Dynamics of bosonic cold atoms in optical lattices.

K. Sengupta

Indian Association for the Cultivation of Science, Kolkata

aborators: ***Anirban Dutta, Christian Trefzger, David Pekker, Shreyoshi Mondal, Brian Clark, and Michael Kobodurez***

Overview

1. *Introduction to ultracold bosons*

2. *Dynamics of the Bose-Hubbard model*

3. *Dynamics induced freezing*

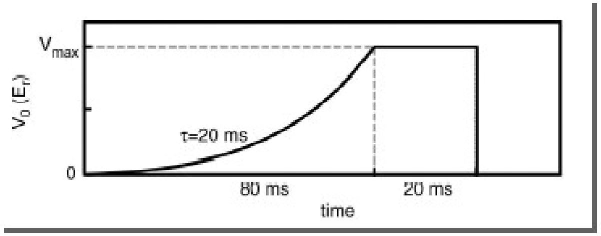
4. *Bosons in an electric field: Theory and experiments*

5. *Dynamics of bosons in an electric field*

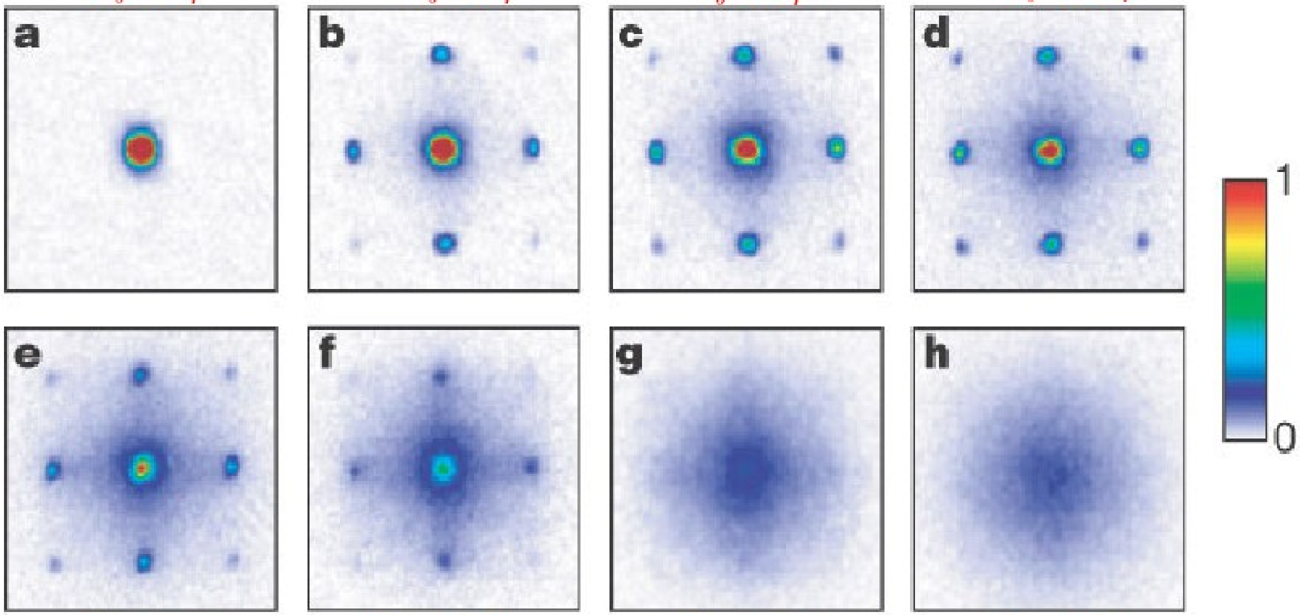
6. *Conclusion*

Introduction to ultracold bosons

State of cold bosons in a lattice: experiment Bloch 2001



$V_0=0E_r$ $V_0=3E_r$ $V_0=7E_r$ $V_0=10E_r$



$V_0=13E_r$ $V_0=14E_r$ $V_0=16E_r$ $V_0=20E_r$

From BEC to the Mott state

Apply counter propagating laser: standing wave of light.

The atoms feel a potential **V = -a |E|2**

Energy Scales

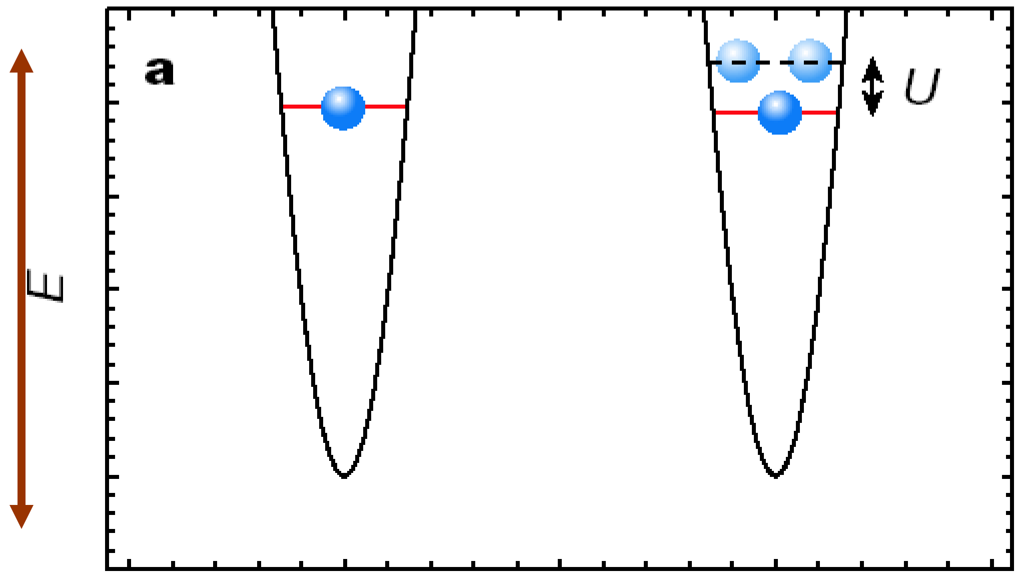
δEn = 5Er ~ 20 U
U ~10-300 t

For a deep enough potential, the atoms are localized : Mott insulator described by single band Bose-Hubbard model.

Model Hamiltonian

$$
\begin{aligned}
\mathcal{H} &= -t \sum_{<i,j>} \left(b_i^\dagger b_j + \text{h.c.} \right) \\
&+ \frac{U}{2} \sum_i n_i (n_i - 1) \\
&- \mu \sum_i n_i
\end{aligned}
$$

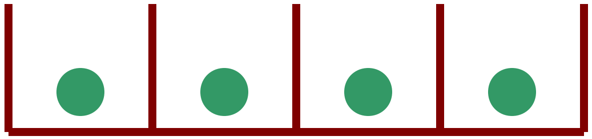
$20E_r$



Ignore higher bands

Mott-Superfluid transition: preliminary analysis

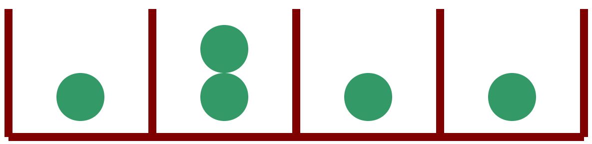
Mott state with 1 boson per site



$$\mathcal{H}_{\text{on-site}} = \frac{U}{2}\sum_i n_i(n_i-1) - \mu \sum_i n_i$$

Stable ground state for 0 < m < U

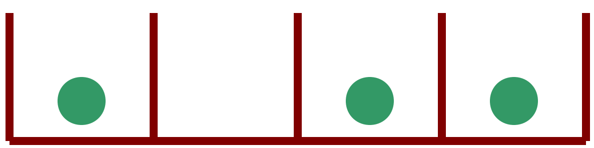
Adding a particle to the Mott state



Mott state is destabilized when
the excitation energy touches 0.

$$\delta E_p = (-\mu + U) - 2zt \qquad t_p^c = (-\mu + U)/2z$$

Removing a particle from the Mott state



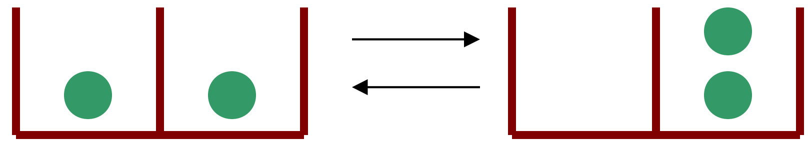
$$\delta E_p = \mu - \_zt$$

$$t_c^h = \mu/\bar{}z$$

tion of the Mott state via addition of particles/hole: onset of su

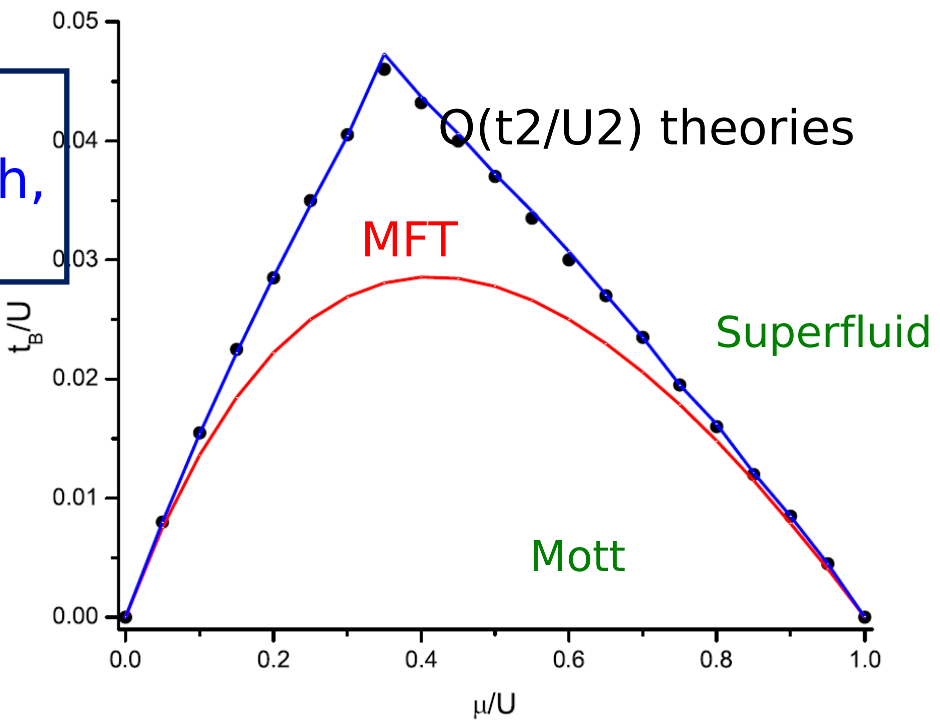
Beyond this simple picture

igher order energy calculation
y Freericks and Monien: Inclusion
f up to O(t3/U3) virtual processes.



