Symmetry constraints in single field models of inflation

A thesis

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by

Ashish Shukla

Department of Theoretical Physics

Tata Institute of Fundamental Research

Mumbai - 400 005, India

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgment of collaborative research and discussions.

The work was done under the guidance of Professor Sandip P. Trivedi, at the Tata Institute of Fundamental Research, Mumbai.

(Ashish Shukla)

In my capacity as the supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Sandip P. Trivedi)

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I would also like to express my gratitude to Prof. Shiraz Minwalla. Every discussion with him, whether on the blackboard or on the lunch table, expanded the horizons of my knowledge and made me think deeply. His energy and enthusiasm for physics motivates others around him. He has no dearth of time for answering even the silliest questions a student may ask. I was lucky to have learnt Advanced QFT and String Theory through the lecture courses he offered.

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Collaborators

This thesis is based on work done in collaboration with several people.

- The work presented in chapters 2 and 3 was done in collaboration with Nilay Kundu and Sandip P. Trivedi, and is based on the publications that appeared in print as *JHEP* **04** (2015) 061 and *JHEP* **01** (2016) 046.

- The work presented in chapter 4 was done in collaboration with Sandip P. Trivedi and V. Vishal, and is based on the publication that appeared in print as *JHEP* **12** (2016) 102.

I dedicate this thesis to
my parents, who made me what I am,
and
my teachers, to whom I owe whatever I know.

को अद्धा वेद क इह प्र वोचत्कुत आजाता कुत इयं विसृष्टिः |
अर्वाग्देवा अस्य विसर्जनेनाथा को वेद यत आबभूव ||

इयं विसृष्टिर्यत आबभूव यदि वा दधे यदि वा न |
यो अस्याध्यक्षः परमे व्योमन्त्सो अङ्ग वेद यदि वा न वेद ||

-- नासदीय सूक्त, ऋग्वेद (१०:१२९)
(लगभग १५०० - १२०० ईसा पूर्व)

But, after all, who knows, and who can say
Whence it all came, and how creation happened?
the gods themselves are later than creation,
so who knows truly whence it has arisen?

Whence all creation had its origin,
He, whether He fashioned it or whether He did not,
He, who surveys it all from the highest heaven,
He knows - or maybe even He knows not.

-- Excerpt from Nasadiya Sukta, Rigveda (10:129)
(c. 1500 - c. 1200 BCE) (Source: Wikipedia)

List of Publications

Papers relevant to the thesis work:

[1] *Constraints from Conformal Symmetry on the Three Point Scalar Correlator in Inflation;*
Nilay Kundu, Ashish Shukla and Sandip P. Trivedi;
JHEP 1504 (2015) 061; arXiv:1410.2606.

[2] *Ward Identities for Scale and Special Conformal Transformations in Inflation;*
Nilay Kundu, Ashish Shukla and Sandip P. Trivedi;
JHEP 1601 (2016) 046; arXiv:1507.06017.

[3] *Symmetry constraints in inflation, α-vacua, and the three point function;*
Ashish Shukla, Sandip P. Trivedi and V. Vishal;
JHEP 1612 (2016) 102; arXiv:1607.08636.

Other papers by the author:

[4] *Kubo formulas for thermodynamic transport coefficients*;
Pavel Kovtun and Ashish Shukla;
JHEP 1810 (2018) 007; arXiv:1806.05774.

[5] *Domain wall moduli in softly-broken SQCD at $\bar{\theta} = \pi$*;
Adam Ritz and Ashish Shukla;
Phys. Rev. D 97, 105015 (2018); arXiv:1804.01978.

[6] *On the Dynamics of Near-Extremal Black Holes*;
Pranjal Nayak, Ashish Shukla, Ronak M. Soni, Sandip P. Trivedi, V. Vishal;
JHEP 1809 (2018) 048; arXiv:1802.09547.

[7] *Gauge transformation through an accelerated frame of reference*;
Ashish Shukla and Kaushik Bhattacharya;
Am. J. Phys. 78 (2010) 627-632; arXiv:0908.1276.

Contents

Chapter 1

Introduction

The mysterious night sky has always been a constant source of amazement and wonder for humans. The prehistoric man would draw the patterns of light he saw in the night sky on the walls of his cave dwellings. As their understanding evolved further, our ancestors started recognizing patterns in the rising and setting of the Sun, the Moon and the stars. The regularity of these heavenly phenomena helped them develop calendars, predict seasons, and navigate better. At the same time, driven by curiosity to understand the cosmos, they were naturally lead to the most fundamental questions surrounding our existence: Where did the universe come from? Does it have a beginning? How did the grand cosmic order that we see emerged? Are their laws that can explain the regularity of the cosmos?

We have come a long way in the search for answers to these fundamental questions: from Ptolemy's epicycles trying to explain the periodicity of planetary motion, to the Heliocentric model of Copernicus, to the modern understanding of galaxies and galactic clusters forming filaments and voids in an infinite expanse of spacetime. Our observations tell us that the universe indeed had a beginning, a very violent one, in which all the matter and energy we see around us came into existence. This violent explosion, or *Big Bang*, took place nearly 13.8 billion years ago, and the universe had been expanding ever since.

The Big Bang model has been very successful in explaining the origin and observed properties of our universe. It had spectacular success in explaining the presence of the Cosmic Microwave Background (CMB) radiation, observed in 1964, as a consequence of the decoupling of photons from the $e^{-}-nuclei$ plasma when the latter started forming neutral atoms. The model also explains the production and relative abundances of various light elements in the universe via nucleosynthesis.

Despite its resounding success the model is incomplete, as it gives no justification for some crucial observations. For instance, it does not explain the isotropy of the CMB on the largest scales we observe today. Also, there is no mechanism for the generation of the minute CMB anisotropies in the Big Bang scenario. It also does not provide any justification for the flatness of our universe, which can arise only if the density of matter and energy equals a certain critical value. These and other fine tuning problems require augmenting the model with further ideas.

The mechanism of *inflation* was proposed in the early 1980s to overcome these issues with the Big Bang model [8, 9, 10, 11, 12, 13]. According to the inflationary scenario, the universe underwent a phase of exponentially fast expansion in its very early history. The rapid expansion smoothed out any traces of inhomogeneities, anisotropies and curvature, giving rise to a very simple universe. The only departures from this smooth state were tiny quantum fluctuations, which became seeds for the formation of galaxies and gave rise to the CMB temperature anisotropies observed today.

In the simplest inflationary models, inflation is brought about by a single homogeneous scalar field slowly rolling down a potential hill. The energy density of the field couples to the background spacetime geometry and drives the inflationary expansion. The model also incorporates perturbations about the homogeneous solution which are quantum in nature. More intricate models may involve multiple scalars fields, more complicated potentials, departures from slow roll, non-canonical kinetic terms etc. The results from the Planck satellite [14, 15, 16] seem to be in good agreement with the simplest models of inflation [17, 18, 19, 20, 21, 22, 23].

During the inflationary epoch the geometry of the universe was approximately de Sitter and had a very high degree of symmetry. The isometry group of de Sitter space is $O(1,4)$. In the present thesis, we explore the idea of how this underlying symmetry can be used to constrain the correlation functions of inflationary perturbations in single-field models of inflation. We also consider departures from the exact de Sitter limit, by taking into account the corrections sourced by the breaking of these symmetries. The resulting constraints take the form of Ward identities, relating n-point correlation functions to $n+1$-point correlation functions under certain limits. In particular, one gets interesting constraints on the scalar three point function, which gives rise to leading non-Gaussianity in the inflationary perturbations and is of great observational significance. The work presented in this thesis is primarily based upon the articles [1, 2, 3], and relies heavily upon the important earlier works [24, 25, 26, 27].

This introductory chapter to the thesis is organized as follows: section 1.1 provides a very brief overview of the Big Bang model and illustrates the horizon and flatness problems associated with it. Section 1.2 introduces inflation and explains how the problems of standard cosmology get resolved in the inflationary framework, and discusses classical dynamics of inflation in the context of single field slow roll models. During inflation, the universe was approximately de Sitter in nature. Section 1.3 provides a very brief overview of the symmetry properties of de Sitter spacetime. This is followed in section 1.4 by a discussion of inflationary perturbations and issues of gauge fixing. Section 1.5 then introduces the wave function of the universe, and talks about some of its properties as well as computation of inflationary correlation functions from it, with the two point function taken as an example. Section 1.6 discusses the subsequent organization of the thesis.

