Modelling Proportionate Growth

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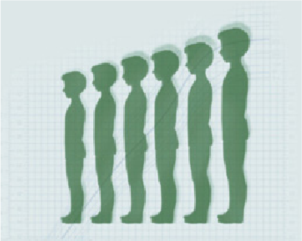
Outline

- Proportionate growth in animals
- Growing sandpiles as models of proportionate growth
- Characterizing the patterns
- Effect of noise
- Anisotropic growth patterns
- Concluding remarks

Proportionate growth in animals



Different body parts in animals grow roughly at the same rate.

- Proportionate growth is typical in animal kingdom.
- Easier problem than development of animal from a single cell.
- Requires regulation, and communication between different parts.
- Same food becomes different tissues in different parts of the body.

Proportionate growth in biology

- The standard biological answer only identifies different chemicals called promoters, inhibitors, growth factors ...., whose production is controlled by the masterplan encoded in DNA.

  "the knife did it "

A 1940's picture of an organized placard display in China.

Mechanism to interpret the detailed instructions in the DNA to generate the spatio-temporal patterns?

Modelling proportionate growth

Follow "On Growth and Form" [D'Arcy Thompson (1917)]. Focus on geometrical structure, ignoring chemical detail.
Qualitaitively different for previously studied models of growth by aggregation in physics, e.g.

- Crystal growth from melt, or supersaturated solution
- Diffusion limited aggregation
- Invasion percolation, Molecular beam epitaxy : KPZ growth

In these models, growth occurs only in the outer regions, with inner parts frozen.

Systems showing proportionate growth outside biology are hard to find.

Self-organization and sandpiles

In 1970's, Haken, Prigogine introduced the idea of living systems being 'self-organized'.

In 1987, Bak et al realized that many natural systems are self-organized to be at the edge of stability, and called these Self-Organized Critical.

They proposed a sandpile model as prototype model of SOC. Many earlier studies about the power-laws in distribution of avalanche sizes.

Proportionate growth in patterns formed by growing sandpiles

Growing patterns formed in Abelian sandpiles show self-organization, and proportionate growth.

- A simple cellular automaton model of proportionate growth
- Complex but beautiful patterns
- Analytically tractable: Exact characterization of patterns
- Involves some interesting mathematics: discrete analytic functions, piece-wise linear functions

Proportionate growth in sandpile patterns

Basic facts from biology:

Food required for growth. Reaches different body parts.
Cell-division occurs only if the cell has enough nutrients.

A well-studied model of threshold dynamics is the Abelian Sandpile Model

Definition of ASM:

- Non-negative integer height z_i at sites i of a square lattice
- Add rule: $z_i \to z_i + 1$
- Relaxation rule : if $z_i > z_c = 3$, topple, and move one grain to each neighbor.

Rule for forming patterns:

Add N particles at one site on a periodic background, and relax.

Sandpile Model: toppling rules

Start with a stable configuration, and add a particle :

0	0	0	0	0
0	0	0	0	0
0	0	4	0	0
0	0	0	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	3	0	0
0	3	4	3	0
0	0	3	0	0
0	0	0	0	0

Finally, we get stable configuration:

0	0	1	0	0
0	2	1	2	0
1	1	0	1	1
0	2	1	2	0
0	0	1	0	0

.

Proportionate growth

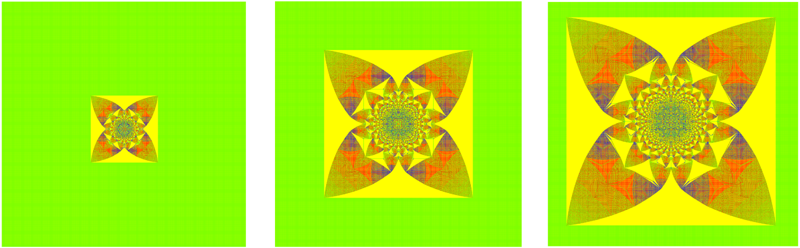


Figure: Patterns formed on a square lattice with initial height 2 at all sites. $N = (a) 4 \times 10^4$ (b) 2×10^5 (c) 4×10^5. Color code $0, 1, 2, 3 =$ R,B,G,Y

Diameter $\sim \sqrt{N}$.

