Fixed-Energy Sandpiles Belong Generically to Directed Percolation

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TIFR, Mumbai

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Outline

- Part- I

  Why things can go wrong in the study of absorbing phase transition ?

  - the difficulties
  - how to avoid them

- Part-II

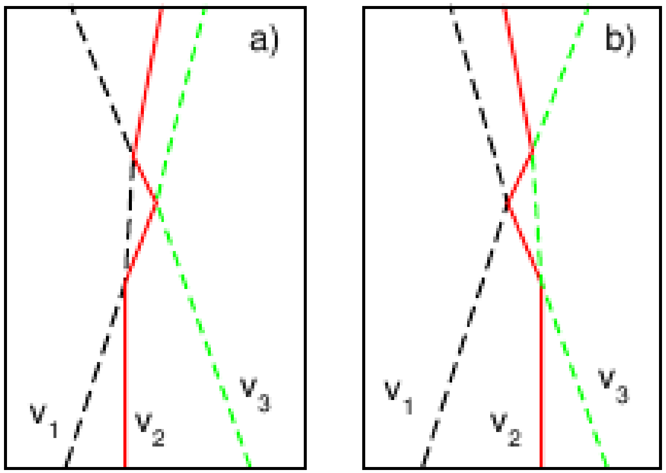
  - fixed energy sandpile models as APT
  - generically belong to DP ?

Classical Systems

- Hamiltonian dynamics

$$\boxed{\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \qquad \dot{q} = \frac{\partial \mathcal{H}}{\partial p}}$$

- Many particles. Predictability ?

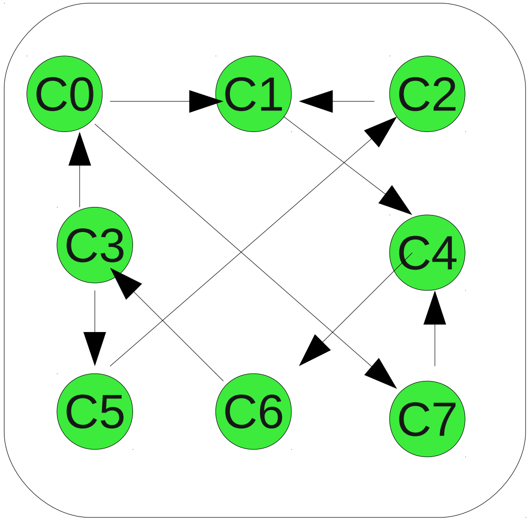


Chaos ?

Denisov, PK, Politi et.al, EPJB 2007

Stochastic dynamics (Markovian)

- System wonders in configuration space



Transition rate : $W(C \to C')$

- How often a configuration is visited ?

Prob(C)= P(C)

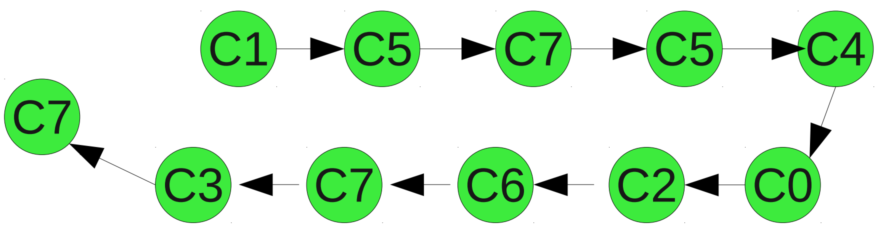
- Stationary state (ss): P(C) do not change with t

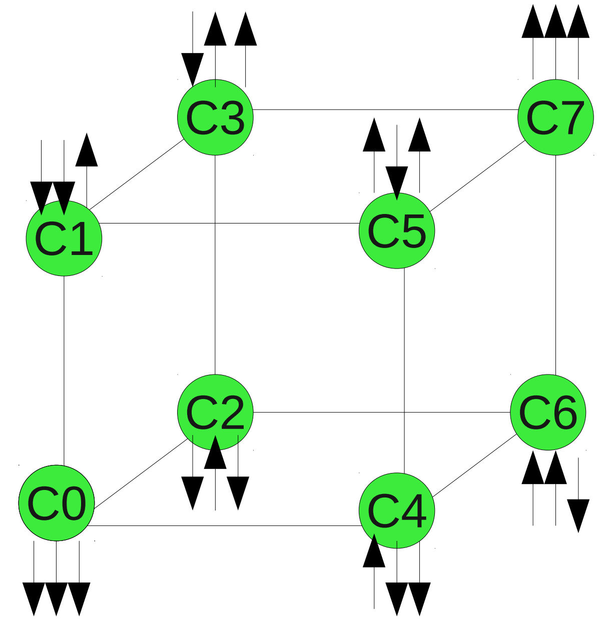
- **Equilibrium :** a special ss, satisfy detailed balance..

$$P(C)\, W(C \to C') = P(C')\, W(C' \to C)$$

Ergodicity

- Ising Model
- Monte Carlo (single spin flip)





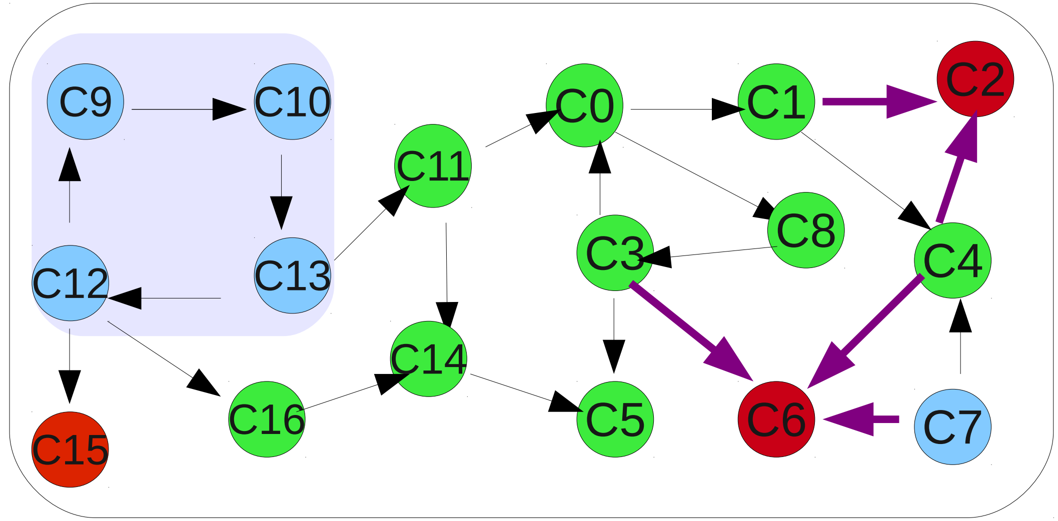
-
- Non-zero chance of reaching
  any configuration

Phase space

- Detailed balance

$$P(C)\, W(C \rightarrow C') = P(C')\, W(C' \rightarrow C)$$

Absorbing and 'transient' configurations



- **Absorbing :**

  - once reached,

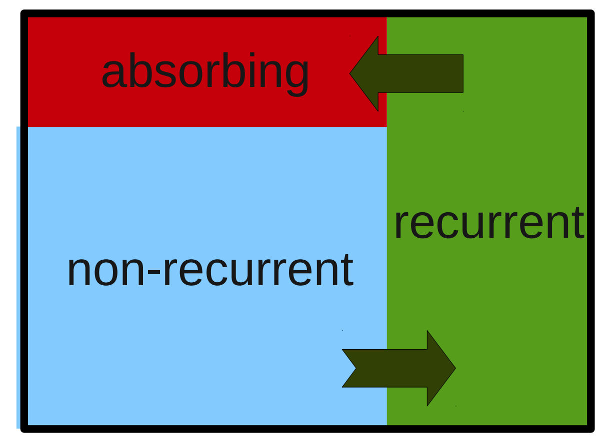
  the system can not escape

- **Transient :**

  a set configurations the system does not visit, once left.

Nontrivial time evolution

Rest of the talk :
I will elaborate these difficulties

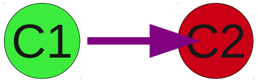


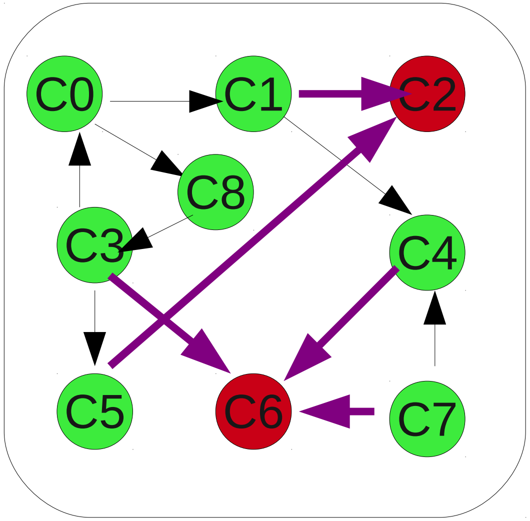
Phase space

Absorbing phase transition (APT)

- One or more absorbing configurations
- No detailed balance :

  $W(C2 \rightarrow C1)=0$





- Question : Will a thermodynamic system

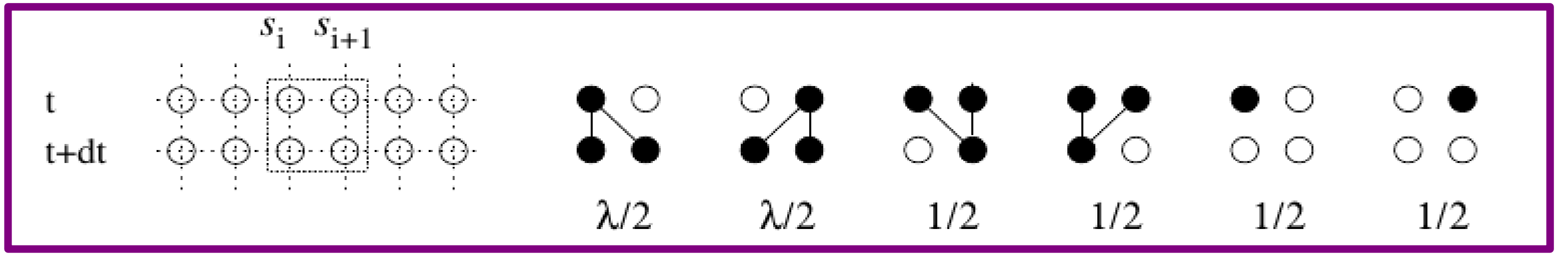
  falls into the absorbing state ?

YES **NO.** Remain active with non-zero probability

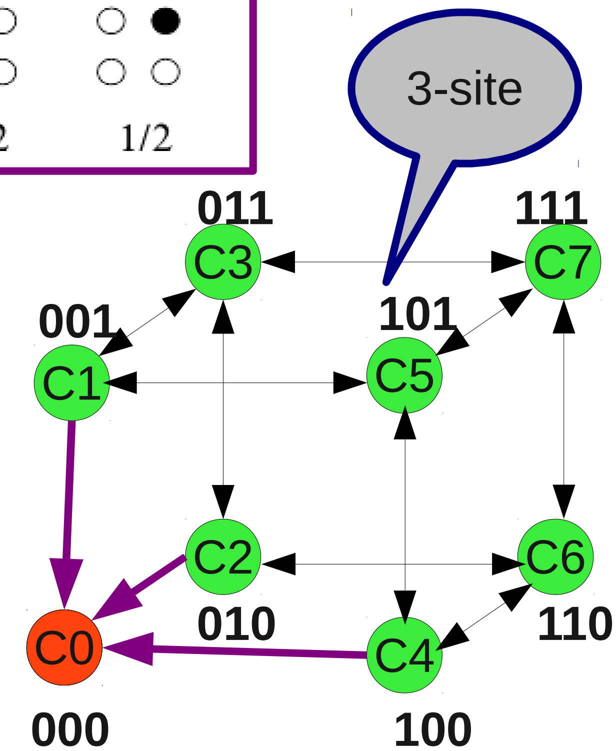
Some control parameter →

•a non-equilibrium phase transition : APT

Example : Contact Process (CP)

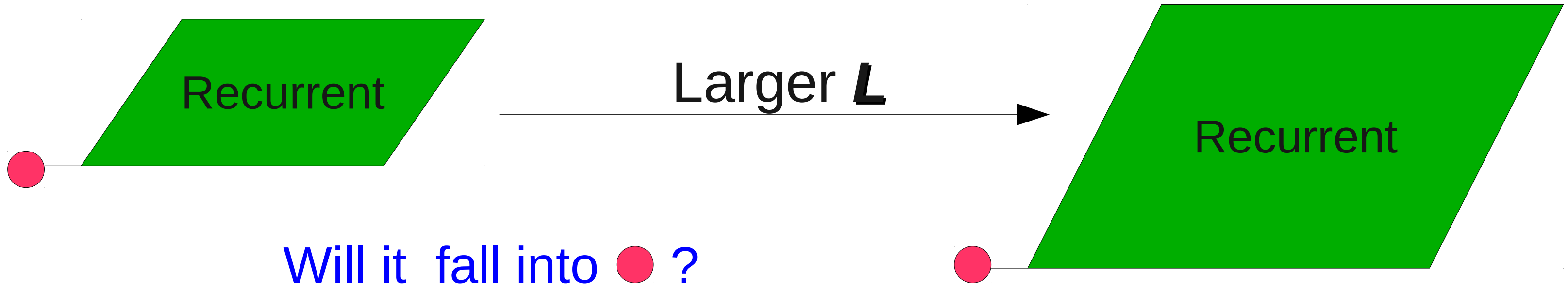


$$0 \xleftarrow{1} A \xrightarrow{\lambda} 2A$$

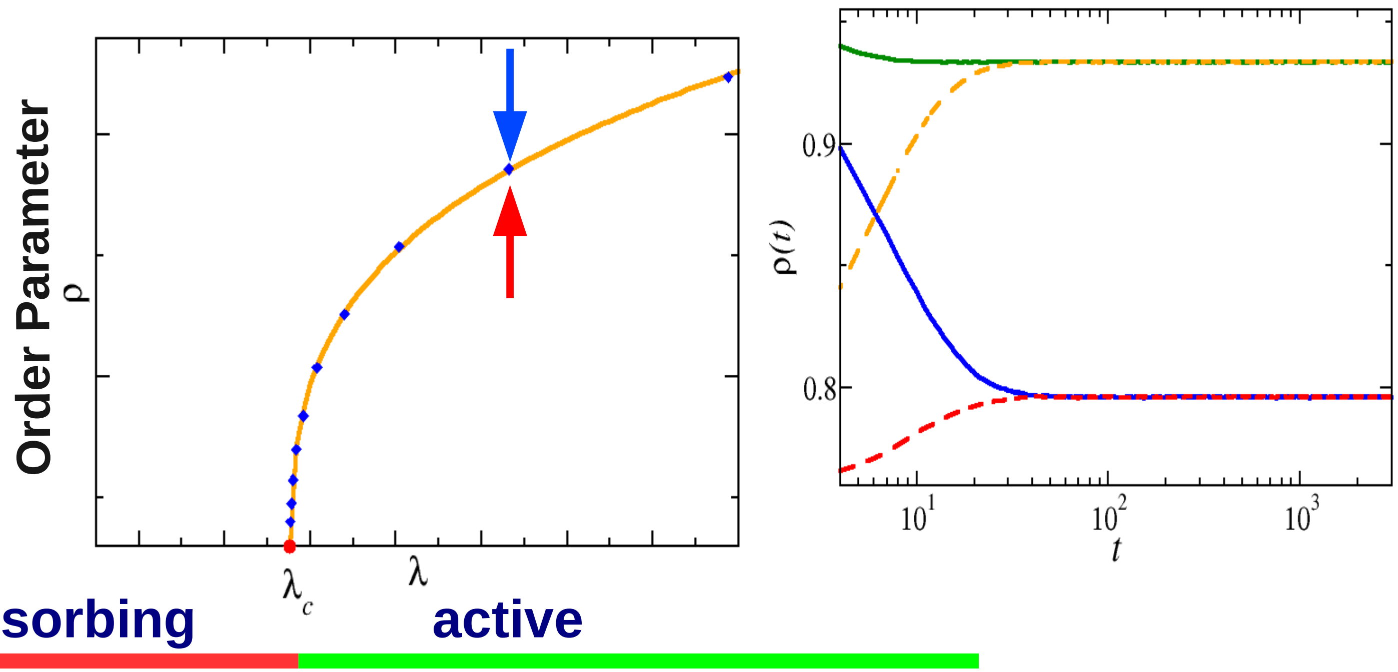


- A particle can create another particle in its (vacant) neighbourhood (rate λ).
- It may die with rate 1
- Only one absorbing configuration : {...0000...}