1.1 The Big Bang model and associated problems

We observe the universe to be expanding uniformly in all directions. This observation naturally gives rise to the idea that at some point of time in the far past, all the matter and radiation we observe today must have been concentrated in a very tiny volume. According to *Big Bang* cosmology, the expansion and subsequent evolution of this dense primordial soup led to the formation of our present universe. Also, to a very good approximation, the universe is spatially flat, and is homogeneous and isotropic on the largest scales. All these observations can be incorporated into the Friedmann-Robertson-Walker (FRW) model of the universe. The FRW model with flat spatial sections has the metric

$$ds^2 = -\,dt^2 + a(t)^2 \left(dr^2 + r^2 d\Omega_2^2\right). \tag{1.1}$$

Here, $a(t)$ is the *scale factor* of the universe and $d\Omega_2^2$ is the metric on the unit 2-sphere. The scale factor essentially measures the growth of the physical size of the universe. The dynamics of the scale factor is governed by the Friedmann equations,

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &\equiv H^2 = \frac{8\pi G}{3}\,\rho(t), \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P). \end{aligned} \tag{1.2}$$

Here, a dot represents a derivative with respect to time. $\rho(t)$ is the time-dependent energy density of the universe, and $P(t)$ is the pressure. As per Big Bang cosmology, the universe after its birth was initially dominated by radiation, and expanded at the rate of $a(t) \sim \sqrt{t}$. Big Bang nucleosynthesis took place in the first few minutes after the birth of the universe, in the radiation dominated era, leading to the formation of light nuclei. This was followed by a period of matter domination, during which the universe grew at $a(t) \sim t^{2/3}$. The decoupling of CMB from the $e^- - nuclei$ plasma took place during the phase of matter domination when the latter started forming neutral atoms, approximately 380,000 years after the Big Bang. The present era is the era of dark energy domination.

One can also express the FRW metric (1.1) in terms of a 'conformal time coordinate' η, where η is related to the ordinary time coordinate t by $dt = a\,d\eta$. Then

$$ds^2 = a(\eta)^2 \left(-\,d\eta^2 + dr^2 + r^2 d\Omega_2^2\right). \tag{1.3}$$

The advantage of working in the conformal coordinates is that radial light rays move along 45° lines. They are also helpful in illustrating the horizon problem associated with the Big Bang cosmology.

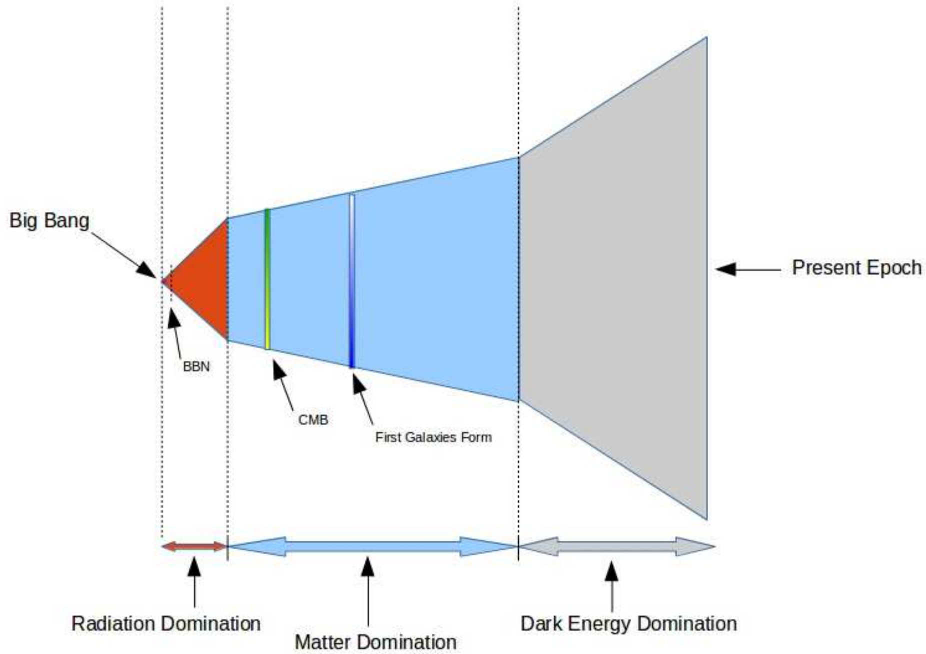


Figure 1.1: A schematic representation of the Big Bang chronology of the universe. The Big Bang birth of the universe took place approximately 13.8 billion years ago. Big Bang nucleosynthesis occurred in the first $\sim$ 20 minutes. After about 47,000 years the universe crossed over from radiation to matter domination. The CMB photons started free streaming at about 380,000 years after the birth of the universe. Somewhere between 500-800 million years after the Big Bang the first galaxies appear. Approximately 9.8 billion years after its birth the universe underwent yet another transition, with the dominant energy contribution now coming from the dark energy, leading to an accelerated expansion.

The horizon and flatness problems

The Big Bang model of cosmology outlined above has been very successful. But it suffers from a few problems also, rendering it incomplete. One of these problems is the *horizon* problem. To understand the problem, we need to introduce the concepts of *horizon size* and *causal connectivity*. The horizon size at any instant of time is defined as the largest distance over which two events could be in causal contact with each other. In other words, it is the maximum distance a photon could have traveled since the birth of the universe. The comoving horizon size at any time t is given by

$$R_h \equiv c\,\eta(t) = \int_0^t \frac{c\,dt}{a(t)}. \tag{1.4}$$

Clearly, the size of the horizon depends upon the evolutionary history of the universe. Events which are beyond the horizons of one-another can not be in causal contact, and hence can not influence each other. One can calculate the size of the comoving horizon at the time of decoupling, when the CMB was emitted. It turns out to be about 284 MPc. Also, the comoving distance to the Last Scattering Surface (LSS), the surface from where we receive CMB today, is about 14 GPc. This leads us to a paradox. Imagine two points on the LSS as shown in the Figure 1.2. Clearly, they are beyond each others comoving horizons at the time of decoupling. Hence, they were never in causal contact with one another, and in particular, they were not in causal contact at the time of CMB emission. Therefore, there is no reason for the temperature of the CMB received from these two points to be the same, as they could not have influenced one another. Yet we observe the CMB temperature to be very isotropic throughout the visible universe: $T_{CMB} = 2.72548 \pm 0.00057$ K. This paradox is termed as the horizon problem. There is no mechanism in Big Bang cosmology to resolve this issue.

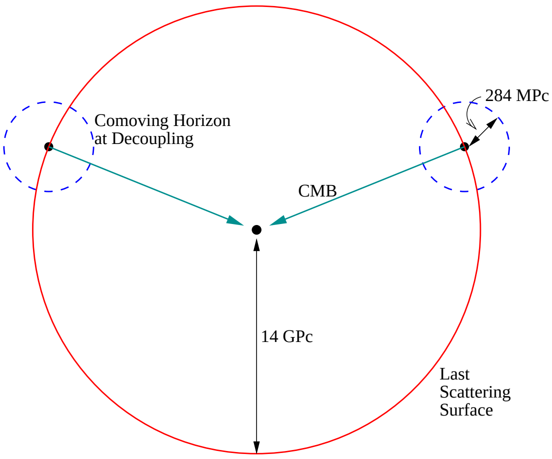


Figure 1.2: Illustrating the Horizon Problem. Calculations suggest that the CMB temperature isotropy should be limited to angular scales of about 2.3°, whereas we observe isotropy at the largest scales.

Another problem associated with the Big Bang model is the *flatness* problem. To illustrate the problem, let us rewrite the first Friedmann equation with the curvature term included,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.5}$$

where the constant k is the curvature parameter: it can be negative, zero or positive depending upon whether the universe is open, flat or closed, respectively. Eq.(1.5) can be rewritten as

$$1 = \frac{\rho}{\rho_c} - \frac{k}{(aH)^2}, \tag{1.6}$$

where $\rho_c = \frac{3H^2}{8\pi G}$ is the critical density for the universe to be flat: if $\rho = \rho_c$ then $k = 0$. Introducing the notation $\Omega = \rho/\rho_c$, the Friedmann equation takes the form

$$\Omega - 1 = \frac{k}{(aH)^2}, \tag{1.7}$$

or equivalently

$$|\Omega - 1| = \frac{|k|}{(aH)^2}. \tag{1.8}$$

Now, our observations tell us that the geometry of the universe is almost flat, $\Omega_{today} \approx 1$. Considering the fact that in standard cosmology the comoving Hubble radius $(aH)^{-1}$ has grown since the Planck epoch by about 60 orders of magnitude, we conclude that the value of Ω immediately after the Big Bang must have been very finely tuned: $|\Omega_{Planck} - 1| \leq 10^{-60}$. This extreme fine tuning of the energy density needed to produce the observed flat universe is what is termed as the flatness problem.

1.2 Inflation - A solution to the horizon and flatness problems

In the early 1980s, the idea of 'inflation' was put forward to resolve the issues of the horizon and flatness problems and other inconsistencies of the Big Bang model [8, 9, 10, 11, 12, 13]; for pedagogical discussions see [28, 29, 30, 31, 32, 33]. Inflation provides a very simple solution to the horizon problem - as per inflation, the comoving horizon size at decoupling was actually much larger than the size of the visible universe today, and hence the points on the Last Scattering Surface (and much beyond) were in mutual causal contact. Thus, it is no surprise that the CMB observed today is highly isotropic.

