SU(N) Gauge Theory Thermodynamics from Lattice

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The theory of Strong interactions: SU(3) gauge theory
Quark fields $\psi_i$: triplet of SU(3), $i = 1, 2, 3$  

Gluon fields: traceless Hermitian matrices $A_{ij}^\mu$

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{(11 - \frac{2N_f}{3}) \ln Q^2/\Lambda^2} \quad \text{in leading order}$$

The interactions become strong at hadron physics scales: no natural small expansion parameter

G. ’t Hooft 1974 Consider SU(N), use $1/N$ as expansion parameter
Leads to some simplifications and qualitative understandings of some properties of hadrons
\[ g \sim 1/\sqrt{N} \text{ for nontrivial large N limit} \]

\[ \lambda_{tH} = g^2 N \]

Quark loops ignored in leading order

unless \( N_f \sim N_c \)

\[ \lambda_{tH}(Q^2) \sim \frac{48\pi^2}{11 \ln Q^2/\Lambda^2} \]
SU(N) Gauge Theory at Large N

- SU(N) gauge theory confines for all N. Nonperturbative physics for $q < \Lambda_{SU(N)}$
- Glueballs, mass $\sim O(N_c^0)$, decay widths $\sim O(1/N_c)$
  Mesons, mass $\sim O(N_c^0)$, scattering $\sim O(1/N_c)$

Lattice study: more expensive. String tension, Glueball spectrum, etc. studied.

  Lucini & Teper; Narayanan & Neuberger; etc.

- At large temperatures, $O(N_c^2)$ gluon degrees of freedom
- 1st order phase transition at $T_c \sim \Lambda_{SU(N)}$ with latent heat $O(N_c^2)$

SU(3): 1st order phase transition at $T_c \sim 270$ MeV
Equilibrium Thermodynamics on Lattice

\[ Z(T) = \int dU \exp(-\beta \int_{0,pbc}^{1/T} d\tau \int d^3x \mathcal{L}(U)) \]

As lattice spacing \( a \to 0 \),
\[ \beta \mathcal{L} \to \text{Tr} F_{\mu\nu}^2 \] with
\[ \beta = 2N/g^2 = 2N^2/\lambda tH \]

- \( A_\mu(\vec{x},0) = A_\mu(\vec{x},1/T) \)
  
  \( Z_N \) group of aperiodic gauge transformation

- \( U_\mu(\vec{x},1/T) = e^{2\pi i/N} U_\mu(\vec{x},0) \)

- Order parameter for deconfinement transition:

\[
L = \frac{1}{N} \text{Tr} \left( \prod_{x_0=1}^{N_\tau} U_{(x_0,x),\hat{0}} \right) \to 1/N \text{Tr} \mathcal{P} e^{\int_{0}^{1/T} d\tau A_0(\vec{x},\tau)}
\]

\[
Z_N : \quad L = \frac{1}{V} \sum_{\vec{x}} L(\vec{x}) \to e^{2\pi i/N} L
\]

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\[ |\langle L \rangle| \sim e^{-F}, \text{ F free energy of a static quark source} \]
\[ \langle L \rangle \neq 0 \rightarrow Z_N \text{ broken and deconfinement} \]
\[ \text{SU}(2): \text{2nd order transition in } Z_2 \text{ universality class} \]

\[ \text{Engels et al. 1982-90} \]

\[ \text{For } N > 2: Z_N: \text{1st order transition expected} \]
\[ \text{The QCD transition } N_f = 2 + 1 \text{ is a crossover.} \]
\[ \text{Expected to be first order for all finite mass at } SU(\infty) \]
\[ \text{Chiral limit may be more involved.} \]
\[ \text{Similarly, complicated phase structures have been predicted} \]
\[ \text{for } \mu_B \sim O(N_c). \]

\[ \text{McLerran & Pisarski '07} \]
Study of SU(N) theory at finite T

Interesting predictions at finite temperature, which may help our understanding of SU(3)

- Symmetry of Polyakov loop → strong 1st order transition?
- How does $T_c/\Lambda_{\text{MS}}$ scale?
- For $N > 3$ the latent heat $\sim N^2$
- Does one reach the asymptotic state soon after the transition?
- Deviation from conformality in the plasma phase?

Deconfinement transition for SU(4)

Gavai; Ohta & Wingate; 2001

Deconfinement transition for SU(N), and pressure of the high temperature phase from coarse lattices.

