Quantum Signatures of Spacetime "Graininess"

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Introduction

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1. Length scales in physics
2. Spacetime noncommutativity from quantum uncertainties
3. Quantum Mechanics on Noncommutative Spacetime
4. Quantum Field Theory on Noncommutative Spacetime
   - Implementing Poincaré Symmetry
   - Hopf Algebras, Drinfel'd Twist and Quantum Theory
5. Gauge Fields on Moyal Space
   - Covariant Derivatives and Field Strength
   - Noncommutative Gauge Theories
6. Signatures of Spin-Statistics Deformation

Introduction

A look at distances

$$
\begin{aligned}
\text{Milky Way} &\sim 10^{21}\ m \\
\text{Solar System} &\sim 10^{12}\ m \\
\text{Car} &\sim 1\ m \\
\text{Atom} &\sim 10^{-10}\ m \\
\text{Proton} &\sim 10^{-15}\ m \\
\text{GUT scale} &\sim 10^{-32}\ m \\
\text{Planck scale} &\sim 10^{-35}\ m
\end{aligned}
$$

How do we measure distances?

- For galactic distances, there are (indirect) techniques involving angular size and standard candle, also red-shift data, and so on.
- For planetary distances, one can use Kepler's laws.
- For even smaller distances (cars, shoes, ...), we can use the tape measure.
- For atomic sizes and smaller, we need to use particles whose Compton wavelength is comparable to the size of the object.

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(Sub-)Atomic scale measurements

- On the scale of atomic distance and smaller, new effects come into play because of quantum mechanics:
- The position and momentum of a particle cannot be measured simultaneously to infinite accuracy.
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Quantum Mechanics and Classical Gravity

- Gravity and quantum mechanics are both important when distances are of the order of the Planck length $\ell_P = \left(\frac{G\hbar}{c^3}\right)^{1/2}$.
- In order to probe physics at the length scale ℓ_P, the Compton wavelength $\hbar/Mc$ of the probe must satisfy

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- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

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Spacetime Uncertainty Relations

Doplicher-Fredenhagen-Roberts, 1995

"Attempts to localize with extreme precision cause gravitational collapse, so spacetime below the Planck scale has no operational meaning."

- More precisely, we get the spacetime uncertainties:

$$\Delta x_0 \left(\sum_i \Delta x_i \right) \gtrsim \ell_P^2, \qquad \sum_{1 \le i < j \le 3} \Delta x_i \Delta x_j \gtrsim \ell_P^2$$

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Noncommutative Spacetime (Moyal Algebra)

- A concrete model for these uncertainties is the algebra generated by operators $\hat{x}_\mu$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is a (fixed) constant antisymmetric matrix.

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Analogy with Quantum Mechanics

- Quantum mechanics emerges because it is operationally meaningless to localize points in classical phase space.
- Classical phase space (a commutative manifold) is replaced in QM by a "noncommutative" manifold $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.
- This leads to "cells" in phase space, giving us Planck's radiation law, and avoiding the ultra-violet catastrophe of Rayleigh-Jeans law.

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The Moyal Algebra

- Our starting point is the set of commutation relations

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- This algebra has the advantage that it can be realized in terms of ordinary functions on Minkowski space, but with a new noncommutative product:

$$\begin{aligned} f(x) * g(x) &= f(x) e^{\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}} g(x) \\ &\simeq f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \cdots \end{aligned}$$

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- Our interest is in understanding quantum theory on this noncommutative space.
- Let us look at a two-dimensional example. The fundamental commutation relations are:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\theta.$$

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Piecewise constant potential

- The Hamiltonian for the circular well problem is

$$H = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m_0} + V(\hat{x}, \hat{y})$$

  where $V(\hat{x}, \hat{y})$ is a "piecewise" constant potential: it is $(-V_0)$ in a circular region of radius R around the origin, and zero elsewhere.

- In the commutative case, the spectrum for the "infinite" circular well is give by the zeros of the Bessel functions:

$$J_m(kR) = 0, \quad m = 0, \pm 1, \pm 2, \cdots, E = \hbar^2 k^2 / 2m_0$$

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L^{m}_{M+1}(\theta k^2/2) &= 0, \quad m \geq 0, \\
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  (M is related to the radius: $R^2 = \theta(2M+1)$.
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Thermodynamics

The most striking effects are brought out by looking at thermodynamics of this system.

- Number of particles in a noncommutative system cannot be made arbitrarily large: there is a “maximal” density!
- Pressure diverges as the maximal density is approached.
- Entropy of the system behaves radically differently: it approaches zero at maximal density.

