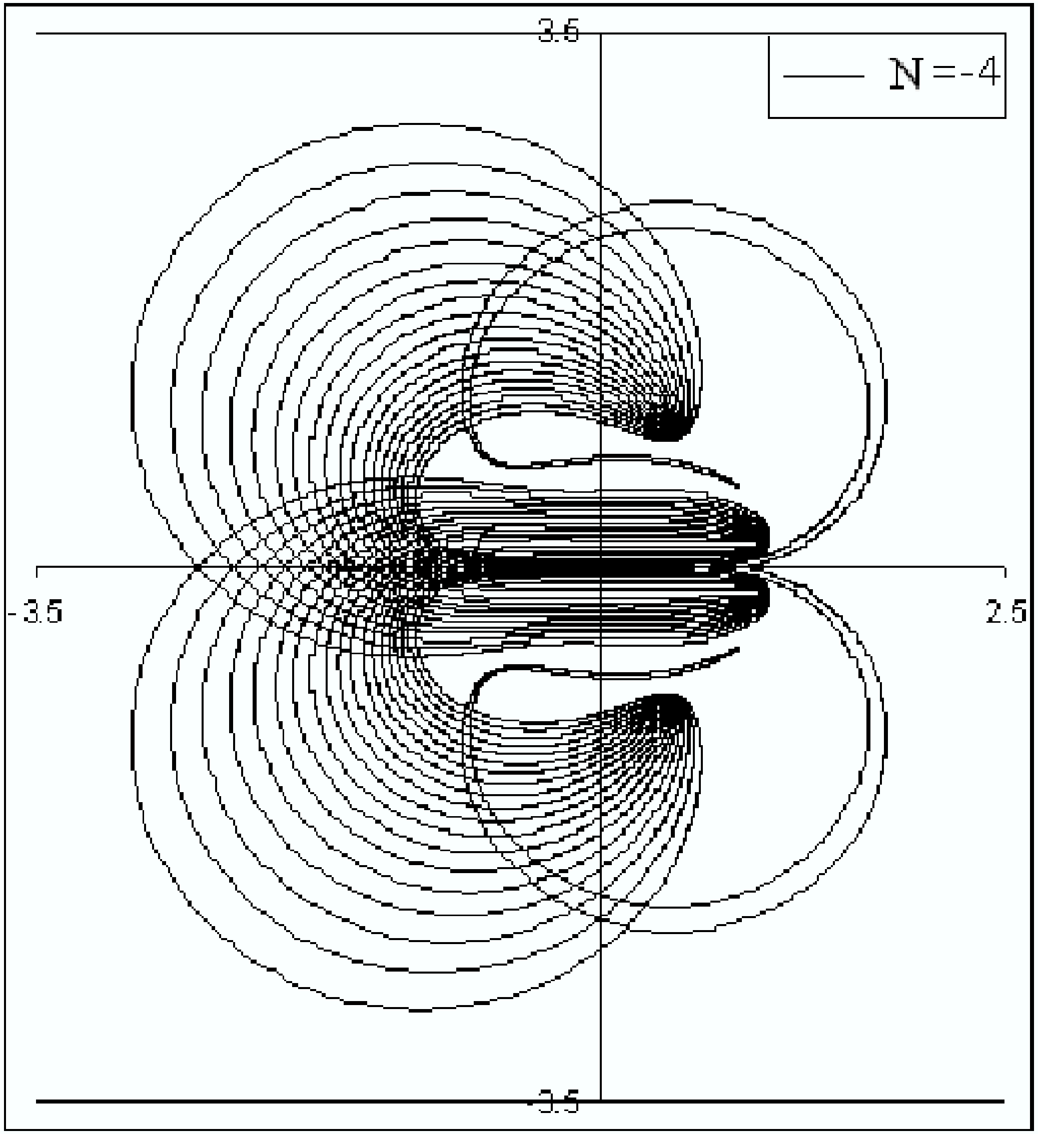
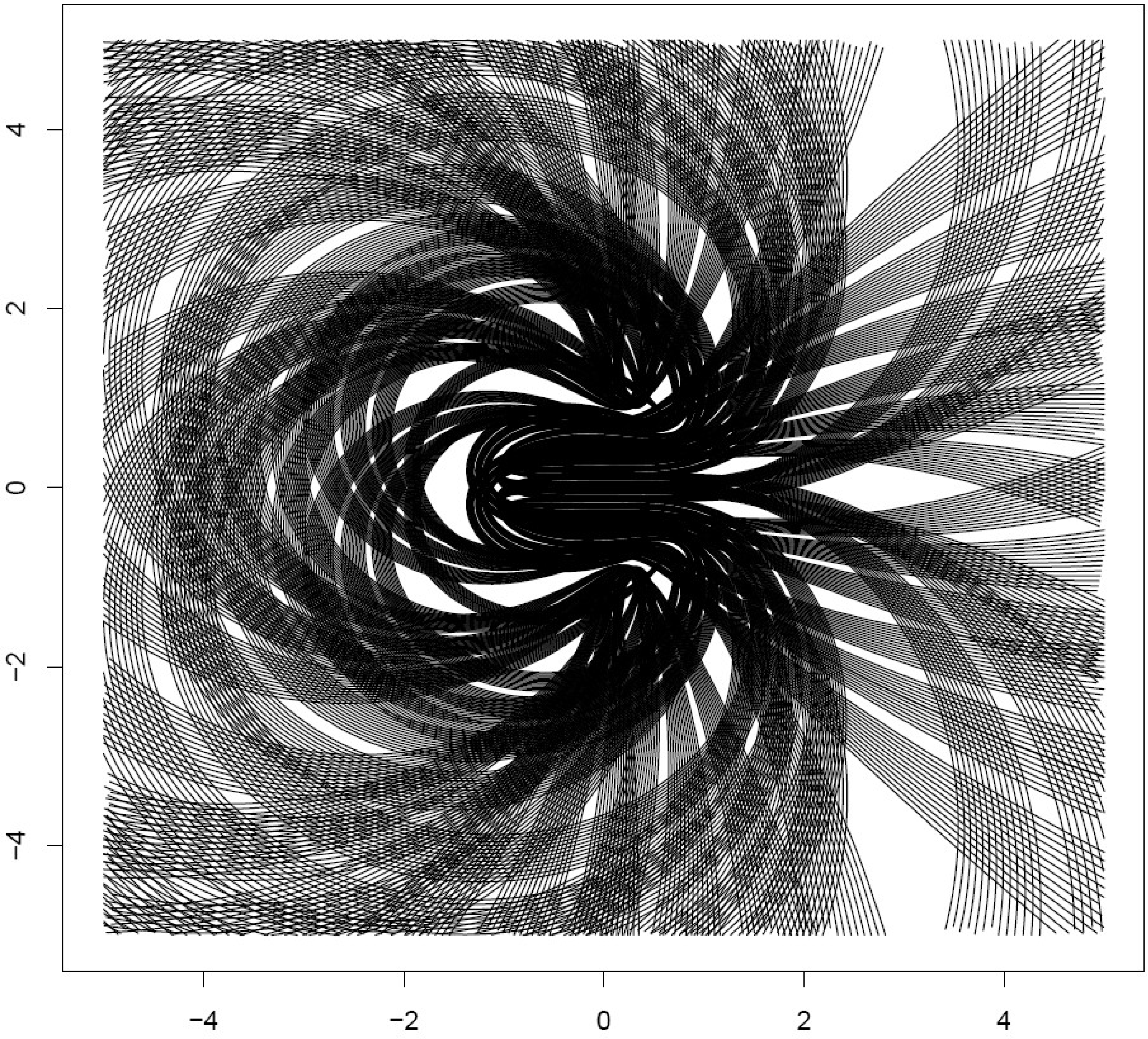


Figure 21. Non-$\mathcal{PT}$-symmetric orbit for $\epsilon = \frac{16}{9}$. This topologically complicated orbit originates at the $N = -4$ turning point but does not reach the $\mathcal{PT}$-symmetric $N = 3$ turning point. Instead, it is reflected back at the complex-conjugate $N = -14$ turning point. The period of this orbit is $T = 186.14$.