Lucini & Teper; Lucini, Teper & Wenger; Bringholtz & Teper 2003–

Need to be careful about discretization errors
Our study

Numerical investigation of SU(4) and SU(6) gauge theories
Study large N theory using results for N=3,4,6


- Study of the cutoff dependence
  Is the lattice spacing small enough for 2-loop running?
- Study of the phase transition in SU(4) and SU(6)
  Finite volume study
- Direct estimation of $T_c/\Lambda_{\overline{MS}}$
- Equation of state for N=4 and 6 theories
  Analyse SU(3) with existing data.
- $\epsilon - 3p$ and conformal symmetry breaking in the deconfined phase
Scaling

\[ T_c(g^2, N_t) = \frac{1}{a(g^2)N_t} \]

Use \( T_c(N_t) \) to predict \( T_c(N'_t) \)

we use for coupling \( V_Q,\bar{Q}(q^2) = -\frac{N^2-1}{2N} \frac{g^2}{q^2} \) (V-scheme)

S. Gupta, PRD 64(2001) 034507
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Bulk and Deconfinement Transitions for SU(6)

\[ P_{\mu,\nu}(x) = \frac{1}{N} \text{Tr} U_\mu(x) U_\nu(x + \hat{m}u) U_\mu^\dagger(x + \nu) U_\nu^\dagger(x) \]

\[ P = \frac{1}{6VT} \sum_{\mu,\nu} \sum_x P_{\mu,\nu}(x) \]

\[ \beta \quad \beta \quad \beta \]

\[ \Omega_x \quad \Omega_x \quad \Omega_x \]

\[ N_t = 4 \quad N_t = 6 \quad N_t = 16 \]

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Bulk and Deconfinement Transition for SU(4)

\[
\left\langle P_t - P_s \right\rangle \text{ associated with deconfinement:}
\]

\[
\frac{\epsilon}{T^4} = 6N^2 N_T^4 \frac{P_t - P_s}{\lambda_{tH}} + \text{corrections}
\]
$\langle P_t - P_s \rangle$ associated with deconfinement:

$$\frac{\epsilon}{T^4} = 6N^2N_T^4 \frac{P_t-P_s}{\lambda_{tH}} + \text{corrections}$$
Degenerate Vacua at $T_c$

SU(4), $N_t=6$, $\beta=10.79$, Hot run, every 400th

$N_s=16, N_s=18, N_s=20, N_s=22$
Susceptibility $\chi_L = V(\langle|L|^2\rangle - \langle|L|\rangle^2) \sim V$ for 1st order.
Deconfinement Transition for SU(6)

SU(6), $L^3 \times 6$ Lattice

\[ \frac{\Lambda}{T} \]

\[ \langle Pt - Ps \rangle \]

$\beta$

$L=16$

$L=20$

$L=24$

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N-dependence of $\beta_c$

\[ \beta = 2 N^2 / \lambda_{tH} \]

\[ \frac{\beta_c}{N^2} = 0.7008(6) - \frac{0.413(6)}{N^2} \quad \text{SU}(N), N_t=6 \]

\[ \frac{\beta_c}{N^2} = 0.7186(5) - \frac{0.406(5)}{N^2} \quad \text{SU}(N), N_t=8 \]
Use 2-loop running to directly calculate \( T_c/\Lambda_{\text{MS}} \).

Definitions of \( g^2 \) through different schemes:
\[
g_E^2 = \frac{8N}{N^2 - 1} (1 - P) \quad \text{(E-scheme)}
\]
besides \( g_V^2 \) and \( g_{\text{MS}}^2 \).

For each \( N \), continuum limit of \( g_{\text{scheme}}^2 \).

Scheme-dependence observed to be stronger than cutoff dependence.

N-dependence using \( c_0 + \frac{c_1}{N^2} \).

\[
\frac{T_c}{\Lambda_{\text{MS}}} \bigg|_{N \to \infty} = 1.17(5)
\]
Define thermodynamic quantities
\[ F(T, V) = T \ln Z(T, V) \quad \text{Z: grand canonical partition function} \]

\[ \epsilon = -\frac{1}{V} \frac{\partial F(T, V)/T}{\partial (1/T)} \]
\[ p = \frac{\partial F}{\partial V} = F/V \quad \text{for homogeneous} \]
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Trace of energy momentum tensor \( \theta^{\mu\mu} = \Delta = \frac{\epsilon-3p}{T^4} \)

measure of breaking of conformal invariance

\[ \Delta/T^4 = T \frac{\partial}{\partial T}(p/T^4) \]
Define thermodynamic quantities
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\[ \Delta/T^4 = T \frac{\partial}{\partial T}(p/T^4) \]

Convenient to calculate this on lattice
\[ T \frac{\partial}{\partial T} \rightarrow -a \frac{\partial}{\partial a} \]
\[ p/T^4 \rightarrow \frac{N_t^3}{N_s^3} \ln Z \]
Define thermodynamic quantities

\[ F(T, V) = T \ln Z(T, V) \]

\[ Z: \text{grand canonical partition function} \]

\[ \epsilon = -\frac{1}{V} \frac{\partial F(T, V)}{\partial (1/T)} \]