One can study patterns on other backgrounds, other lattices, higher dimensions

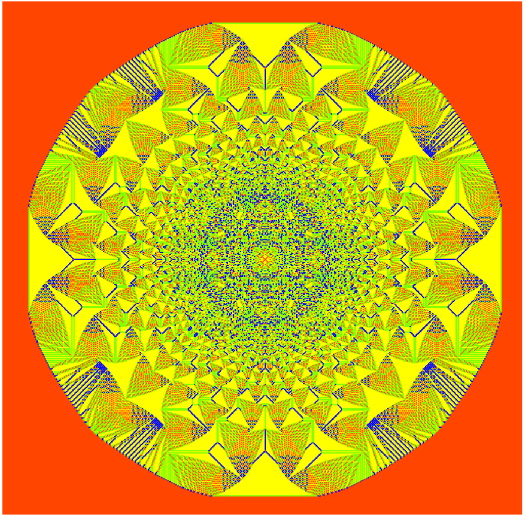


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0. Color code 0, 1, 2, 3 = R,B,G,Y

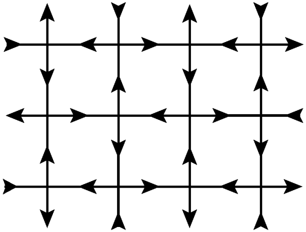


Figure: The F-lattice : A square lattice with directed bonds

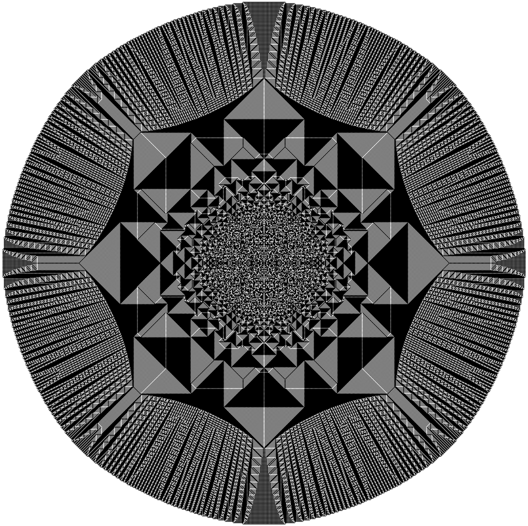


Figure: Pattern produced by adding 10^5 particles at the origin, on the F-lattice ASM, with initial state alternating columns of 1's and 0's. Color code: $B = 0, W = 1$

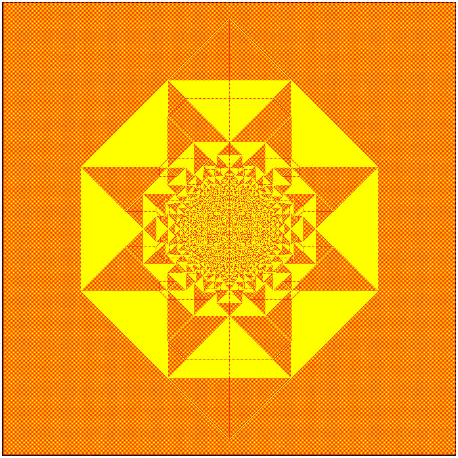
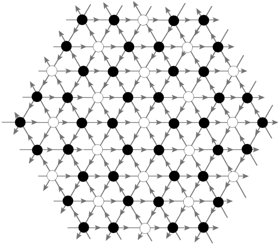