According to the simplest inflationary scenario, there was a phase prior to the era of radiation domination during which the energy density of the universe was dominated by a scalar field, the *inflaton*. The inflaton is hypothesized to have a very special kind of potential function, as depicted in Figure 1.3. During inflation, the inflaton evolves along the almost flat plateau region of the potential, rolling slowly towards the minimum. The energy density during inflation thus stays approximately constant. From the Friedmann equation (1.2), this tells us that the Hubble rate is almost a constant during inflation, and hence the universe expands exponentially rapidly: $a(t) \sim e^{Ht}$.

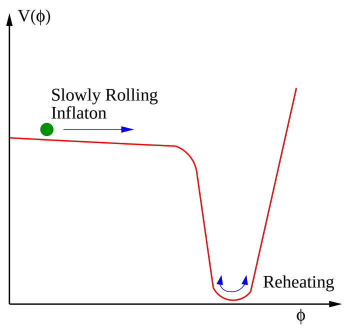


Figure 1.3: Slow roll inflation. The kinetic energy contribution for the inflaton is negligible compared to the potential energy contribution. The energy density of the universe is thus roughly constant.

During inflation, the comoving horizon size also increases exponentially. All of the visible universe and regions much beyond as well come into mutual causal contact. This is schematically shown in Figure 1.4. Thus, by postulating an inflationary phase in the evolutionary history of the universe, one can explain the remarkable isotropy of the CMB.

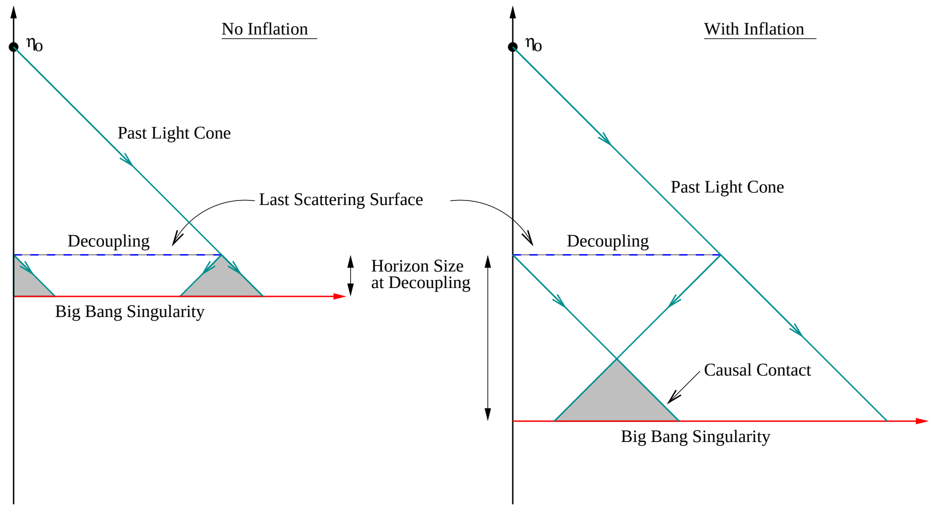


Figure 1.4: Solution to the Horizon Problem. Inflation pushes the Big Bang singularity much farther in conformal time, allowing the past light cones of points on the Last Scattering Surface to intersect. This brings all of the visible universe into causal contact.

Inflation also provides a simple and elegant resolution to the flatness problem. During inflation, the comoving Hubble radius $(aH)^{-1}$ shrinks exponentially rapidly. Inflation thus naturally drives the universe towards flatness. This removes the sensitive dependence of the geometry of the universe on the primordial energy density and any specific value of k: if during inflation the comoving Hubble radius shrinks sufficiently to generate $|\Omega - 1| \approx 10^{-60}$, then the subsequent post-inflationary evolution will automatically lead to the present universe being observed flat with the correct value for Ω, thus resolving the flatness problem.

Classical dynamics of single-field inflation

In the previous section, we illustrated the idea that how the horizon and flatness problems in the Big Bang model can be solved by postulating an exponentially expanding inflationary phase in the very early history of the universe. We would now like to illustrate that how can we bring about this exponential expansion from the dynamics of a scalar field.

In the simplest models of inflationary cosmology, we assume the universe to be filled by a real scalar field $\phi(t, \boldsymbol{x})$. This scalar field is called the 'inflaton'. The inflaton is coupled to gravity minimally via the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right) . \tag{1.9}$$

Here, $V(\phi)$ is the potential energy term for the inflaton, which is assumed to have the generic slow roll form shown in Figure 1.3. The energy-momentum tensor for the inflaton is given by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right). \tag{1.10}$$

The equation of motion for the field ϕ is

$$\frac{\delta S}{\delta\phi} = 0 \Rightarrow \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,\partial^\mu\phi) + V'(\phi) = 0. \tag{1.11}$$

Here, a $'$ denotes a derivative with respect to the field ϕ.

In the homogeneous limit, one assumes that the inflaton is purely a function of time: $\phi(t,\boldsymbol{x}) \equiv \bar{\phi}(t)$. Inserting the FRW form of the metric $g_{\mu\nu}$ (1.1), one gets the energy-momentum tensor for the inflaton to be of the perfect fluid form $T^\mu_\nu = diag(-\rho, P, P, P)$, with the energy density and pressure given by

$$\rho(t) = \frac{1}{2}\,\dot{\bar{\phi}}^2 + V(\phi), \tag{1.12}$$

$$P(t) = \frac{1}{2}\,\dot{\bar{\phi}}^2 - V(\phi). \tag{1.13}$$

The dynamics of the universe is governed by the Friedmann equation (1.2), which takes the form

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\,\dot{\bar{\phi}}^2 + V(\phi)\right). \tag{1.14}$$

From (1.11), we find that the dynamics of the inflaton is governed by

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\phi) = 0. \tag{1.15}$$

In the slow roll approximation (see Figure 1.3), we can neglect the kinetic energy term $\frac{\dot{\bar{\phi}}^2}{2}$ in comparison to the potential energy term $V(\phi)$. Equation (1.14) then gives us

$$H^2 \approx \frac{8\pi G}{3}\,V(\phi) \Rightarrow H = \sqrt{\frac{8\pi G}{3}\,V(\phi)}. \tag{1.16}$$

As the potential energy is almost a constant during inflation, from (1.16) we can see that the Hubble rate H is also a constant. This tells us that the scale factor of the universe grows exponentially at the rate of $a(t) \sim e^{Ht}$. Thus, inflationary dynamics can be easily brought about by a scalar field slowly rolling down a potential hill.

Slow roll inflation is generally quantified in terms of two parameters ϵ and η. They are given by

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{1.17}$$

$$\eta = \epsilon - \frac{\ddot{\phi}}{H\dot{\phi}} \approx \frac{1}{8\pi G}\frac{V''}{V}. \tag{1.18}$$

During slow roll inflation, we have $\epsilon \ll 1$ and $\eta \ll 1$.

1.3 de Sitter spacetime and its symmetries

Assuming the Hubble rate to be constant, the metric during the inflationary regime was

$$ds^2 = -\,dt^2 + e^{2Ht}\,dx_i dx^i. \tag{1.19}$$

If the Hubble rate H is an exact constant, then such a spacetime is called the *de Sitter* spacetime. It is the maximally symmetric solution to the Einstein's equations with a positive cosmological constant. In practice, the Hubble rate varies slowly during inflation, and hence the geometry of spacetime is only approximately de Sitter.

The de Sitter spacetime has a large symmetry group $O(1,4)$, with ten generators. For our purposes we will be interested only in the connected subgroup of $O(1,4)$. The de Sitter spacetime is invariant under spatial translations and rotations, scale transformations and special conformal transformations. In other words, the de Sitter spacetime is *conformally invariant*, as the above transformations form the group of conformal transformations. See [34, 35] for pedagogical discussions on some key properties of de Sitter space.

From the form of the metric in (1.19), spatial translations and rotations are an obvious symmetry of the de Sitter geometry. It is also easy to see that the scale transformation

$$x^i \to \lambda x^i,\ t \to t - \frac{1}{H}\,ln(\lambda) \tag{1.20}$$

where λ is a constant, is a symmetry of the de Sitter spacetime. Finally, infinitesimal special conformal transformations of the form

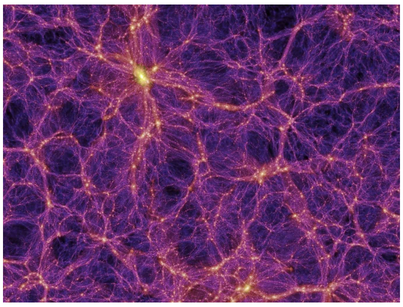
$$\begin{aligned} x^i &\to x^i - 2(\boldsymbol{b}\cdot\boldsymbol{x})x^i + b^i\left(\boldsymbol{x}^2 - \frac{1}{H^2}e^{-2Ht}\right) \\ t &\to t + \frac{2}{H}(\boldsymbol{b}\cdot\boldsymbol{x}) \end{aligned} \tag{1.21}$$

also leave the metric (1.19) unchanged. Here, $\boldsymbol{b}$ parametrizes the infinitesimal special conformal transformation.