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Trace of energy momentum tensor \( \theta^\mu\mu = \Delta = \frac{\epsilon - 3p}{T^4} \)

measure of breaking of conformal invariance

\[ \Delta / T^4 = T \frac{\partial}{\partial T} \left( \frac{p}{T^4} \right) \]

Convenient to calculate this on lattice

\[ T \frac{\partial}{\partial T} \rightarrow -a \frac{\partial}{\partial a} \]

\[ p / T^4 \rightarrow \frac{N_t^3}{N_s^3} \ln Z \]

\[ \Delta / T^4 = \frac{\partial}{\partial a} \beta \frac{N_t^3}{N_s^3} \langle \frac{dS}{d\beta} \rangle \]
Eqn. of State (Contd.)

\[ \frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_0^\beta d\beta \left( P(\beta) - P(\beta_0) \right) \]

\[ \frac{\epsilon}{T^4} = 3 \frac{p}{T^4} + \frac{\Delta}{T^4} \quad \frac{s}{T^3} = \frac{\epsilon}{T^4} + \frac{p}{T^4} \]

For free gas, Stefan-Boltzmann limit

\[ \frac{\epsilon}{T^4} = 3 \frac{p}{T^4} = (N^2 - 1) \frac{\pi^2}{15} R(N_\tau) \]

Here \( R(N_\tau) \) discretization error

\[ = 1 + \frac{10}{21} \left( \frac{\pi}{N_\tau} \right)^2 + ... \]

Boyd et al., Nucl.Phys. B 469(’96) 419;
Cutoff and Volume Dependence of EOS for SU(4)

SU(4), $N_t=6$

SU(4), $N_t=8$

$N_s=24$

$N_s=18$

$N_t=6$

$N_t=8$

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Thermodynamics for SU(4)

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Thermodynamics for SU(6)

- $s/T^3$
- $\epsilon/T^4$
- $p/T^4$

SU(6)

$T/T_c$
Scaling of Thermodynamic Quantities with $N$

![Graph showing scaling of thermodynamic quantities with $T/T_c$.]
Scaling of Thermodynamic Quantities with $N$

SU(3)  
SU(4)  
SU(6)

$p/p_{SB}$  
$s/s_{SB}$
Interaction Measure $\epsilon - 3p$

$SU(N)$ gauge theory classically scale invariant $\rightarrow T^{\mu\mu} = 0$
Quantum theory may break this symmetry
$\epsilon - 3p$ measure of breaking of scale invariance

Substantial conformal symmetry breaking in plasma
Peak moves from $\sim 1.09 T_c$ at $N=3$ to $\sim 1.025 T_c$ at $N=6$
$\sim 1/T^2$ fall-off at intermediate temperature
Conformal vs. Weak Coupling Limit

Conformal vs. Weak Coupling Limit

Conformal vs. Weak Coupling Limit

SU(N) gauge theories studied at finite temperature for N=3,4,6

Important to check for discretization errors. Under control for $N_\tau > 6$.

First order deconfinement transition confirmed for N=4,6

$T_c/\Lambda_{\overline{MS}} = 1.17(5)$ with small $N$ dependence

Nice $N^2$ scaling for thermodynamic quantities for $T/T_c > 1.2$.

Large deviation from Stefan-Boltzmann limit

Considerable conformal symmetry breaking till deep in the plasma

No evidence of a window for strongly coupled conformal theory
Extra: \( (\epsilon - 3p)/T^4 \) scaling
Extra: $T_c/\Lambda_{MS}$
Extra: $T_c/\Lambda_{\text{MS}}$

![Graph showing $T_c/\Lambda_{\text{MS}}$ vs. $1/N_t^2$ for SU(6)]

- E-scheme
- V-scheme
- MS-scheme
Extra: $T_c/\Lambda_{\overline{MS}}$
Extra: $\beta_c$ for $N_T = 8$

**SU(4), $L^3 \times 8$ lattices**

- $L=22$
- $L=24$
- $L=28$
- $L=30$

**SU(4), $L^3 \times 8$ lattices, Cold start**

- $L=22$
- $L=24$
- $L=28$
- $L=30$

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Extra: Integration Method in Pressure

SU(4), $18^3 \times 6$ Lattice

SU(4), $24^3 \times 8$ Lattice

$P/T^4$

$\beta$
Extra: Volume Dependence of $\epsilon - 3p$

SU(6), $N_t=6$ Lattice

$(\epsilon-3p)/T^4$

$T/T_c$

L=24
L=16
Extra: Volume Dependence of $\epsilon - 3p$
Extra: Latent Heat for $N_t=5$

\[
\frac{1}{T_c} \left\{ \frac{L_h}{N^2} \right\}^{\frac{1}{4}}
\]