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard. Color code: 0 =R, 1=Y



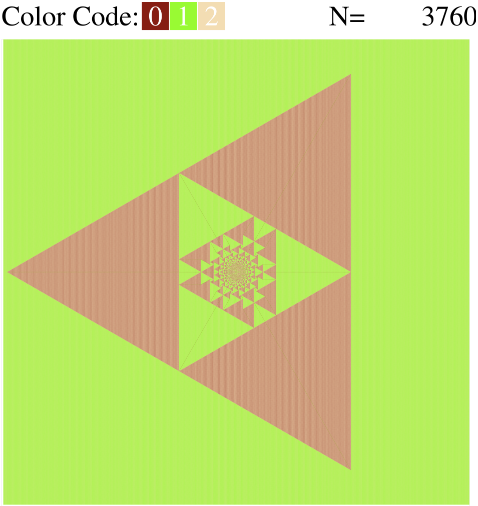
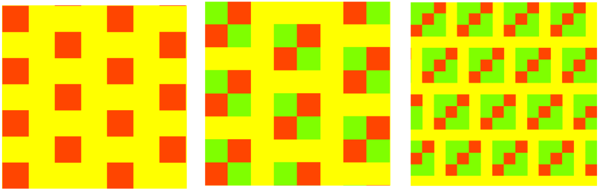


Figure: Directed triangular lattice. In the background configuration, filled and unfilled circles denote $z = 1$ and 2. Here diameter $\sim N$

The Key Observation
S. Ostojic (2003).

- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]

Examples of periodic patterns in patches



Characterizing the patterns

Let $T_N(\vec{R})$ = the number of topplings at point $\vec{R}$.

Define reduced coordinate $\vec{r} = \vec{R}/\Lambda$, Λ = diameter

Proportionate growth $\Leftrightarrow$ Scaling $T_N(\vec{R}) \sim \Lambda^a \phi(\vec{r})$.

A non-trivial $\phi(\vec{r})$ defines the asymptotic pattern.

The excess density of grains $\nabla^2\phi(\vec{r})$ is bounded, for $\vec{r} \neq \vec{0} \implies a \leq 2$.

In addition, we have $N \sim \Lambda^b$.

The Main Result

In each patch with a periodic height pattern, we can only have

$a = 2$, and $\phi(x, y)$ is a quadratic function of x and y,
Or
$a = 1$, and $\phi(x, y)$ is a linear function of x and y.

Proof:
Expand $\phi(x_0 + \Delta x, y_0 + \Delta y)$ in a Taylor series:
$\phi(x_0 + \Delta x, y_0 + \Delta y) = \phi(x_0, y_0) + A\Delta x + B\Delta y + .. + K(\Delta x)^3 + ..$
Equivalently,
$T_N(X, Y) = \ldots + K(\Delta X)^3/\Lambda^{3-a}$
For finite ΔX integer, T is also integer, and no proliferation of defect lines $\Rightarrow K = 0$.
Same is true for all higher powers.
For a non-trivial dependence on x, y, if quadratic term is not zero, $a = 2$. Else, $a = 1$. Independent of dimension.

Dependence of the diameter Λ with N

This is much less constrained.

- If the initial background density is low enough everywhere,
  $$\Lambda \sim N^{1/d}$$
- If many sites have large heights
  $$\Lambda = \infty \quad \text{for finite N}$$
- For an in-between set of periodic backgrounds
  $$\Lambda \sim N^{\alpha} \qquad \text{for } 1/\mathrm{d} < \alpha \leq 1$$

If $\Lambda \sim N^{\alpha}$, with $\alpha > 1/2$
We construct an infinite family of periodic backgrounds on the F-lattice that seem to have a different α for each member.

Patterns with non-compact proportionate growth

Directed traingular lattice with honeycomb background pattern

Diameter $\sim N$

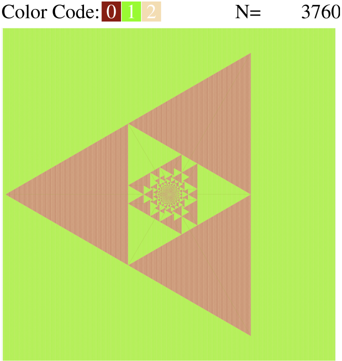
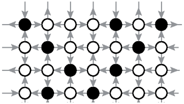