Mean-field theory (Fisher 89,
Seshadri 93)

Phase diagram for n=1 and d=3



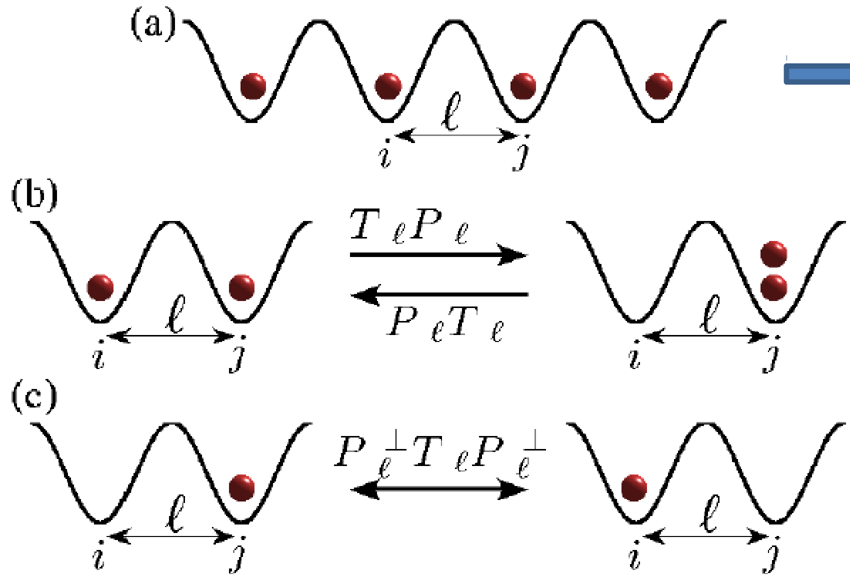
uantum Monte Carlo studies for
D & 3D systems: Trivedi and Krauth,
. Sansone-Capponegro

redicts a quantum phase ansition with z=2 (except at e tip of the Mott lobe where =1).

No method for studying dynamics beyond mean-field theory

Projection Operator Method

Distinguishing between hopping processes



Mott state

Distinguish between two types of hopping processes using a projection operator technique

Define a projection operator $\boxed{P_\ell = |\bar{n}\rangle\langle\bar{n}|_{\mathbf{r}} \times |\bar{n}\rangle\langle\bar{n}|_{\mathbf{r}'}}$

Divide the hopping to classes (b) and (c)

$$\boxed{T = \sum_{\langle\mathbf{r}\mathbf{r}'\rangle} -J b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} = \sum_\ell T_\ell = \sum_\ell [(P_\ell T_\ell + T_\ell P_\ell) + P_\ell^\perp T_\ell P_\ell^\perp]}$$

Building fluctuations over MFT

gn a transformation which eliminate hopping esses of class (b) perturbatively in J/U.

$$S \equiv S[J] = \sum_\ell i[P_\ell, T_\ell]/U$$

btain the effective Hamiltonian

$$H^* = \exp(iS)\mathcal{H}\exp(-iS)$$

$$\begin{aligned} H^* &= H_0 + \sum_\ell P_\ell^\perp T_\ell P_\ell^\perp - \frac{1}{U}\sum_\ell \Big[P_\ell T_\ell^2 + T_\ell^2 P_\ell \\ &- P_\ell T_\ell^2 P_\ell - T_\ell P_\ell T_\ell\Big] - \frac{1}{U}\sum_{\langle \ell\ell' \rangle}\Big[P_\ell T_\ell T_{\ell'} - T_\ell P_\ell T_{\ell'} \\ &+ \frac{1}{2}\Big(T_\ell P_\ell P_{\ell'} T_{\ell'} - P_\ell T_\ell T_{\ell'} P_{\ell'}\Big) + \text{h.c.}\Big] \qquad (2) \end{aligned}$$

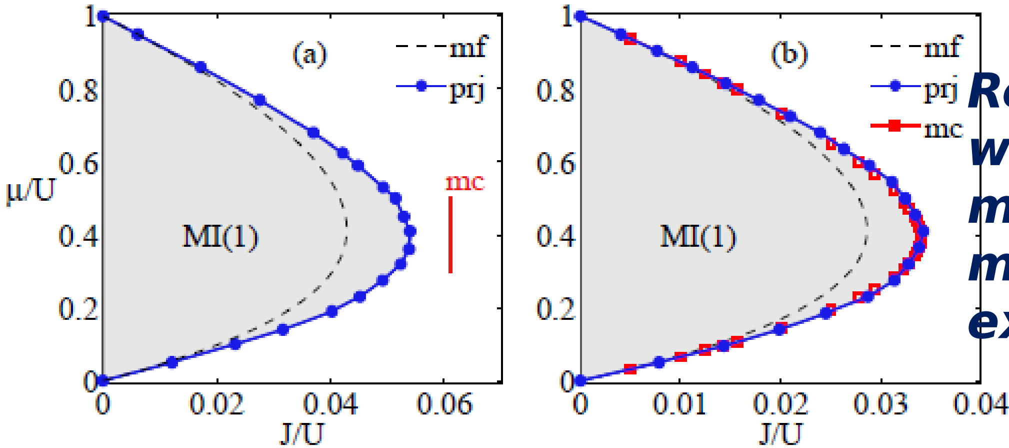
Use the effective Hamiltonian o compute the ground state energy and hence the phase diagram

$$E = \langle\psi|\mathcal{H}|\psi\rangle = \langle\psi'|H^*|\psi'\rangle + \mathrm{O}(z^3J^3/U^2)$$

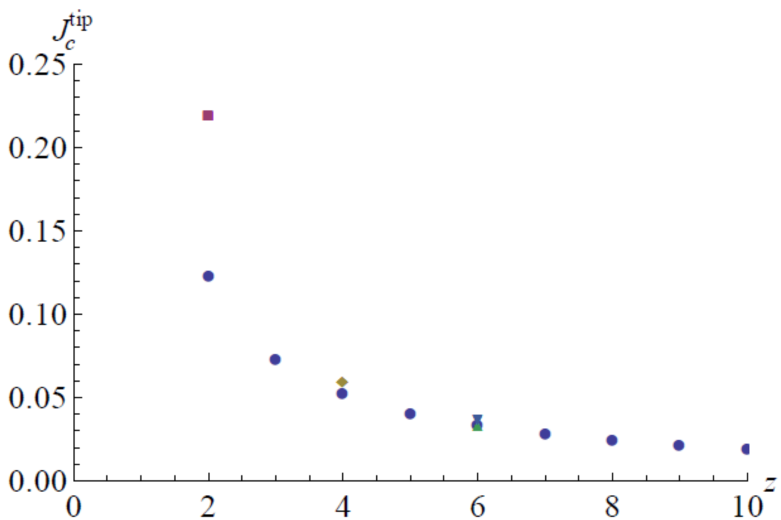
$$|\psi'\rangle = \exp(iS)|\psi\rangle$$

$$|\psi'\rangle = \prod_{\mathbf{r}} \sum f_n^{(\mathbf{r})} |n\rangle$$

Equilibrium phase diagram



Reproduction of the phase diagr
with remarkable accuracy in d=3
much better than standard
mean-field or strong coupling
expansion in d=2 and 3.