Absorbing phase transition in CP



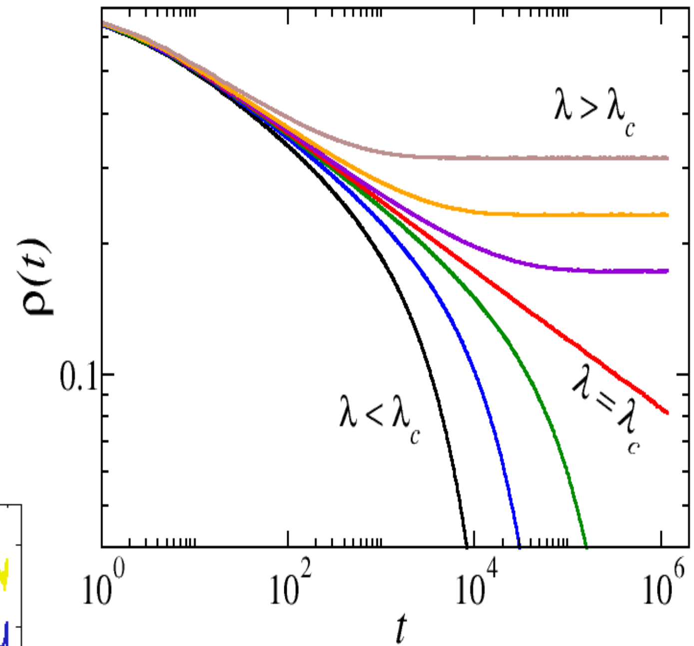
- Absorbing phase transition



$$\lambda_c = 3.2978$$

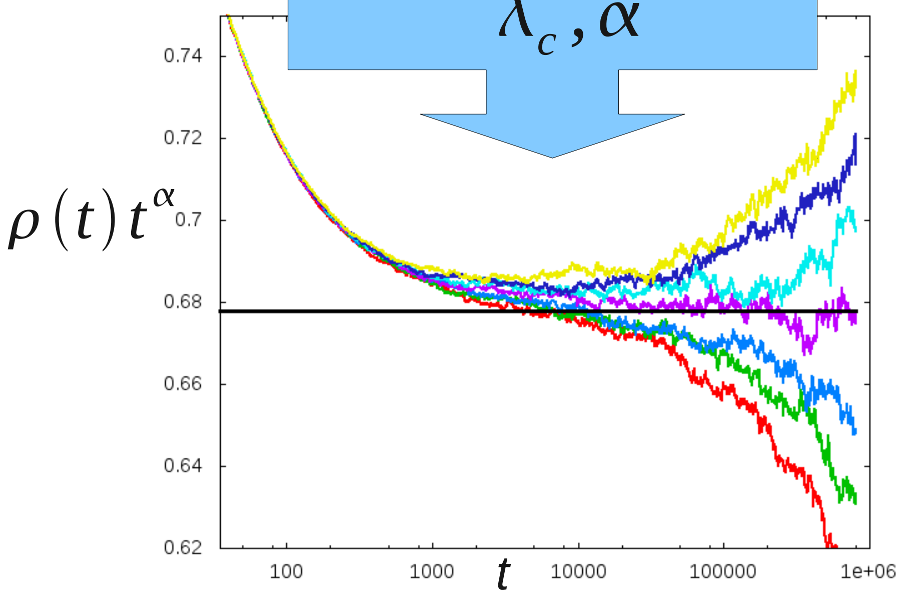
Critical behaviour (numerical study)

- Decay of OP from a highly active state
- At $\lambda=\lambda_c, \quad \rho(t)\sim t^{-\alpha}$



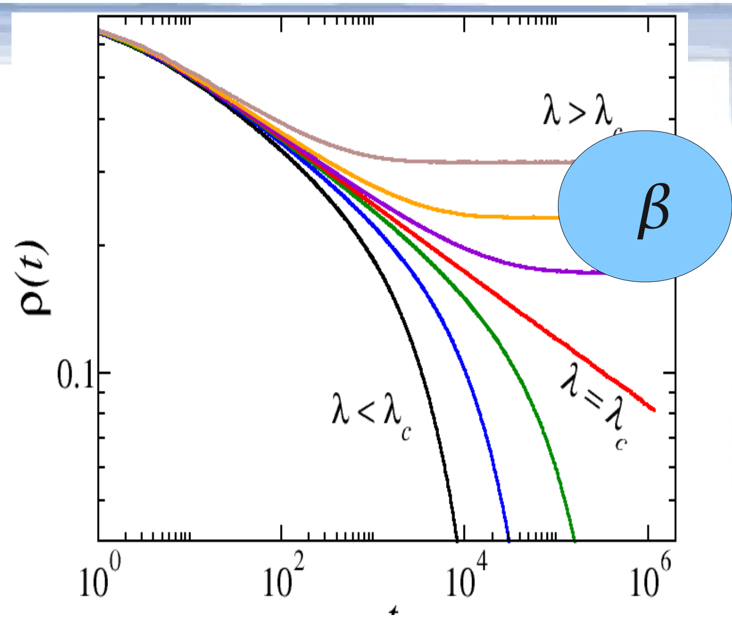
For large t, $\rho(t)t^{\alpha}=const.$

Determination of λ_c, α

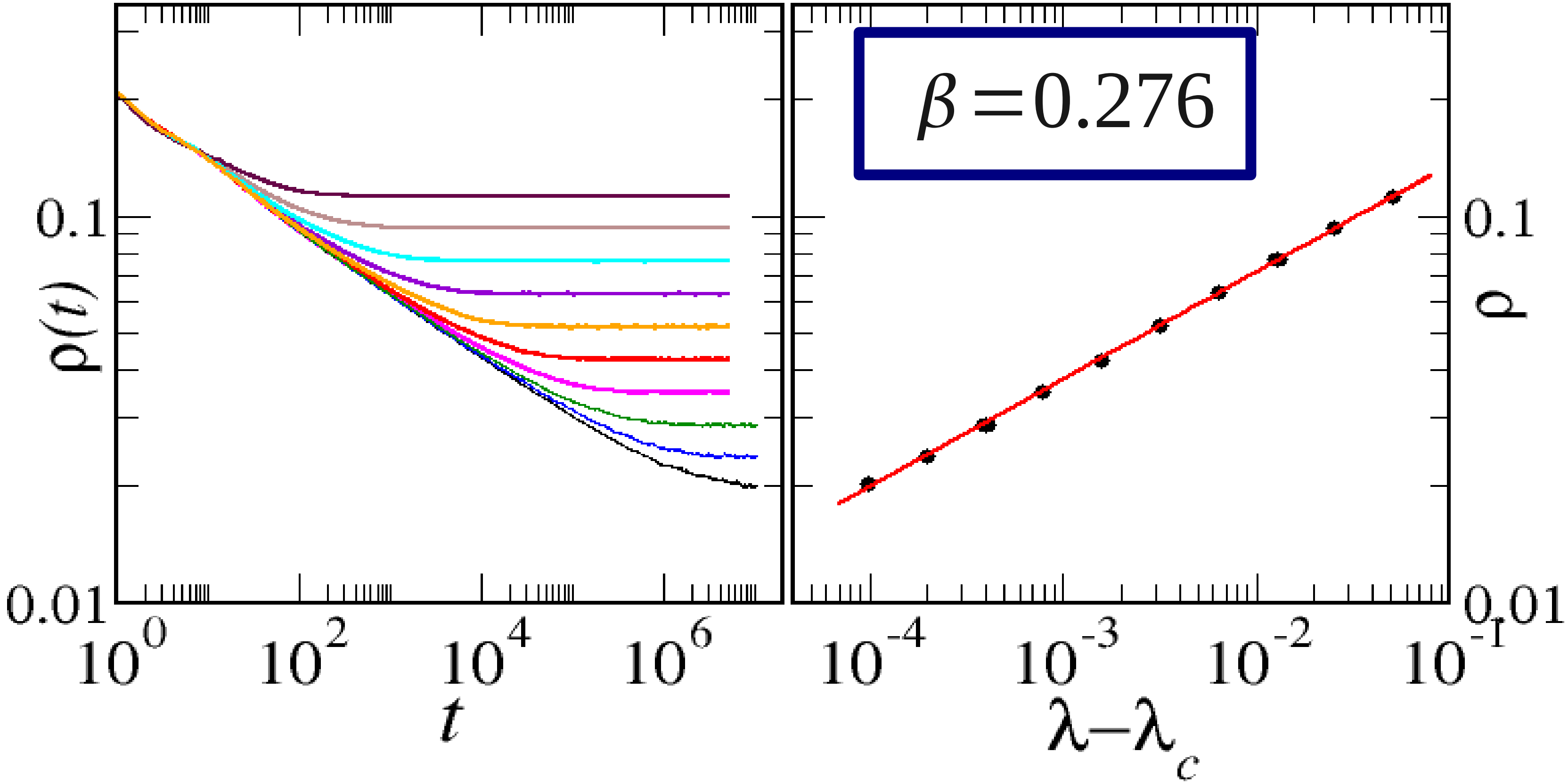


$$\alpha = 0.159$$

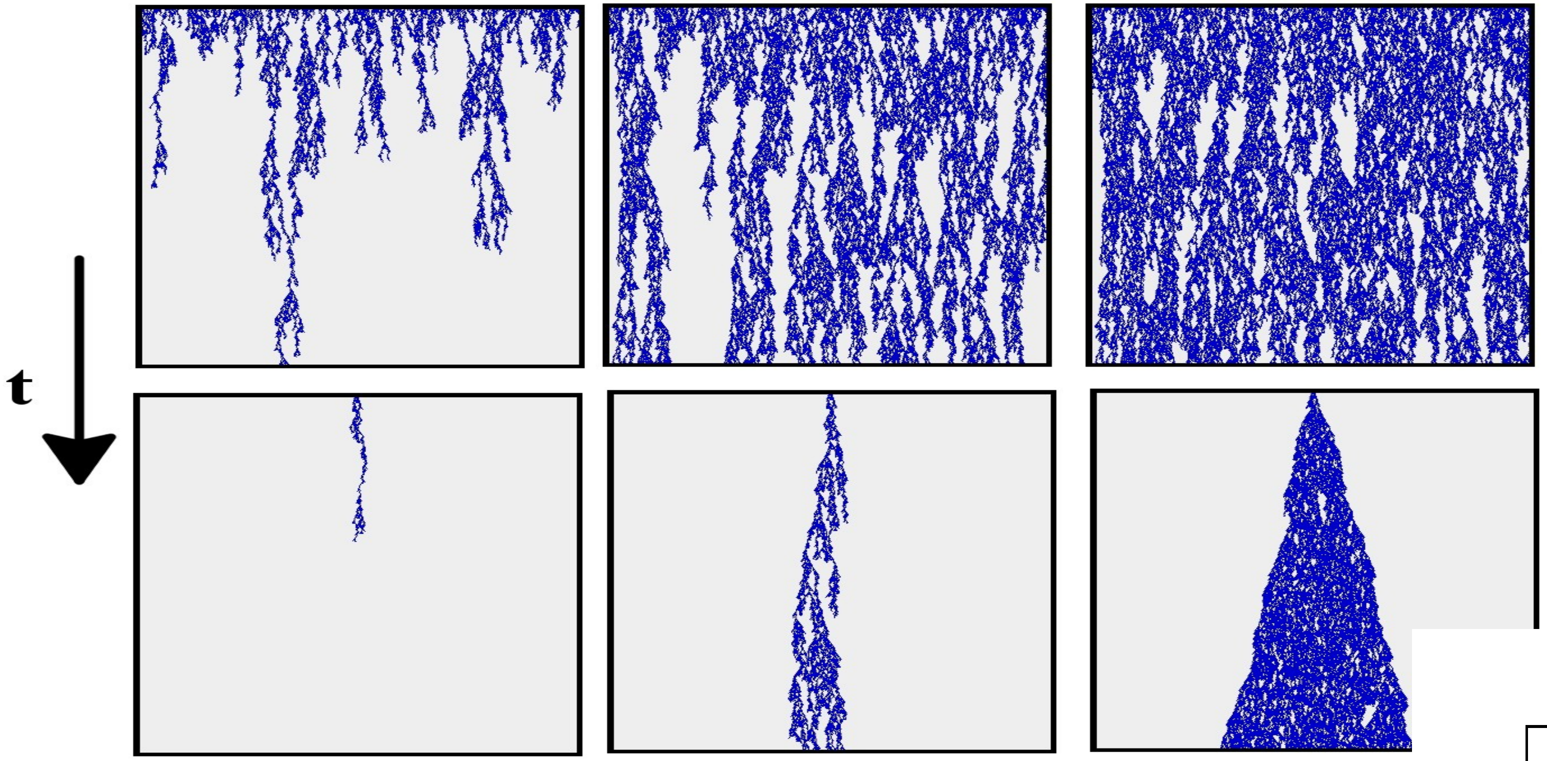
Critical exponents



- For $\lambda > \lambda_c , \quad \rho \sim \epsilon^{\beta}$



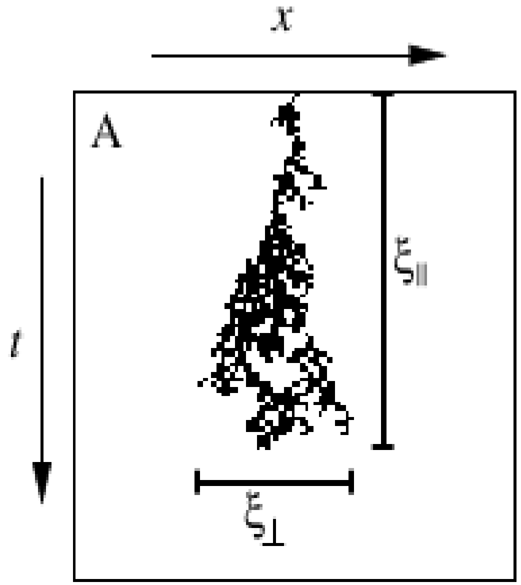
Other exponents



Correlation lengths :

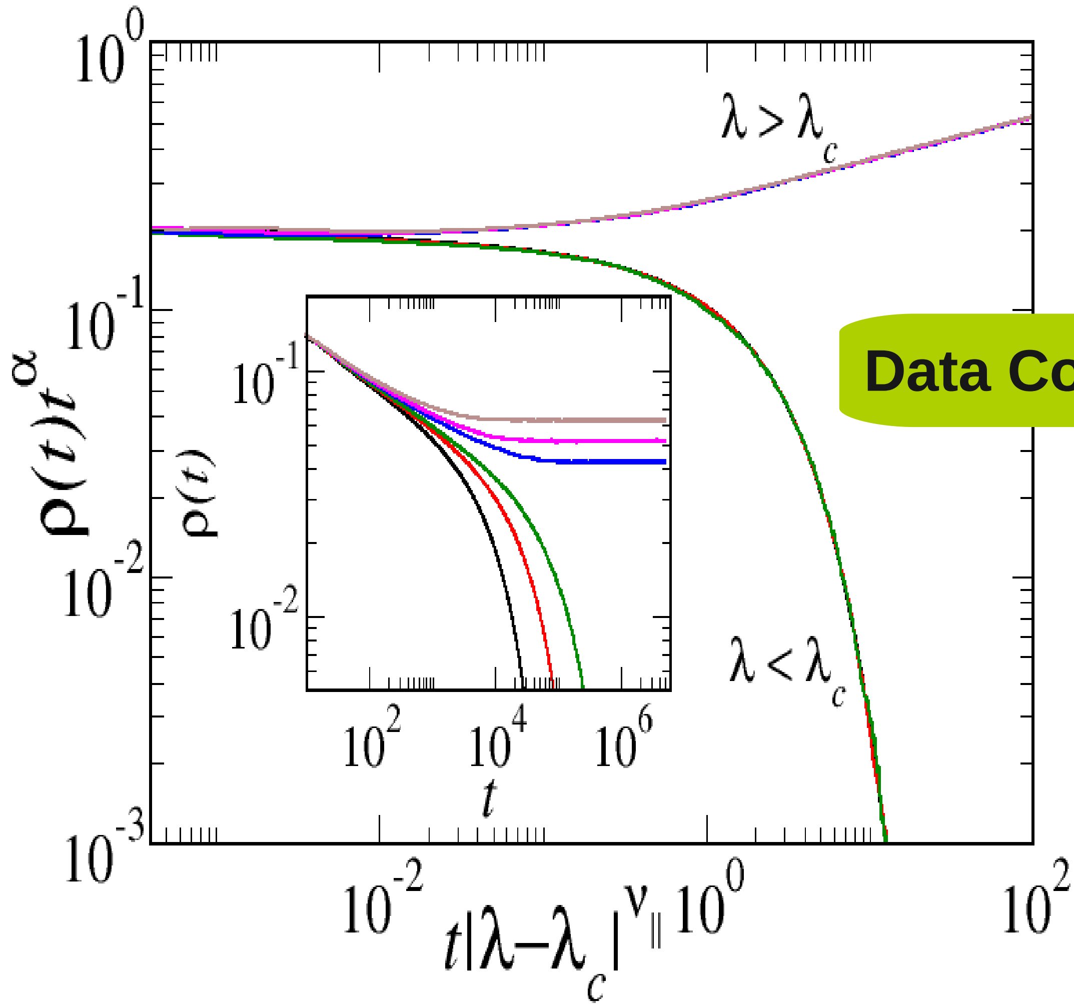
$$\xi_\perp \sim (\lambda - \lambda_c)^{-\nu_\perp}$$

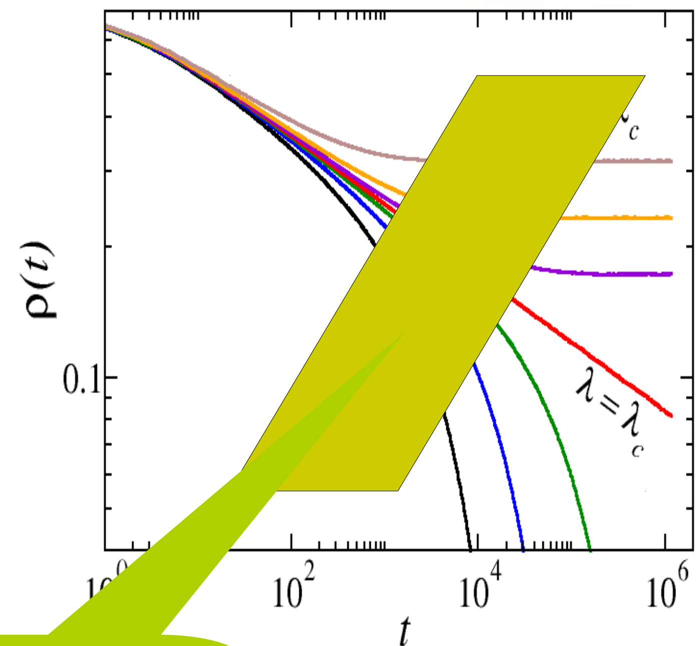
$$\xi_\parallel \sim (\lambda - \lambda_c)^{-\nu_\parallel}$$