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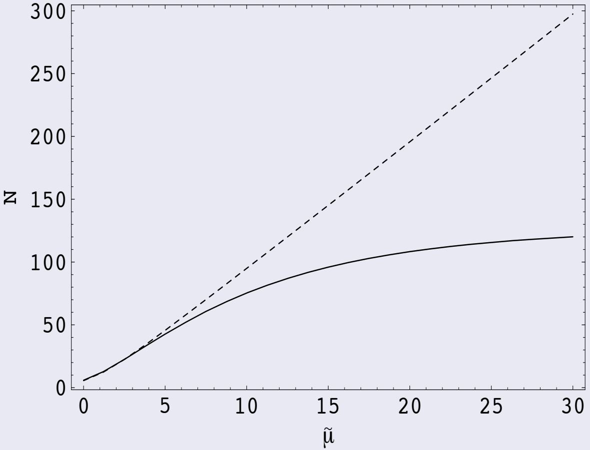
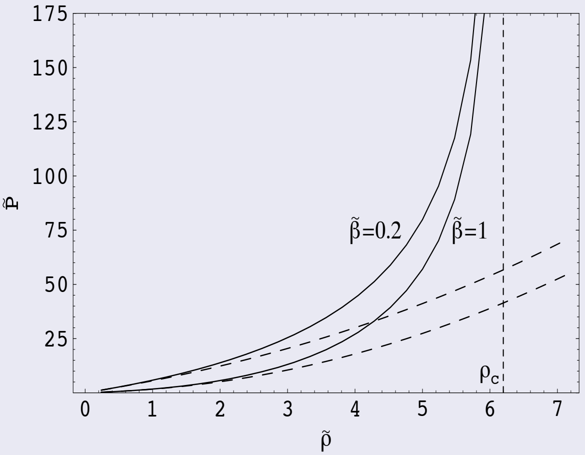
Particle in a noncommutative circular well: thermodynamics

Average number of particles as a function of chemical potential (Dashed line is the commutative case).



Particle in a noncommutative circular well: thermodynamics

Pressure as a function of density (Dashed line is the commutative case).



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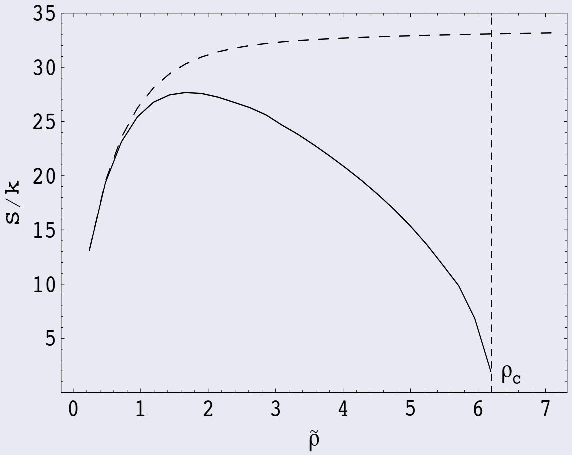
Entropy as a function of density (Dashed line is the commutative case).



QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
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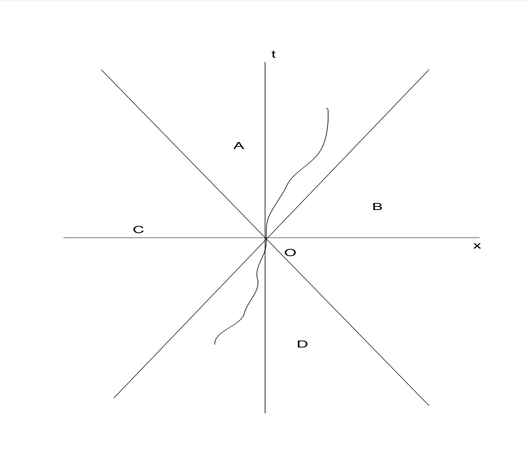
Light Cone

A is in the future of *O* and *D* is in its past, whereas *B* and *C* are causally unrelated to *O*.



- Mathematically, we say this by requiring that observables ρ at *spacelike* separation commute:

$$[\rho(x), \rho(y)] = 0 \quad \text{if} \quad \mathrm{x} \sim \mathrm{y}.$$

- This condition is enforced by requiring that the *quantum fields* at points x and y satisfy

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- A deep theorem in QFT: for particles with integer spin, we must choose the minus sign (use the commutator), and for particles with half-integer spin, we must choose the plus sign (use anti-commutator).
- So causality, statistics, and spin are intimately related.
- Relativistic invariance implies that the notions of fermions and bosons are not frame-dependent – e.g. a two-fermion state will be anti-symmetric in all reference frames.