Figure: Unfilled circles=2 and filled circles=1

Examples of patterns with non-compact growth

Pattern on F-lattice showing $\alpha = 0.55$



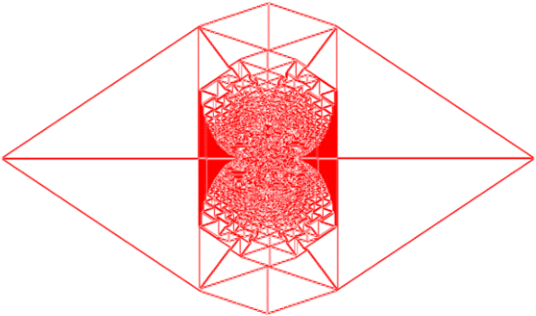
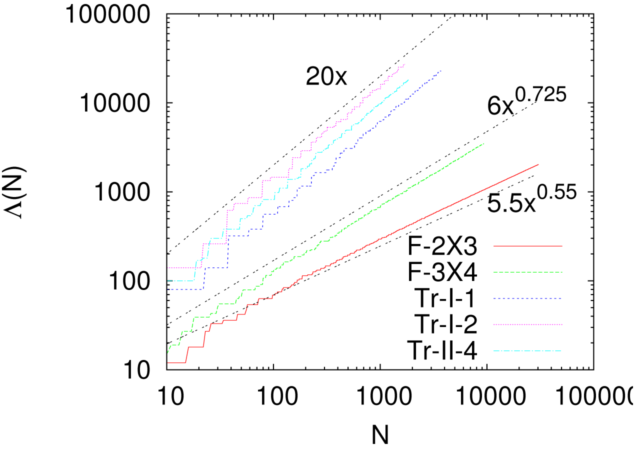


Figure: Only the boundaries of patches are shown.

Graph of Diameter Λ vs N



Quantitative characterization of the compact pattern

Consider the case with $a = 1$, say the triangular lattice pattern.
The exact characterization of the patterns involves four steps:

- $\phi(\xi, \eta)$ is a piece-wise linear function, with rational slopes. Parameterize as $\phi_P(\xi, \eta) = a_P\, \xi + b_P\, \eta + C_P$
- The allowed values of (a_P, b_P) for different patches form a periodic hexagonal lattice.
- The condition that three patches meet at a point implies that C_P satisfies a Laplace equation on the adjacency graph of patches.
- Exact solution of these equations gives the exact boundaries of patches

For $a = 2$, the procedure is similar, but quadratic functions need six parameters per patch.

Effect of noise in the background pattern

In presence of noise, the function ϕ is no longer polynomial, but the proprtionate growth still holds.

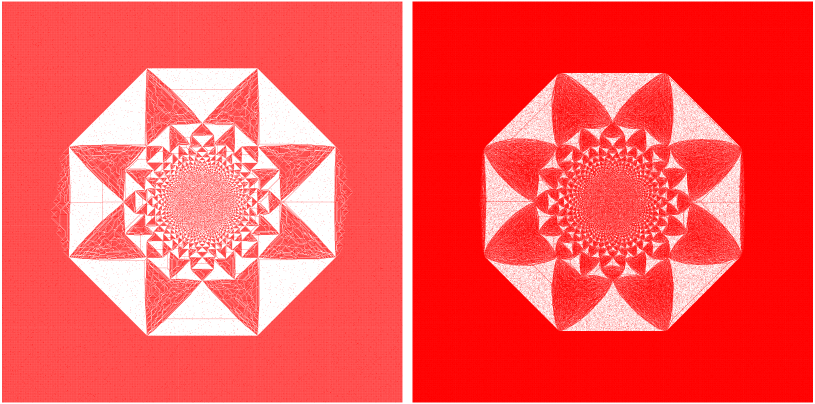


Figure: Pattern grown on the F-lattice with some heights 1 replaced by 0's. (a) 1% sites changed, N=228,000 , (b) 10% changed, N= 896,000.

If some 0's are also repalced by 1's, the effect is more dramatic.

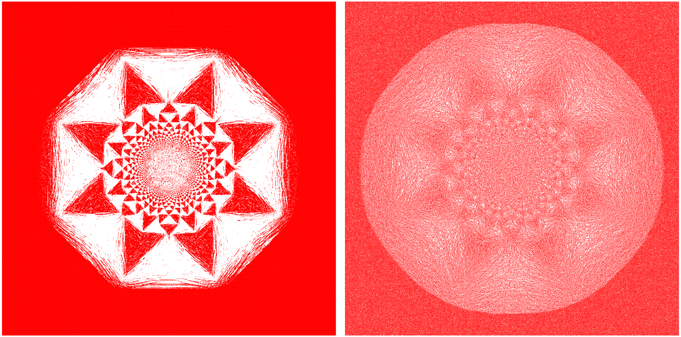


Figure: Pattern grown on the F-lattice with some heights flipped. (a) 1% sites changed (b) 10% sites changed

At higher noise level, the details of the pattern are not easy to see, but averaging over different realizations of noise brings out the pattern clearly.