Off-critical scaling

- For $\lambda \neq \lambda_c, \quad \rho(t) \sim t^{-\alpha} F(t\,\epsilon^{1/\nu_\parallel})$



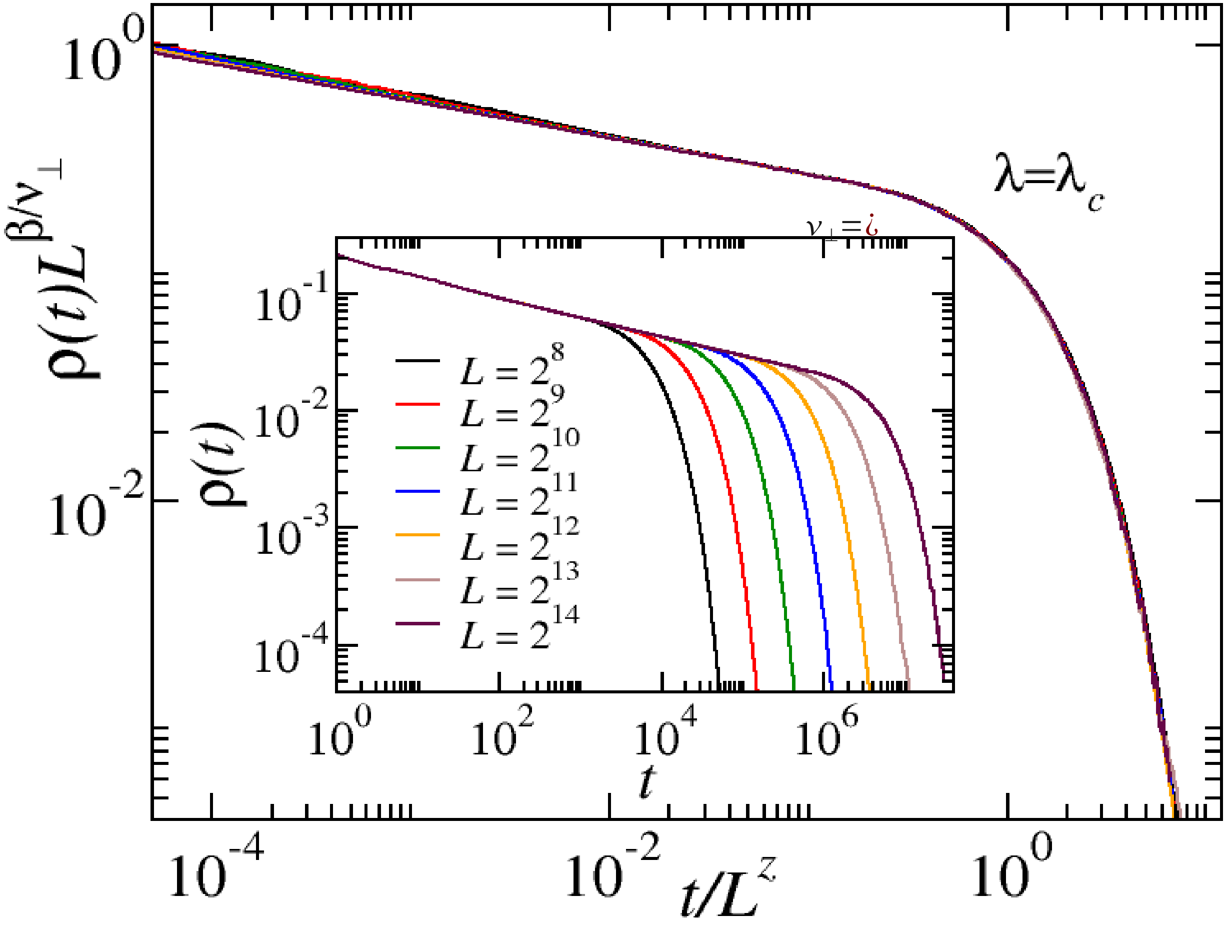


Data Collapse

$$\boxed{\nu_\parallel = 1.73}$$

Finite size scaling :

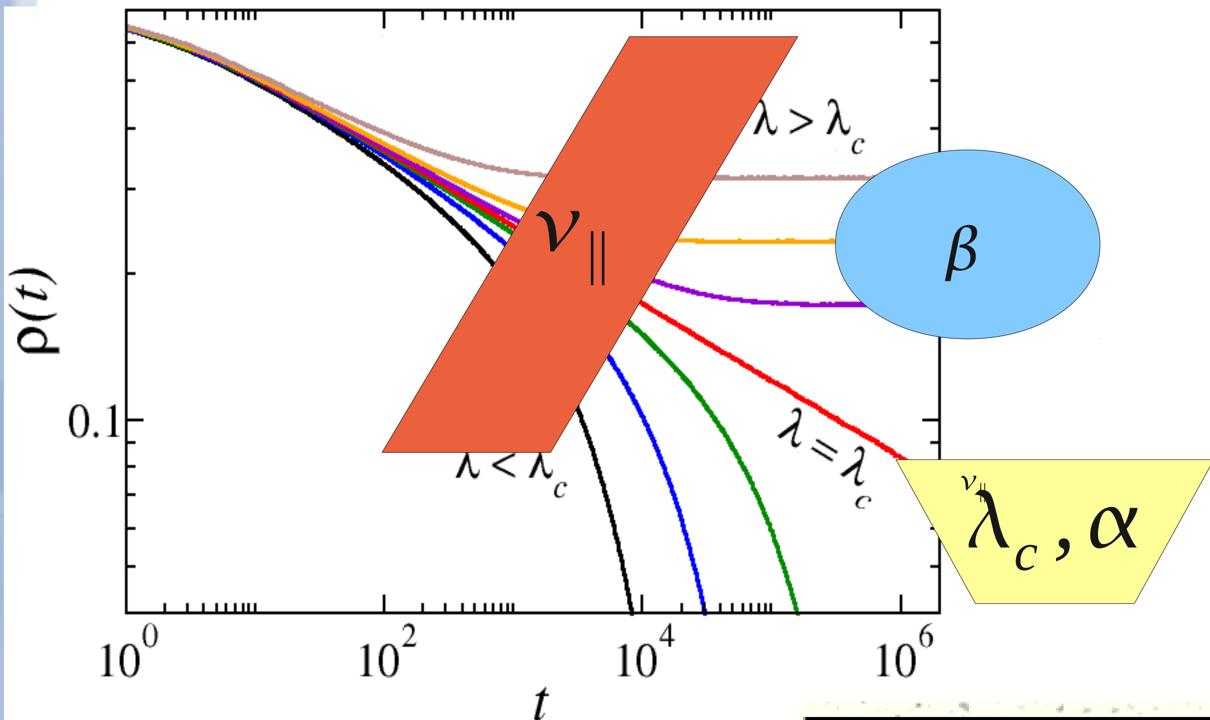
- For finite L and $\lambda = \lambda_c, \quad \rho(t, L) \sim L^{-\beta/\nu_\perp} F(t/L^z)$

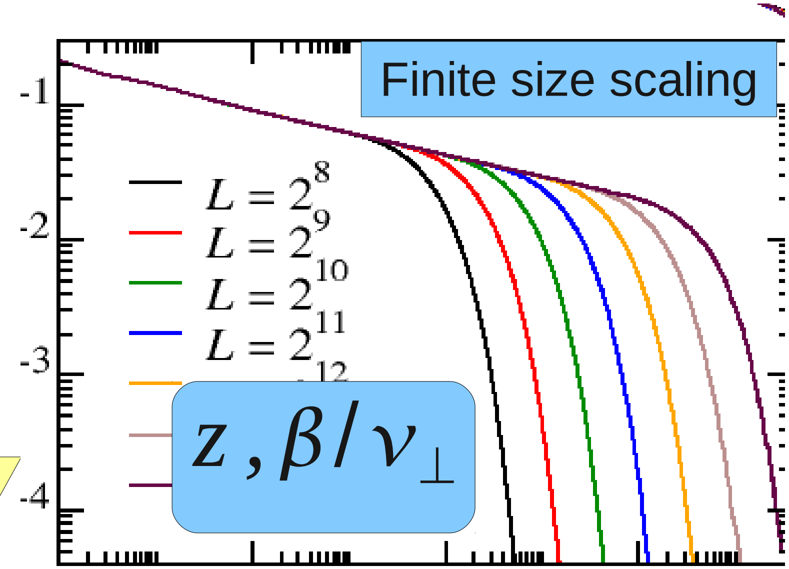


$\beta/\nu_\perp = 0.25$

z=1.59

Summary : numerical study of critical behaviour (from decay of $\rho_a(t)$)



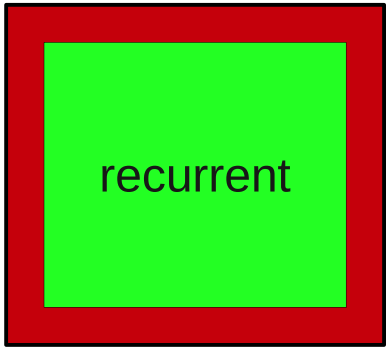


critical exponent	MF	IMF [155]	$d = 1$ [168]
β	1	1/2	0.276486(8)
$\nu_\perp$	1/2	1	1.096854(4)
$\nu_\parallel$	1	3/2	1.733847(6)
z	2	3/2	1.580745(10)

Model with many absorbing states

- Pair contact process (PCP)

$$0 \xleftarrow{1} 2A \xrightarrow{\lambda} 3A$$



Phase space

- Isolated particles are inactive

- Absorbing = “all particles are isolated”

  Infinitely many absorbing configurations

- APT in DP class

DP Universality :

Critical behaviour of APT in CP, PCP, DK, DP and may other models are identical. **DP** is known to be the most robust universality class of APT.

DP experiment :

TABLE II. Summary of the measured critical exponents (see the remark [36] for the range of errors shown in the list).

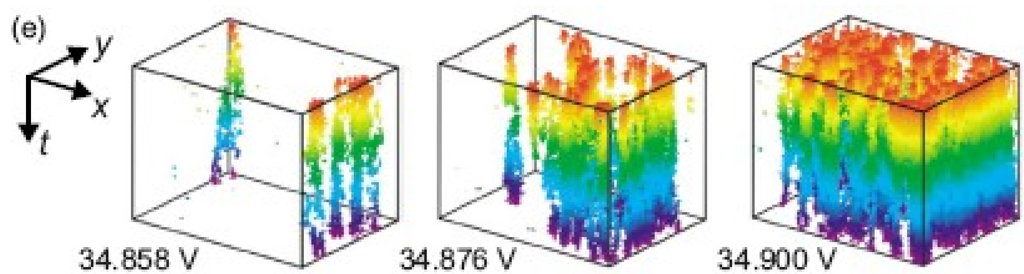
Exponent		DSM1-DSM2		DP[a]
Density order parameter	β	0.59(4)		0.583(3)
Correlation length[b]	$\nu_\perp$	0.75(6)	0.78(9)	0.733(3)
Correlation time	$\nu_\parallel$	1.29(11)		1.295(6)
Inactive interval in space[b]	$\mu_\perp$	1.08(18)	1.19(12)	1.204(2)[c]
Inactive interval in time	$\mu_\parallel$	1.60(5)		1.5495(10)[c]
Density decay	α	0.48(5)		0.4505(10)
Local persistence	θ_1	1.55(7)		1.611(7)[d]
Aging in autocorrelator	b	0.9(1)		0.901(2)
	λ_C/z	2.5(3)		2.583(14)
Survival probability	δ	0.46(5)		0.4505(10)
Cluster volume	θ	0.22(5)		0.2295(10)
Cluster mean square radius	ζ	1.15(9)		1.1325(10)

DP Conjecture

Absorbing Phase Transitions generically belong to DP if the system has

- a fluctuating scalar order parameter
- short range interaction
- no unconventional symmetry
- no quenched disorder

Janssen, Z Phys B 1981
Grassberger, Z Phys B 1982



Takeuchi et al. PRE 2009

Non-recurrent configurations...

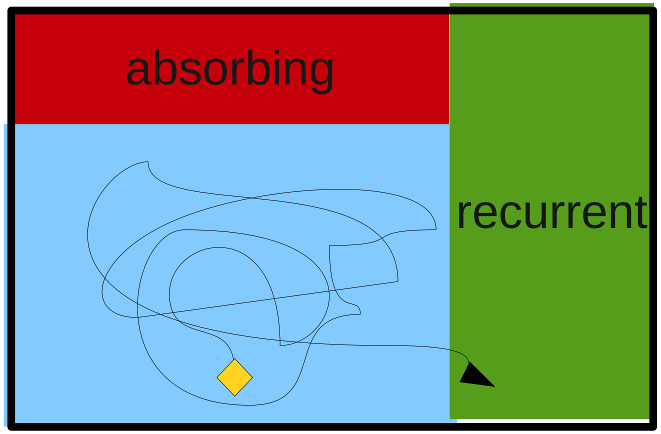
- More complicated !
- Stationary state (active phase):

- system wonders

in 'recurrent space'

- may fall into an absorbing one,

with probability<1



Phase space

- time evolution may depend strongly on initial condition !

Example : Conserved Lattice Gas (1D)

- N hardcore particles on a ring (size L)
- Infinite nearest neighbour repulsion

110 $\longrightarrow$ **101**
011 $\longrightarrow$ **101**

Particle conserving dynamics $\rho = N/L$

- Isolated particles can not move
- Active particle : has only one occupied neighbour
- Total active particles : N_a , activity density $\rho_a = N_a / L$

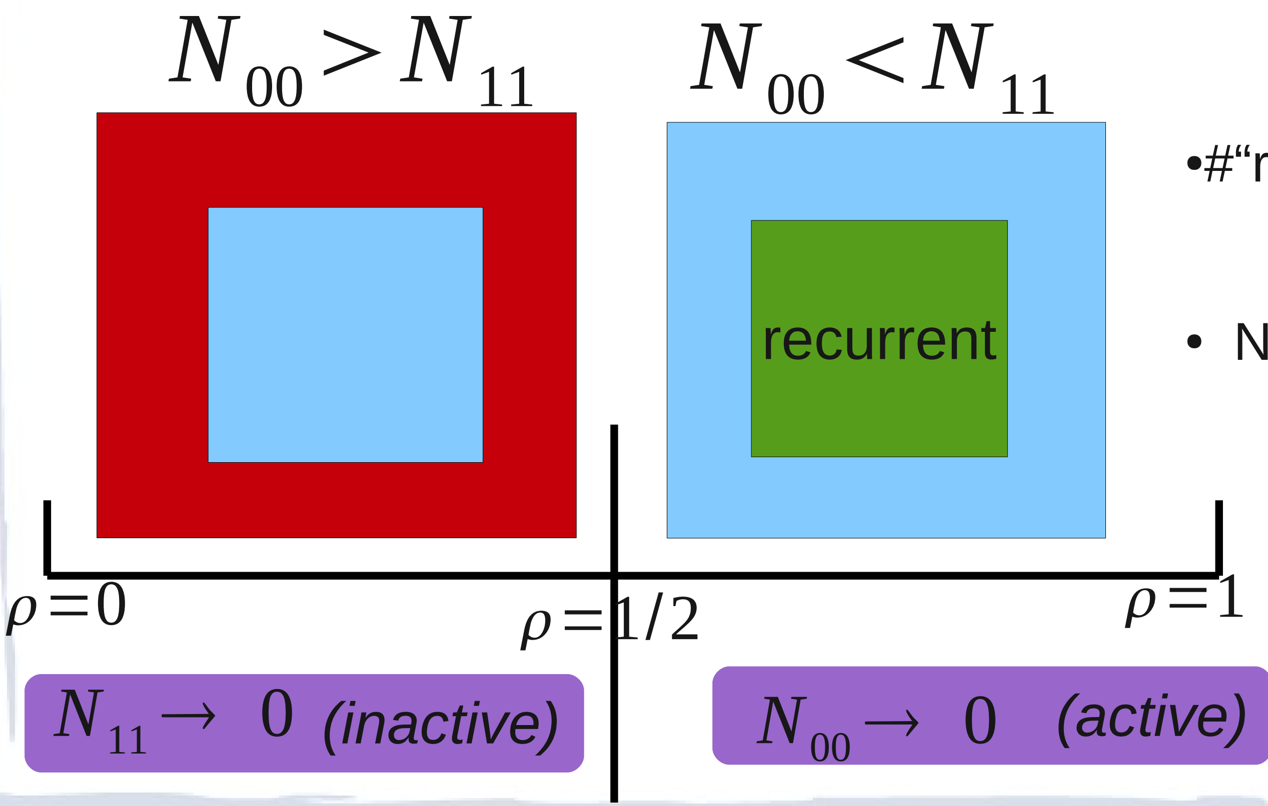
110 → 101

011 → 101

1100 → 1010

1101 → 1011

- N_{00} and N_{11} can *only* decrease.
- Simultaneously



- #"recurrent configurations" → 0 as $\rho \to \rho_c$
- No recurrent configurations at $\rho_c = 1/2$

⇒ $\rho_c = 1/2$

Critical exponents of CLG (1D)

	Exact Results				Numerical Results	
	β	$\nu_{\perp}$	z	β'	α	$\nu_{\parallel}$
CLG	1	1	2	0	1/4	4
DP	0.276	1.09	1.58	0.276	0.159	1.732

- Oliveira PRE 2005
- U. Basu and PKM, PRE 2009

-Lee and Lee, PRE 2008

- Scaling violation :

$$\boxed{\begin{array}{c} \beta = \alpha \nu_{\parallel} \\ z \neq \nu_{\parallel} / \nu_{\perp} \end{array}}$$

VOLUME 85, NUMBER 9 PHYSICAL REVIEW LETTERS 28 AUGUST 2000

Universality Class of Absorbing Phase Transitions with a Conserved Field

Michela Rossi,[1,2] Romualdo Pastor-Satorras,[2] and Alessandro Vespignani[2]

2D CLG

Measured: $z, \nu_\perp, \beta, \alpha$

Assumed : $z = \nu_{\|} / \nu_\perp$

Violation : $\beta \neq \alpha \nu_{\|}$

L. It follows that data collapse in time is not achievable with standard scaling forms, and that θ violates the usual scaling relation. Albeit its origin is not clear, it is noteworthy that this anomaly is common to all APT with conserved fields inspected so far [10,11], irrespective of the updating rules employed, either parallel or sequential.