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Noncommutative QFT

- Our experience with noncommutative quantum mechanics suggests that particles are not point-like – there is a certain graininess/discreteness.
- Quantum field theories on such a space should somehow retain information of this discreteness.
- We also want relativistic invariance to be compatible with this discreteness – not easy! For example, if we replace $\mathbb{R}^3$ by a discrete lattice, we lose translational and rotational symmetry.

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- Fortunately for us, there are mathematical structures (known as twisted Hopf symmetries) that allow us to implement relativistic symmetries on the noncommutative spacetime.
- This procedure of “twisting” can be used to define properties of quantum fields.
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(Naive) Lorentz Transformations

Moyal (or star) product in terms of commutative product

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\begin{aligned}
& (f * g)(x) = m_\theta(f \otimes g)(x) = m_0(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g)(x), \\
= \; & m_0(\mathcal{F} f \otimes g)(x) = (f \cdot g)(x) + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f \cdot \partial_\nu g)(x) + \cdots
\end{aligned}
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- Under a Lorentz transformation Λ, functions f and g transform as

$$f(x) \to f^\Lambda(x) = f(\Lambda^{-1}x), \quad g(x) \to g^\Lambda(x) = g(\Lambda^{-1}x)$$

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(f \cdot g)^\Lambda(x) &= (f^\Lambda \cdot g^\Lambda)(x), \quad \text{BUT} \\
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A Closer Look at the Moyal Algebra

- Left- and right- multiplications are not the same:

$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

- Define (a commuting) $\hat{x}^c_\mu$ in terms $\hat{x}^L_\mu, \hat{x}^R_\mu$ as

$$\hat{x}^c_\mu \equiv \frac{1}{2}\left(\hat{x}^L_\mu + \hat{x}^R_\mu\right), \quad [\hat{x}^c_\mu, \hat{x}^c_\nu] = 0,$$

$$\hat{x}^c_\mu f = \frac{1}{2}(x_\mu * f + f * x_\mu) = x_\mu \cdot f$$

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$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

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A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu\hat{p}_\nu - \hat{x}^c_\nu\hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
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Modified Leibnitz rule

- Although the $M_{\mu\nu}$ correctly generate the Lorentz algebra, their action on the star product of two functions is different:

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\begin{aligned}
& M_{\mu\nu}(\alpha * \beta) = (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) \\
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Suppose a group G acts on a vector space V as

Group Action

$$v \to \rho(g)v, \quad v \in V, \text{ and } \rho \text{ a representation of } G.$$

- On a tensor product $V \otimes W$, the group acts as

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- Suppose V is also an algebra (we can multiply two vectors to get another vector).
- Then our coproduct better be compatible with multiplication in V!
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- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
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- Indeed, $m_\theta[\Delta_\theta(\Lambda)f \otimes g] = \rho(\Lambda)m_\theta(f \otimes g)$.
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Implications for Quantum Statistics

- In usual quantum mechanics, the wavefunction of two identical particles is the (anti-)symmetrized tensor product of single particle wavefunctions:

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2}(\phi \otimes \chi \pm \chi \otimes \phi) = \left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi)$$

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Twisted Quantum Fields

- Suppose $\Phi(x)$ is a second-quantized field, and $a_p^\dagger$ the creation operator with momentum p. As usual we require that

$$\begin{aligned}\langle 0|\Phi^{(-)}(x)a_p^\dagger|0\rangle &= e_p(x), \\ \langle 0|\Phi^{(-)}(x_1)\Phi^{(-)}(x_2)a_q^\dagger a_p^\dagger|0\rangle &= (\mathbf{1}\pm\tau_\theta)\,(e_p\otimes e_q)(x_1,x_2) \\ &\equiv (e_p\otimes_{S_\theta,A_\theta} e_q)(x_1,x_2)\end{aligned}$$

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$$c_p c_q - c_q c_p = 0, \quad c_p c_q^\dagger - c_q^\dagger c_p = 2p_0 \delta(\vec{p} - \vec{q}).$$

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- The notion of identical particles can still be defined, but now there is a scale dependence – for example, a two-fermion wavefunction is anti-symmetric at low energies, but picks us a symmetric piece at high energies.
- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

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- An experimental signature would be a transition between a "Pauli-allowed" and a "Pauli-forbidden" state.