Accurate for large z as can be checked by comparing with QMC data for 2D triangular (z=6) and 3D cubic lattice

Allows for straightforward generalization for treatment of dynami

Non-equilibrium dynamics: Linear ramp

sider a linear ramp of J(t)=Ji +(Jf - Ji) t/t.
dynamics, one needs to solve the Sch. Eq.

$$i\hbar\partial_t|\psi\rangle = \mathcal{H}[J(t)]|\psi\rangle$$

e a time dependent transformation
ddress the dynamics by projecting on
instantaneous low-energy sector.

$$|\psi'\rangle = \exp(iS[J(t)])|\psi\rangle$$

$$(i\hbar\partial_t + \partial S/\partial t)|\psi'\rangle = H^*[J(t)]|\psi'\rangle$$

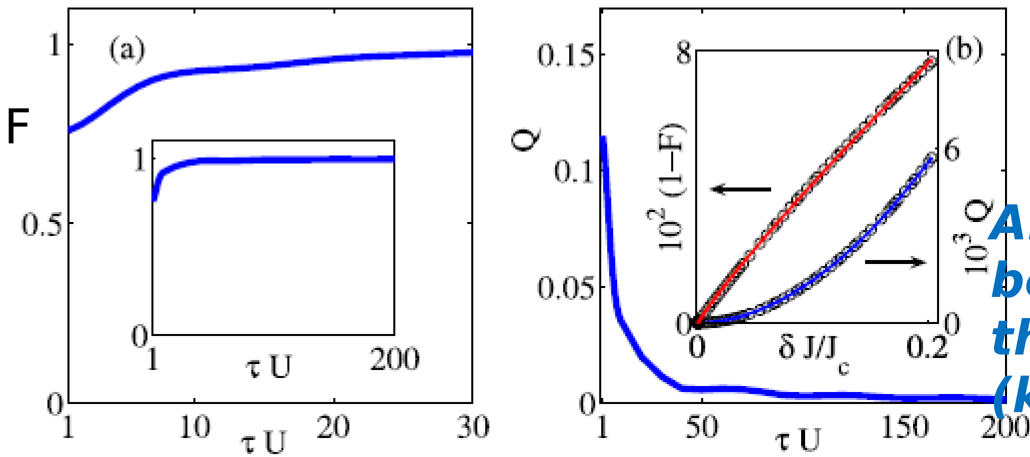
method provides an accurate descriptio
he ramp if J(t)/U <<1 and hence can
t slow and fast ramps at equal footing.

Takes care of particle/hole productio
due to finite ramp rate

$$\begin{aligned} i\hbar\partial_t f_n^{(\mathbf{r})} &= \delta E[\{f_n(t)\}; J(t)]/\delta f_n^{*(\mathbf{r})} + i\hbar\frac{(J_f - J_i)}{U\tau} \\ &\times \sum_{\langle \mathbf{r}'\rangle_{\mathbf{r}}} \sqrt{n} f_{n-1}^{(\mathbf{r})}\left[\delta_{n\bar{n}}\varphi_{\mathbf{r}'\bar{n}} - \delta_{n,\bar{n}+1}\varphi_{\mathbf{r}',\bar{n}-1}\right] \\ &+ \sqrt{n+1} f_{n+1}^{(\mathbf{r})}\left[\delta_{n\bar{n}}\varphi^*_{\mathbf{r}',\bar{n}-1} - \delta_{n,\bar{n}-1}\varphi^*_{\mathbf{r}'\bar{n}}\right] \end{aligned}$$

$$\varphi_{\mathbf{r}}[\Phi_{\mathbf{r}}] = \langle\psi'|b_{\mathbf{r}}|\psi'\rangle[\langle\psi'|b_{\mathbf{r}}^2|\psi'\rangle]$$

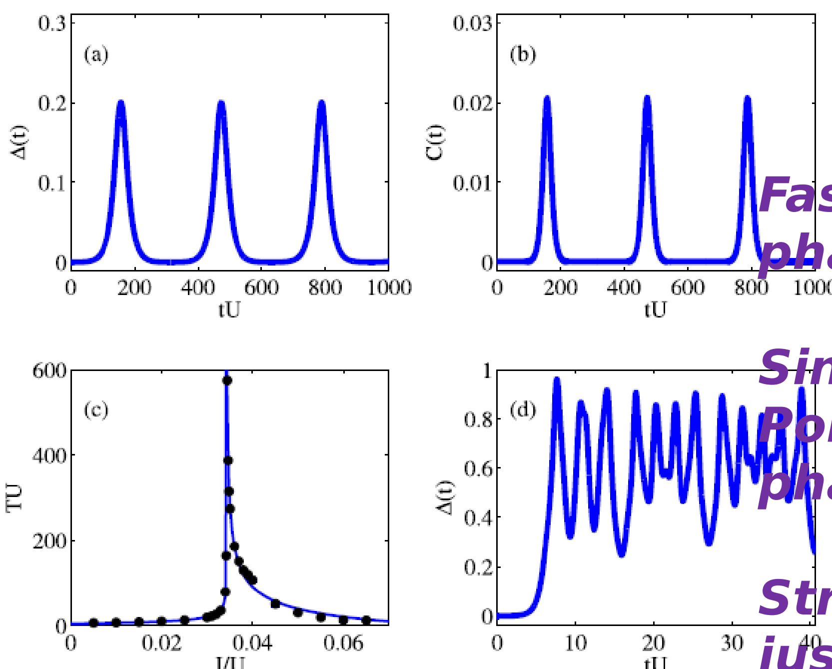
$$E = \langle\psi|\mathcal{H}|\psi\rangle$$



$$P = 1 - |\langle \psi_G|\psi(t_f)\rangle|^2$$

$$Q = \langle \psi_c|\mathcal{H}[J_f]|\psi_c\rangle - E_G[J_f]$$

Absence of critical scaling: may be understood as the inability of the system to access the critical (k=0) modes.



$$\Delta_{\mathbf{r}}(t) = \langle \psi(t)|b_{\mathbf{r}}|\psi(t)\rangle = \langle \psi'(t)|b'_{\mathbf{r}}|\psi'(t)\rangle$$

$$C_{\mathbf{r}}(t) = \langle \psi'(t)|b'_{\mathbf{r}}b'_{\mathbf{r}}|\psi'(t)\rangle - \Delta^2_{\mathbf{r}}(t)$$

Fast quench from the Mott to the SF phase; study of superfluid dynamics.

Single frequency pattern near the critica Point; more complicated deeper in the S phase.

Strong quantum fluctuations near the Q justification of going beyond mft.

Experiments with ultracold bosons on a lattice: finite rate dynamics

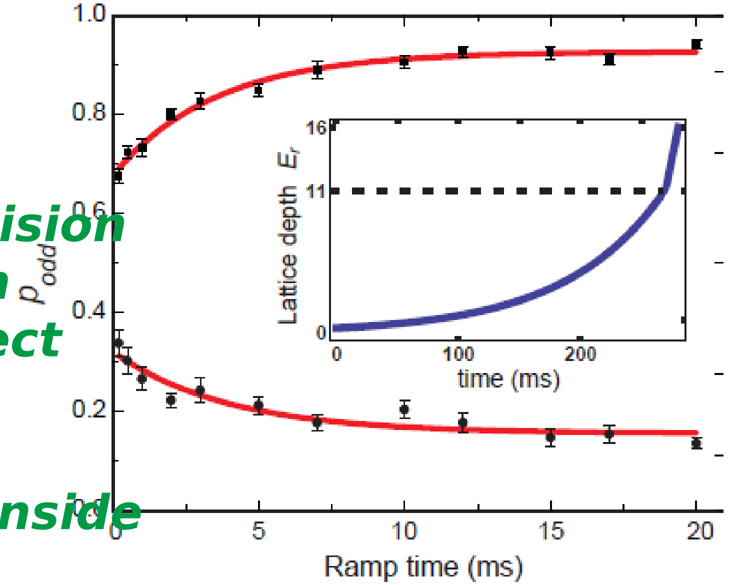
BEC confined in a trap and in the presence
n optical lattice.

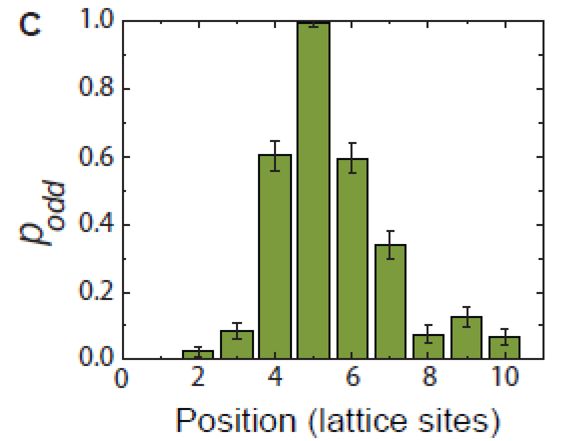
le site imaging done by light-assisted collision
h can reliably detect even/odd occupation
site. In the present experiment they detect
s with n=1.

p from the SF side near the QCP to deep inside
Mott phase in a linear ramp with different
p rates.

no. of sites with odd n displays plateau like
avior and approaches the adiabatic limit
n the ramp time is increased asymptotically.

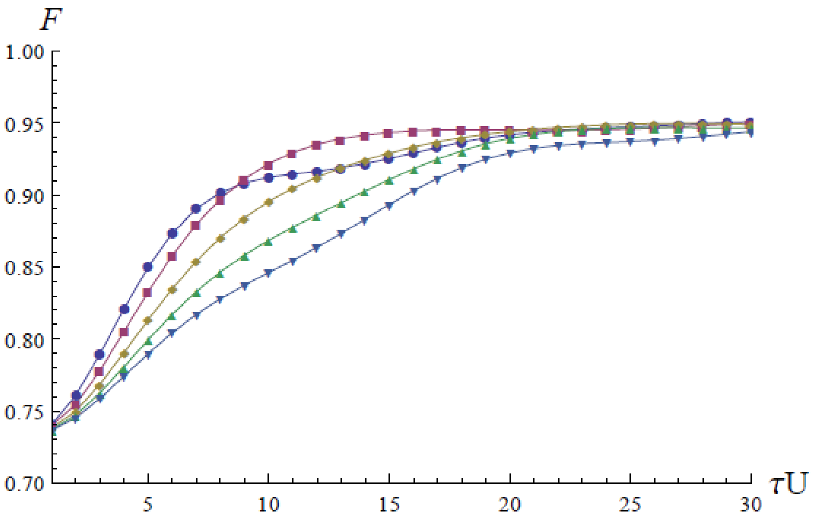
ignature of scaling behavior. Interesting
ial patterns.