PHYSICAL REVIEW E 77, 021113 (2008)

Absorbing phase transition in a conserved lattice gas model in one dimension

Sang-Gui Lee and Sang B. Lee*

1D CLG

Measured : $z, \nu_\perp, \beta\ \alpha, \nu_{\|}$

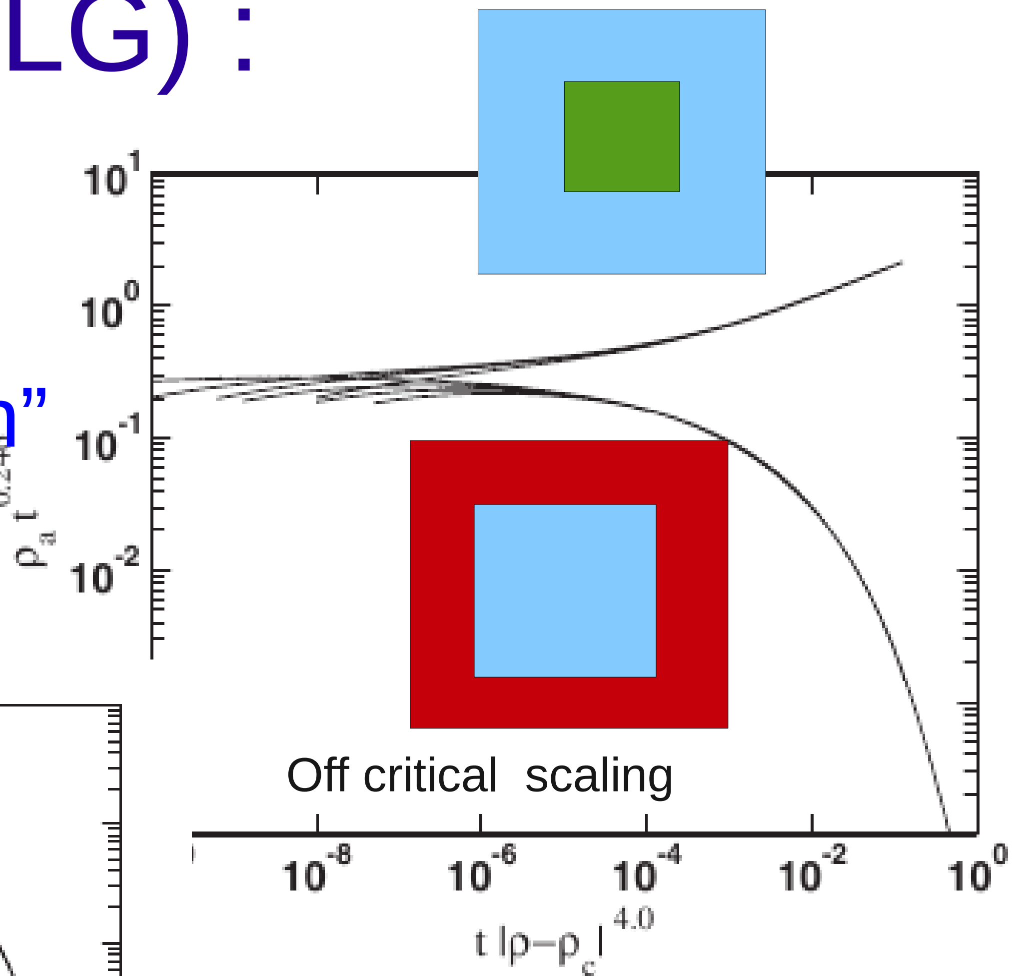
Correct : $\beta = \alpha \nu_{\|}$

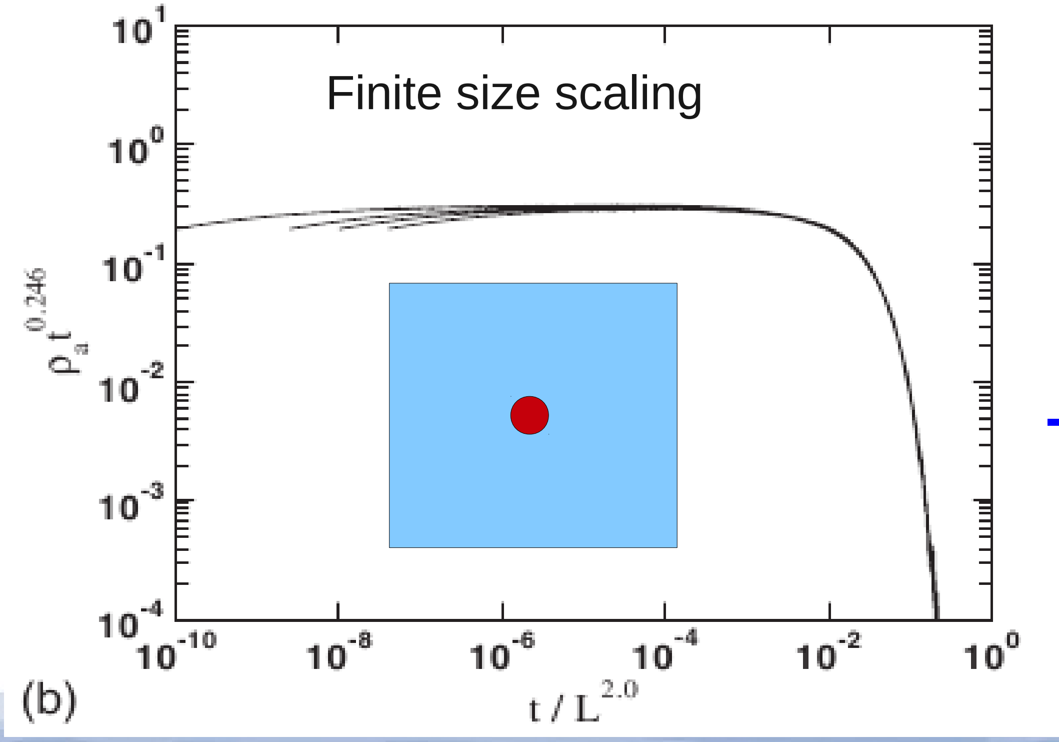
Violation : $z \neq \nu_{\|} / \nu_\perp$

with these values. A similar violation of the scaling relation was previously found by Rossi *et al.* [18]. They found that, with the relation in Eq. (4), simple scaling behavior was broken with their data. They, instead, obtained $\nu_{\|}$ from the relation in Eq. (6) using the value of z estimated from the finite-size scaling plot of Eq. (5). If we do similarly, we would get $\nu_{\|}=2$; however, this value does not yield the data collapsing.

Closer look (1D CLG) :

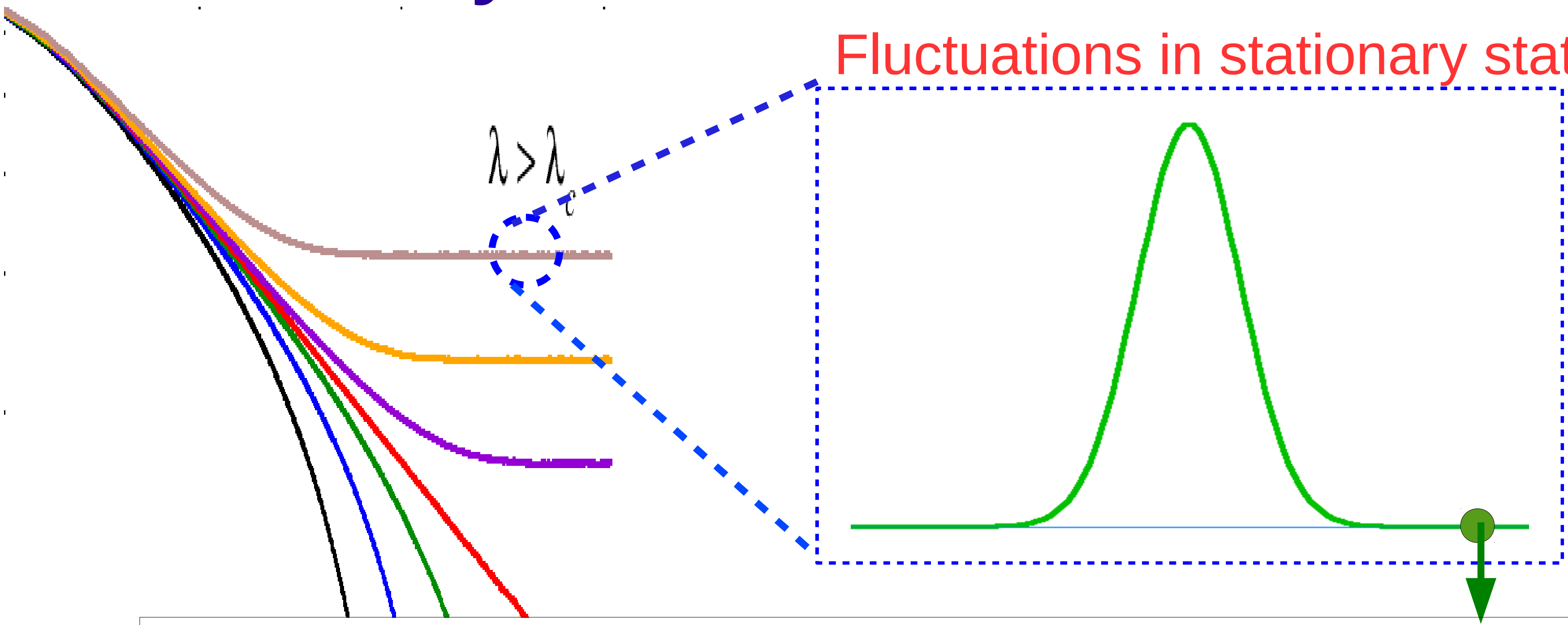
- Great scaling collapse
- Starting from "random initial condition"





- No "recurrent" config. at $\rho = \rho_c$

Decay of ρ_a, revisited.



- **Initial condition :**
- Choose randomly from a set of
  - recurrent configurations
  - which has large number of active sites

How to do that ?

- Exactly solvable models : recurrent space is known

- Numerical simulation:

  -choose a configuration from stationary state (recurrent)

  - *reactivate* it "suitably" for a short time

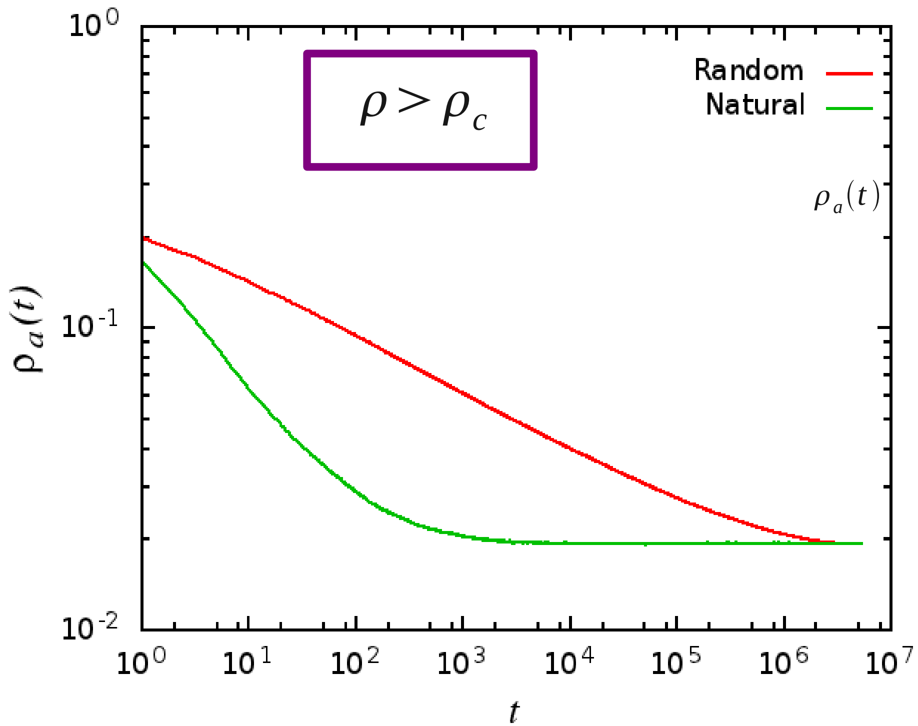
    [a highly active configuration, but possibly non-recurrent.

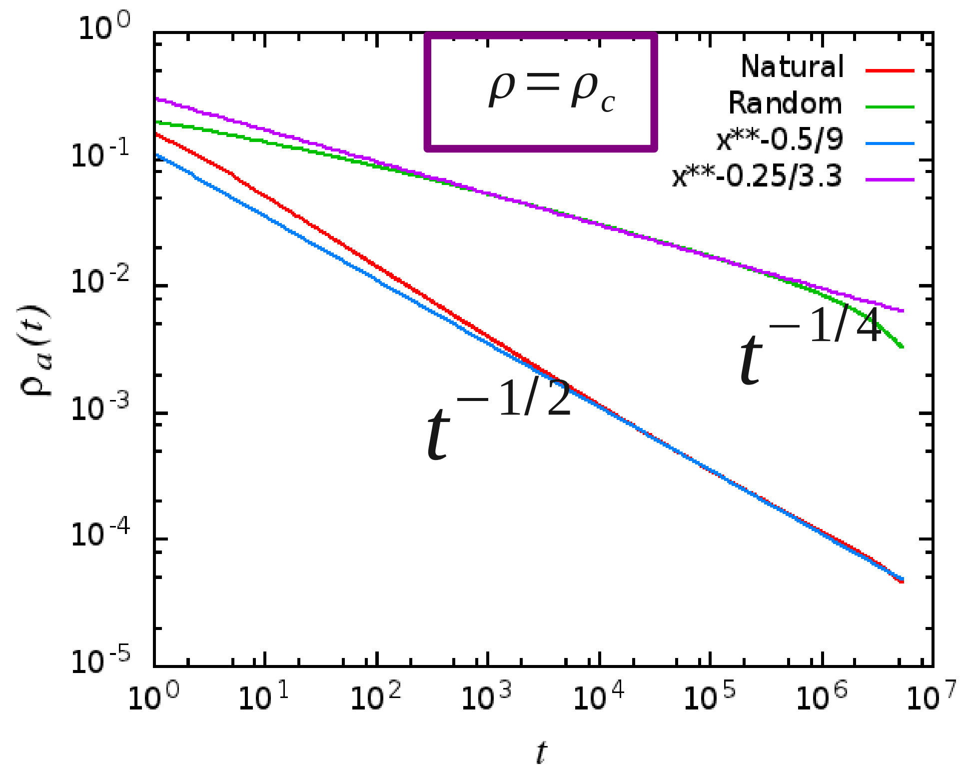
    but it is not far from the "recurrent space"]

- **Suitably reactive initial state** = "natural initial condition"

CLG 1D (again)

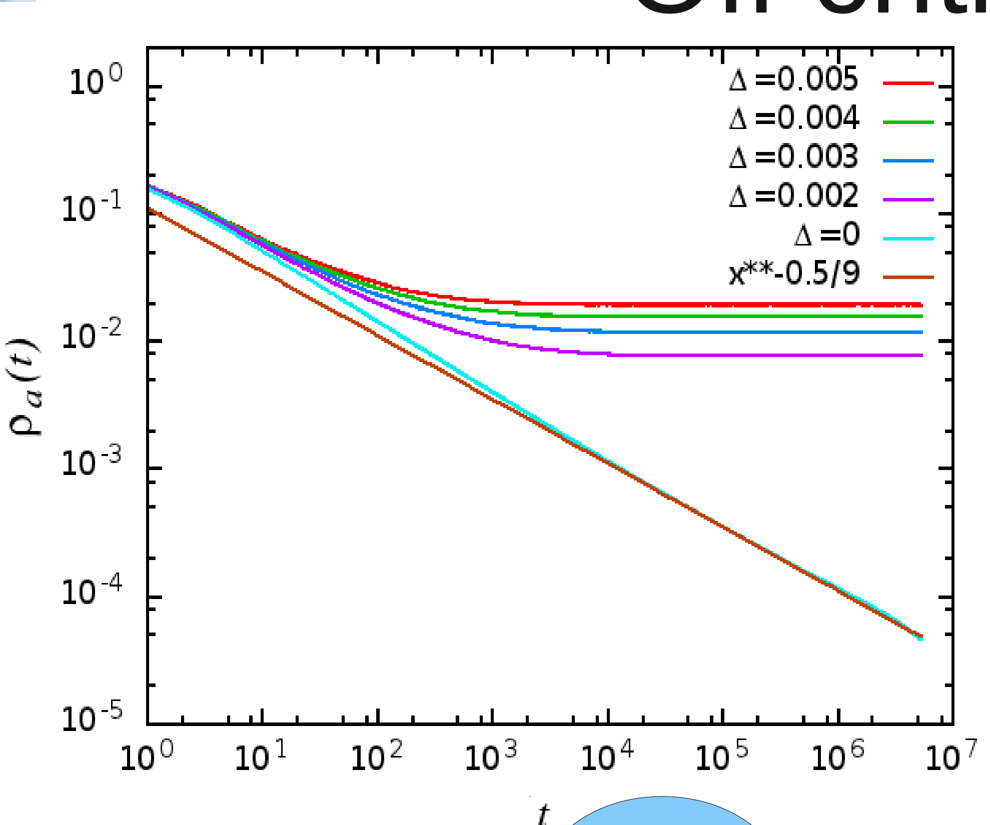
- Exactly solvable, but no “recurrent config”
- We use natural initial condition