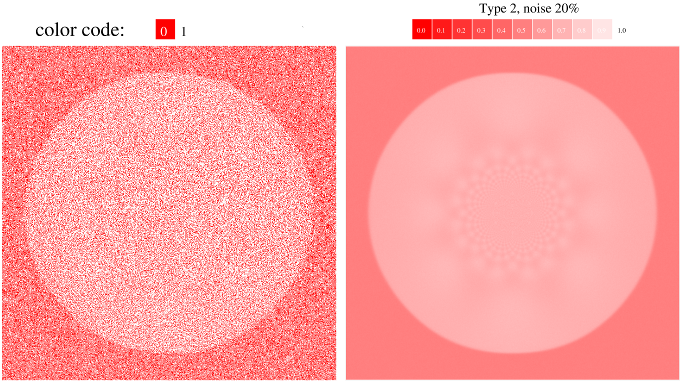


Figure: F-lattice, checkerboard with 20% sites flipped. $N = 57000$. (a) single realization (b) averaged height over 10^5 realizations.

On increasing noise, the amplitude of the density perturbation decreases. Suggests a linearized perturbation theory analysis.

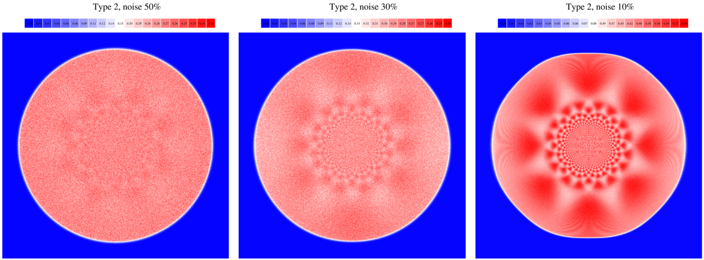


Figure: Averaged change in height with decreasing noise strength $50\%, 30\%, 10\%$ The color code for each pattern representing the height values are shown in the colorbar.

This suggests that we can write the change in density as

$$\Delta\rho(x,y) = B(\epsilon)g(x,y) + \text{ higher order in } \epsilon, \tag{1}$$

Where ϵ is a measure of difference of noise strength from 50%
If we apply a $z \to 1/z^2$ transformation to these figures, we get

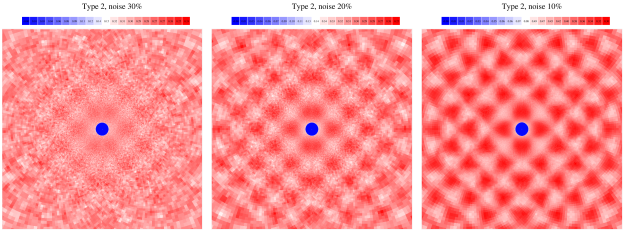


Figure: Result of applying $1/z^2$ transformation. Note the nearly gridlike pattern

This suggests that the simplest perturbation to the density field in the high noise limit is a periodic perturbation in the z'-coordinates.

$$g(x,y) = -\cos\frac{\pi x'}{2}\cos\frac{\pi y'}{2}, \qquad (2)$$

where $x' = \frac{2xy}{(x^2+y^2)^2}$, and $y' = \frac{x^2-y^2}{(x^2+y^2)^2}$.
A pictorial representation of this function is given below.

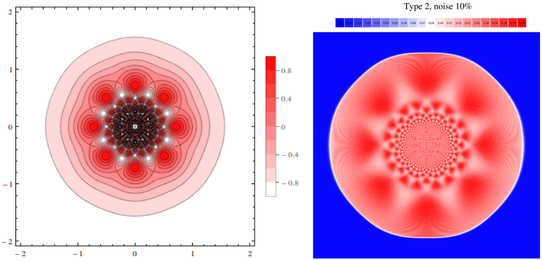


Figure: Density pattern using the function $g(x,y)$, compared to actual pattern. The black lines are contours of constant density.