W. Bakr et al. Science 2010

Power law ramp

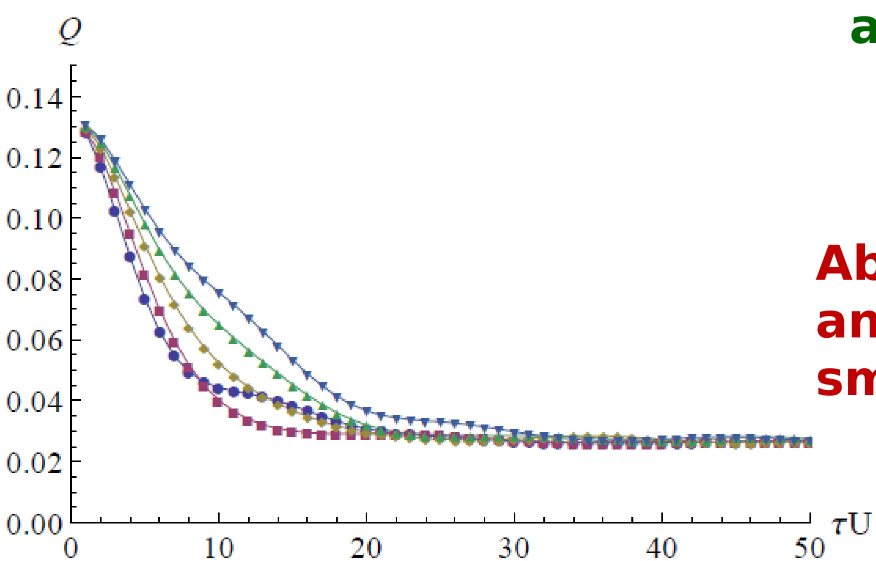


Use a power-law ramp protocol

$$J(t) = J_i + (J_f - J_i)(t/\tau)^\alpha$$

Slope of both F and Q depends on a; however, the plateau-like behavio at large t is independent of a.

Absence of Kibble-Zureck scaling for any a due to lack of contribution of small k (critical) modes in the dynami

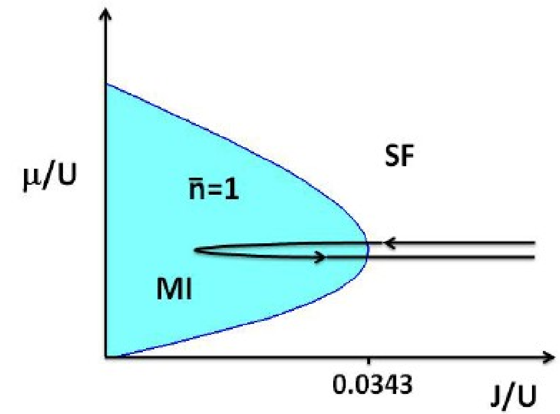


Periodic protocol: dynamics induced freezing

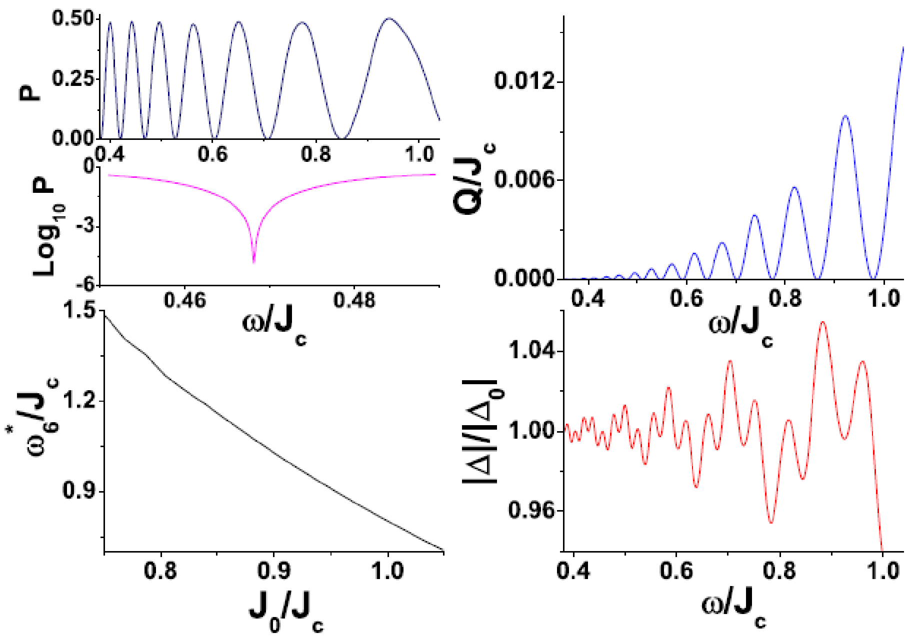
Dynamics induced freezing

une J from superfluid phase to Mott
nd back through the tip of the Mott lobe.

$$J(t) = J_0 + \delta J \cos(\omega t)$$



Key result



There are specific frequencies at which the wavefunction of the system comes back to itself after a cycle of the drive leading to P=1-F -> 0.

Dynamics induced freezing

Mean-field analysis: A qualitative picture

Choose a gutzwiller wavefunctic $|\psi(\mathbf{r},t)\rangle_{\rm mf} \quad = \prod_{\mathbf{r}} \sum_n f_n(t)|n\rangle$

. The mean-field equations for fn $(i\partial_t - E_n) f_n = \tilde{\Delta}(t)\sqrt{n} f_{n-1} + \tilde{\Delta}^*(t)\sqrt{n+1} f_{n+1},$

En is the on-site energy of the state |n>

$$\tilde{\Delta}(t) \ = \ -zJ(t)\sum_n \sqrt{n} f^*_{n-1} f_n,$$

Numerical solution of this equatic
dicates that fn vanishes for n>2 fc
nges of drive frequencies studied

$$i\partial_t f_0 \ = \ -zJ(t)[|f_1|^2 f_0 + \sqrt{2} f_2^* f_1^2]$$

$$i\partial_t f_2 \ = \ E_2 f_2 - zJ(t)[2|f_1|^2 f_2 + \sqrt{2} f_0^* f_1^2]$$

$$i\partial_t f_1 \ = \ E_1 f_1 - zJ(t)[(2|f_2|^2 + |f_0|^2) f_1 + 2\sqrt{2} f_1^* f_2 f_0].$$

Analysis of these equations leads to the
relation involving

$$f_n = r_n(t)\exp[i\phi_n(t)].$$

$$\partial_t |f_0|^2 = \partial_t |f_2|^2 = -\partial_t |f_1|^2/2.$$

$$r^2_{2[0]}(t) \ = \ -(r_1^2(t) - 1)/2 + [-]\eta,$$

hus one can describe the system
erms of three real variables : ampli
tate |1> and the sum and differenc
he relative phases.

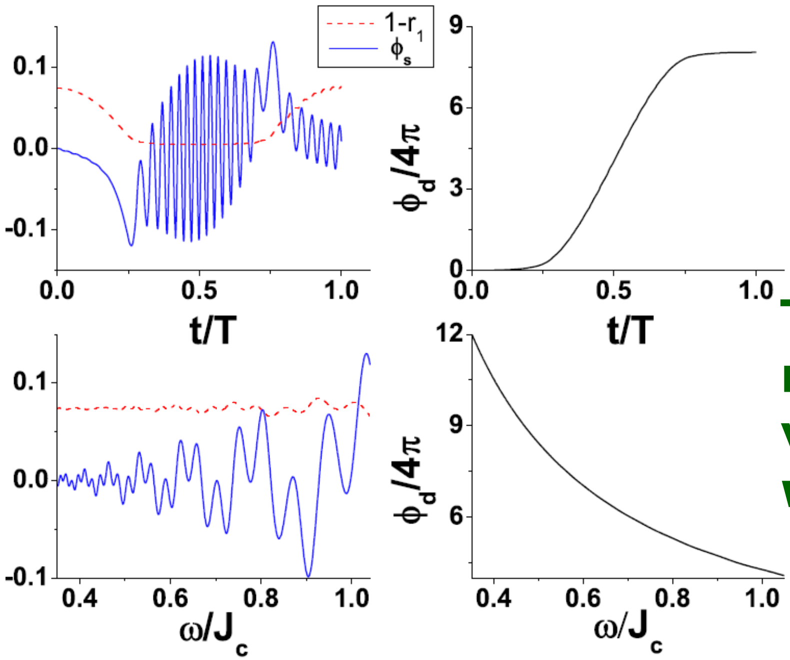
$$
\begin{aligned}
\partial_t r_1 &= -\sqrt{2} z J(t) \sin(\phi_s) r_1 g_0(r_1), \\
\partial_t \phi_s &= -U + zJ(t)\left[g_1(r_1) - g_2(r_1)\cos(\phi_s)\right], \qquad (6) \\
\partial_t \phi_d &= -U + 2\mu + zJ(t) r_1^2 \left[1 - 4\sqrt{2}\eta \cos(\phi_s)/g_0(r_1)\right],
\end{aligned}
$$

fs=f0+f2-2f1   fd=f2-f0

$$
\begin{aligned}
g_0(r_1) &= \sqrt{(1-r_1^2)^2 - 4\eta^2}, \quad g_1(r_1) = 6r_1^2 - 3 - 2\eta, \\
g_2(r_1) &= 2\sqrt{2}\left[r_1^2(r_1^2-1)/g_0(r_1) + g_0(r_1)\right]. \qquad (7)
\end{aligned}
$$