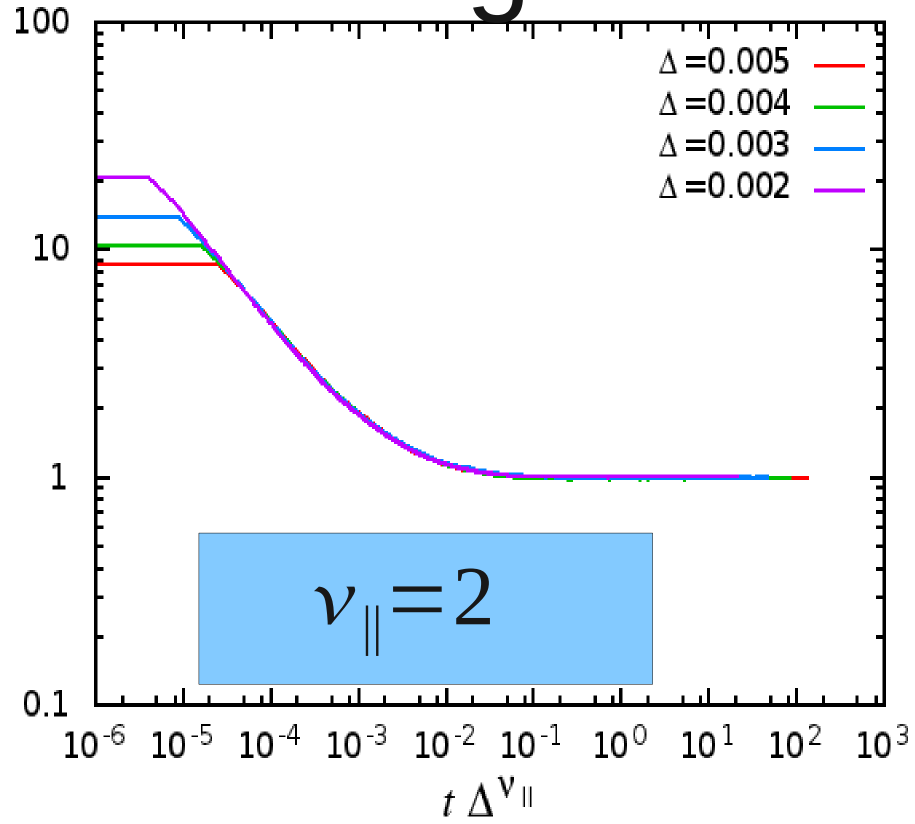


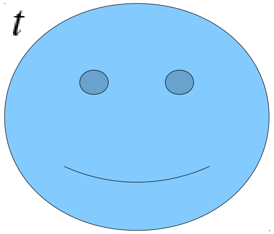


CLG (1D) : from natural ic

Off critical scaling

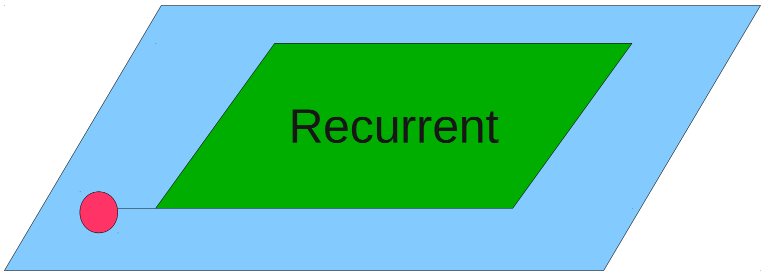






$$z = \nu_\parallel / \nu_\perp$$

Even CP shows apparent *scaling violation* in presence of transient states



Model : CLG guided CP

- ★ 1D Lattice (PBC). Each site

1	1	O	O
0	A	0	A

- ★ **A** is active if corresponding **1** is active (1**1**O or O**1**1).

CLG	**....O**	**1**	**1**	**1**	**O**	**1**	**1**	**O**	**1**	**O....**
CP	**....A**	**0**	**A**	**A**	**A**	**A**	**A**	**0**	**A**	**0.....**

Dynamics:

- "1, O" follow CLG dynamics (density conserving)
- "A, 0" follow CP dynamics

11O → 1O1 ← O11

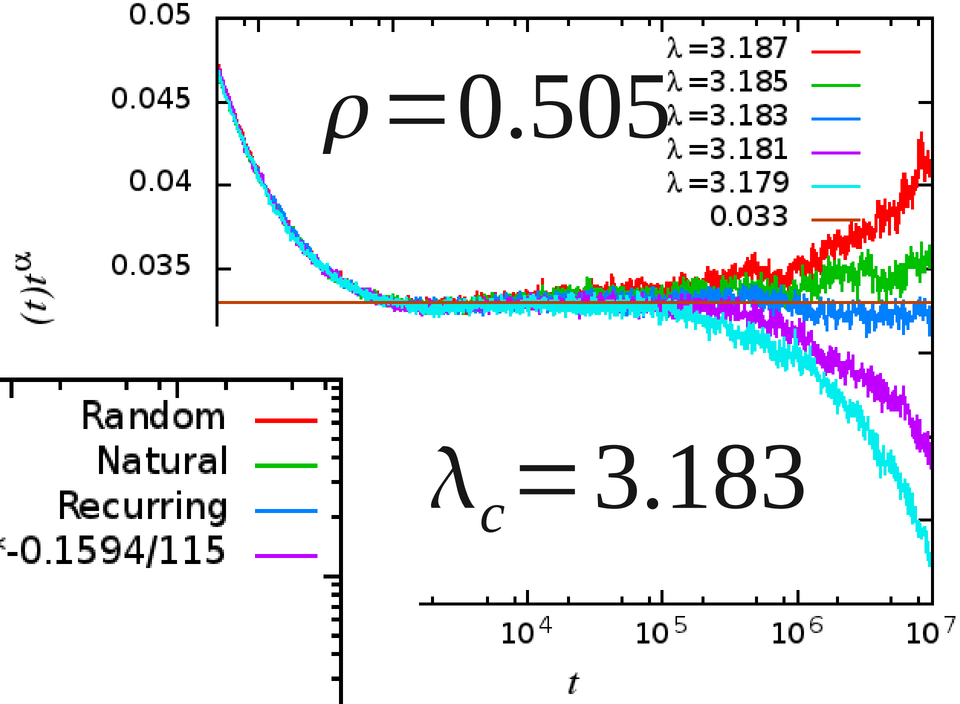
$0 \xleftarrow{1} A \xrightarrow{\lambda} 2A$

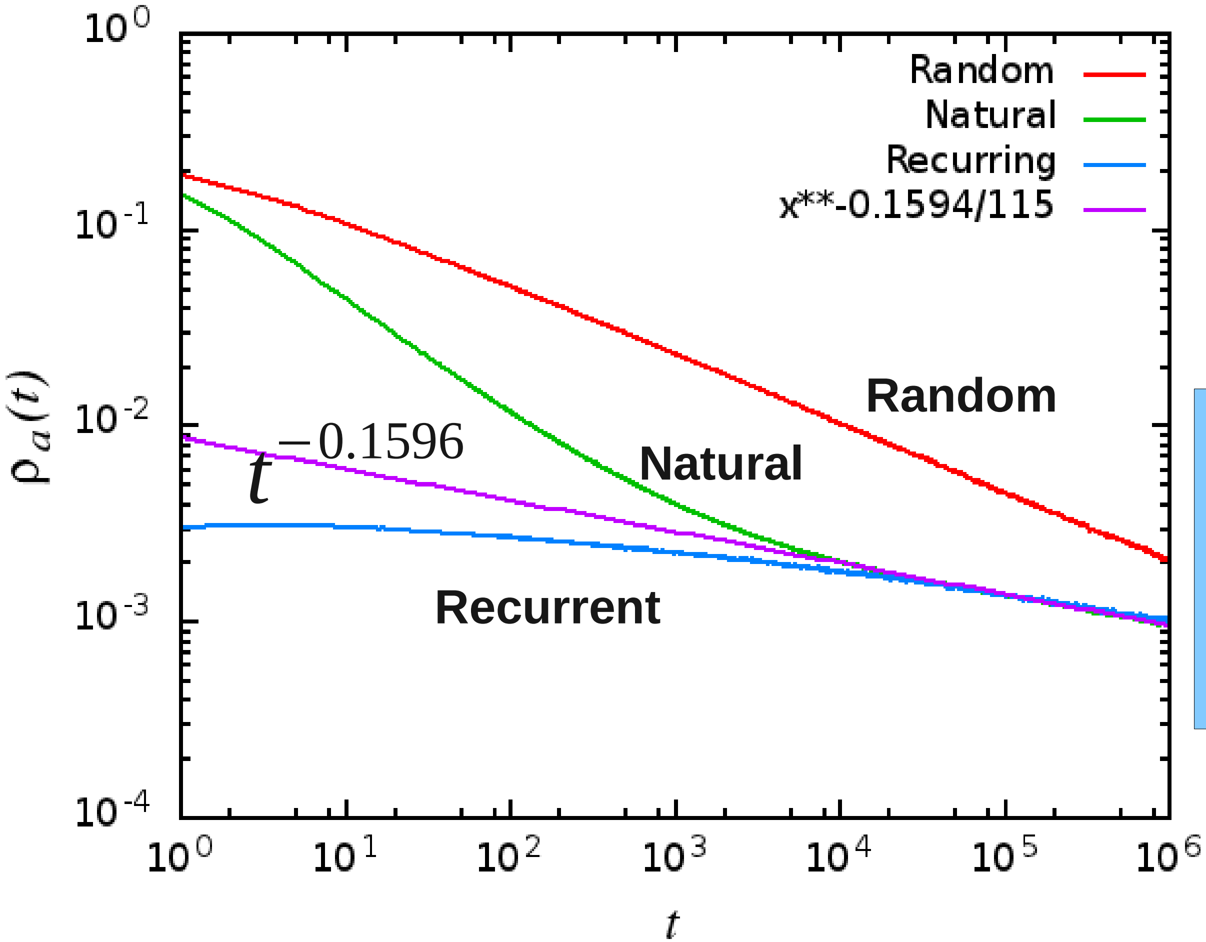
- Any configuration with OO is transient

CLG	**....O 1 1 OO 1 1 O 1 O....**
CP	**....A 0 A A A A A 0 A 0.....**

Activity density : $\rho_A = N_A / L$

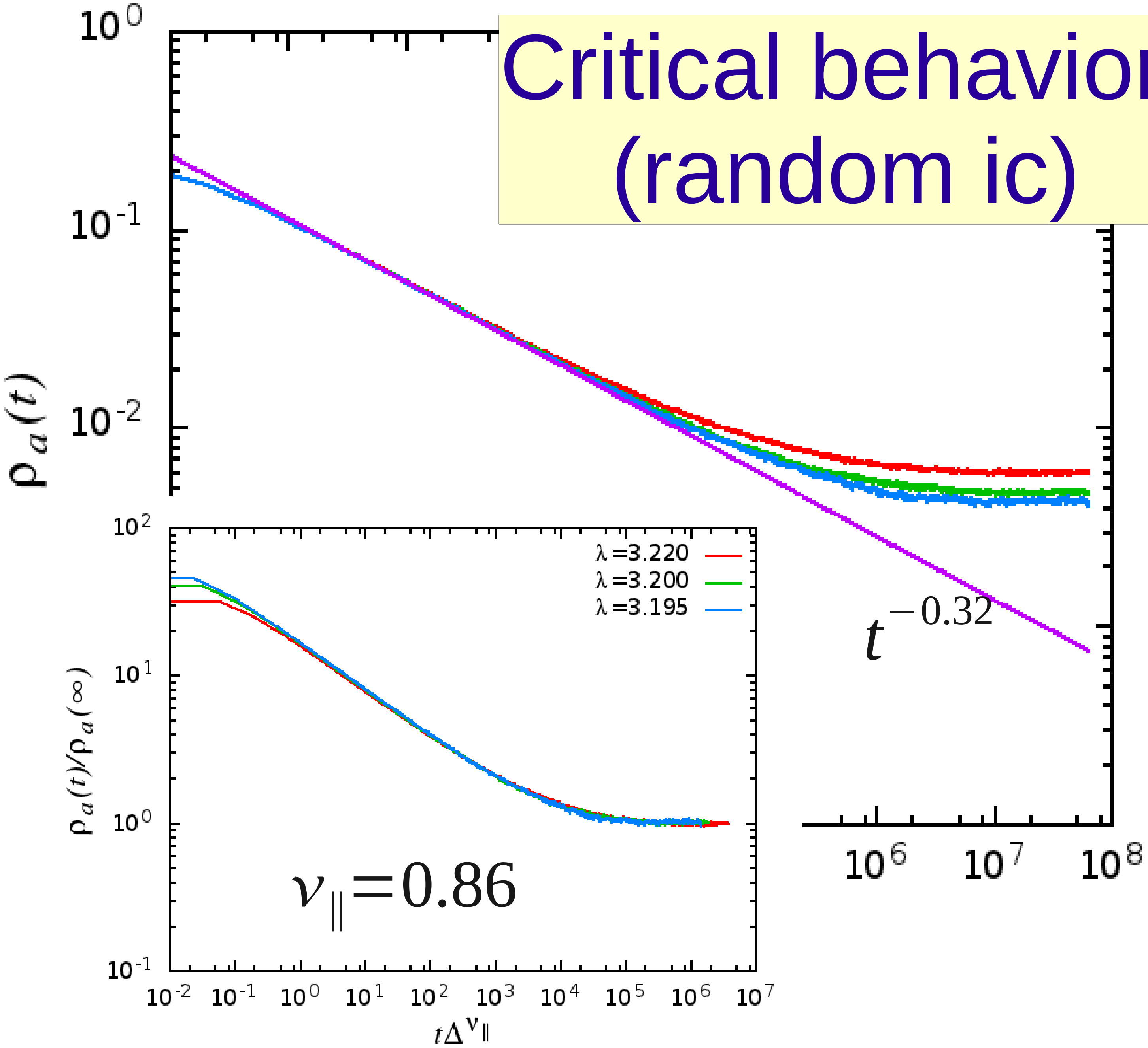
Critical behaviour (natural ic)





Other Exponents
are
consistent with DP

Critical behavior (random ic)



$$z \neq \nu_{\|} / \nu_{\perp}$$

$$\beta = \nu_{\|} \alpha$$

Part-II

Fixed-Energy Sandpiles Belong Generically to Directed Percolation

M. Basu, U. Basu, S. Bondyopadhyay, PKM and H. Hinrichsen, PRL 2012

Outline

- Sandpile models & Self organized criticality (SOC)
- Fixed energy sandpile model (as APT)
  Equivalence of SOC and FES ?
- FES has non-recurring states
- Difficulty in studying critical behaviour from random ic
- Natural initial condition heals

BTW model

Bak, Tang, and Weizenfield

Phys. Rev. Lett. 59, 381 - 384 (1987)

In 2 dimension

1. Slow driving

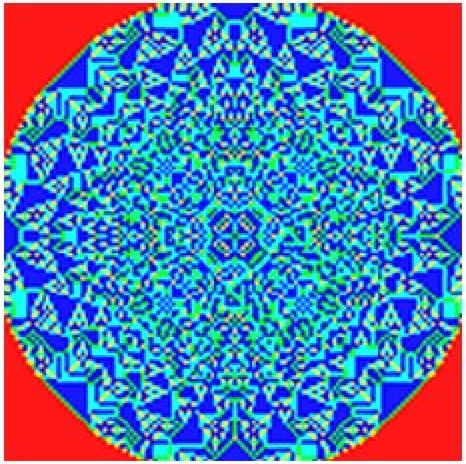
$$h(x,y) \rightarrow h(x,y)+1$$

2. Avalanches

If $h(x,y) \geq 4$ then

$$h(x,y) \rightarrow h(x,y)-4$$
$$h(x+1,y) \rightarrow h(x+1,y)+1$$
$$h(x-1,y) \rightarrow h(x-1,y)+1$$
$$h(x,y+1) \rightarrow h(x,y+1)+1$$
$$h(x,y-1) \rightarrow h(x,y-1)+1$$