The Beast

$\epsilon = 16/15$



Lotka-Volterra equations

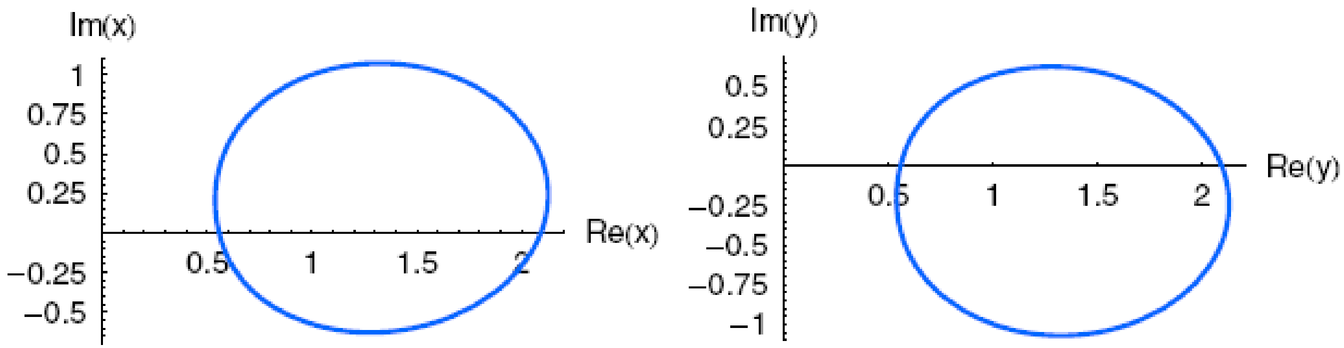


Figure 5. Periodic $\mathcal{PT}$-symmetric complex solutions to the Lotka–Volterra equations (8). For the initial conditions $x(0) = 1 + \mathrm{i}$ and $y(0) = 2.112\,21 - 0.403\,243\mathrm{i}$ the complex trajectories $x(t)$ (left plot) and $y(t)$ (right plot) are shown. Observe that the trajectories are periodic and $\mathcal{PT}$ symmetric, where $\mathcal{P}$ reflection interchanges x and y and $\mathcal{T}$ reflection consists of complex conjugation.

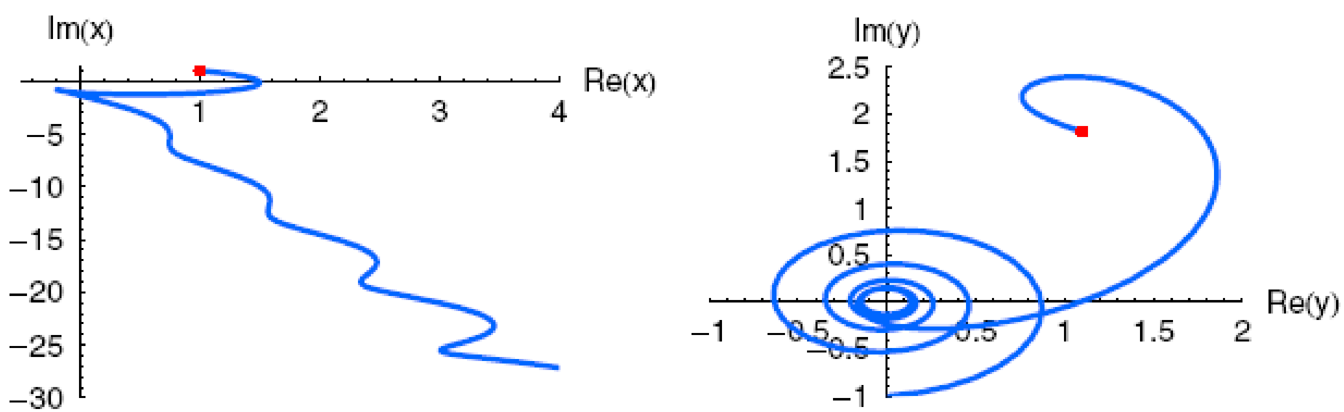
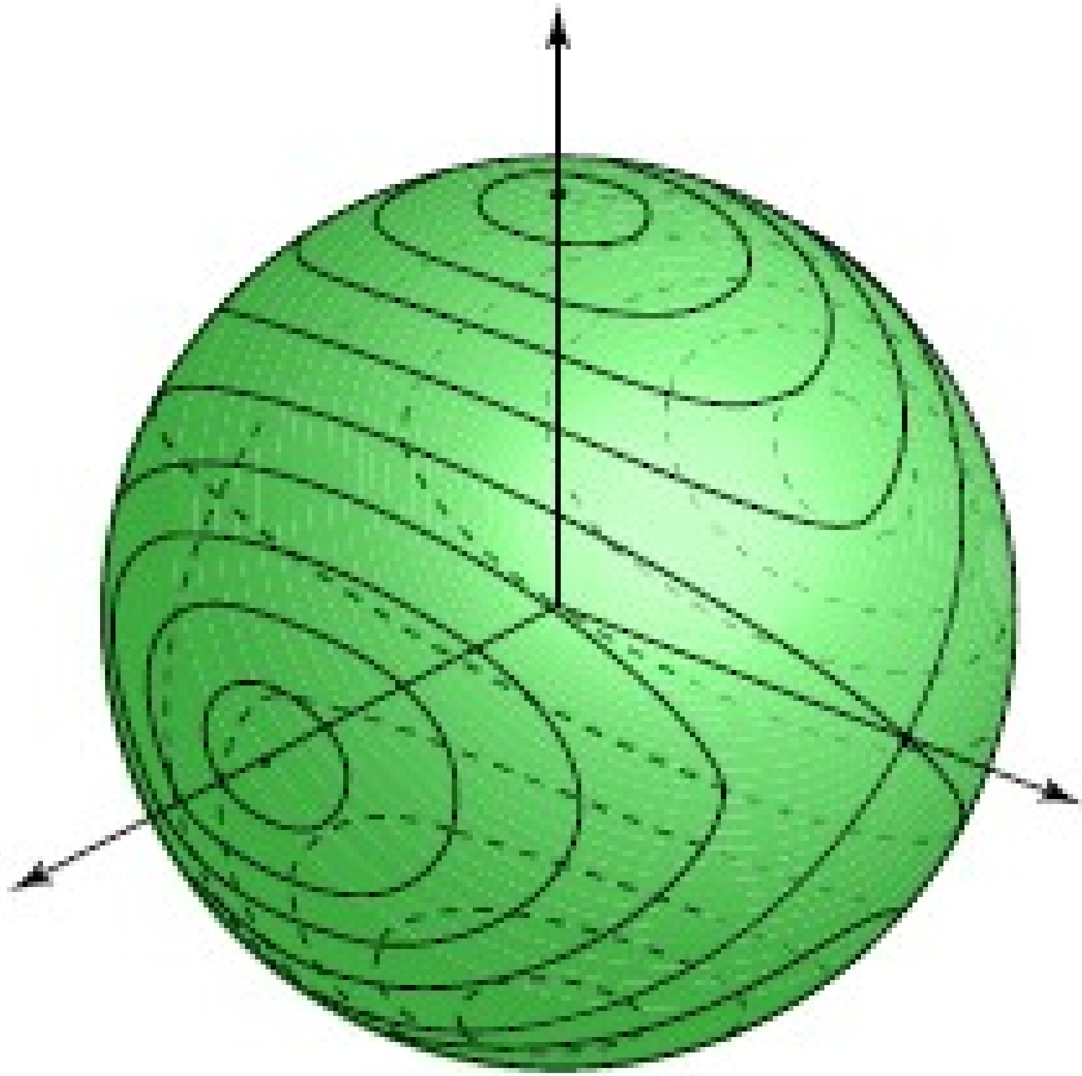
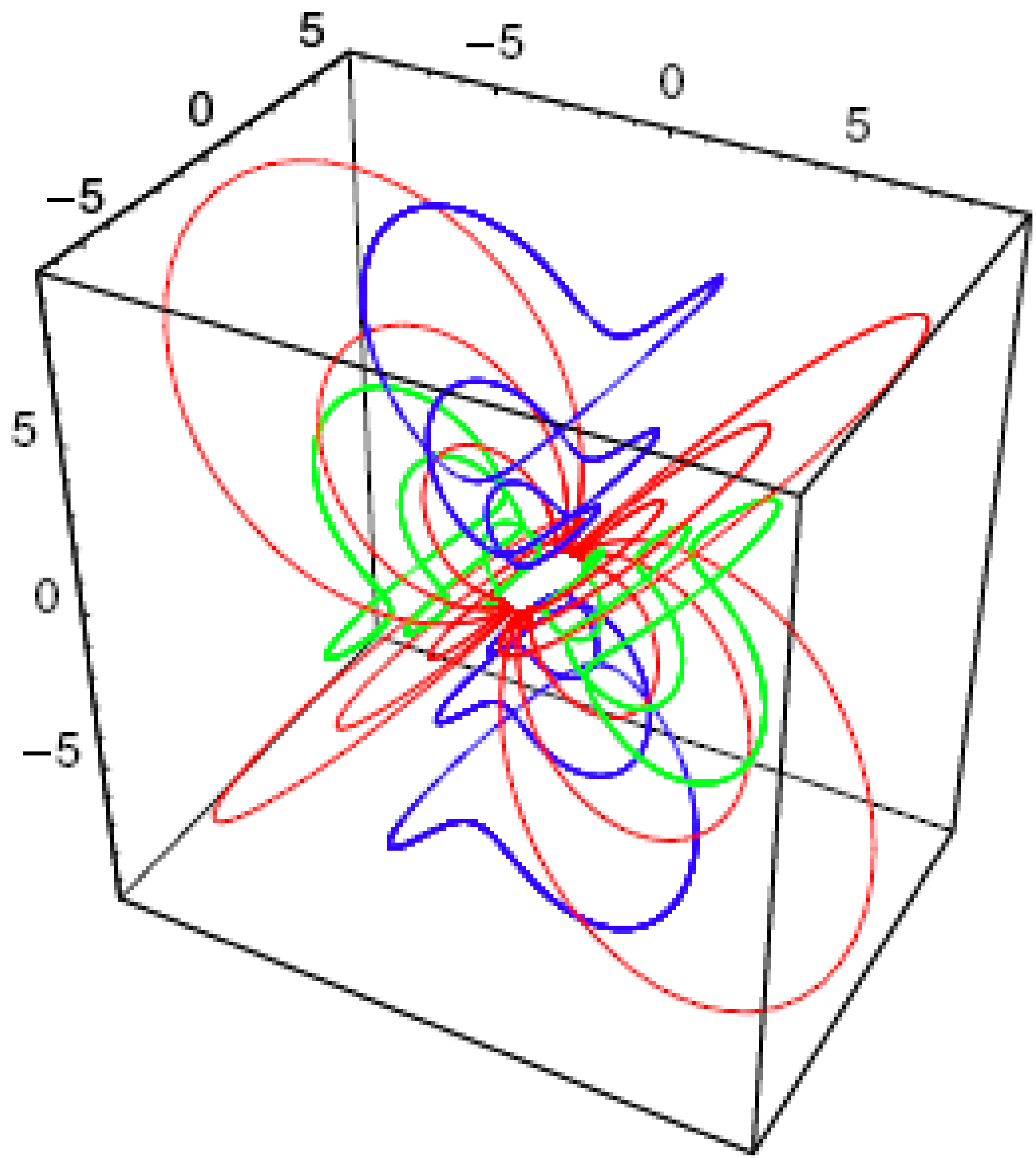