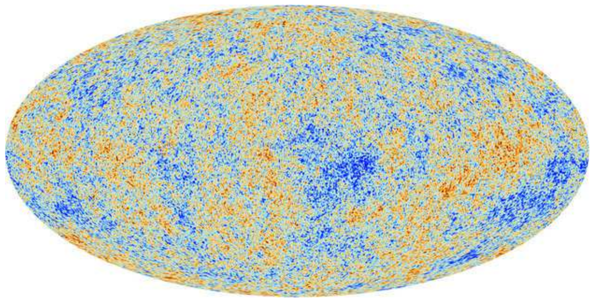
During inflation, the Hubble rate is not a constant and varies slowly with time. The spatial translational and rotational symmetries of the spacetime stay intact, but the scaling and special conformal symmetries get broken. However, as long as the slow roll parameters ϵ and η are small, the breaking of conformal symmetries is small, and the geometry is still approximately that of the de Sitter space. In the subsequent chapters of this thesis, we explore the constraints imposed on inflationary correlation functions due to this large underlying approximate symmetry group.

1.4 Perturbations in the early universe

In the previous sections, we introduced the basic inflationary mechanism driven by a homogeneous scalar field. We will now go beyond the homogeneous limit and introduce perturbations about the smooth background. These perturbations generated during the inflationary era ultimately give rise to the CMB temperature anisotropies and the large scale structure of the universe. The presentation here is elementary and serves the purpose of setting up the ideas and notation involved in subsequent chapters of the thesis; for detailed discussions see [29, 36, 37, 38, 39, 40].



(a) Millennium Run



(b) Planck CMB Map

Figure 1.5: Perturbations in the very early universe give rise to the large scale structure as well as the temperature anisotropies in the CMB. Fig. 1.5a shows the Millennium Run simulation [41, 42], which simulated the large scale structure of the universe using billions of dark matter clouds. Fig. 1.5b shows the CMB temperature anisotropy map obtained by the Planck satellite [43, 44, 45].

The perturbations are introduced in the inflaton as well as the metric. We can classify the perturbations using the underlying spatial rotation symmetry of de Sitter spacetime into scalars, vectors and tensors. Out of these only the scalar and tensor perturbations are relevant for our simple setup, any perturbations vector in nature die out exponentially at late times. The scalar perturbations are physically sourced by the inflaton, but can be transferred to the metric by a suitable choice of coordinates. The tensor perturbations are purely in the metric.

The perturbations in the inflaton are denoted by $\delta\phi(t, \boldsymbol{x})$. Thus the complete inflaton field is

$$\phi(t, \boldsymbol{x}) = \bar{\phi}(t) + \delta\phi(t, \boldsymbol{x}), \tag{1.22}$$

where $\bar{\phi}(t)$ is the homogeneous background. For introducing perturbations in the metric, we work with the ADM decomposition [46, 47]. The ADM form of the metric is

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{1.23}$$

where N, N^i are the lapse and shift functions respectively. The lapse and shift functions are non-dynamical variables and act as Lagrange multipliers, their equations of motion providing the constraint equations of general relativity. We make the gauge choice $N = 1, N^i = 0$, sometimes referred to as the synchronous gauge. The perturbed metric to the linear order in this gauge has the form

$$ds^2 = -\,dt^2 + a(t)^2\,[(1+2\zeta)\delta_{ij} + \widehat{\gamma}_{ij}]\,dx^i dx^j. \tag{1.24}$$

By definition, the tensor perturbations are traceless: $\widehat{\gamma}_{ii} = 0$; the trace has essentially been absorbed in ζ.

In bringing the metric into the ADM form (1.24), we have not completely exhausted the spacetime reparametrization invariance of the theory. Consider a spatial reparametrization of the form

$$x^i \to x^i + v^i(\boldsymbol{x}). \tag{1.25}$$

This does not alter the gauge choice we have already made. It can be shown that at late times, when the modes of interest have exited the horizon, the tensor perturbations become time independent. Using the leftover spatial reparametrization freedom, we can impose the condition

$$\partial_i \widehat{\gamma}_{ij} = 0 \tag{1.26}$$

on the tensor perturbations. Thus, the tensor perturbations become transverse and traceless at late times.

One can also perform a time reparametrization of the form

$$t \to t + \epsilon(\boldsymbol{x}) \tag{1.27}$$

which, when compensated by an accompanying spatial reparametrization

$$x^i \to x^i + v^i(t,\boldsymbol{x}),\; v^i(t,\boldsymbol{x}) = \partial_i \epsilon(\boldsymbol{x}) \int^t dt' \frac{1}{a^2(t')}, \tag{1.28}$$

leaves the gauge choice unaffected. But in the late time limit, $v^i(t,\boldsymbol{x}) \to 0$ in (1.28), and hence we are left only with the time reparametrizations (1.27). Using this, at late times, one can choose a gauge in which the scalar perturbations in the inflaton vanish, $\delta\phi = 0$.[1] In this gauge, the scalar perturbations are thus purely in the metric, ζ. We call this gauge choice as Gauge 1. Alternatively, one can set the scalar perturbations in the metric to vanish, $\zeta = 0$, and have the scalar perturbations purely in the inflaton. This is called as Gauge 2. The two gauges are related to one another by a time reparametrization. Suppose

[1]To keep the discussion simple, here we consider only the leading order in slow roll, where one can conveniently move between setting either $\delta\phi = 0$ or $\zeta = 0$ by appropriate time reparametrizations at late times. At higher orders in slow roll, the late time behaviour of the perturbations is such that it is natural to set $\delta\phi = 0$, and after this choice ζ becomes time independent. See Chapter 2 for a detailed discussion of the leading order case, and Chapter 3 for higher orders in slow roll.

we start in gauge 1, in which the perturbations are given by ζ and $\widehat{\gamma}_{ij}$, and then perform a time reparametrization

$$t \to t - \frac{\zeta}{H}. \tag{1.29}$$

To the leading order in perturbations, this sets the perturbation ζ to vanish. The tensor perturbation $\widehat{\gamma}_{ij}$ is unchanged under the time reparametrization (1.29). Now, if the background value of the inflaton is $\bar{\phi}(t)$, then the resulting value of the perturbation this time reparametrization gives rise to is

$$\delta\phi = -\frac{\dot{\bar{\phi}}}{H}\zeta. \tag{1.30}$$

Similarly one can go from gauge 2 to gauge 1 by performing the inverse transformation.

As far as observational consequences are concerned, the CMB temperature anisotropies as well as the density perturbations in matter distribution that give rise to the large scale structure are seeded by the scalar perturbations [48, 49], whereas the tensor fluctuations give rise to gravitational waves, which can leave a unique imprint on the B-mode polarization of the CMB [50, 51, 52, 53].

1.4.1 Coherent phases of inflationary perturbations

Before we proceed, let us present an argument, following [54], that provides very strong evidence for the claim that inflation must have been the mechanism which seeded the primordial density fluctuations.

A generic property of the quantum fluctuations produced during inflation is that they freeze out once they exit the horizon, i.e. there is no time evolution once $k \ll aH$ for the modes with comoving wave number k. However, after inflation ends, the frozen modes re-enter the horizon and become dynamical again. In particular, the curvature perturbations $\zeta(\boldsymbol{k})$ source the density perturbations $\delta(\boldsymbol{k})$ in the tightly coupled photon-baryon fluid. The density perturbations $\delta(\boldsymbol{k})$ qualitatively obey an equation of the form

$$\ddot{\delta} + F_1\dot{\delta} + c_s^2 k^2 \delta = F_2,$$

where F_1, F_2 are respectively functions of the baryon-to-photon energy density ratio and the curvature perturbations $\zeta(\boldsymbol{k})$, and c_s is the speed of sound in the photon-baryon fluid. Thus inflationary perturbations source acoustic oscillations in the photon-baryon fluid.

When the frozen super-horizon modes $\zeta(\boldsymbol{k})$ re-enter the horizon, they start evolving slowly, and satisfy $\dot{\zeta}(\boldsymbol{k}) \approx 0$. If one considers each Fourier mode of the fluctuations as a linear combination of a sine mode and a cosine mode, then it is as if inflation excites only the cosine modes. Thus all Fourier modes with a given wave number k and differing amplitudes start oscillating again from their extremas once they re-enter the horizon. This *phase coherence* of different modes re-entering the horizon leaves an imprint in the CMB temperature anisotropies generated at the time of recombination, as we discuss below.