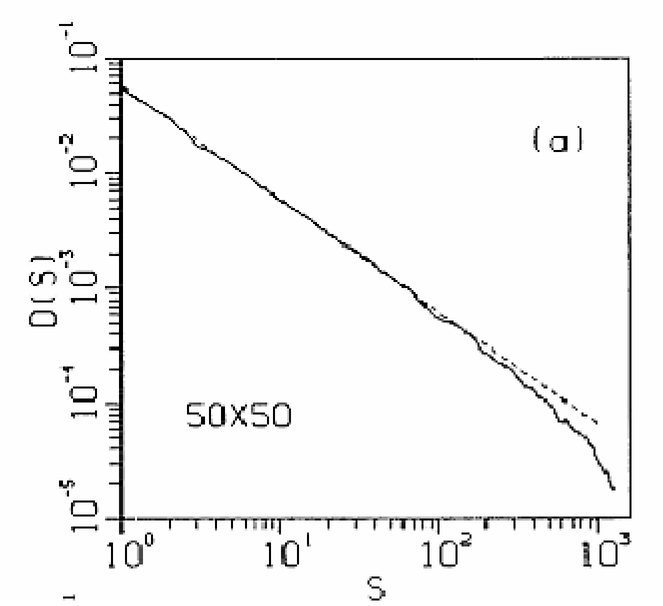
3. Dissipation at the boundaries



Avalanche size distribution

- Shows a power-law distribution
- A stochastic version (2D) was introduced later

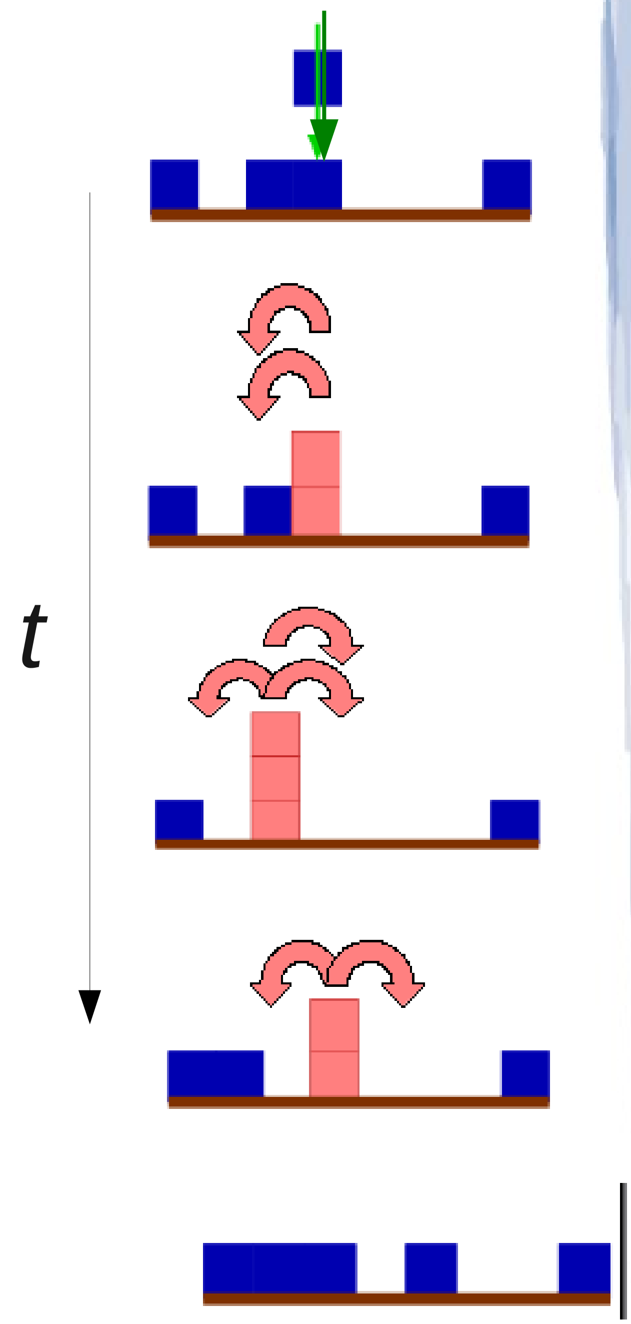
S. S. Manna, J. Phys. A 24, L363 (1991)



- which shows a critical behaviour different from BTW

Manna Model (1D interval)

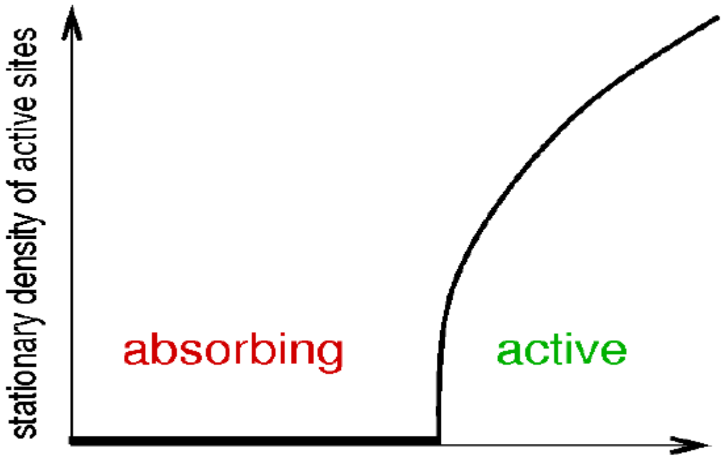
- Sites with more than **2** grains, move each grain individually to a randomly selected neighbor
- If all sites have one or no grains, *add* a grain at a random site
- Dissipation at boundaries



Fixed energy sandpile (FES)

- Use the same update rules for toppling
- No driving
- No dissipation (use periodic b.c.)
- Take conserved density as control parameter

Absorbing phase transition



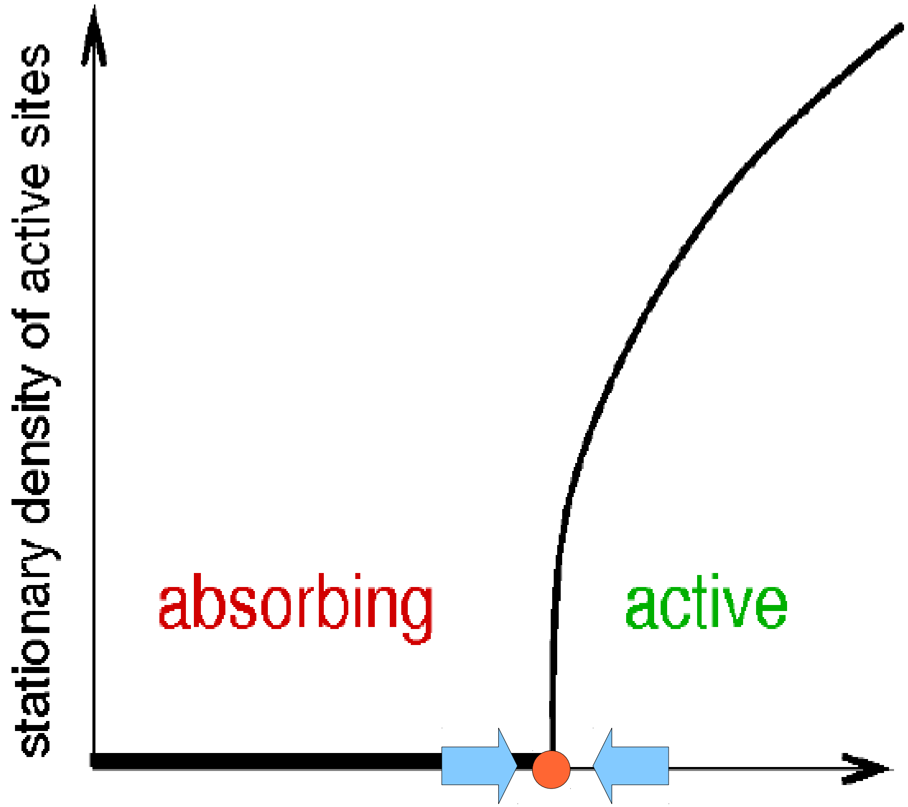
Munoz, Dickman, Vespignani and Zapperi, Phys. Rev. E 59, 6175 (1999)

SOC and FES equivalence

- Slow drive and dissipation in SOC makes density $\longrightarrow$ critical density.
-
- Avalanche exponents

$$\tau = \frac{1+\theta+2\delta}{1+\theta+\delta}$$

$$\tau_{\parallel} = 1+\delta$$



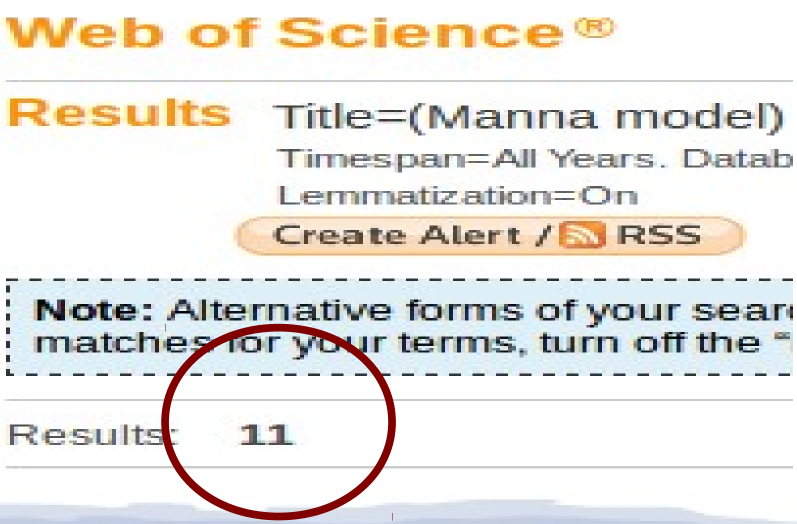
Manna Universality Class

- APT in stochastic FES (Manna Model) and several other models (CTTP, CLG, ...), where orderpameter is coupled to a conserved field form a new universality class-

$$\partial_t \rho_a = r\rho_a - b\rho_a^2 + \nabla^2 \rho_a + \sigma\sqrt{\rho_a}\,\eta + \omega\rho_a\phi$$
$$\partial_t \phi = D\nabla^2 \rho_a\,,$$

Conserved DP or Manna Class (MC)

- MC is believed to be one of the fundamental universality class of absorbing APT



Doubts about an independent MC

- Existing evidence are numerical
- Scattered exponents

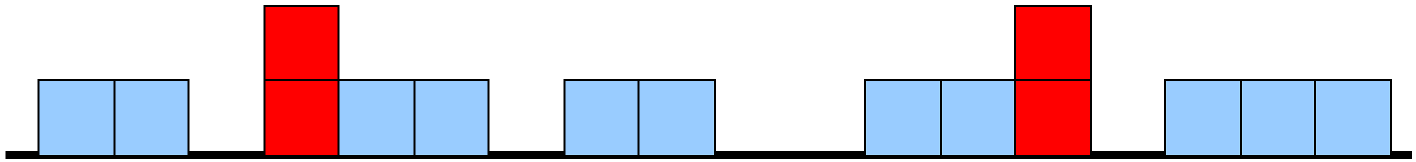
Ref.	ϕ_c	α	β	$\nu_\parallel$	$\nu_\perp$	z
Lübeck [1, 19]	0.96929(3)	0.141	0.382	2.452	1.760	1.393
Dickman *et. al.* [20]	0.92965(3)	-	0.412	2.41	1.66	1.45
Lee [21]	0.98285(5)	0.118	0.396	3.36	2.26	1.49
Dickman [22]	0.92978(2)	0.141	0.289	2.03	1.36	1.50
this work	0.929735(15)	0.155(5)	0.308(2)	1.74(1)	1.13(1)	1.52(2)
DP		0.159	0.277	1.733	1.096	1.580

- About 15% error in estimation of $\alpha, \nu_\parallel, \beta$
- MC and DP have same mean-field theory
- Universality splitting in 1D:

  CTTP has different $(\nu_\perp, \nu_\parallel)$ but same (α, β, z) as MC

FES have non-recurrent states

- In steady state, activity lives in odd and even sub lattices alternatively



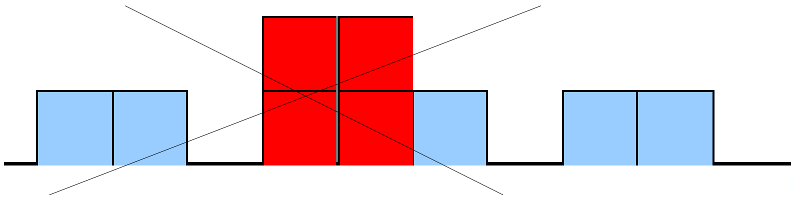
Two consecutive sites can not be active.

Random initial condition :

macroscopic number of

consecutive active sites

(non-recurrent configurations)

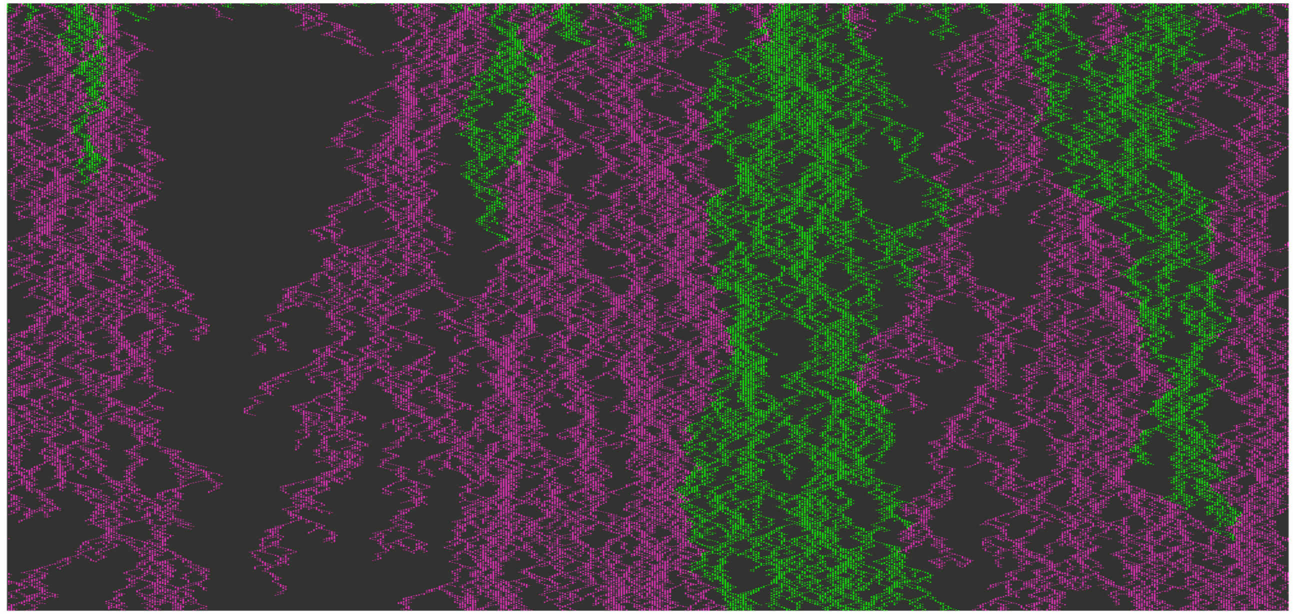


- Unusual journey to steady state

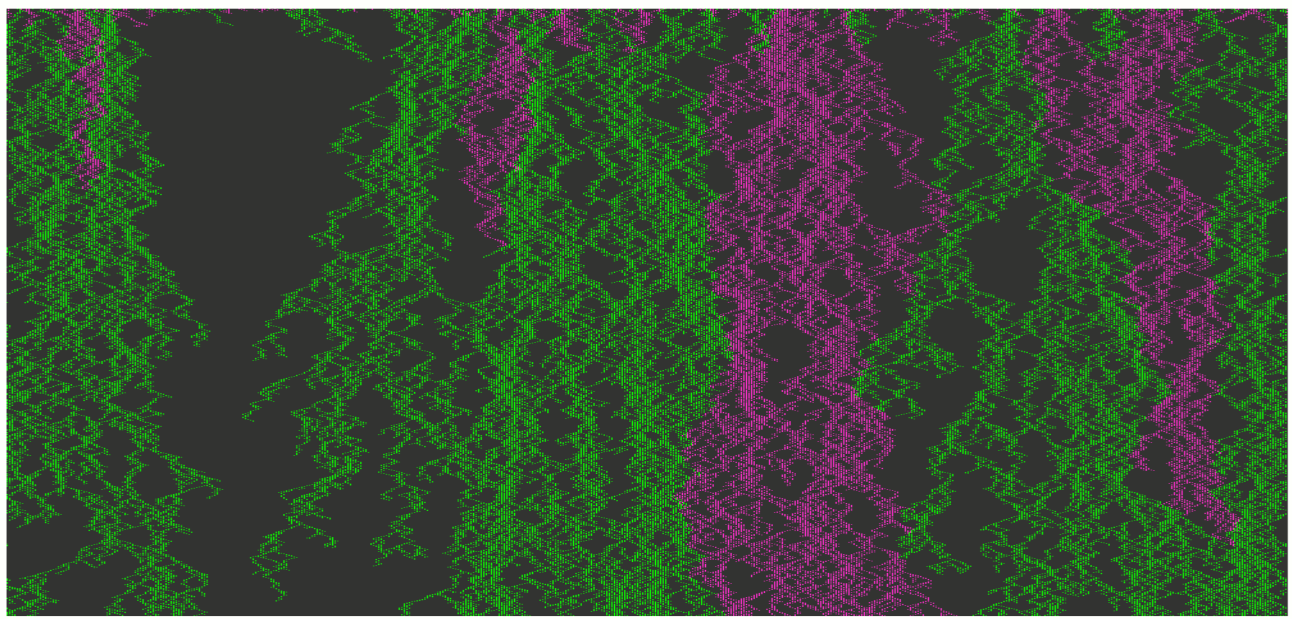
- Background inhomogeneity has long life