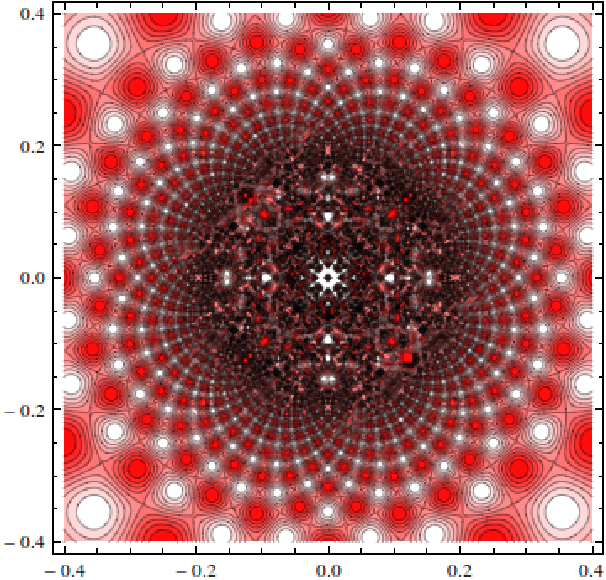


Figure: A zoom-in on the theoretical lowest -mode density perturbation $g(x, y)$.

One can get patterns that show a striking similarity to natural ones.

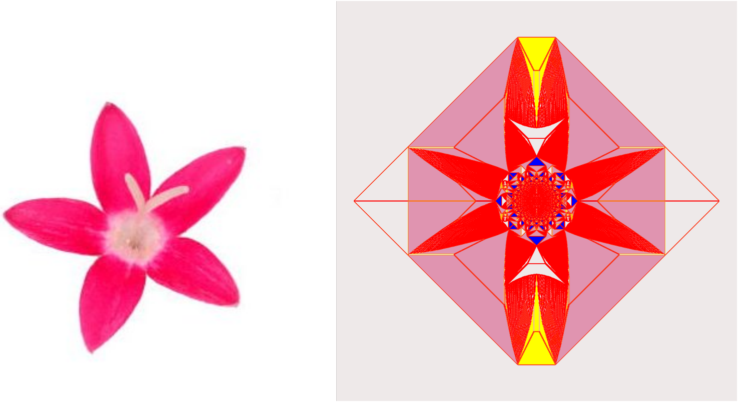


Figure: (a) A flower. (b) pattern produced by adding 256k particles on the F-lattice, with tilted squares backgound with spacing 4. Different colours denote different densities of particles, averaged over the unit cell of the background pattern

Anisotropic growth patterns

We can also study patterns that different rates of growth in different directions.

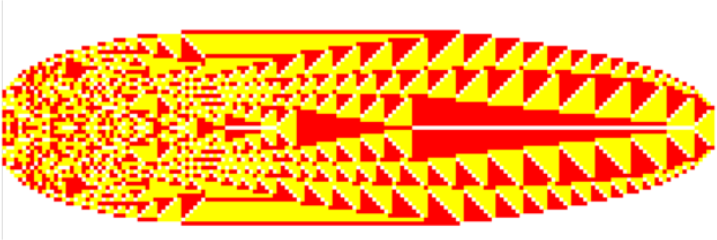


Figure: A 'larva' pattern. Produced on square lattice, with particle transfer on toppling only to up, down, right neighbors. Here $N = 10^4$. Particles are added at the left column center. Color code: 0=white, 1= red, 2=yellow.