Figure 6. Nonperiodic non-$\mathcal{PT}$-symmetric complex solutions to the Lotka–Volterra equations (8). For the initial conditions $x(0) = 1 + \mathrm{i}$ and $y(0) = 1.097\,04 + 1.811\,73\mathrm{i}$ the complex trajectories $x(t)$ (left plot) and $y(t)$ (right plot) are clearly not periodic and not $\mathcal{PT}$ symmetric. (The initial conditions are indicated by dots.)



$$\dot{L}_1 = L_2 L_3, \qquad \dot{L}_2 = -2L_1 L_3, \qquad \dot{L}_3 = L_1 L_2.$$


5
0
−5
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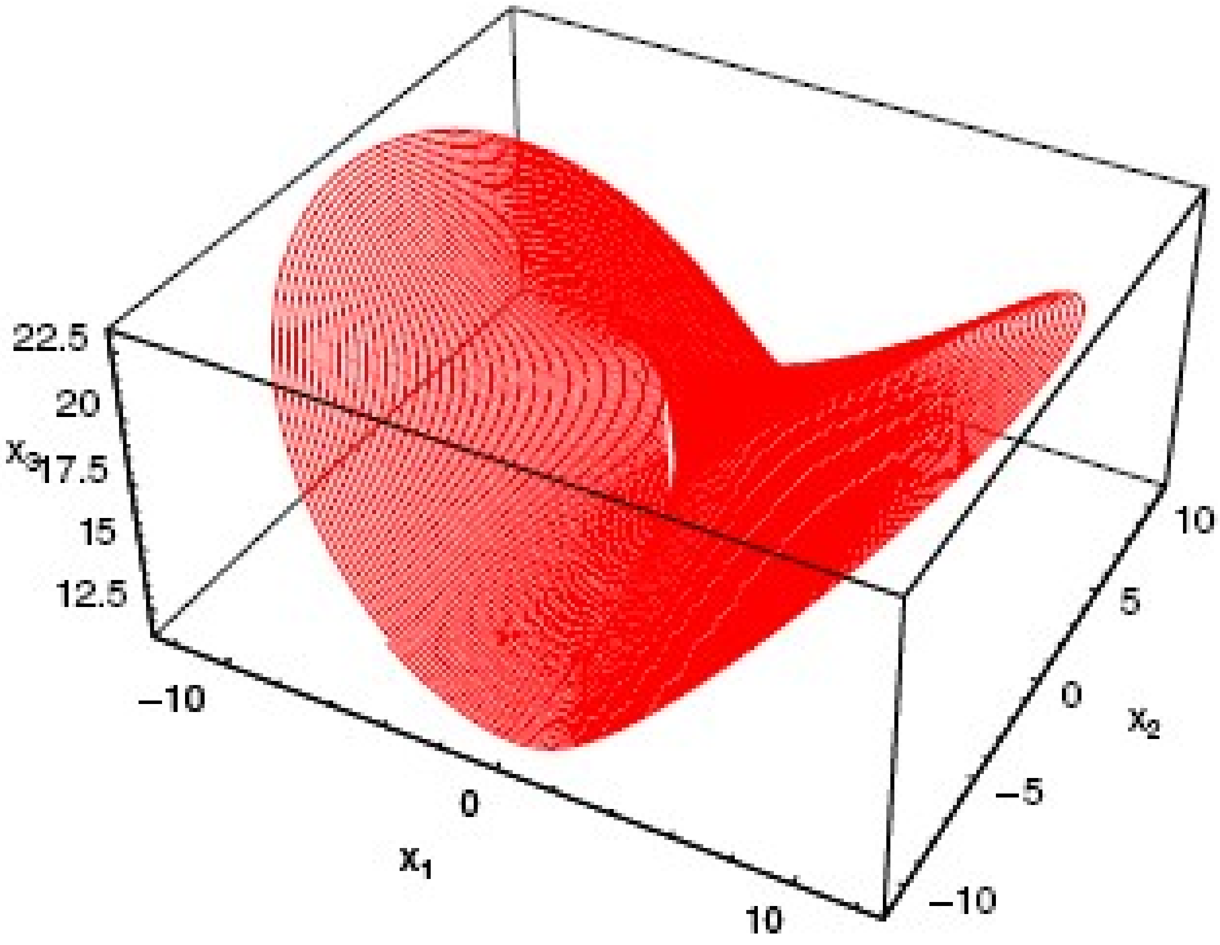


Figure 12. A single nonperiodic butterfly-shaped open trajectory that arises when the constants of motion $H = 1 + \mathrm{i}$ and $C = 1 + \mathrm{i}$ in (13) are complex. The trajectory is not $\mathcal{PT}$ symmetric and not closed. Rather, it spirals out to infinity.