One can construct a frequency-
ependent relation between r1 and

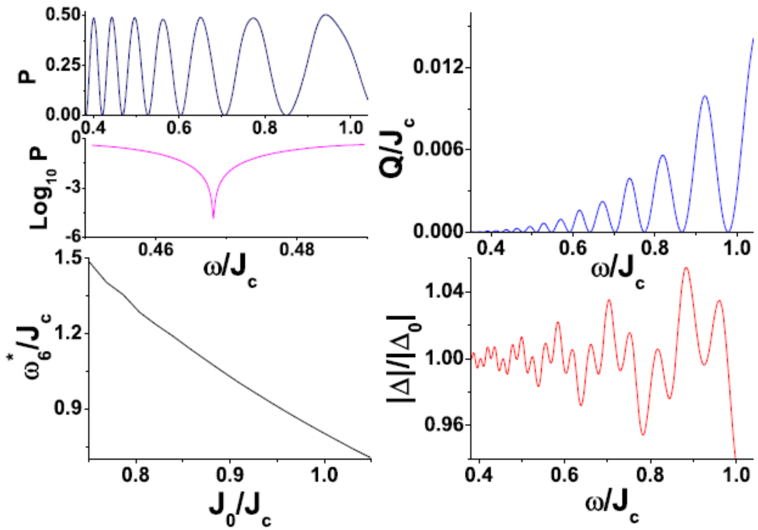
$$
dr_1/d\phi_s = \frac{-\sqrt{2}\sin(\phi_s) r_1 g_0(r_1)}{[g_1(r_1) - g_2(r_1)\cos(\phi_s)] - U/zJ(t')}.
$$

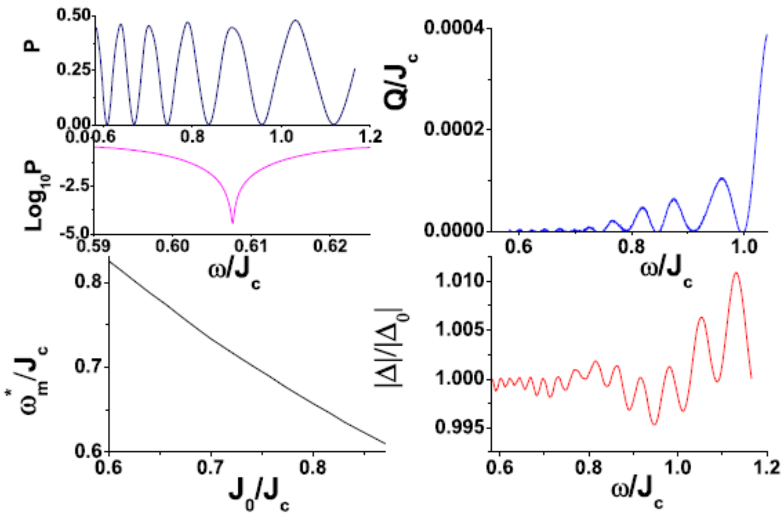


Numerical solution of (6)

There is a range of frequency for which r1 and fs remain close to their original values; dynamics induced freezing occur when fd/4p = n within this range.

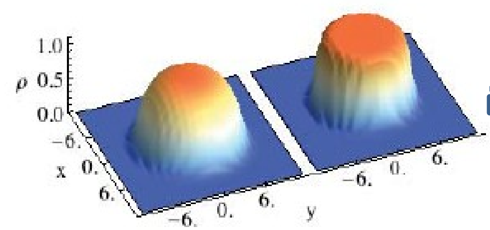
Robustness against quantum fluctuations and presence of a trap





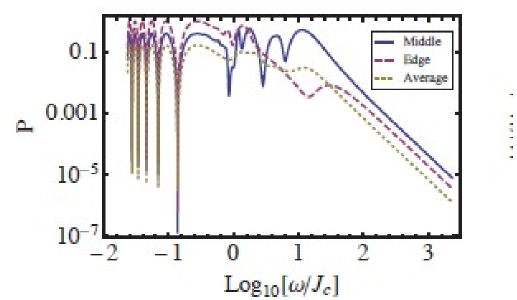
Mean-field theory

Projection operator formalism



Density distribution of the bosons inside a trap

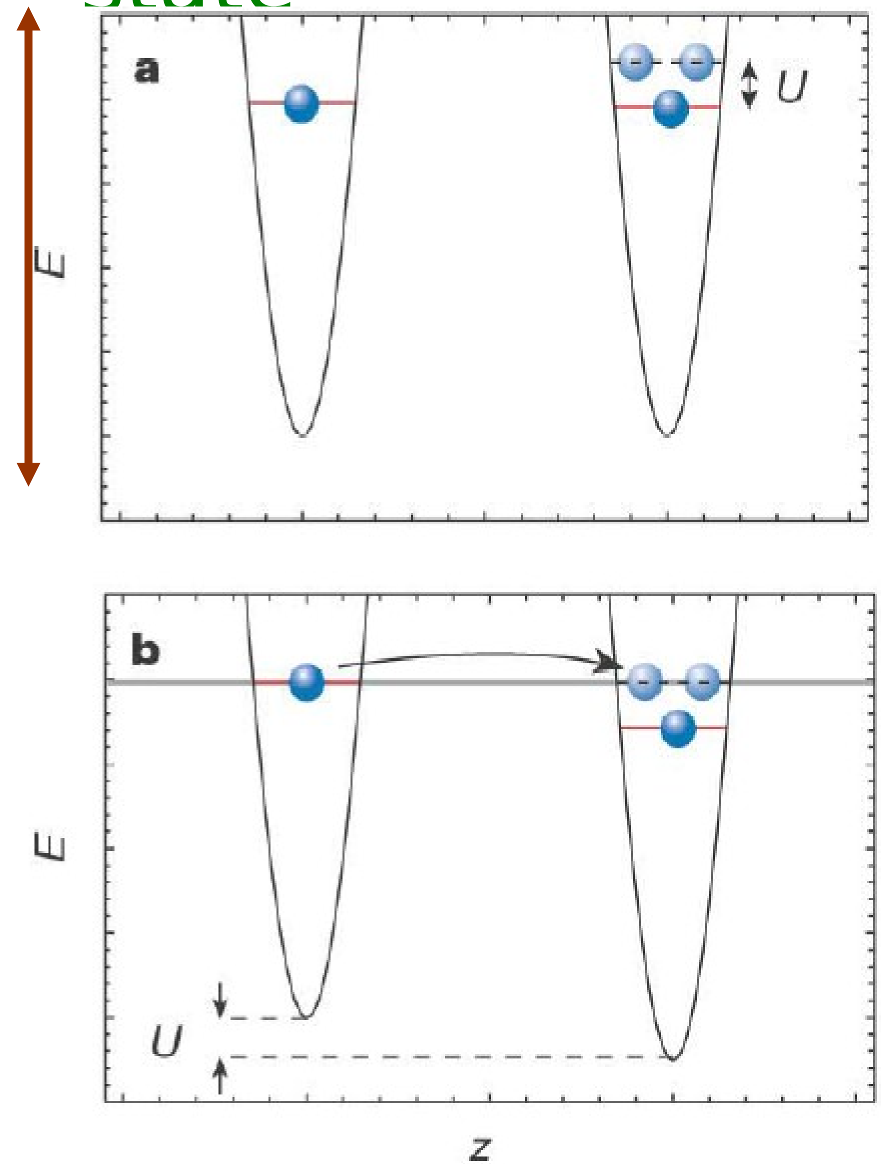
Robust freezing phenomenon



Bosons in an electric field

Applying an electric field to the Mott state

$$20E_r$$

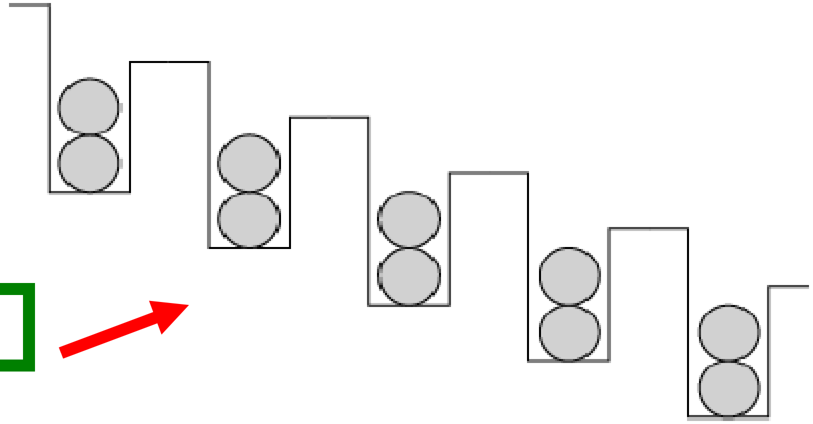


Energy Scales:

$h\omega\nu = 5Er \;\approx 20\, U$

$U \approx 10\text{–}300\; J$

Construction of an effective model: 1D



Parent Mott state

$$H = -w\sum_{\langle ij\rangle}\left(b_i^\dagger b_j + b_j^\dagger b_i\right) + \frac{U}{2}\sum_i n_i\left(n_i - 1\right) - \sum_i \boldsymbol{E}\cdot\boldsymbol{r}_i n_i$$