A 'larva' pattern formed on a directed cubic lattice

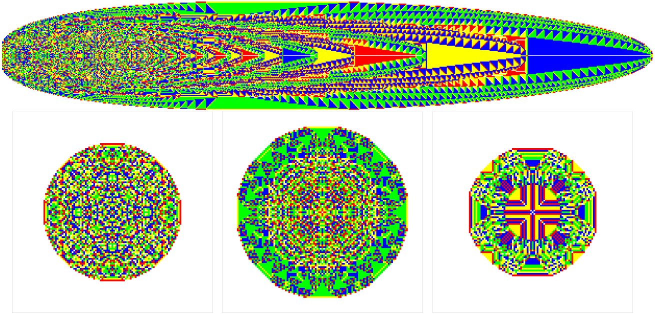


Figure: $N = 10^7$ grains added on an initial background of all heights zero. The particles were added at the central point of the left end. The first shows the mid-saggital section, and the next three show different transverse sections. Colour code : $0, 1, 2, 3, 4 =$ white, red, yellow, blue and green.

The theoretical understanding of these patterns is very limited.

- In general, we expect that $\Lambda_{longitudinal} \sim \Lambda^2_{transverse}$.
- Patterns do not show proportionate growth, but there are easily identified sub-structures, independent of N.
- For large N, $T_N(X, Y) \sim \Lambda^2 \phi(X/\Lambda, Y/\Lambda^2)$, where $\phi(\xi, \eta)$ defines the asymptotic pattern.
- In each periodic patch, $\phi(\xi, \eta)$ is a polynomial function of ξ and η, of degree 2 in ξ, and degree 1 in η.
- A repeated motif here is a layer of square patches of slowly varying sizes.

Detailed characterization ?

Concluding remarks

- Growing sandpiles give a simple model showing complex patterns, and proportionate growth.
- Exact characterization of asymptotic pattern in a simple cases.
- For more complex patterns, calculation of size exponent α is an open problem.
- Connection to discrete analytic functions, tropical polynomials, Appolonian circles packings
- The striking similarity of patterns generated to real organisms is perhaps not an accident.

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Thank You.

Connection to Tropical Mathematics

Define

$$a \oplus b = Max[a, b] \tag{3}$$

$$a \otimes b = a + b \tag{4}$$

Then standard properties of usual addition and multiplication (commutative, identity, distributive ..) contiue to hold.

Example: $3 \oplus 5 \oplus 2 = 5$

$3 \otimes 4 = 7$

Tropical polynomials: $a \otimes x \otimes x \oplus b \otimes x \oplus c$

Example: $x \otimes x \oplus 2 \otimes x \oplus 5 = Max[2x, x + 2, 5]$.

Fundamental theorem of tropical algebra.

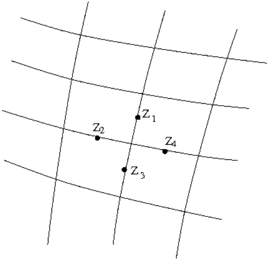
A piecewise -linear convex function can be represented as a tropical polynomial.

Hence useful for describing the toppling function function in growing sandpiles where toppling function is piece-wise linear. Say, for the linearly growing triangular pattern,

$$\phi(\xi, \eta) = \oplus_{l,m=0}^{\infty} \ F_{l.m} \otimes \xi^{l} \otimes \eta^{m}$$

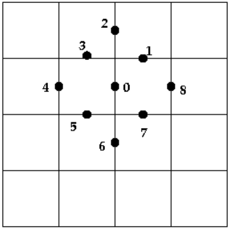
Discrete Analytic Functions

Functions defined only on discrete points in the complex $z-$ plane.



Discrete Cauchy-Riemann conditions:

$$\frac{F(z_1) - F(z_3)}{z_1 - z_3} = \frac{F(z_2) - F(z_4)}{z_2 - z_4}$$



On a square grid :

$$\Delta F_{13} + \Delta F_{35} + \Delta F_{57} + \delta F_{71} = 0$$

is equivalent to

$$\Delta F_{02} + \Delta F_{04} + \Delta F_{06} + \Delta F_{08} = 0$$

Discrete Laplace Equation.
Sum, but not product, of discrete analytic functions is also DA

We find that the coefficients of the linear terms in the toppling function define a discrete analytic function $d + ie$ of the complex variable $m + in$, where (m, n) is the patch label.
These conditions, and the asymptotic behavior for large $|m + in|$ determine the toppling function, and hence the pattern, completely.

Appolonian Circles