Other PT-symmetric classical systems

- KdV equation
- Camassa-Holm equation
- Sine-Gordon equation
- Boussinesq equation

Making sense of non-Hermitian Hamiltonians

Crab Lender
Practised Nymphets
Washing Nervy Tuitions

Making sense of non-Hermitian Hamiltonians

Carl Bender
Physics Department
Washington University

Quantum Mechanics

- "Anyone who thinks he can contemplate quantum mechanics without getting dizzy hasn't properly understood it." – Niels Bohr
- "Anyone who thinks they know quantum mechanics doesn't." – Richard Feynman
- "I don't like it, and I'm sorry I ever had anything to do with it." – Erwin Schrödinger

Axioms of Quantum Mechanics

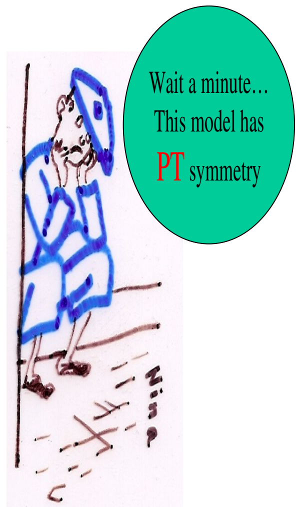
- Causality
- Locality
- Stability of the vacuum
- Relativity
- Probabilistic interpretation

Dirac Hermiticity …

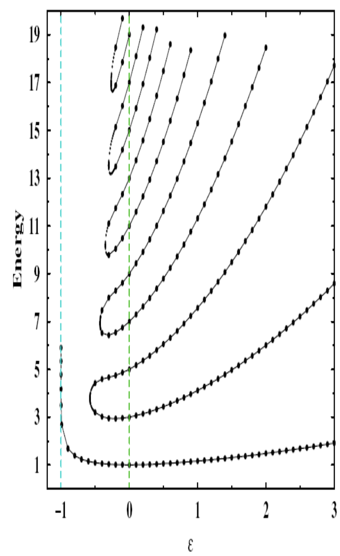
- guarantees real energy and conserved probability
- but … is mathematical and not physical

$$H = p^2 + ix^3$$

$$H = p^2 + ix^3$$



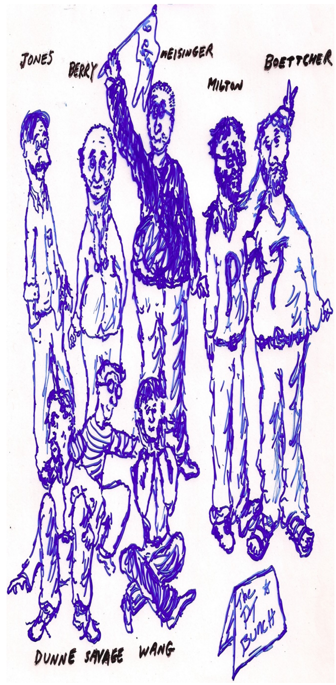
$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Some references …

- CMB and S. Boettcher, PRL **80**, 5243 (1998)
- CMB, D. Brody, H. Jones, PRL **89**, 270401 (2002)
- CMB, D. Brody, and H. Jones, PRL **93**, 251601 (2004)
- CMB, D. Brody, H. Jones, B. Meister, PRL **98**, 040403 (2007)
- CMB and P. Mannheim, PRL **100**, 110402 (2008)
- CMB, Reports on Progress in Physics **70**, 947 (2007)
- P. Dorey, C. Dunning, and R. Tateo, JPA **34**, 5679 (2001)
- P. Dorey, C. Dunning, and R. Tateo, JPA **40**, R205 (2007)

The original discoverers of PT symmetry:


JONES
BERRY
MEISINGER
MILTON
BOETTCHER
DUNNE
SAVAGE
WANG


PT
Posteitaliane
POSTAMAT

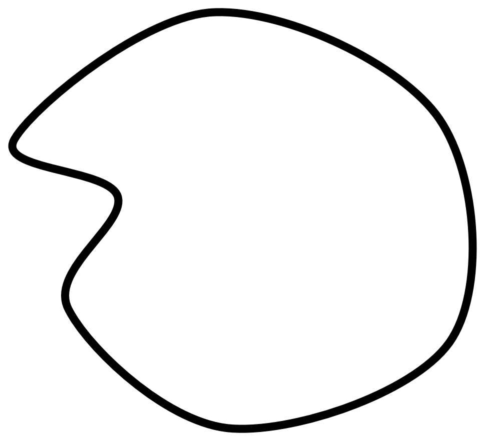

PALAZZO
PITTI
PT
INGRESSO PRINCIPALE
MAIN ENTRANCE
1 BIGLIETTERIA
TICKETS
2 INGRESSO MUSEI
MUSEUMS ENTRANCE


Lake
TRAILHEAD
SAN ISABEL
National Forest
U.S. DEPARTMENT OF AGRICULTURE


PT HIGHWAY
LITTER CONTROL
NEXT 2 MILES
SEROPTIMIST INT
OF SARATOGA


PT OF
INTEREST


CRUISER
SE VOCÊ



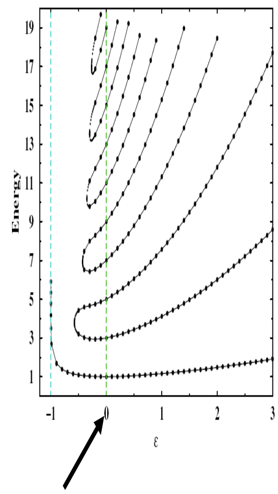

MY TALK

Outline of Talk

- Beginning
- Middle
- End
- (applause)

How to "prove" that the eigenvalues are real

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$

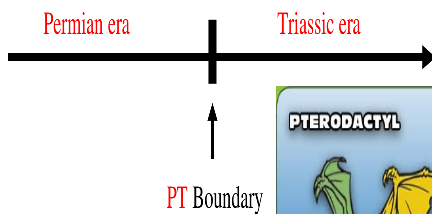


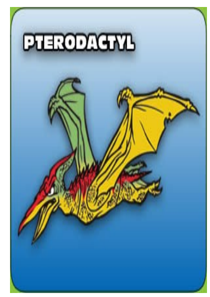
How to "prove" that the eigenvalues are real

The proof is really hard!
You need to use
(3)Bethe ansatz
(4)Monodromy group
(5)Baxter T-Q relation
(6)Functional Determinants

PT Boundary

Greatest murder mystery of all time…
Extinction of over 90% of species!





OK, so the eigenvalues are real ...
But is this quantum mechanics??

- Probabilistic interpretation??
- Hilbert space with a positive metric??
- Unitarity??

Dirac, Bakerian Lecture 1941, Proceedings of the Royal Society A

Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results. The physical interpretation of relativistic quantum mechanics that one gets by a natural development of the non-relativistic theory involves these things and is thus in contradiction with experiment. We therefore have to consider ways of modifying or supplementing this interpretation.

The Hamiltonian determines its own adjoint

$$[\mathcal{C}, \mathcal{PT}] = 0,$$

$$[\mathcal{C}^2 = 1],$$

$$[\mathcal{C}, H] = 0$$

Replace $\dagger$ by $\mathcal{CPT}$

Unitarity

With respect to the *CPT* adjoint the theory has UNITARY time evolution.

Norms are strictly positive!
Probability is conserved!

OK, we have unitarity…
But is PT quantum mechanics useful??

- It revives quantum theories that were thought to be dead
- It is beginning to be observed experimentally

The Lee Model

$$V \to N + \theta, \qquad N + \theta \to V.$$

$$H = H_0 + g_0 H_1,$$

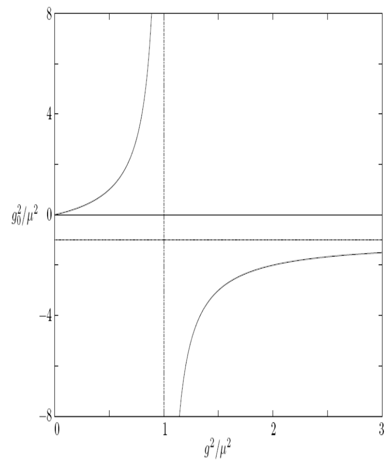
$$H_0 = m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a,$$

$$H_1 = V^\dagger N a + a^\dagger N^\dagger V.$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

The problem:



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

MR0076639 (17,927d) 81.0X
Källén, G.; Pauli, W.
On the mathematical structure of T. D. Lee's model of a renormalizable field theory.
Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), *no. 7*, 23 *pp.*

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR0064658 (16,317b)] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V \rightleftarrows N + \theta$ but forbidding $N \rightleftarrows V + \theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/{g_0}^2 = 1 - Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