$t \longrightarrow 0$

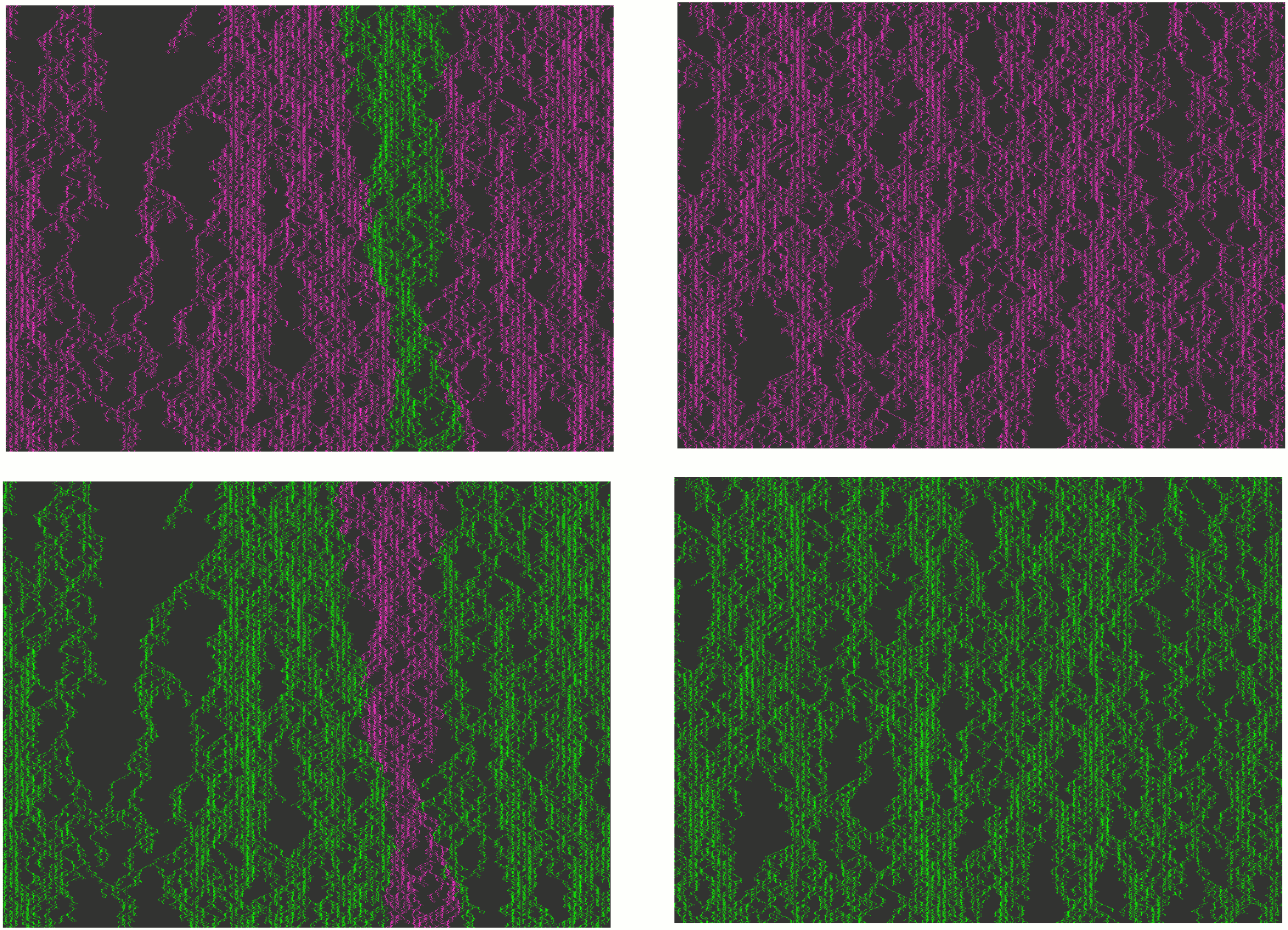
Odd time



Even time



2000 **10000**

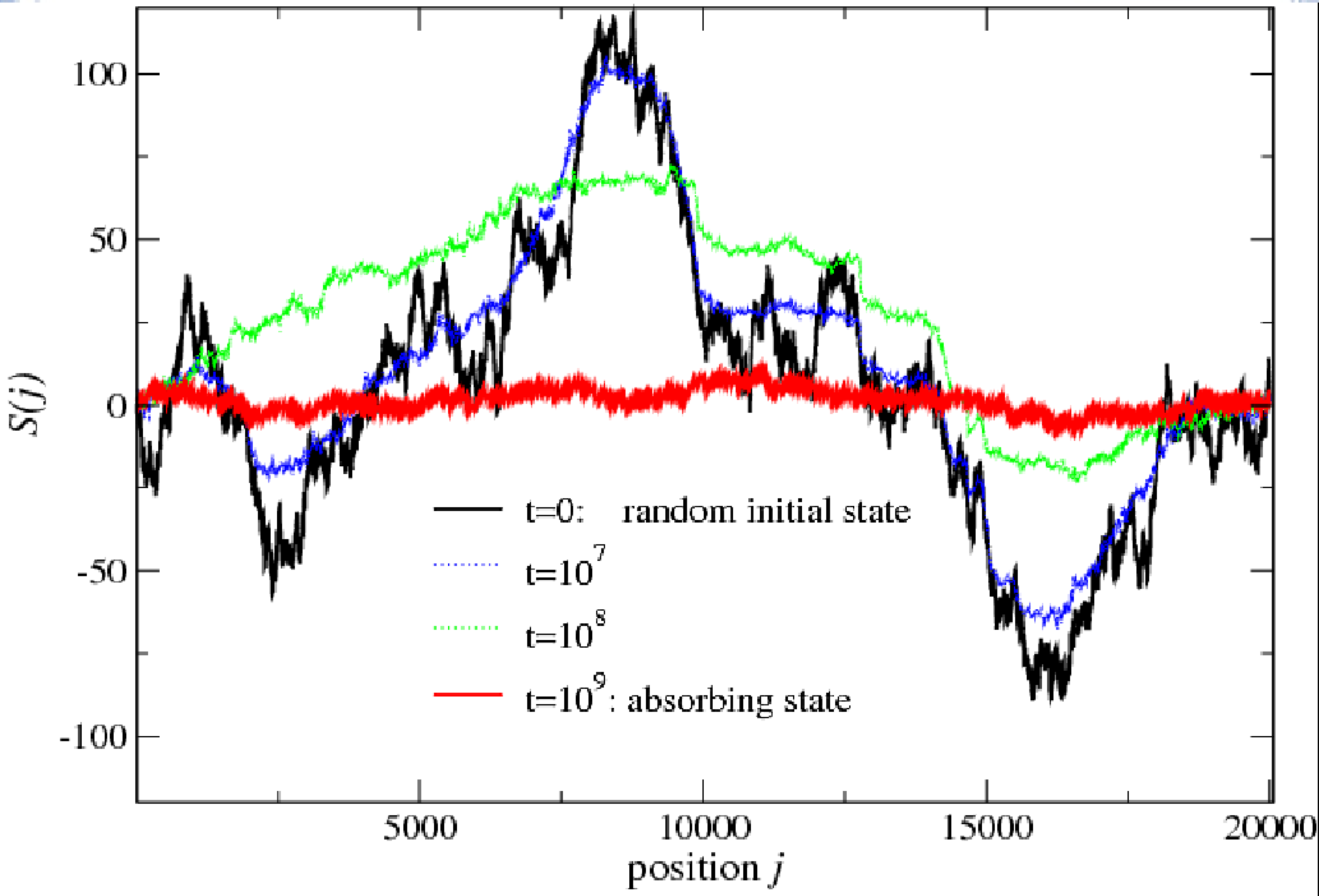


Background disorder

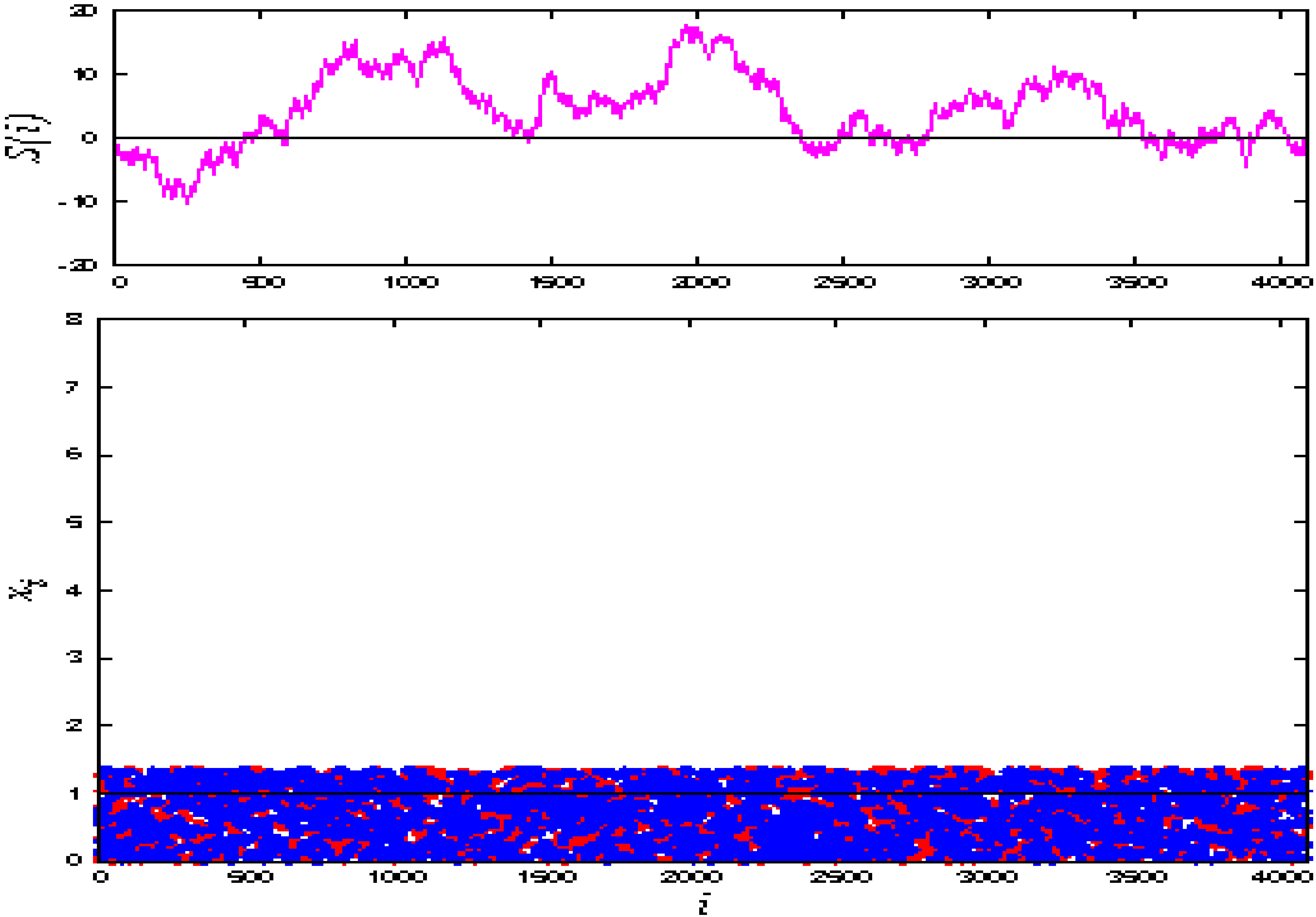
- The relative excess of particles, to the left of position j.

$$S(j) = \left(\sum_{i=1}^{j} h_i\right) - j\phi$$

Density $\phi = \frac{1}{N} \sum_{i=1}^{N} h_i$



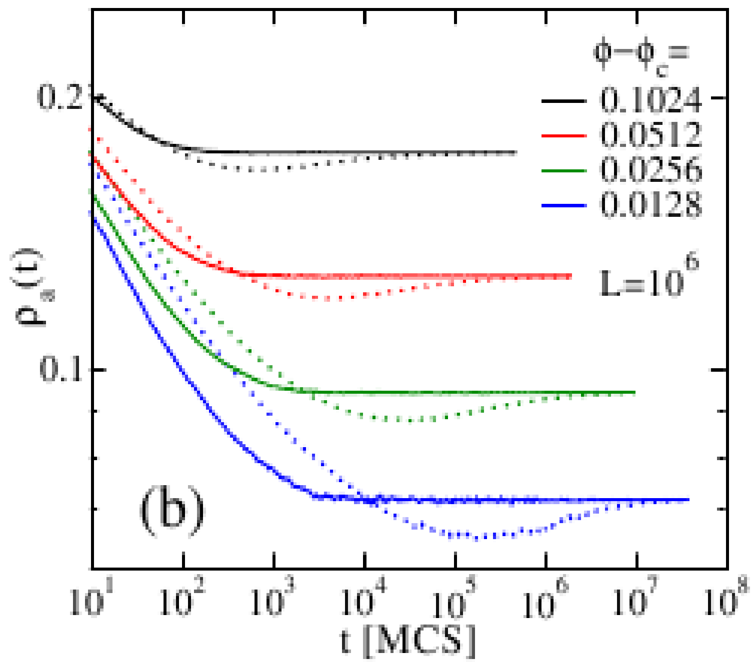
100
50
0
-50
-100
S(j)
5000
10000
15000
20000
position j
t=0: random initial state
t=10^7
t=10^8
t=10^9: absorbing state



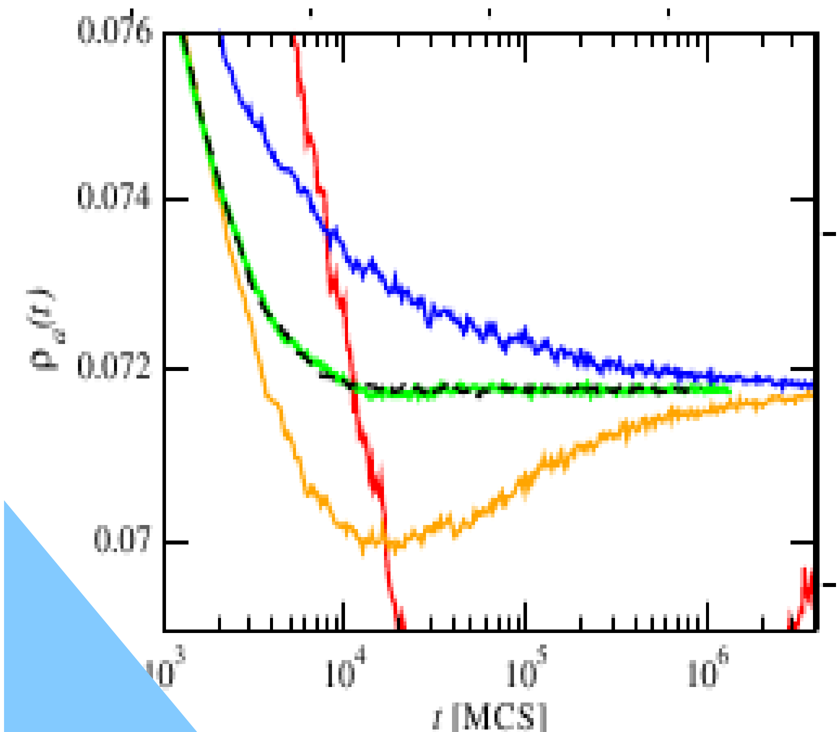
20
10
0
-10
-20
S(i)
0
500
1000
1500
2000
2500
3000
3500
4000
8
7
6
5
4
3
2
1
0
X_i^2
0
500
1000
1500
2000
2500
3000
3500
4000
i

Decay of activity density

- Undershooting in random ic

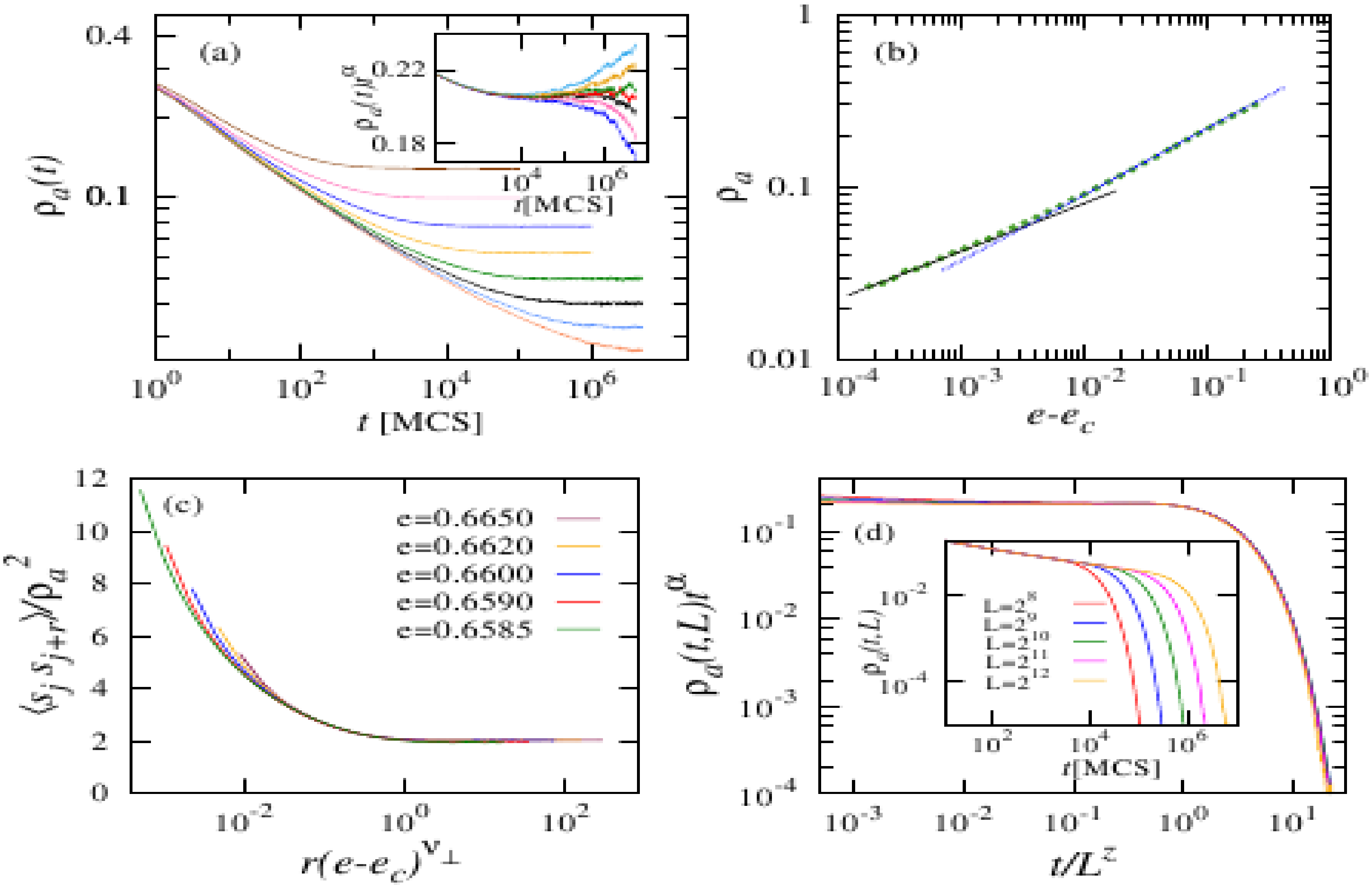


- Healed by “natural ic”