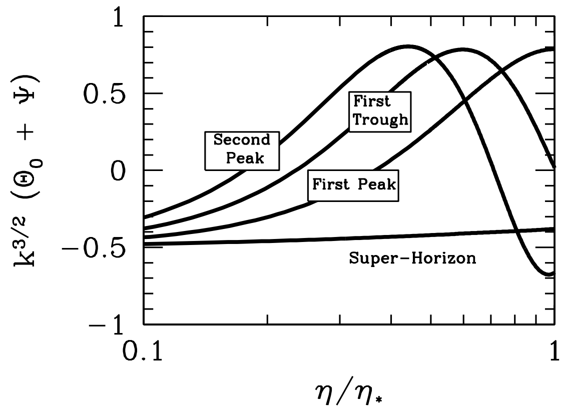


Figure 1.6: Evolution of four different wave numbers influencing the CMB anisotropy spectrum. $\eta = \eta_*$ corresponds to the time at which recombination took place.

Consider modes with different wave numbers k which re-entered the horizon before recombination took place and the CMB photons started free streaming. The smaller the wavelength of a mode, the earlier it re-entered the horizon, and the more acoustic oscillations it had undergone by the time of recombination. Figure 1.6 shows four different wave numbers in different states of their oscillations at the time of recombination. The modes labeled "Super-Horizon" have not yet re-entered the horizon and are insensitive to causal physics. The modes labeled "First Peak" re-entered the horizon sufficiently early, so that at the time of recombination they have made half an oscillation and are at the maximum of their amplitudes. We therefore expect large anisotropies in the CMB spectrum corresponding to the scales for these modes. The modes labeled "First Trough" are the ones which have completed three-fourths of an oscillation, and are at the zero of their oscillation cycle, implying very low anisotropies at the scales corresponding to these wave numbers. This pattern gets repeated as we keep on going to shorter and shorter wavelengths.

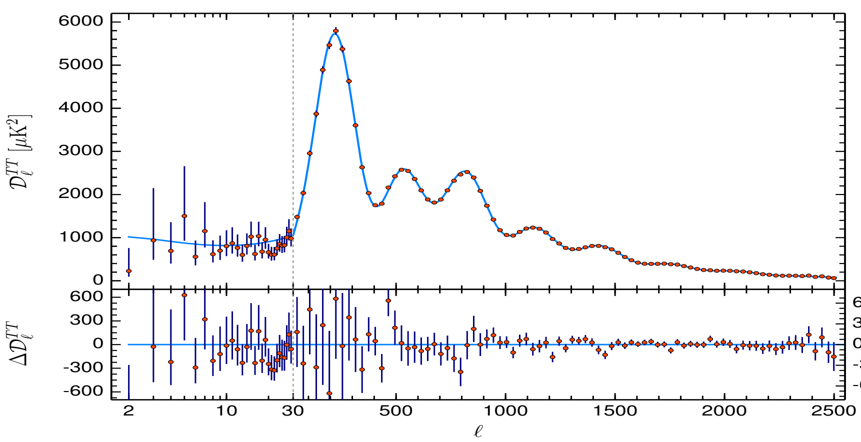


Figure 1.7: The CMB temperature anisotropy spectrum measured by the Planck satellite [55]. Notice the distinctive appearance of peaks and troughs in the spectrum for $l \geq 200$, or $\theta < 1°$.

It is now important to note that had it not been for the coherent phases of the different Fourier modes re-entering the horizon, one would not see the tell-tale peaks and troughs in the CMB temperature anisotropy spectrum, as seen in figure 1.7. As mentioned earlier, due to the property of phase coherence, all Fourier modes with different amplitudes but the same wave number oscillate coherently after horizon re-entry. Figure 1.8 shows the modes which correspond to the first peak and first trough in the CMB anisotropy spectrum. Notice that although the magnitudes of their extremas are different, they oscillate coherently. Had it not been for inflation giving rise to phase coherence, different modes with the same wave number will oscillate incoherently, and will interfere destructively at the time of recombination, as shown in figure 1.9. This will destroy the peaks and troughs and will give rise to a flat featureless CMB anisotropy spectrum.

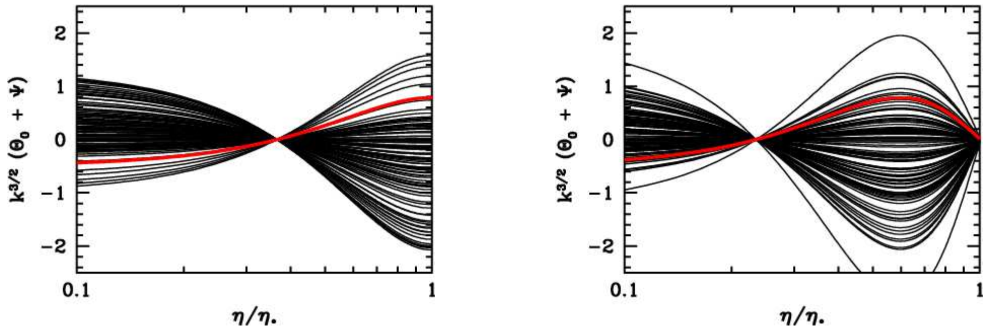


Figure 1.8: The coherent oscillations in the Fourier modes corresponding to wave numbers which give rise to the first peak and the first trough in the CMB anisotropy spectrum of figure 1.7.

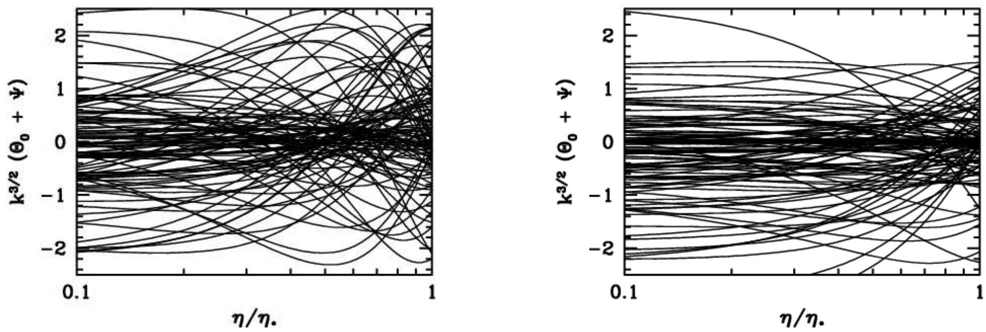


Figure 1.9: Incoherent oscillations in the Fourier modes corresponding to the same wavelengths as in figure 1.8. Due to the destructive interference at the time of recombination, one does not get peaks and troughs in the CMB temperature anisotropy spectrum for this scenario.

One may be tempted at this stage to consider alternatives to inflation which excite only the 'cosine' mode and give rise to the observed phase coherence. However, most of these alternatives are ruled out once one takes the CMB temperature-polarization cross correlation data into account.[2] The cross correlation data, figure 1.10, shows a strong anti-correlation between temperature and polarization for angular scales $50 \leq l \leq 200$. These scales were

[2]The only alternatives that are not ruled out are the ones in which the universe undergoes a contracting phase prior to the hot big bang, during which it decelerates; see [56, 57] for reviews of such models.

super-horizon at the time of recombination in alternatives to inflation, and any alternative involving causal physics can not explain this observed anti-correlation.

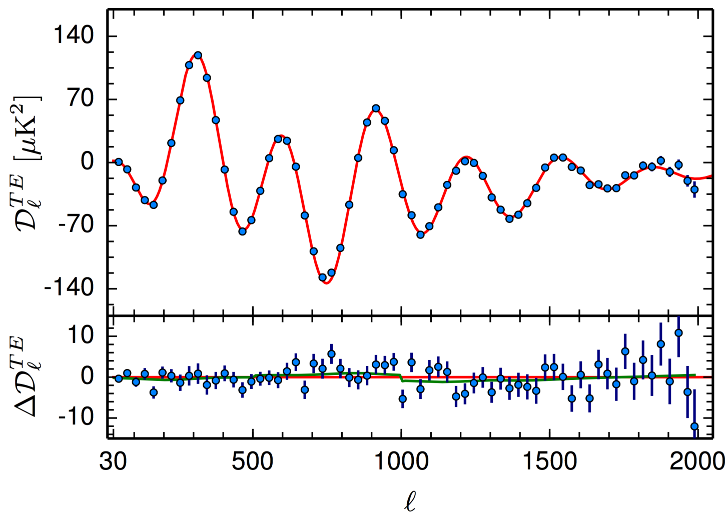


Figure 1.10: The CMB temperature-polarization cross power spectrum measured by the Planck satellite [43]. Note the anti-correlation for $50 \leq l \leq 200$, corresponding to angular scales $1^\circ < \theta < 5^\circ$.

1.5 The wave function of the universe

Another important idea relevant to the thesis is that of the *wave function* of the universe [58], defined as a functional over the late time values of inflationary perturbations. Our discussion of symmetries will rely heavily upon the wave function and its invariance.