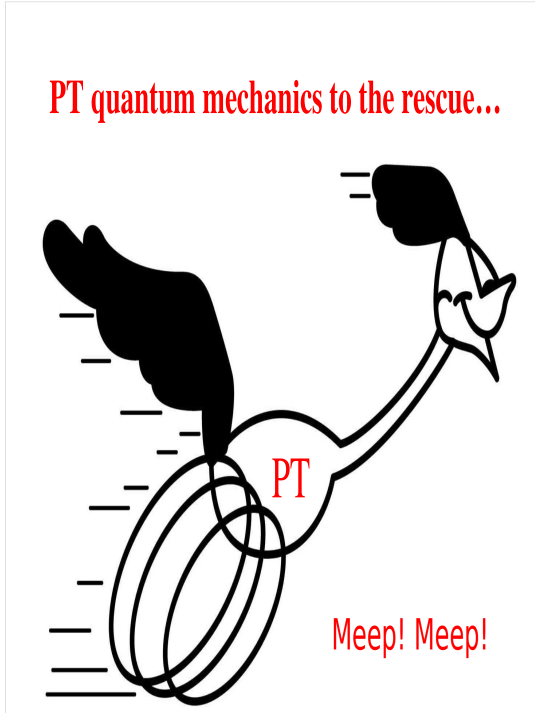
The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio $g^2/{g_0}^2$ should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR0046306 (13,713d); Källén, Helv. Phys. Acta **25** (1952), 417–434; MR0051156 (14,435l); Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR0072756 (17,332e); Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR0064652 (16,315e)]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value $g_{\rm crit}$. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

The authors discover the remarkable result that if $g > g_{\rm crit}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g > g_{\rm crit}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

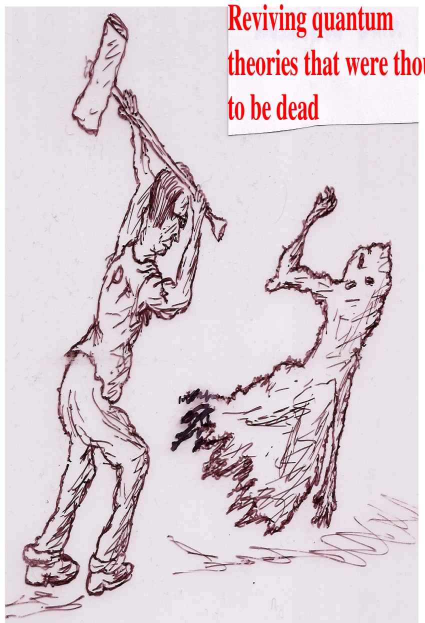
Reviewed by *A. Salam*

“A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)


PT quantum mechanics to the rescue...
PT
Meep! Meep!

GHOSTBUSTING:
Reviving quantum theories that were thought to be dead



Pais-Uhlenbeck action

$$I = \frac{\gamma}{2} \int dt \left[\ddot{z}^2 - \left(\omega_1^2 + \omega_2^2\right) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2 \right]$$

Gives a fourth-order field equation:

$$z''''(t) + (\omega_1^2 + \omega_2^2) z''(t) + \omega_1^2 \omega_2^2 z(t) = 0$$

CMB and P. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008)
CMB and P. Mannheim, Phys. Rev. D **78**, 025002 (2008)

The problem: A fourth-order field equation gives a propagator like

$$G(E) = \frac{1}{(E^2 + m_1^2)(E^2 + m_2^2)}$$

$$G(E) = \frac{1}{m_2^2 - m_1^2}\left(\frac{1}{E^2 + m_1^2} - \frac{1}{E^2 + m_2^2}\right)$$

GHOST!

There are now two possible realizations...

(I) If a_1 and a_2 annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2|\Omega\rangle = 0,$$

then the energy spectrum is real and bounded below. The state $|\Omega\rangle$ is the ground state of the theory and it has zero-point energy $\frac{1}{2}(\omega_1 + \omega_2)$. The problem with this realization is that the excited state $a_2^\dagger|\Omega\rangle$, whose energy is ω_2 above ground state, has a *negative Dirac norm* given by $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$.

(II) If a_1 and $a_2^\dagger$ annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2^\dagger|\Omega\rangle = 0,$$

then the theory is free of negative-norm states. However, this realization has a different and equally serious problem; namely, that the energy spectrum is unbounded below.

There can be many realizations!

$$H = p^2 - x^4$$

$$-\psi''(x) - x^4\psi(x) = E\psi(x)$$

Equivalent Dirac Hermitian Hamiltonian:

$$\mathcal{C} = e^{\mathcal{Q}}\mathcal{P} \qquad \tilde{H} = e^{-\mathcal{Q}/2}He^{\mathcal{Q}/2}$$

$$\tilde{H} = p^2 + 4x^4 - 2\hbar x$$

$$\mathcal{Q} = \alpha pq + \beta xy$$

$$\beta = \gamma^2\omega_1^2\omega_2^2\alpha \quad \text{and} \quad \sinh(\sqrt{\alpha\beta}) = \frac{2\omega_1\omega_2}{\omega_1^2 - \omega_2^2}$$

$$\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2}$$

$$\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2$$

No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck model, CMB and P. Mannheim, PRL **100**, 110402 (2008)

TOTALITARIAN PRINCIPLE

"Everything which is not forbidden is compulsory."

---M. Gell-Mann

And there are now observations in table-top optics experiments!

Observing *PT* symmetry using wave guides:

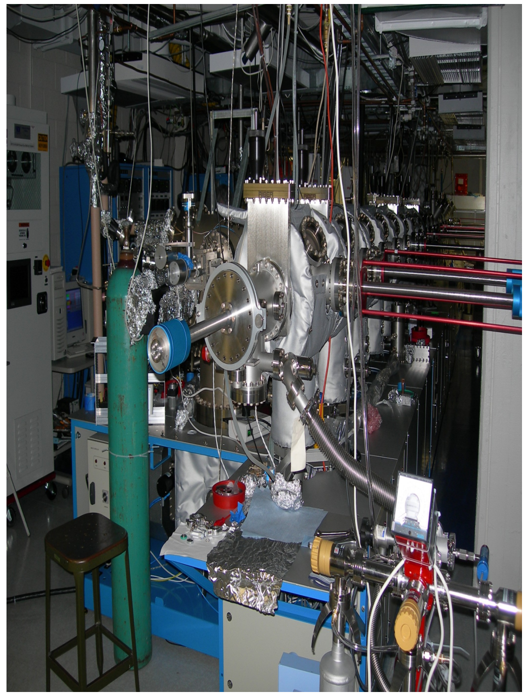
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, PRL **100**, 030402 (2008)
- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, PRL **100**, 103904 (2008)

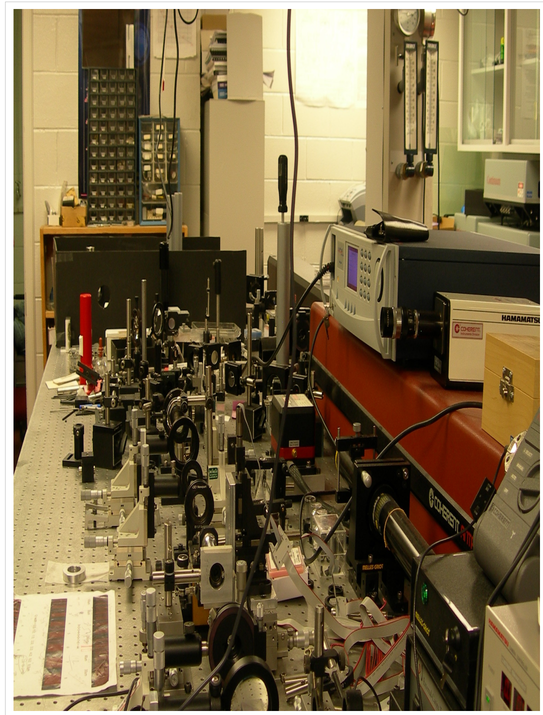
Date: Thu, 13 Mar 2008 23:04:45 -0400
From: Demetrios Christodoulides <demetri@creol.ucf.edu>
To: Carl M. Bender <cmb@wuphys.wustl.edu>
Subject: Re: Benasque workshop on non-Hermitian Hamiltonians

Dear Carl,
I have some good news from Greg Salamo (U. of Arkansas). His students (who are now visiting us here in Florida) have just observed a PT phase transition in a passive AlGaAs waveguide system. We will be submitting soon these results as a post-deadline paper to CLEO/QELS and subsequently to a regular journal. We are still fighting against the Kramers-Kronig relations, but the phase transition effect is definitely there. We expect even better results under TE polarization conditions. I will bring them over to Israel.