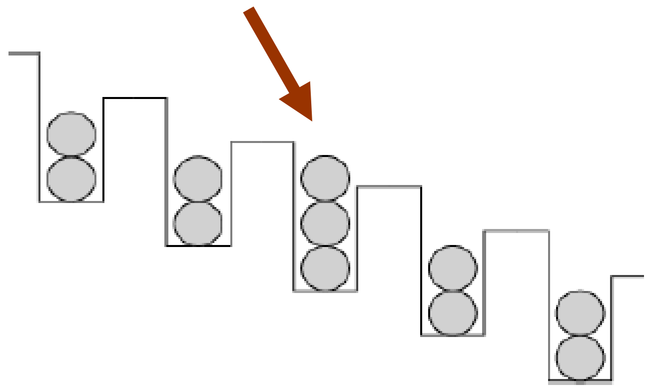
$$n_i = b_i^\dagger b_i$$

$$\left|U - E\right|, w \ll E, U$$

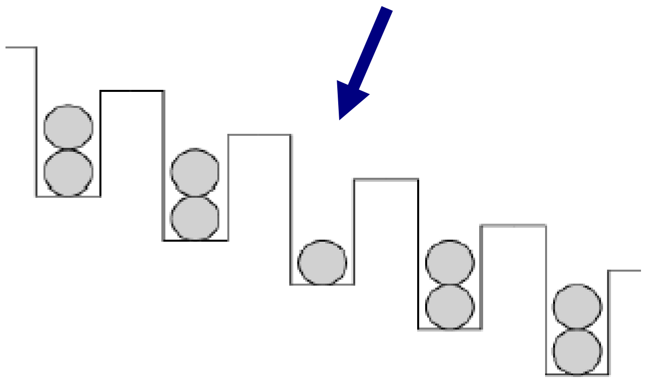
Describe spectrum in subspace of states resonantly coupled to the Mott insulator

Charged excitations

quasiparticle



quasihole



Effective Hamiltonian for a quasiparticle in one dimension (similar for a quasihole):

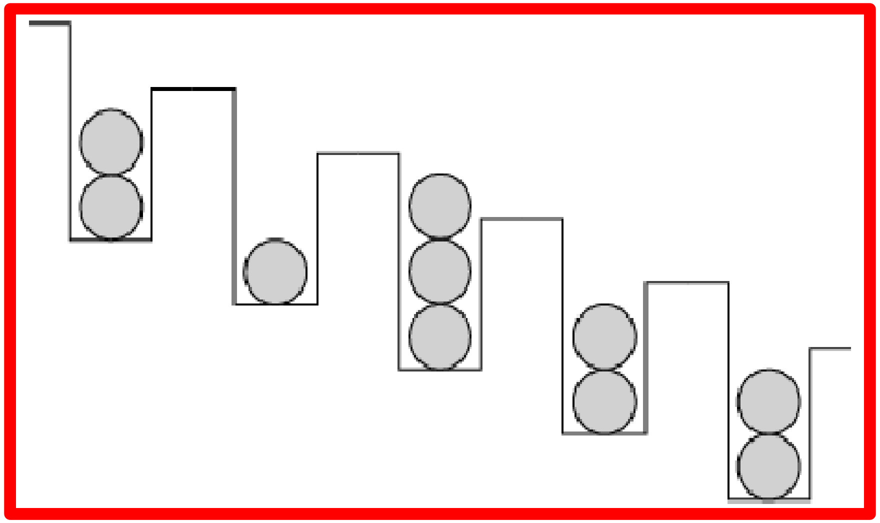
$$\boxed{H_{\mathrm{eff}} = -\sum_j \left[3w\left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + E j b_j^\dagger b_j \right]}$$

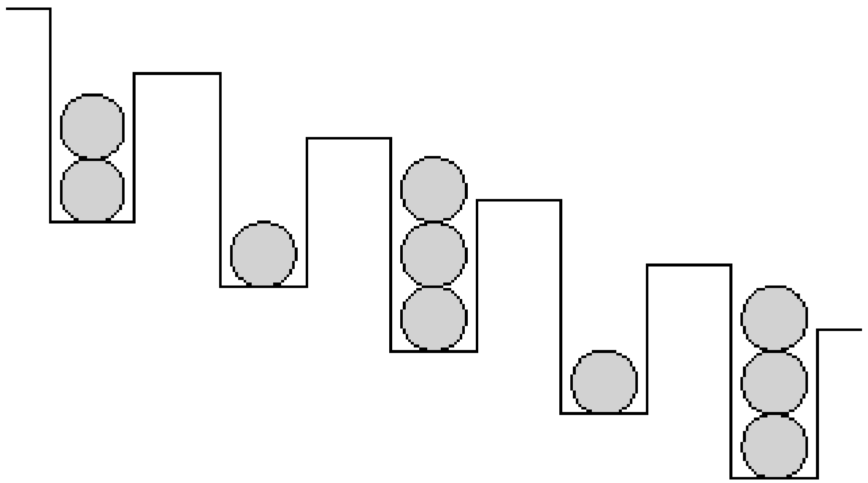
Exact eigenvalues $\varepsilon_m = Em \quad ; \quad m = -\infty \cdots \infty$

Exact eigenvectors $\psi_m(j) = J_{j-m}(6w/E)$

All charged excitations are strongly localized in the plane perpendicular electric field.
Wavefunction is periodic in time, with period h/E (Bloch oscillations)
Quasiparticles and quasiholes are not accelerated out to infinity

Neutral dipoles





Resonantly coupled to the parent Mott state when U=E.

Neutral dipole state with energy U-E.

Two dipoles which are not nearest neighbors with energy 2(U-E).

Effective dipole Hamiltonian: 1D

$d_\ell^\dagger \Rightarrow$ Creates dipole on link ℓ

$$H_d = -\sqrt{6}w\sum_\ell \left(d_\ell^\dagger + d_\ell\right) + (U-E)\sum_\ell d_\ell^\dagger d_\ell$$

Constraints: $d_\ell^\dagger d_\ell \leq 1 \quad ; \quad d_{\ell+1}^\dagger d_{\ell+1} d_\ell^\dagger d_\ell = 0$

Determine phase diagram of H_d as a function of $(U\text{-}E)/w$

Note: there is no explicit dipole hopping term.

However, dipole hopping is generated by the interplay of terms in H_d and the constraints.

Weak Electric Field

For weak electric field, the ground state is dipole vaccum and the low-energy excitations are single dipole

- The effective Hamiltonian for the dipoles for weak E:

$$\mathcal{H}_{d,\mathrm{eff}} = (U-E)\sum_l \Big[|l><l| + \frac{w^2 n_0(n_0+1)}{(U-E)^2}\Big(|l><l| + |l+1><l| + |l><l+1|\Big)\Big]$$

- Lowest energy excitations: Single band of dipole excitations.

- These excitations soften as E approaches U. This is a precursor of the appearance of Ising density wave with period 2.

- Higher excited states consists of multiparticle continuum.

Strong Electric field

The ground state is a state of maximum dipoles.

Because of the constraint of not having two dipoles on consecutive sites, we have two degenerate ground states

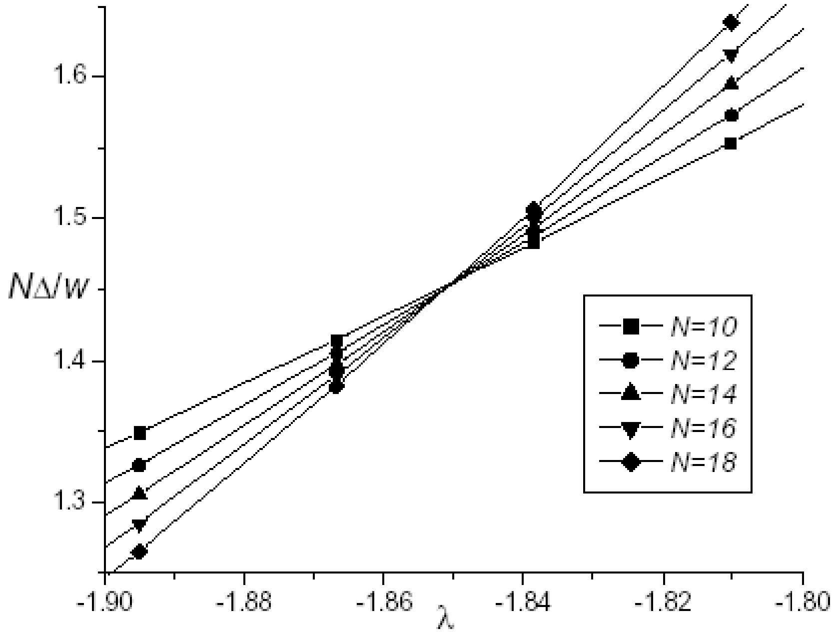
$$\boxed{\cdots d_1^\dagger d_3^\dagger d_5^\dagger d_7^\dagger d_9^\dagger d_{11}^\dagger \cdots |0\rangle \qquad or \qquad \cdots d_2^\dagger d_4^\dagger d_6^\dagger d_8^\dagger d_{10}^\dagger d_{12}^\dagger \cdots |0\rangle}$$

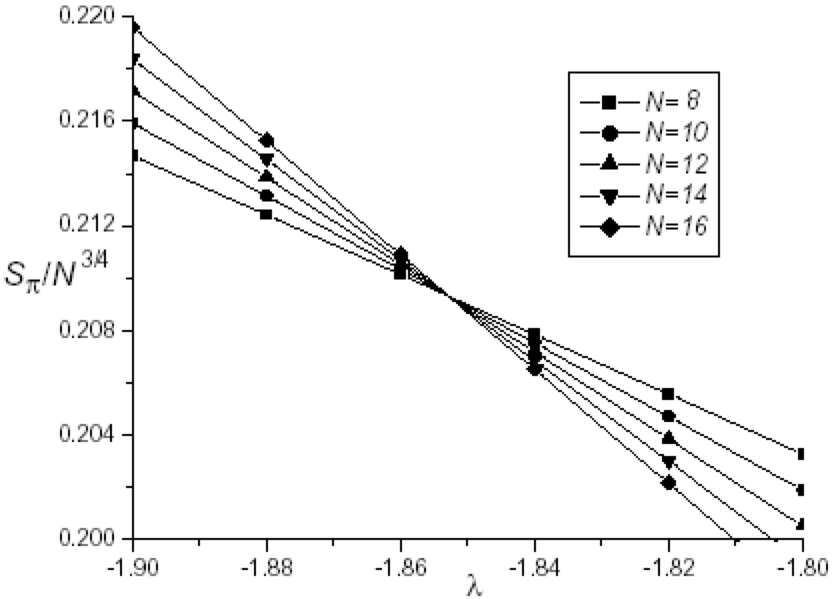
The ground state breaks Z2 symmetry.