The wave function is defined as a functional integral over the inflationary perturbations via

$$\psi[\chi(\boldsymbol{x})] = \int_{initial}^{\chi(\boldsymbol{x})} D\chi\, e^{iS[\chi]}, \tag{1.31}$$

where $S[\chi]$ is the action and $\chi(\boldsymbol{x})$ denotes collectively the physical scalar and tensor perturbations on a late time slice on which the wave function is being constructed. Note that the perturbations have frozen out after appropriate gauge fixing at late times. Thus the wave function (1.31) also becomes independent of time. The action S can be evaluated to the desired order in the perturbations: the leading term is quadratic, the cubic and higher order terms depict interactions. The wave function can be schematically expanded as [24]

$$\begin{aligned}\psi[\chi(\boldsymbol{x})] = \exp\Big[&-\frac{1}{2}\int d^3x\, d^3y\, \chi(\boldsymbol{x})\chi(\boldsymbol{y})\, \langle\hat{\theta}(\boldsymbol{x})\hat{\theta}(\boldsymbol{y})\rangle \\ &+\frac{1}{6}\int d^3x\, d^3y\, d^3z\, \chi(\boldsymbol{x})\chi(\boldsymbol{y})\chi(\boldsymbol{z})\, \langle\hat{\theta}(\boldsymbol{x})\hat{\theta}(\boldsymbol{y})\hat{\theta}(\boldsymbol{z})\rangle + \ldots\Big].\end{aligned} \tag{1.32}$$

Here $\langle\hat{\theta}(\boldsymbol{x})\hat{\theta}(\boldsymbol{y})\rangle$, $\langle\hat{\theta}(\boldsymbol{x})\hat{\theta}(\boldsymbol{y})\hat{\theta}(\boldsymbol{z})\rangle$ etc. are coefficient functions that determine the correlation functions of the inflationary perturbations $\chi(\boldsymbol{x})$, following the standard quantum mechanical recipe for computing expectation values from the wave function. For instance, the two point function of the perturbations is given by

$$\langle\chi(\boldsymbol{x})\chi(\boldsymbol{y})\rangle = \mathcal{N}\int\mathcal{D}[\chi]\,\chi(\boldsymbol{x})\chi(\boldsymbol{y})\,|\psi[\chi]|^2, \tag{1.33}$$

where $\mathcal{N}$ is the normalization

$$\mathcal{N}^{-1} = \int\mathcal{D}[\chi]\,|\psi[\chi]|^2. \tag{1.34}$$

We have written the coefficient functions in (1.32) in a very suggestive form. As we will see below, they behave identically as the correlation functions of marginal operators in a three dimensional Euclidean conformal field theory.

One also needs to impose appropriate boundary conditions on the inflationary perturbations in the far past to make the functional integral eq.(1.31) well defined. We work with the *Bunch-Davies* boundary conditions [59, 60, 61]. The Bunch-Davies boundary conditions correspond to the physical condition that all the modes of interest were deep within the horizon before inflation, and their wavelengths were much smaller compared to the Hubble scale. Thus, they were indifferent to the curvature on larger scales and behaved as if they were living in Minkowski space. The initial vacuum state for the inflationary perturbations is thus defined in a way analogous to the vacuum state for quantum fields in flat spacetime. An interesting point to note is that the Bunch-Davies state is invariant under the de Sitter isometries [62, 63, 64].

One can, for instance, choose to express the wave function in terms of the scalar perturbation $\delta\phi(\boldsymbol{x})$, and the tensor perturbations $\widehat{\gamma}_{ij}(\boldsymbol{x})$, as[3]

$$\begin{aligned}\psi[\delta\phi,\widehat{\gamma}_{ij}] = \exp\Bigg[\frac{M_{Pl}^2}{H^2}\Bigg(&-\frac{1}{2}\int d^3x\,d^3y\,\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\,\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle\\ &-\frac{1}{2}\int d^3x\,d^3y\,\widehat{\gamma}_{ij}(\boldsymbol{x})\widehat{\gamma}_{kl}(\boldsymbol{y})\,\langle T^{ij}(\boldsymbol{x})T^{kl}(\boldsymbol{y})\rangle\\ &+\frac{1}{6}\int d^3x\,d^3y\,d^3z\,\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\delta\phi(\boldsymbol{z})\,\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})\rangle\\ &-\frac{1}{4}\int d^3x\,d^3y\,d^3z\,\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\widehat{\gamma}_{ij}(\boldsymbol{z})\,\langle O(\boldsymbol{x})O(\boldsymbol{y})T^{ij}(\boldsymbol{z})\rangle+\dots\Bigg)\Bigg].\end{aligned} \tag{1.35}$$

Notice that for every factor of the scalar perturbation $\delta\phi(\boldsymbol{x})$, we introduce a factor of $O(\boldsymbol{x})$,

[3]The discussion here assumes that we are working to the leading order in slow roll, where only corrections of $\mathcal{O}(\dot{\phi}/H)$ are incorporated. As discussed earlier, as well as in Chapter 2, at this order it is trivial to switch between the two gauges $\zeta=0$ and $\delta\phi=0$, and the wave function can be expressed in either of the two. However, for higher orders in slow roll, it is natural to express the late time wave function in the $\delta\phi=0$ gauge, a point discussed in considerable detail in Chapter 3.

and for every factor of $\widehat{\gamma}_{ij}(\boldsymbol{x})$, we introduce a factor of $T^{ij}(\boldsymbol{x})$ in the wave function.

The wave function can equivalently be expressed in the momentum space as [4]

$$\begin{aligned}\psi[\delta\phi,\widehat{\gamma}_{ij}] = \exp\Bigg[\frac{M_{Pl}^2}{H^2}\Bigg(&-\frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\,\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle \\ &-\frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\gamma_s(\boldsymbol{k_1})\gamma_{s'}(\boldsymbol{k_2})\,\langle T^s(-\boldsymbol{k_1})T^{s'}(-\boldsymbol{k_2})\rangle \\ &+\frac{1}{6}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3}) \\ &\qquad\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})O(-\boldsymbol{k_3})\rangle + \ldots\Bigg)\Bigg]\end{aligned} \tag{1.36}$$

Here, we have expanded the tensor perturbations $\widehat{\gamma}_{ij}(\boldsymbol{k})$ in terms of the two polarization tensors e^s_{ij} as

$$\widehat{\gamma}_{ij}(\boldsymbol{k}) = \sum_{s=1}^{2} e^s_{ij}\,\gamma_s(\boldsymbol{k}) \tag{1.37}$$

where the polarization tensors are normalized according to

$$\sum_{i,j} e^s_{ij}\,e^{s',ij} = 2\,\delta^{s,s'}. \tag{1.38}$$

1.5.1 The coefficient functions and their relation to CFT

We now investigate the nature of the coefficient functions in the wave function (1.35). Note that spatial translations and rotations are symmetries of the background that remain unbroken. Also, although we have made the gauge choice $N = 1, N^i = 0$, we still need to impose the equations of motion of these variables. In particular, the N^i equation of motion corresponds to the spatial reparametrization invariance of the wave function, and the N equation of motion gives rise to time reparametrization invariance. Let us exploit these invariance properties to understand the nature of the coefficient functions appearing in the wave function.

The translational invariance of the wave function implies that the coefficient functions are also translationally invariant. Similarly, the rotational invariance of the wave function implies that $O(\boldsymbol{x})$ should behave like a scalar and $T^{ij}(\boldsymbol{x})$ should behave like a rank-2 tensor under spatial rotations.

Consider next the scaling transformations of the form given in (1.20) at late times. The perturbation $\delta\phi(\boldsymbol{x})$ transforms as a scalar under the transformation. Hence

$$\delta\phi(\boldsymbol{x}) \to \delta\phi'(\boldsymbol{x}) = \delta\phi\left(\frac{\boldsymbol{x}}{\lambda}\right). \tag{1.39}$$

[4]Our convention for the Fourier transform is: $f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$, and $f(\boldsymbol{k}) = \int d^3x\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})$.

The condition that the wave function stays invariant under this transformation is

$$\psi[\delta\phi'(\boldsymbol{x})] = \psi[\delta\phi(\boldsymbol{x})]. \tag{1.40}$$

Now, every additional power of the perturbation $\delta\phi(\boldsymbol{x})$ in the wave function is accompanied by an extra $O(\boldsymbol{x})$ in the coefficient function. Thus, in a schematic way, the condition for the invariance of the wave function (1.40) translates to

$$\int d^3x\, \delta\phi'(\boldsymbol{x})\, O(\boldsymbol{x}) = \int d^3x\, \delta\phi(\boldsymbol{x})\, O(\boldsymbol{x}), \tag{1.41}$$

which by using (1.39) becomes

$$\begin{aligned} &\int d^3x\, \delta\phi\left(\frac{\boldsymbol{x}}{\lambda}\right) O(\boldsymbol{x}) = \int d^3x\, \delta\phi(\boldsymbol{x})\, O(\boldsymbol{x}) \\ \Rightarrow &\int d^3x\, \delta\phi(\boldsymbol{x})\, \lambda^3 O(\lambda\boldsymbol{x}) = \int d^3x\, \delta\phi(\boldsymbol{x})\, O(\boldsymbol{x}). \end{aligned} \tag{1.42}$$

From (1.42), we can conclude that under a scaling transformation, $O(\boldsymbol{x})$ transforms as

$$O(\boldsymbol{x}) \to \lambda^3 O(\lambda\boldsymbol{x}), \tag{1.43}$$

and if we choose the scaling parameter λ to be small, $\lambda = 1 + \epsilon$, then

$$\begin{aligned} &O(\boldsymbol{x}) \to O(\boldsymbol{x}) + \epsilon\, \delta O(\boldsymbol{x}), \\ &\delta O(\boldsymbol{x}) = 3O(\boldsymbol{x}) + x^i \partial_i O(\boldsymbol{x}). \end{aligned} \tag{1.44}$$

This is exactly the condition that will arise for scale invariance if $O(\boldsymbol{x})$ is a scalar operator of dimension 3 in a three dimensional Euclidean conformal field theory, and the coefficient functions $\langle OO\rangle$ etc. are correlation functions of these operators [26].