In close collaboration with us, more teams (also best friends!) are moving ahead in this direction. Moti Segev (from Technion) is planning an experiment in an active-passive dual core optical fiber -- fabricated in Southampton, England. More experiments will be carried later in Germany by Detlef Kip. Christian (his post doc) just left from here with a possible design. If everything goes well, with a bit of luck we may have an experimental explosion in the PT area. I wish the funding situation was a bit better. So far everything is done on a shoe-string budget (it is subsidized by other projects). Let us see...

All the best
Demetri

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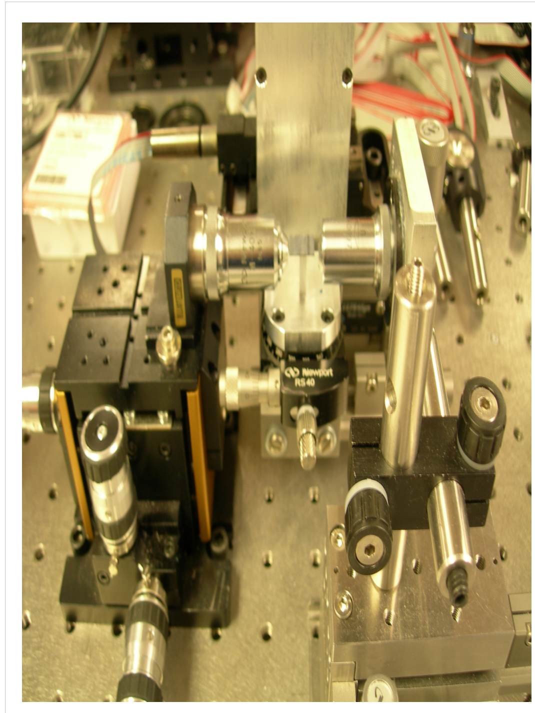
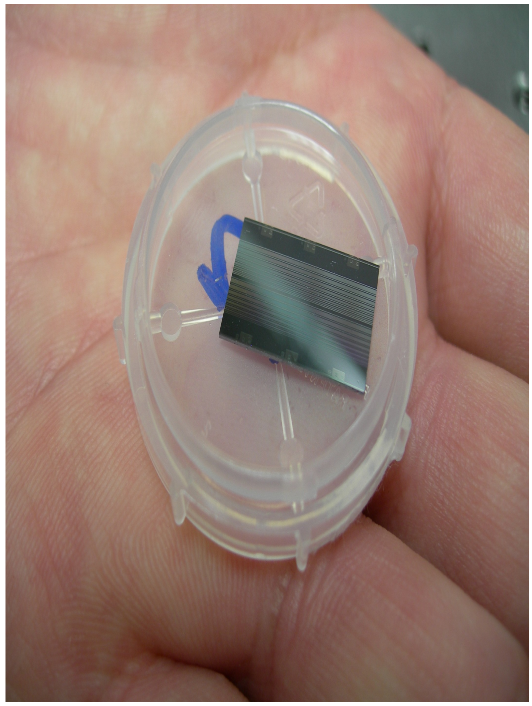

Newport
RS40



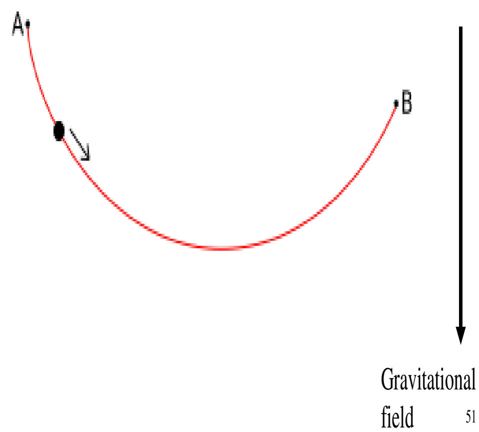
OK, but how do we interpret a non-Hermitian Hamiltonian??

Solve the quantum brachistochrone problem…

Classical Brachistochrone

- Newton
- Bernoulli
- Leibniz
- L'Hôpital

Classical Brachistochrone is a cycloid



Quantum Brachistochrone

$$|\psi_I\rangle \to |\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle$$

Constraint: $\omega = E_{\max} - E_{\min}$

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Hermitian case

$$H = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix} \qquad (r,\ s,\ u,\ \theta \text{ real})$$

$$H = \tfrac{1}{2}(s+u)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n}$$

$$\mathbf{n} = \frac{1}{\omega}(2r\cos\theta, 2r\sin\theta, s-u) \qquad \omega^2 = (s-u)^2 + 4r^2$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\exp(i\phi\,\sigma\cdot\mathbf{n}) = \cos\phi\,\mathbf{1} + i\sin\phi\,\sigma\cdot\mathbf{n}$$

$$|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$$

becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-\frac{1}{2}i(s+u)t/\hbar} \begin{pmatrix} \cos\frac{\omega t}{2\hbar} - i\frac{s-u}{\omega}\sin\frac{\omega t}{2\hbar} \\ -i\frac{2r}{\omega}e^{i\theta}\sin\frac{\omega t}{2\hbar} \end{pmatrix}$$

$$t = \frac{2\hbar}{\omega}\arcsin\frac{\omega|b|}{2r}$$

Minimize t over all positive *r* while maintaining constraint

$$\omega^2 = (s-u)^2 + 4r^2.$$

Minimum evolution time:

$$\tau\omega = 2\hbar \arcsin |b|.$$

Looks like uncertainty principle but is merely
rate times time = distance

Note that if $a = 0$ and $b = 1$, we have $\tau = \pi\hbar/\omega$ for the smallest time required to transform $\binom{1}{0}$ to the orthogonal state $\binom{0}{1}$. The time τ required to transform a vector into an orthogonal vector is called the *passage time*.

Non-Hermitian PT-symmetric Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (r,\ s,\ \theta \text{ real})$$

$\mathcal{T}$ is complex conjugation and $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$E_{\pm} = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \quad \text{real if } s^2 > r^2\sin^2\theta$$

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}$$

where $\sin\alpha = (r/s)\sin\theta.$

Exponentiate H

$$H = (r\cos\theta)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n},$$

where

$$\mathbf{n} = \frac{2}{\omega}(s, 0, ir\sin\theta)$$

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta.$$

$$e^{-iHt/\hbar}\begin{pmatrix}1\\0\end{pmatrix} = \frac{e^{-itr\cos\theta/\hbar}}{\cos\alpha}\begin{pmatrix}\cos(\frac{\omega t}{2\hbar}-\alpha)\\ -i\sin\left(\frac{\omega t}{2\hbar}\right)\end{pmatrix}$$

Consider the pair of vectors used in the Hermitian case: $|\psi_I\rangle = \binom{1}{0}$ and $|\psi_F\rangle = \binom{0}{1}$. (Note that these two vectors are not orthogonal with respect to the $\mathcal{CPT}$ inner product.) Observe that the evolution time needed to reach $|\psi_F\rangle$ from $|\psi_I\rangle$ is $t = (2\alpha - \pi)\hbar/\omega$. Optimizing this result over allowable values for α as α approaches $\frac{1}{2}\pi$, the optimal time τ tends to zero!

Interpretation...

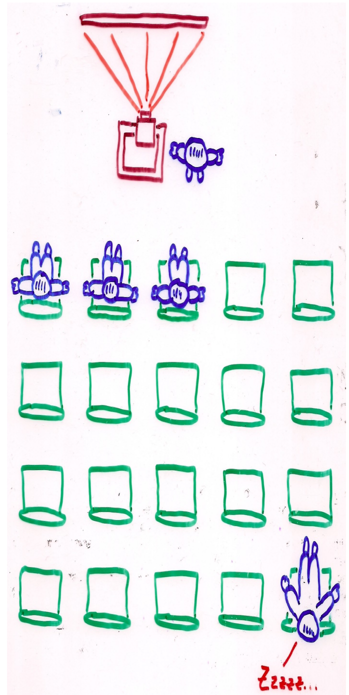
Finding the optimal PT-symmetric Hamiltonian amounts to constructing a wormhole in Hilbert space!

“The shortest path between two truths in the real domain passes through the complex domain.”