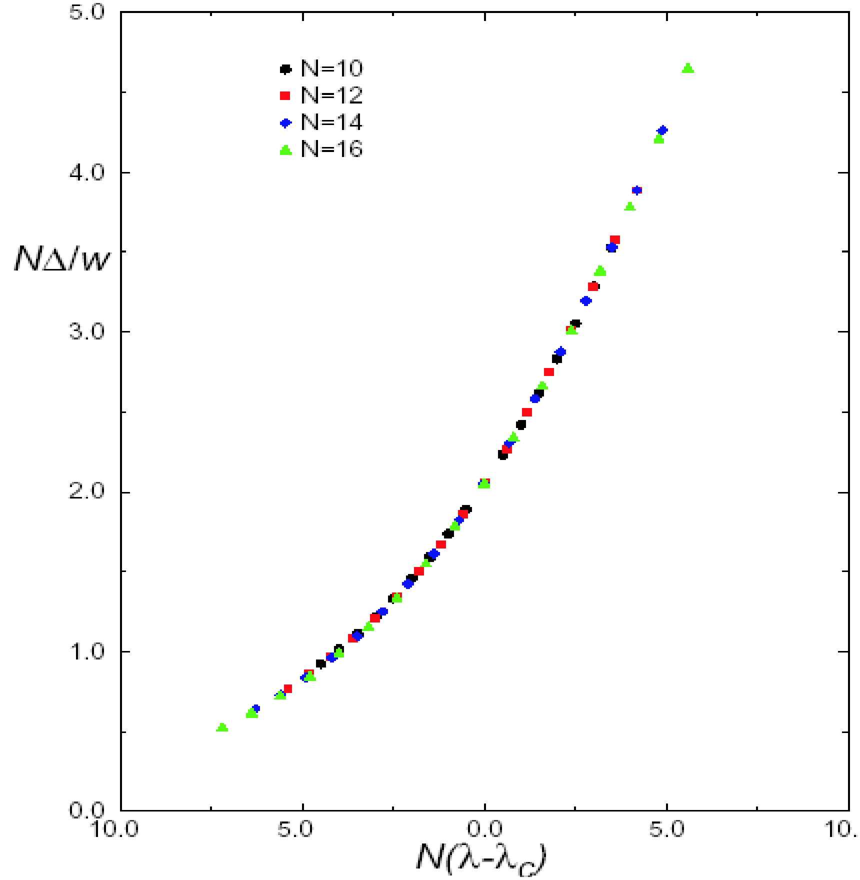
The first excited state consists of band of domain walls between the two filled dipole states.

Similar to the behavior of Ising model in a transverse field.

Intermediate electric field: QPT

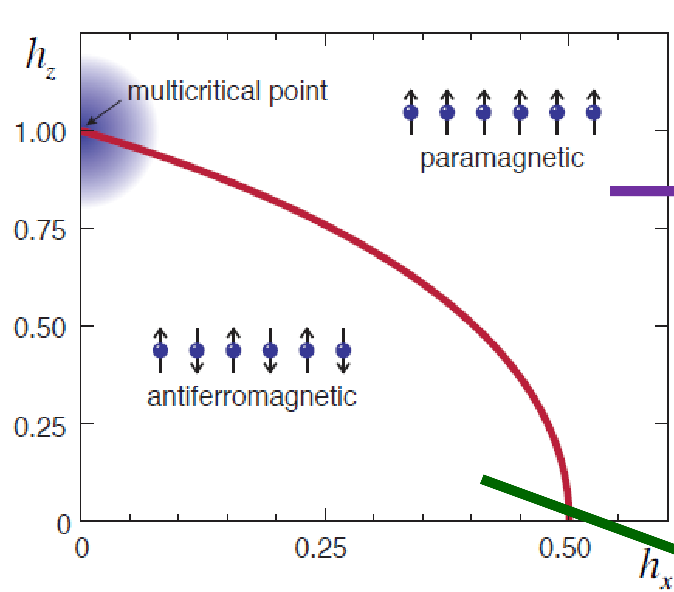


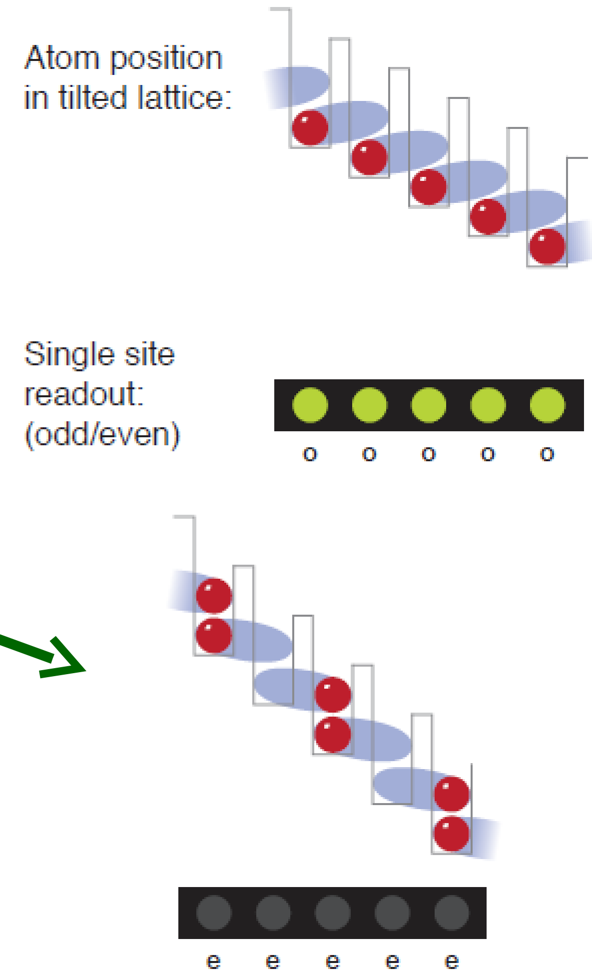




Quantum phase transition at E-U=1.853w. Ising universality.

Recent Experimental observation of Ising order (Bakr et al Nature 20





(b)

$$H = J\sum_i \overbrace{S_z^i S_z^{i+1}}^{\text{realizes constraint}} \overbrace{- \underbrace{(1-\tilde{\Delta})\, S_z^i}_{h_z\ \text{longitudinal}} - \underbrace{2^{3/2}\tilde{t}\, S_x^i}_{h_x\ \text{transverse}}}^{\text{drives quantum phase transition}}$$

magnetic fields:

$$S_z^j = \frac{1}{2} - d_j^\dagger d_j\,,\, S_x^j = \frac{1}{2}\left(d_j^\dagger + d_j\right),\text{and } S_y^j = \frac{i}{2}\left(d_j^\dagger - d_j\right)$$

experimental realization of effective Ising model in ultracold atom syst

Quench dynamics across the quantum critical point

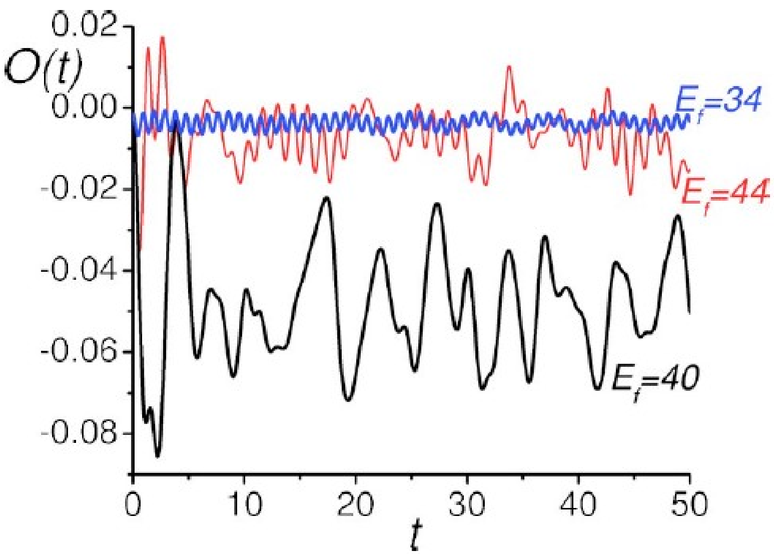
une the electric field from
i to Ef instantaneously