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Gauge transformations

- Gauge fields A_λ transform as one-forms under diffeos generated by vector fields. They could be functions of $\hat{x}^c$ or $\hat{x}^L$.
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- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

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Matter-Gauge Interactions

- The interaction Hamiltonian is of the form

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  $\mathcal{H}^{MG}$ has all matter-matter and matter gauge couplings, $\mathcal{H}^G$ has only gauge field terms.
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Signatures of Noncommutativity

- Any high-energy phenomenon involving identical particles is expected to carry signatures of noncommutativity, through the deformation of the spin-statistics connection.
- New physics at high densities – potential implications for neutron star physics, Chandrasekhar limit, and early cosmology.
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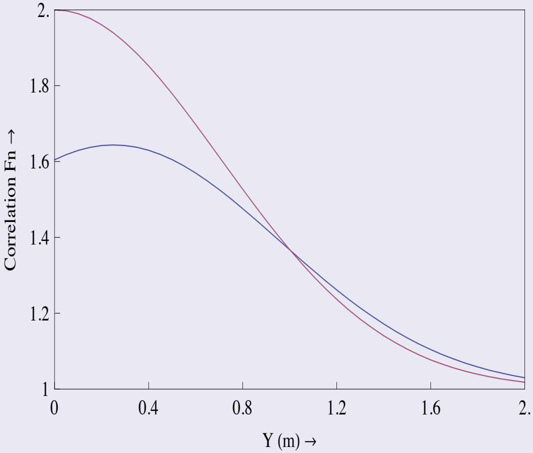
Two-particle distribution function

The commutative distribution (red) and the noncommutative distribution (blue), with $\sqrt{\theta} \sim 10^{-13} m$



QED from Spontaneously Broken $SU(2) \times U(1)$

- The gauge group for the Standard Model is non-Abelian, and will show similar effects.
- In particular, signatures of Lorentz (or spin-statistics) violation can be seen in QED.
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- We can calculate the scattering amplitude $\mathcal{T}_\theta$ in the centre-of-momentum frame, with the spins of the electrons aligned. It depends on scattering angle Θ_M, dimensionless c.m energy $x = E/m$, and $t = m^2\theta_{ij}\epsilon^{ijk}(\hat{p}_F \times \hat{p}_I)^k$.

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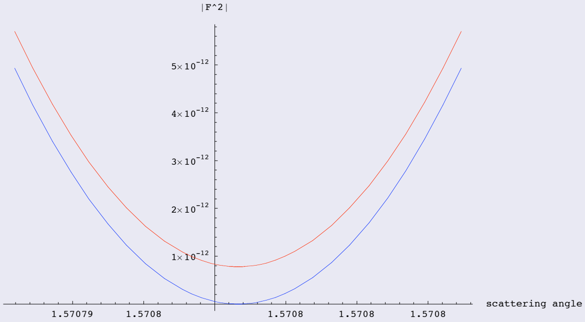
Normalized scattering cross-section for $t = 10^{-5}$ and $x = 100$

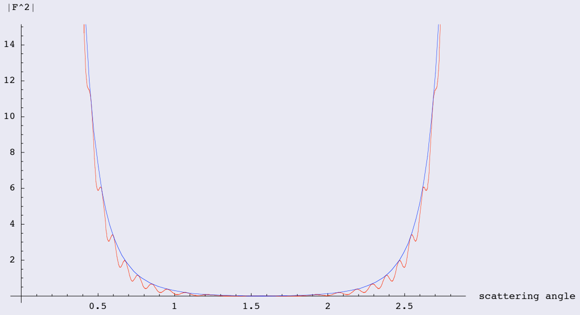
Moller1.nb 1



Möller scattering in QED

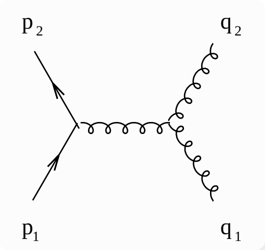
Normalized scattering cross-section for $t = 10^{-2}$ and $x = 100$

Moller2.nb 1



Non-Abelian Gauge Theories

- In non-Abelian gauge theories, there are even more (conceptually) dramatic effects: these theories lose relativistic invariance at the quantum level.
- Processes like $qg \to qg$ (quark-gluon scattering) violate Lorentz invariance (the propagator is "frame-dependent").



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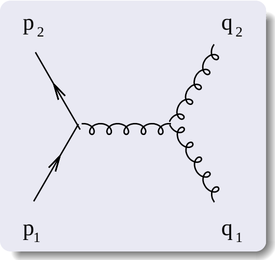

Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
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Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure “fuzzy”, leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a “free” fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

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Collaborators I

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- Fedele Lizzi, Patrizia Vitale
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