-- Jacques Hadamard

[The Mathematical Intelligencer **13** (1991)]

Overview of talk:



Classical PT symmetry

Provides an intuitive explanation of what is going on…

Motion on the Real Axis

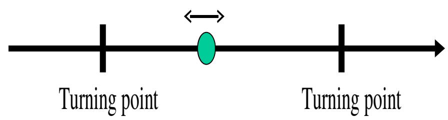
Motion of particles is governed by Newton's Law:

F=ma

In freshman physics this motion is restricted to the REAL AXIS.

Harmonic Oscillator: Particle on a Spring

Back and forth motion
on the real axis:



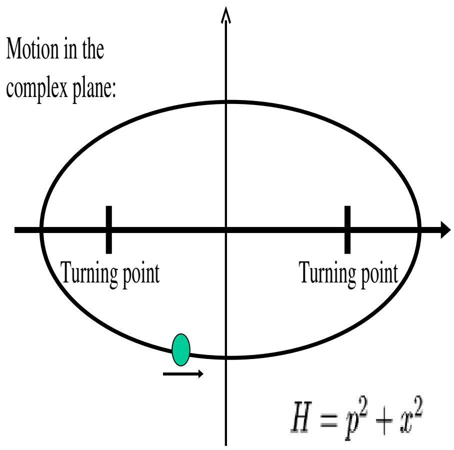
$$H = p^2 + x^2 \qquad (\epsilon = 0)$$

Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p}$$

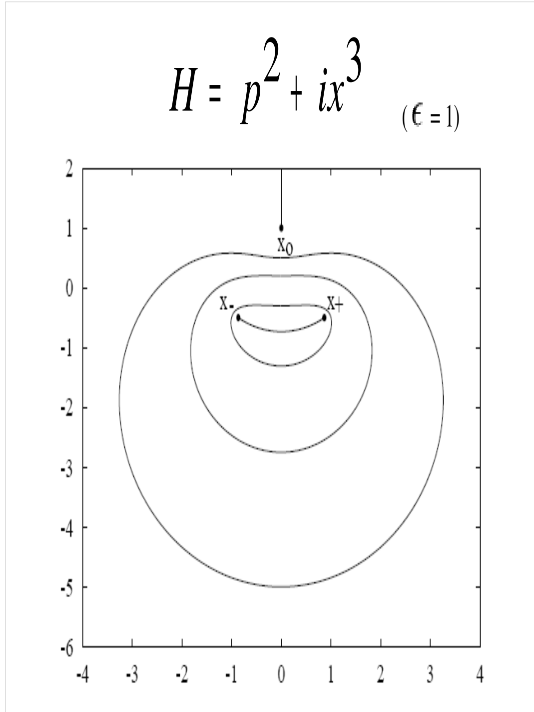
$$\dot{p} = -\frac{\partial H}{\partial x}$$

Harmonic Oscillator:



$$H = p^2 + x^2$$

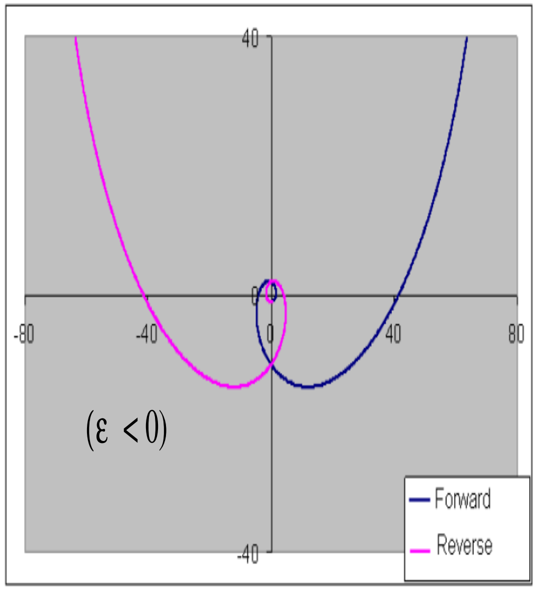
$(\epsilon = 0)$

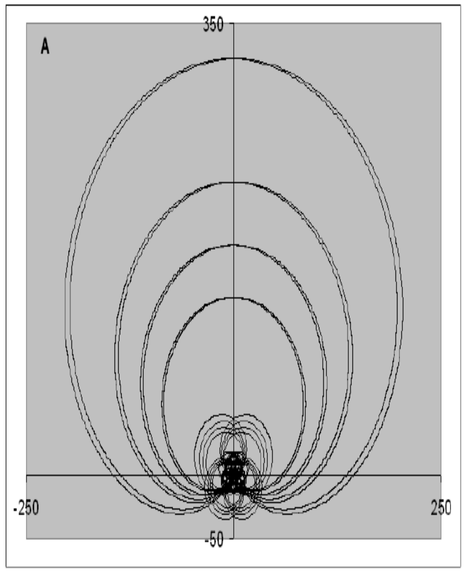

$H = p^2 + ix^3$
($\epsilon = 1$)
x_0
x_-
x_+
2
1
0
-1
-2
-3
-4
-5
-6
-4
-3
-2
-1
0
1
2
3
4

Bohr-Sommerfeld Quantization of a complex atom

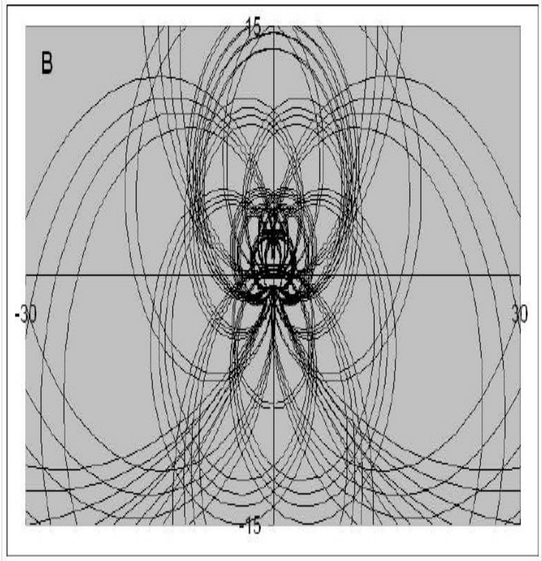
$$\oint dx\, p = \left(n + \frac{1}{2}\right)$$