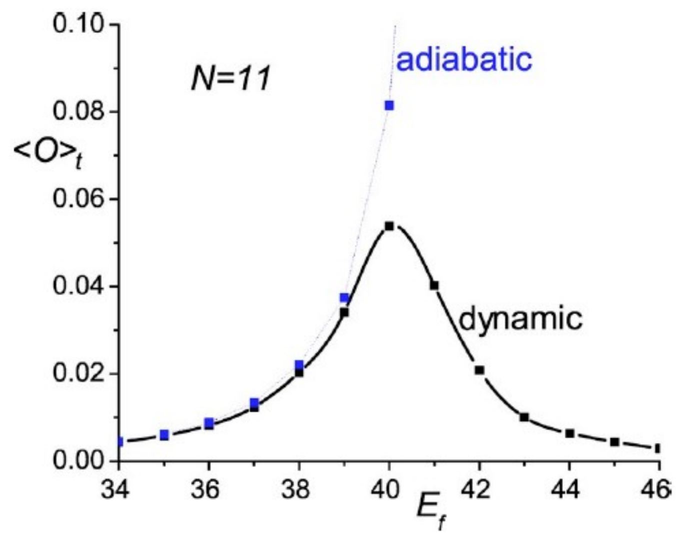
mpute the dipole order
rameter as a function of time

$$|\Psi(t)\rangle = \sum_n c_n \exp(-i\epsilon_n t/\hbar)|n\rangle$$

$$O = \frac{1}{N}\langle\Psi|\sum_\ell (-1)^\ell d_\ell^\dagger d_\ell|\Psi\rangle$$

$$O(t) = \frac{1}{N}\sum_{m,n} c_m c_n \cos[(E_m - E_n)t/\hbar]\langle m|\sum_\ell (-1)^\ell d_\ell^\dagger d_\ell|n\rangle.$$





e time averaged value of the order parameter is maximal near the QCP

Generic critical points: A phase space argument

e system enters the impulse region when
e of change of the gap is the same order
the square of the gap.

$$d\ln(\Delta_{\vec{k}})/dt \;\geq\; \Delta_{\vec{k}}$$

 slow dynamics, the impulse region is a
all region near the critical point where
ling works

$$\Delta_{\vec{k}} \;\sim\; \lambda^{z\nu}|t/\tau|^{z\nu}$$

The system thus spends
a time T in the impulse
region which depends on
the quench time

$$T \sim \tau^{z\nu/(z\nu+1)}$$

this region, the energy gap scales as

$$\Delta_{\mathbf{k}} \sim \tau^{-z\nu/(z\nu+1)}$$

us the scaling law for the defect
nsity turns out to be

$$\Omega_n \sim |\mathbf{k}|^d \sim \Delta_{\mathbf{k}}^{d/z} \sim \tau^{-\nu d/(z\nu+1)}$$

eneralization to finite system size
ite-size scaling

$$Q \sim L^d v^{(d+z)\nu/(z\nu+1)} g_r(vL^{1/\nu+z})$$

$$F \sim L^d v^{d\nu/(z\nu+1)} f_r(vL^{1/\nu+z}),$$

Dynamics with a finite rate: Kibble-Zureck scaling

**hange the electric field linearly
time with a finite rate v**

$$H_d(t) \;=\; [U - \mathcal{E}(t)] \sum_\ell d_\ell^\dagger d_\ell - J \sum_\ell (d_\ell^\dagger + d_\ell),$$

Quantities of interest

Theory

Experiment

$$Q(t) = \langle \psi(t)|H(t)|\psi(t)\rangle - E_G(t),$$

$$F(t) = \log[|\langle \psi(t)|\psi_G(t)\rangle|^2],$$

$$n_d(t) = \langle \psi(t)|\sum\nolimits_\ell (1 + 2S_\ell^z)\,|\psi(t)\rangle,$$

$$D(t) = n_d(t) - \langle \psi_G(t)|\sum\nolimits_\ell (1 + 2S_\ell^z)\,|\psi_G(t)\rangle,$$

$$C_{ij}(t) = \langle \psi(t)|S_i^z S_j^z|\psi(t)\rangle,$$

**Scaling laws for
finite –size systems**

$$Q \sim L^d v^{(d+z)\nu/(z\nu+1)} g_r(vL^{1/\nu+z}),$$

$$F \sim L^d v^{d\nu/(z\nu+1)} f_r(vL^{1/\nu+z}),$$

$$g_r(x \;\ll\; 1) \;\sim\; x^{2-(d+z)\nu/(z\nu+1)}$$

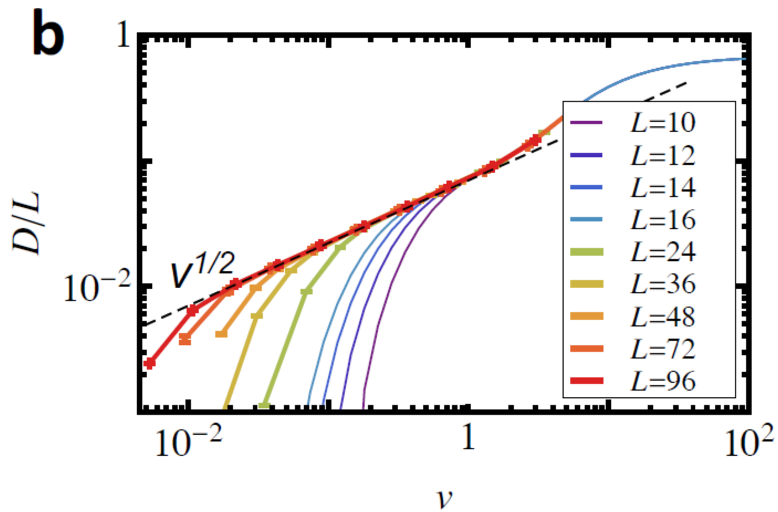
$$f_r(x \;\ll\; 1) \;\sim\; x^{2-d\nu/(z\nu+1)}$$

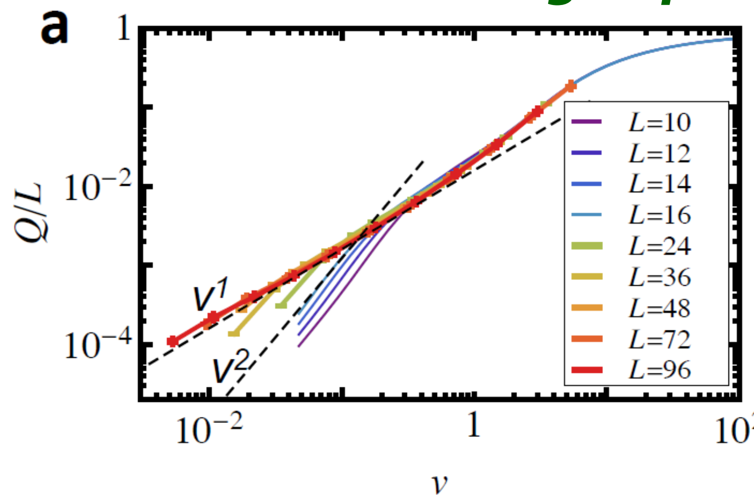
Q ~ v2 (v) for slow (intermediate) quench. These are termed as LZ(KZ) regimes for finite-size systems.

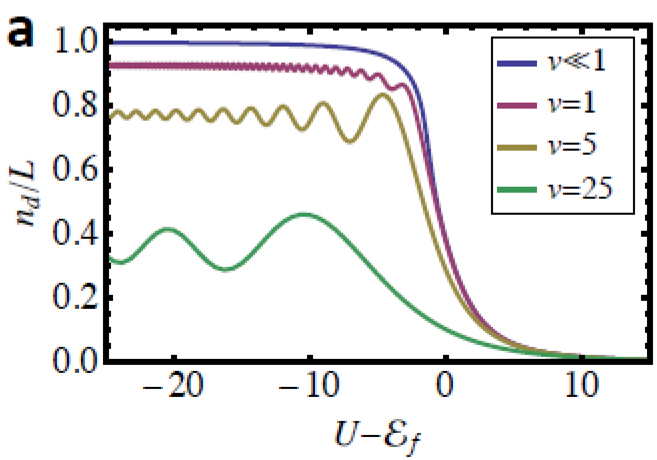
Kibble-Zureck scaling for finite-sized system

Expected scaling laws for Q and F

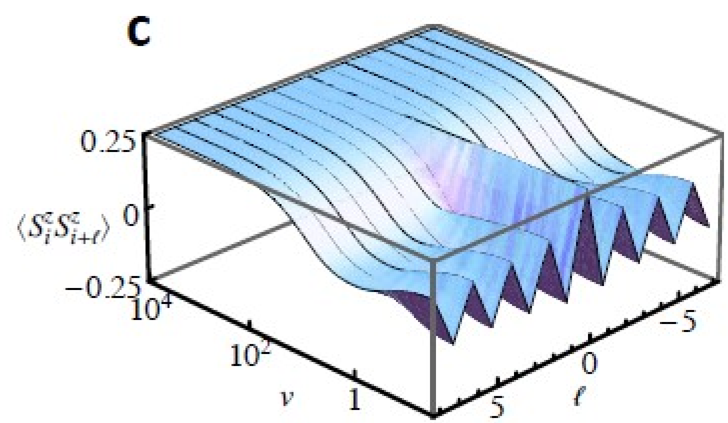
Observation of Kibble-Zurek law for intermediate v with Ising exponents.







Dipole dynamics



Correlation function