A very similar calculation for the tensor perturbations $\widehat{\gamma}_{ij}$ tells us that the objects $T^{ij}(\boldsymbol{x})$ also behave as operators of dimension 3 in a three dimensional Euclidean CFT.

We next consider the invariance of the wave function under special conformal transformations. A infinitesimal special conformal transformation at late times is given by

$$\begin{aligned} &x^i \to x^i + \delta x^i, \\ &\delta x^i = \boldsymbol{x}^2 b^i - 2x^i(\boldsymbol{b}\cdot\boldsymbol{x}). \end{aligned} \tag{1.45}$$

The scalar perturbation transforms as

$$\delta\phi(\boldsymbol{x}) \to \delta\phi'(\boldsymbol{x}) = \delta\phi(x^i - \delta x^i). \tag{1.46}$$

Imposing the condition (1.41) for the invariance of the wave function then leads to the following transformation of the operator $O(\boldsymbol{x})$ under a special conformal transformation

$$\begin{aligned}
&O(\boldsymbol{x}) \to O(\boldsymbol{x}) + \delta O(\boldsymbol{x}),\\
&\delta O(\boldsymbol{x}) = -6(\boldsymbol{b}\cdot\boldsymbol{x})O(\boldsymbol{x}) + \hat{D}O(\boldsymbol{x}),\\
&\hat{D} = x^2(\boldsymbol{b}\cdot\boldsymbol{\partial}) - 2(\boldsymbol{b}\cdot\boldsymbol{x})(\boldsymbol{x}\cdot\boldsymbol{\partial}).
\end{aligned} \tag{1.47}$$

This is same as the transformation rule for a scalar operator of dimension 3 under a special conformal transformation in a CFT. Similarly, the tensor operator $T^{ij}(\boldsymbol{x})$ transforms under the special conformal transformation (1.45) as

$$\begin{aligned}
&T_{ij}(\boldsymbol{x}) \to T_{ij}(\boldsymbol{x}) + \delta T_{ij}(\boldsymbol{x}),\\
&\delta T_{ij}(\boldsymbol{x}) = -6(\boldsymbol{b}\cdot\boldsymbol{x})T_{ij}(\boldsymbol{x}) + 2\hat{M}_i^k T_{jk} + 2\hat{M}_j^k T_{ik} + \hat{D}T_{ij}(\boldsymbol{x}),\\
&\hat{M}_i^k = x^k b^i - x^i b^k,\ \hat{D} = x^2(\boldsymbol{b}\cdot\boldsymbol{\partial}) - 2(\boldsymbol{b}\cdot\boldsymbol{x})(\boldsymbol{x}\cdot\boldsymbol{\partial}).
\end{aligned} \tag{1.48}$$

This agrees with the transformation rule for the stress-energy tensor of a three dimensional CFT.

Thus, we have established that $O(\boldsymbol{x})$ and $T^{ij}(\boldsymbol{x})$ behave as dimension three operators in a 3D Euclidean conformal field theory. The coefficient functions appearing in the wave function behave as the correlation functions of these operators, a point discussed in greater detail in Chapter 3, see also [26].

1.5.2 The two point function

The wave function approach provides a very elegant methodology for calculating the correlation functions of perturbations generated during inflation. We illustrate the method below by computing the correlator $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ in terms of the CFT correlator $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$. The same technique can be generalized to find similar relations between other correlators.

To derive the relationship between $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ and $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$, we start by considering the wave function in momentum space in the form

$$\psi[\delta\phi] = \exp\left[\frac{M_{Pl}^2}{H^2}\left(-\frac{1}{2}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\,\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle\right)\right]. \tag{1.49}$$

By definition, the correlation function $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ is given by

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle = \mathcal{N}\int\mathcal{D}[\delta\phi]\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\,\big|\psi[\delta\phi]\big|^2, \tag{1.50}$$

where $\mathcal{N}$ is the normalization factor given by

$$\mathcal{N}^{-1} = \int\mathcal{D}[\delta\phi]\,\big|\psi[\delta\phi]\big|^2. \tag{1.51}$$

Define the quantity $W[J]$ as

$$W[J] = \mathcal{N} \int \mathcal{D}[\delta\phi] \exp\left[\frac{M_{Pl}^2}{H^2}\left(- \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\, \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\, \langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle \right.\right.$$
$$\left.\left. - \int d^3k\, J(\boldsymbol{k})\, \delta\phi(\boldsymbol{k})\right)\right], \quad (1.52)$$

where $J(\boldsymbol{k})$ is a source term. $W[J]$ acts like a generating functional for the correlation functions. In terms of $W[J]$, we can write the correlation function $\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ as

$$\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle = \frac{H^4}{M_{Pl}^4}\, \frac{\delta^2 W[J]}{\delta J(\boldsymbol{k_1})\, \delta J(\boldsymbol{k_2})}\bigg|_{J=0}. \quad (1.53)$$

The expressions for $\mathcal{N}$ and $W[J]$ can be evaluated exactly by using Gaussian integration techniques. This gives us

$$\mathcal{N} = \exp\left\{\frac{1}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(\frac{2}{(2\pi)^2}\,\frac{M_{Pl}^2}{H^2}\,\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle\right)\right]\right\}, \quad (1.54)$$

and

$$W[J] = \exp\left\{(2\pi)^6\,\frac{M_{Pl}^2}{4H^2}\int d^3k_1\, d^3k_2\, J(\boldsymbol{k_1})\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle^{-1} J(\boldsymbol{k_2})\right\}. \quad (1.55)$$

Putting the expression (1.55) into the equation (1.53) we get a relationship between $\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ and $\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle$ as

$$\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle' = \frac{1}{2}\,\frac{H^2}{M_{Pl}^2}\,\frac{1}{\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle'}\,, \quad (1.56)$$

where a $'$ on a correlator denotes the suppression of the momentum conserving factor of $(2\pi)^3\,\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})$. It is straight forward to obtain similar relations for higher point correlation functions.

Let us now move further and compute the correlation function $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ in a CFT. The conformal symmetries of the CFT fix this completely. We start by considering translational invariance. This will require that the correlator $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ be proportional to the delta function $(2\pi)^3\,\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})$, as translational invariance implies conservation of momentum. Next consider the invariance under rotations. This will require the correlator $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ to be a function only of the magnitude of $\boldsymbol{k_1}$, which we denote by k_1. Thus translational and rotational invariance fix the form of the correlation function $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ to be

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle = (2\pi)^3\,\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\, f(k_1), \quad (1.57)$$

where $f(k_1)$ is for now an arbitrary function. Finally, we impose the constraint due to scaling. Recall that the position space operator $O(\boldsymbol{x})$ has dimension 3. Since the momentum space operator $O(\boldsymbol{k})$ is related to $O(\boldsymbol{x})$ by a Fourier transform, it must be dimensionless.

Thus the correlation function $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ should also be dimensionless. Now, in equation (1.57), the delta function has a scaling dimension -3. To compensate for this, the function $f(k_1)$ should have dimension 3. This tells us that $f(k_1) \sim k_1^3$, and hence the form of the correlator $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$ is

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle = (2\pi)^3\, \delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\, k_1^3. \tag{1.58}$$

Thus conformal symmetries alone can fix the scalar two point function $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle$. In particular, (1.58) tells us that

$$\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle = \langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle. \tag{1.59}$$

Equation (1.56) then tells us that

$$\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle = (2\pi)^3\, \delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\, \frac{H^2}{2M_{Pl}^2}\,\frac{1}{k_1^3}. \tag{1.60}$$

Using the expression (1.30) for the change of gauge between Gauge 2 and Gauge 1, we get

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\, \delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\, \frac{H^2}{\dot{\phi}^2}\,\frac{H^2}{2M_{Pl}^2}\,\frac{1}{k_1^3}. \tag{1.61}$$

This is a very well known and useful result, with important observational consequences. Note that in reaching up to eq.(1.60) we had been assuming exact conformal invariance, with H being a constant. This, however, is not true during inflation. The conformal symmetries are slightly broken, and H is no more a constant. This results into a modification of the result in eq.(1.61), with the momentum dependence getting modified to $\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle' \sim k_1^{-3+n_s}$, where n_s is called the *tilt* of the two point function, and is sensitive to the departures from exact conformal invariance. The tilt can be expressed in terms of the slow roll parameters as[5] $n_s = 2\eta - 6\epsilon$, [24]. A similar computation can also be done for the tensor two point function, which also deviates from the exact scale invariant form due to breaking of the conformal symmetries. This inflationary prediction for departures from the exact scale invariant form is in excellent agreement with observations.[6]

The example above illustrates the elegance of the computational technique based upon the wave function, which is quite straight forward and simple. It can be used to calculate correlation functions of other inflationary perturbations in a similar manner as well [3, 27].