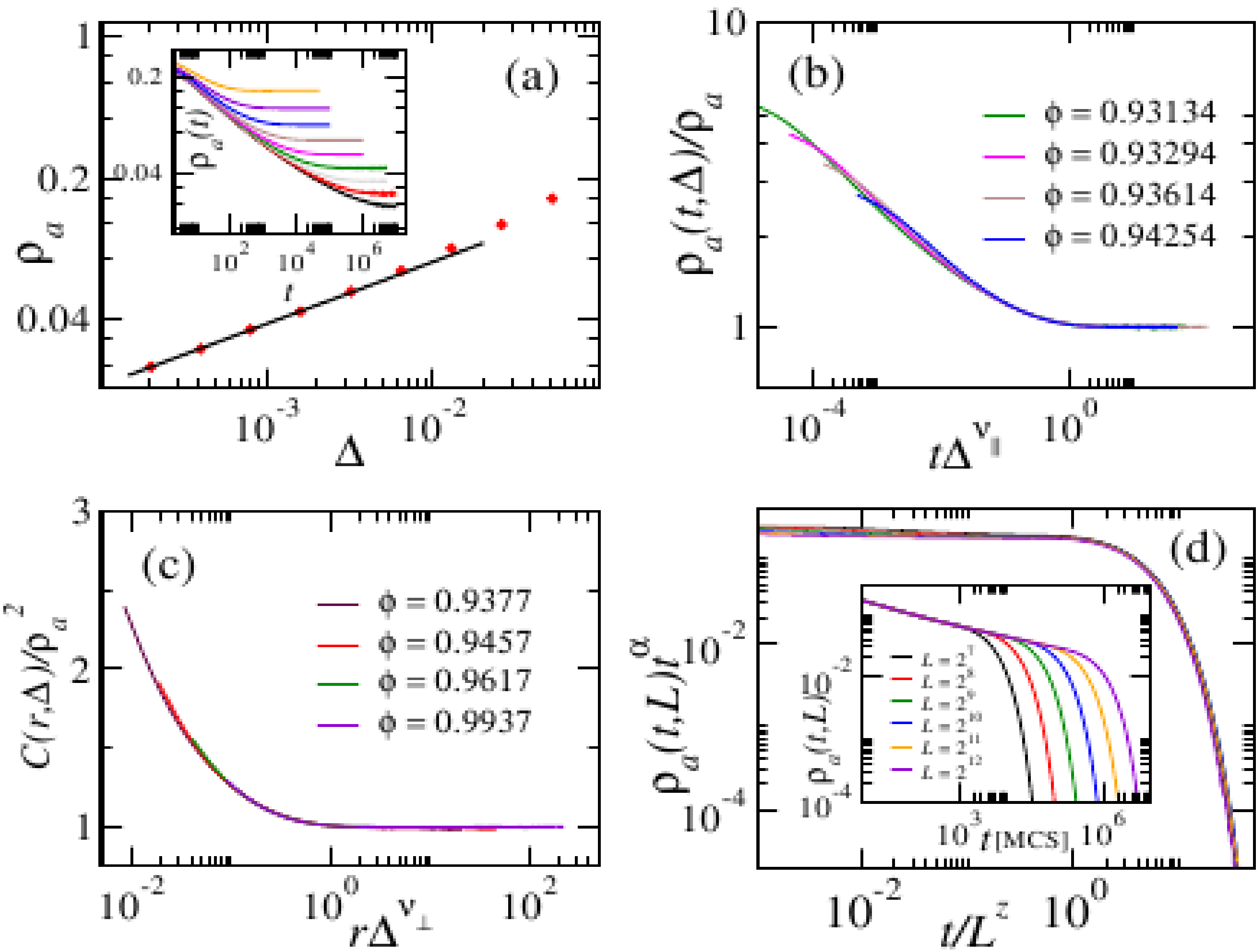


Matches with decay of ρ_a of DP with same ϵ

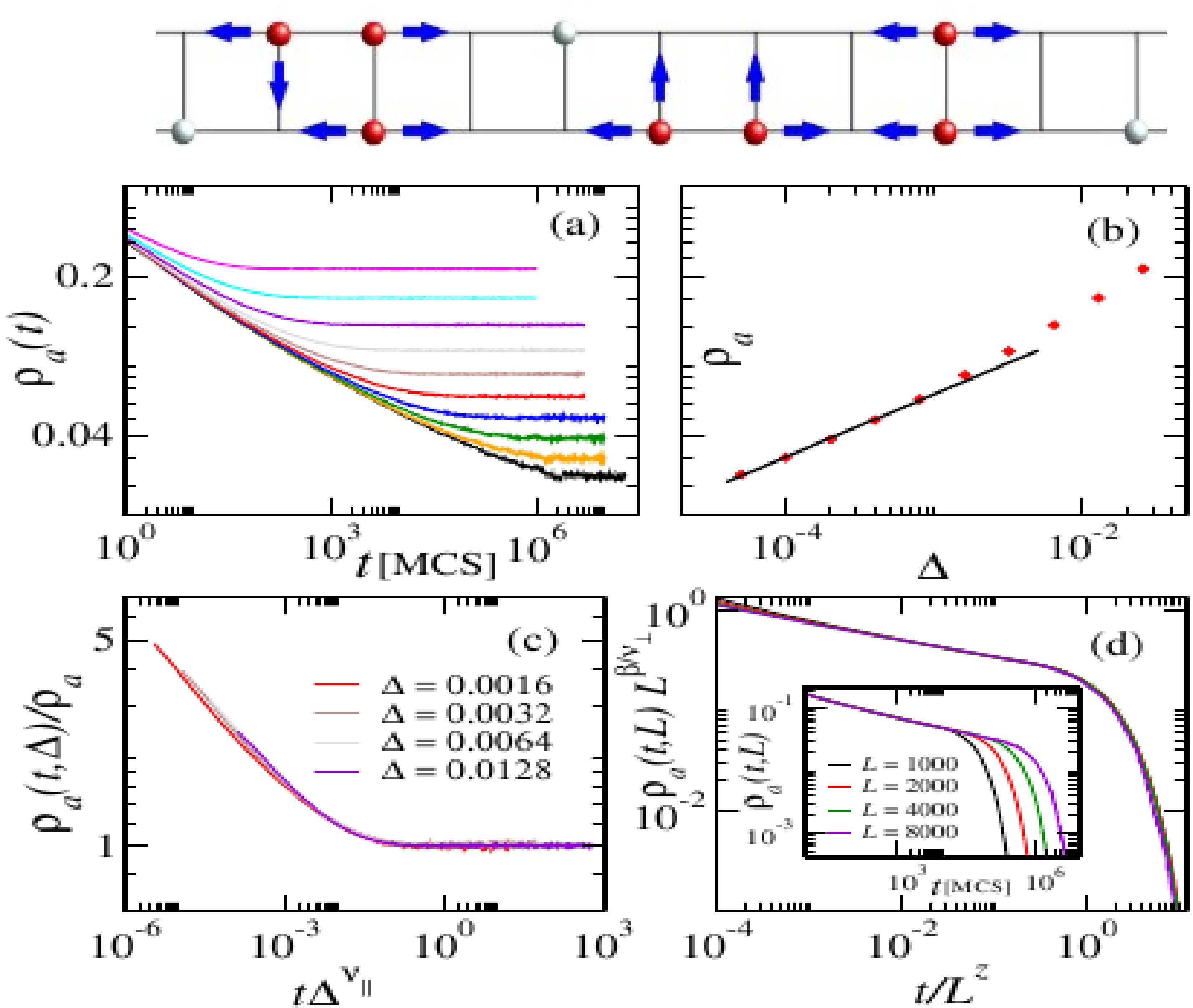
Scaling with natural ic (Manna FES)



Scaling with natural ic (CTTP)



Scaling with natural ic CLG on a ladder



Critical exponents

Manna Model (FES)

Model	α	β	$\nu_\perp$	$\nu_\parallel$	z
DCMM[9]	0.141(24)	0.382(19)	1.347(91)	1.87(13)	1.393(37)
DCMM*	0.159(3)	< 0.31	1.095(5)	1.75(5)	1.51(5)
CCMM*	0.1596(2)	0.277(18)	1.096(4)	1.74(1)	1.52(1)

CTTP

Ref.	ϕ_c	α	β	$\nu_\parallel$	$\nu_\perp$	z
Lübeck [1, 19]	0.96929(3)	0.141	0.382	2.452	1.760	1.393
Dickman *et. al.* [20]	0.92965(3)	-	0.412	2.41	1.66	1.45
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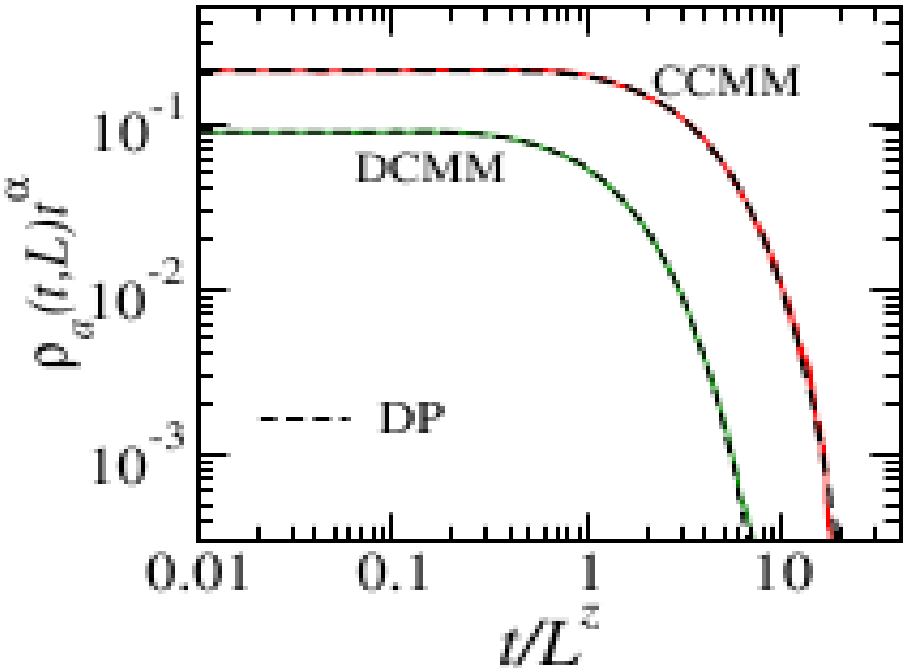
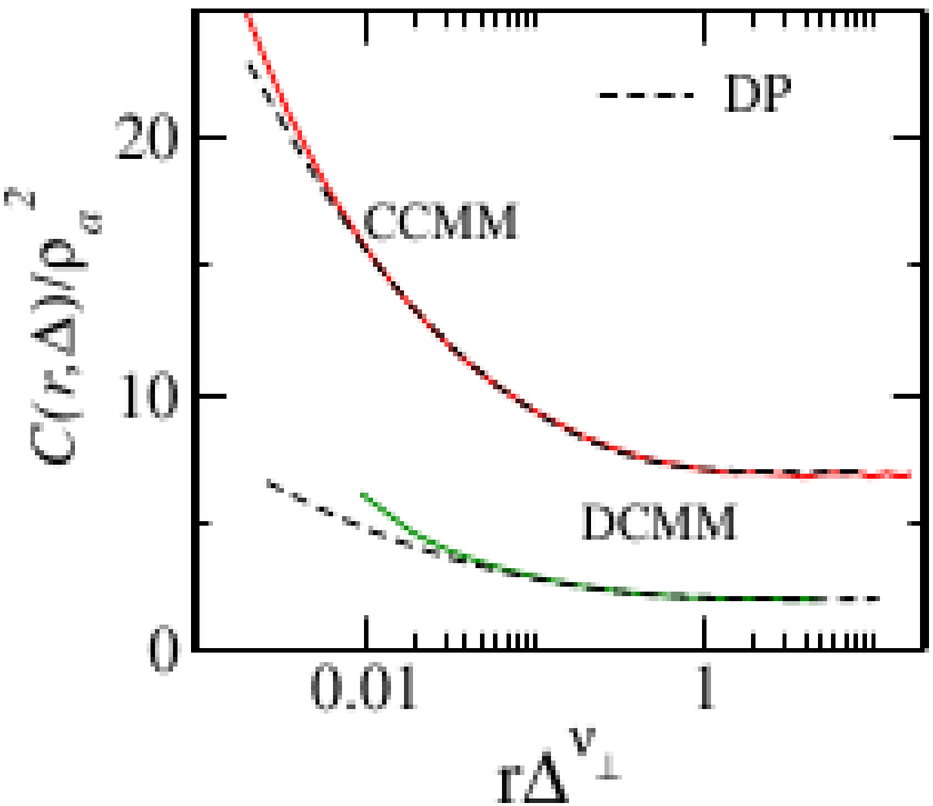
Dickman [22]	0.92978(2)	0.141	0.289	2.03	1.36	1.50
this work	0.929735(15)	0.155(5)	0.308(2)	1.74(1)	1.13(1)	1.52(2)

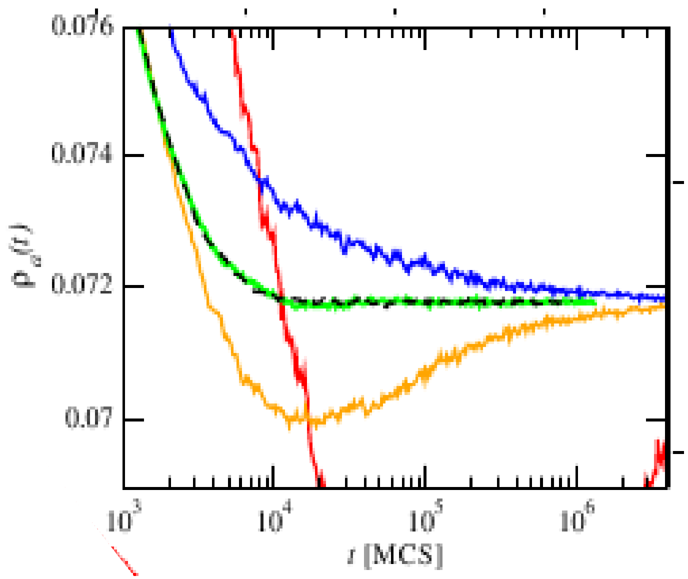
CLG Ladder

Ref.	α	β	$\nu_\parallel$	$\beta/\nu_\perp$	z
CLG 1D[25]	1/4	1	4	1	2
modified [28]	0.13(1)	0.277(3)	2.41	0.223(5)	-
ladder [26]	-	0.40(1)	-	-	-
this work	0.1553(1)	0.275(6)	1.76(2)	0.251(4)	1.50(2)
DP	0.159	0.277	1.73	0.252	1.58

Scaling functions...







Conclusion

- Dynamical behaviour in transient (non-recurrent) state could be very different from the same in stationary state.
- "Recurrent" initial conditions give consistent scaling.
- If recurrent space is not identified (in absence of exact solution), one may use

  "natural" ic : *suitably re-activated stationary state*

APT in fixed energy sandpiles

- Fixed energy sandpile models have “non-recurrent” configurations.

  - suffer from undesirable“transients” and

  - unusual undershooting in decay of activity

- Natural initial condition “heals” the ill-effect

- The critical behaviour generically belong to DP

SOC..... ?

- Manna class turned out to be DP.

  A simple conclusion, "SOC is just DP" would be premature.

- Drive and dissipation produce non-trivial background

- Equivalence of SOC and FES is still debated

[30] A. Fey, L. Levine, and D. B. Wilson, Phys. Rev. Lett. **104**, 145703 (2010).

[31] A. Fey, L. Levine, and D. B. Wilson, Phys. Rev. E **82**, 031121 (2010).

[32] Su.S. Poghosyan, V.S. Poghosyan, V.B. Priezzhev, and P. Ruelle, eprint arXiv:1104.3548

Thank you.

Thank you.