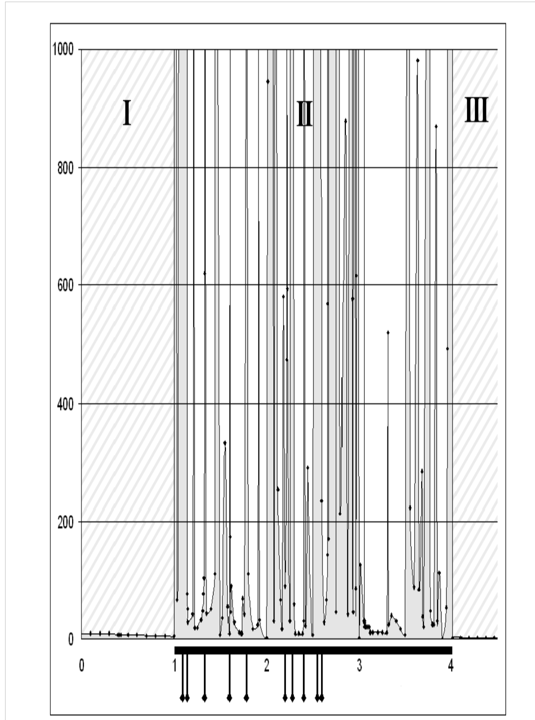
Broken PT symmetry



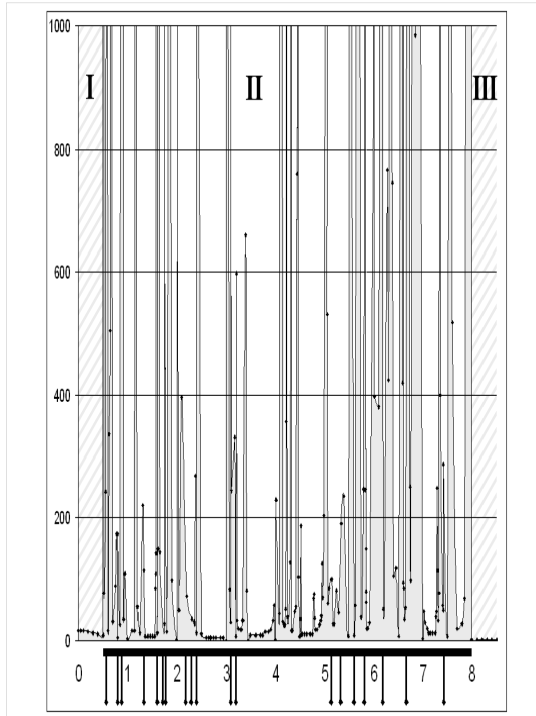


$\epsilon = \pi - 2$ 11 sheets

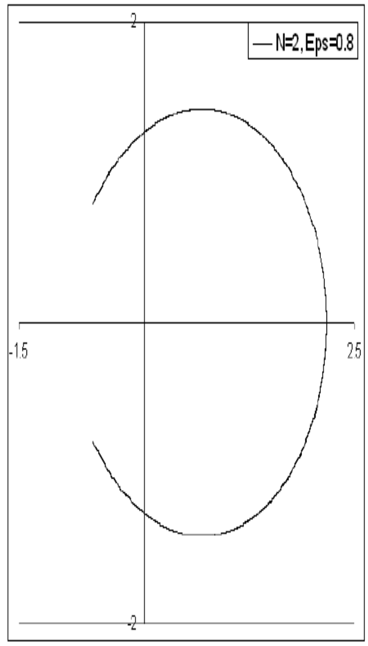

B
15
-30
30
-15



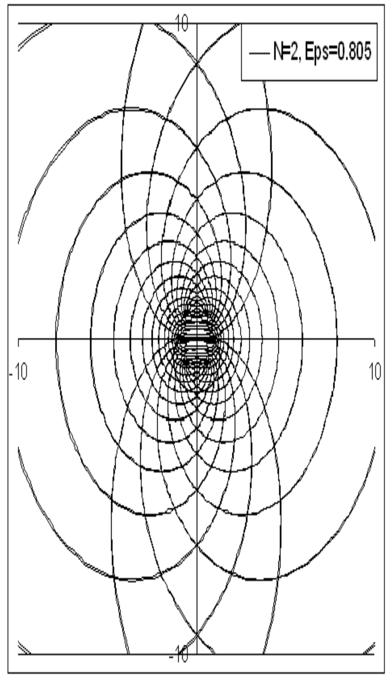
1000
800
600
400
200
0
I
II
III
0
1
2
3
4



1000
800
600
400
200
0
I
II
III
0
1
2
3
4
5
6
7
8



2
N=2, Eps=0.8
-1.5
2.5
-2



N=2, Eps=0.805
10
-10
-10
10

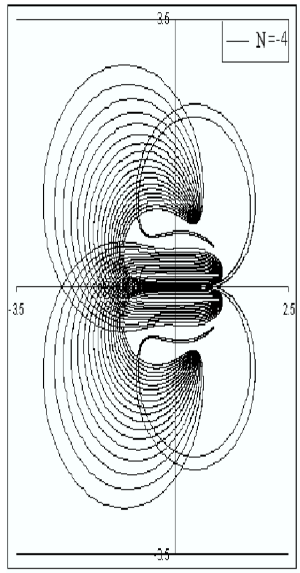
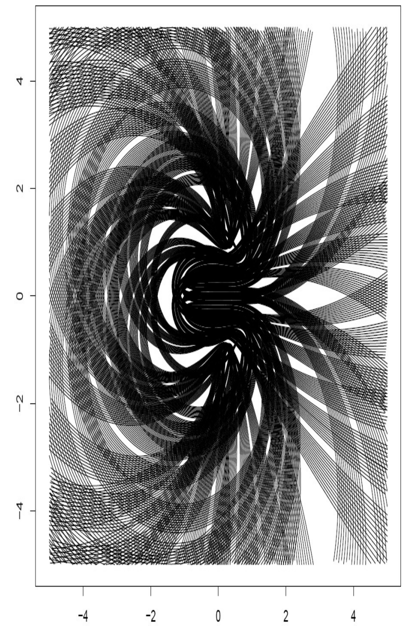


Figure 21. Non-$\mathcal{PT}$-symmetric orbit for $\epsilon = \frac{16}{9}$. This topologically complicated orbit originates at the $N = -4$ turning point but does not reach the $\mathcal{PT}$-symmetric $N = 3$ turning point. Instead, it is reflected back at the complex-conjugate $N = -14$ turning point. The period of this orbit is $T = 186.14$.

The Beast $\epsilon = 16/15$



Lotka-Volterra equations

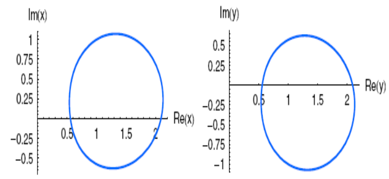


Figure 5. Periodic $\mathcal{PT}$-symmetric complex solutions to the Lotka–Volterra equations (8). For the initial conditions $x(0) = 1 + \mathrm{i}$ and $y(0) = 2.112\,21 - 0.403\,243\mathrm{i}$ the complex trajectories $x(t)$ (left plot) and $y(t)$ (right plot) are shown. Observe that the trajectories are periodic and $\mathcal{PT}$ symmetric, where $\mathcal{P}$ reflection interchanges x and y and $\mathcal{T}$ reflection consists of complex conjugation.

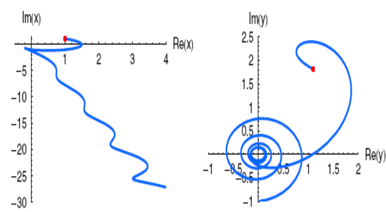
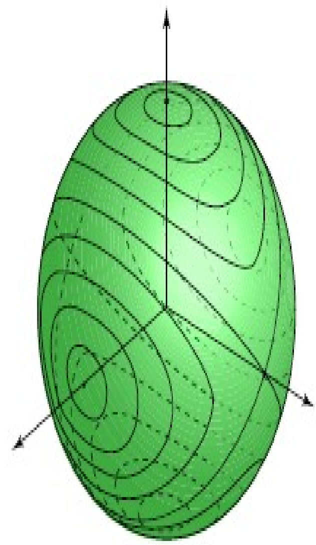
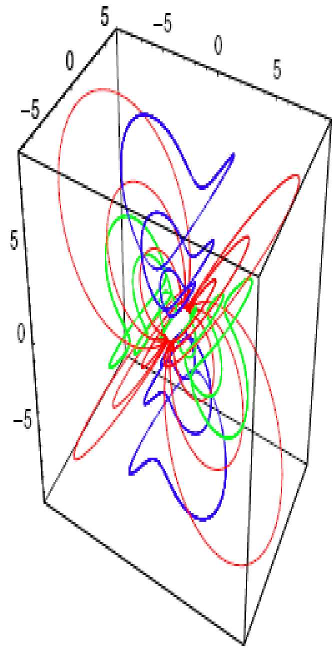


Figure 6. Nonperiodic non-$\mathcal{PT}$-symmetric complex solutions to the Lotka–Volterra equations (8). For the initial conditions $x(0) = 1 + \mathrm{i}$ and $y(0) = 1.097\,04 + 1.811\,73\mathrm{i}$ the complex trajectories $x(t)$ (left plot) and $y(t)$ (right plot) are clearly not periodic and not $\mathcal{PT}$ symmetric. (The initial conditions are indicated by dots.)



$$\dot{L}_1 = L_2 L_3, \qquad \dot{L}_2 = -2 L_1 L_3, \qquad \dot{L}_3 = L_1 L_2.$$


5
−5
0
5
0
−5
5
0
−5

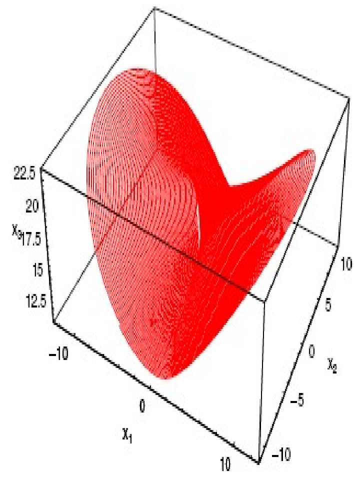


Figure 12. A single nonperiodic butterfly-shaped open trajectory that arises when the constants of motion $H = 1 + \mathrm{i}$ and $C = 1 + \mathrm{i}$ in (13) are complex. The trajectory is not $\mathcal{PT}$ symmetric and not closed. Rather, it spirals out to infinity.

Other PT-symmetric classical systems

- KdV equation
- Camassa-Holm equation
- Sine-Gordon equation
- Boussinesq equation