[5]The scalar tilt, which parametrises the modification in momentum dependence of the two point function from the exact scale invariant form, arises because the values of $H, \dot{\phi}$ appearing in eq.(1.61) depend upon the exact moment when the momentum mode of interest exits the horizon via $k \approx aH$, and is given by

$$n_s = \frac{d}{d\ln k}\ln\left(\frac{H^4}{\dot{\phi}^2}\right) = 2\left(\frac{2\dot{H}}{H^2} - \frac{\ddot{\phi}}{H\dot{\phi}}\right) = 2(\eta - 3\epsilon).$$

[6]The latest data from Planck [14, 16] gives $n_s = 0.968 \pm 0.006$.

1.6 Organization of the thesis

Most of this thesis deals with the idea of using the underlying approximate de Sitter symmetries to derive constraints on the correlation functions of inflationary perturbations, by utilizing the coordinate reparametrization invariance of the late time wave function. We have presented a very basic overview of the ideas needed for the rest of the thesis in the present chapter. In Chapter 2, we work to the leading order in slow roll, where deviations from exact de Sitter invariance only to $\mathcal{O}(\dot{\bar{\phi}}/H)$ are relevant, and derive symmetry constraints in the form of Ward identities for the correlation functions of inflationary perturbations. In Chapter 3 we generalize these Ward identities to incorporate corrections to all orders in the slow roll expansion. Chapter 4 discusses some applications and consequences of the Ward identities. Appendices associated with each chapter provide additional supplementary material.

Chapter 2

Symmetry constraints in inflation: the leading order in slow roll

2.1 Introduction

Inflation is an attractive idea which explains the approximate homogeneity and isotropy of our universe. It also leads to the genesis of small perturbations required for the observed anisotropy in the Cosmic Microwave Background and for the growth of structure.

There has been considerable effort in developing many theoretical models that can give rise to inflation in the early universe. However, relatively little work has been done on understanding the nature of the perturbations which are produced during inflation in a model independent manner. More recently, such a model independent analysis has been developed based on symmetry considerations.

During inflation, spacetime is approximately described by de Sitter space. The essential idea of some of the symmetry based analysis is to use the $O(1,4)$ symmetry of de Sitter space, which is also the symmetry group of three dimensional Euclidean Conformal Field Theories, to constrain correlation functions of the perturbations. Of course, the universe is not exactly described by de Sitter space during inflation, but the corrections which are quantified in terms of the slow roll parameters are small, being of order 1% or so. The $O(1,4)$ symmetry should therefore be useful in constraining the correlation functions. In the following discussion, we shall refer to this symmetry group as the de Sitter group or the conformal group interchangeably.

In the present chapter, we carry out such a symmetry based analysis for the scalar three point function, by including the first non-vanishing corrections in the slow roll approximation. Among all the three point correlations, the three point scalar correlator is expected to be of the biggest magnitude, and therefore of most significance for observational tests of non-Gaussianity. It is therefore clearly important to understand what constraints can be imposed on it from symmetry considerations alone. This is the motivation underlying the present chapter. The constraints obtained can be straight forwardly generalized to higher point correlation functions as well.

One of our main results in this chapter is a set of Ward identities relating the three point function to the scalar four point function in a particular limit. The coefficient of proportionality between the two is the parameter $\frac{\dot{\bar{\phi}}}{H}$, defined in section 3.2.

It is well known that the three point function is suppressed in the canonical model of slow roll inflation (for a definition of this model see eq.(2.20)) so that, in a sense which we make precise below, it can be thought of as vanishing to leading order in the slow roll approximation. We argue that this feature is more generally valid. In addition, the Ward identities allow us to estimate the magnitude of the leading non-vanishing contribution to the three point function, in the slow roll approximation. We find that generically it is of the same order as the three point function in the canonical slow roll model. To get a rough idea, this means that quite generally, as long as conformal symmetry is approximately valid, $f_{NL} \sim O\big((\frac{\dot{\bar{\phi}}}{H})^2\big)$, although the detailed functional form is not the same as assumed in the standard f_{NL} parametrization, so this is only an estimate.

While the small magnitude for the three point function is disappointing from the point of view of observations, this result can be turned around in an interesting way as follows. If observationally a three point function of bigger magnitude is observed then it would rule out not only the canonical model of slow roll inflation, but in fact all models where the dynamics is approximately conformally invariant, and the slow roll approximation holds.[1]

We also show that the Ward identities determine the three point function, nearly completely, upto one constant, in terms of the four point function. To leading order, the latter can be computed in the de Sitter limit and is thus constrained by the full de Sitter symmetry group. In this way, we can make precise the extent to which conformal symmetry constrains the scalar three point correlator.

Unfortunately, as is well known, the four point function itself is not significantly constrained in a conformal field theory. In position space there are three invariant cross ratios in three dimensions, and conformal symmetry allows the four point scalar correlator to be a general function of these three variables. This is a rather weak constraint. It follows from our analysis then that conformal invariance also constrains the three point scalar correlator only weakly.

Directly checking the Ward identities through observations seems very challenging, although it cannot be ruled out, perhaps. A more interesting angle might be the following. In the canonical slow roll model, the four point function in the de Sitter limit arises from a tree diagram with single graviton exchange, see [3, 27, 65]. If the three point function is observed

[1]Strictly speaking, in a non-generic case, approximate conformal invariance and the slow roll approximation do allow the magnitude to be bigger, as we discuss below. But in this case the functional form is completely fixed, so one should be able to test for this possibility as well.

and found to depart from the functional form it has in the canonical slow roll model, then it would follow from the Ward identities that the four point function must also have a different form. This would suggest that perhaps higher spin fields might have been involved during inflation, a possibility explored in [66, 67, 68, 69, 70, 71].

The approach followed in this chapter is based on the important work of [24] and [25] and also the subsequent papers, [26] and [27]. As was emphasized in these works, symmetry considerations are conveniently discussed in terms of the wave function of the universe at late times. In the de Sitter limit, the Ward identities of conformal invariance can be obtained from the constraints of spatial reparametrization and time reparametrization invariance, which the wave function must satisfy. The time reparametrization constraint in particular is the same as the Wheeler-DeWitt equation. These constraints must continue to hold even when we go beyond the de Sitter limit. In this way, the spatial and time reparametrization invariance can be used to obtain the corrected Ward identities which now include the breaking of conformal invariance.

It is worth explicitly mentioning that while the analysis we carry out draws on techniques developed in the study of the AdS/CFT correspondence [72, 73, 74],[2] we do not assume that there is a hologram for de Sitter space or for inflation. We use the techniques drawn from AdS/CFT only as a way of efficiently organizing the analysis of symmetry constraints for perturbations which are generated during inflation in the gravitational system.

The analysis we carry out assumes, as was mentioned above, that the full inflationary dynamics, including the scalar sector, preserves approximate conformal invariance. Some of our conclusions therefore do not apply to models like DBI inflation [81, 82] or Ghost inflation [83], in which the scalar sector breaks the full conformal symmetry badly. In addition, it assumes that only one inflaton was present during inflation, and that the initial state was the Bunch-Davies vacuum. We also assume that the slow-roll conditions hold; these are more precisely discussed in section 2.2.1. Besides these assumptions, our conclusions are robust, and as was emphasized above, model independent. For example, they should hold even if higher derivative corrections to Einstein gravity become important.[3]

This chapter is organized as follows. In section 3.2 we discuss some of the introductory material. The Ward identities are derived in section 2.3. In section 2.4 we analyze these identities further and derive various consequences. Finally, we conclude in section 2.5. Appendices A.1, A.2 and A.3 contain additional important details.

The present chapter is largely based on [1]. Early work on using conformal symmetry to constrain inflationary correlators includes [84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95]. More recent work, where the conformal symmetries are often thought of as being non-linearly realized, include [96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112,

[2] For reviews see [75, 76, 77, 78, 79, 80]

[3] More correctly, these results should apply also to models where quantum effects are small but classical higher derivative corrections are important. As would happen, for example, if the Hubble scale is of order the string scale, M_{st}, but much smaller than the Planck scale, M_{Pl}.

113, 114]. Many interesting Ward identities have already been derived using this approach. Additional related work is in [115, 116, 117], see also [118, 119, 120, 121, 122, 123]. Our discussion in section 2.4 is closely related to [124], see also [125]. The basic approach of using time and spatial reparametrizations to derive Ward identities that we follow was first discussed in the AdS context in [126]. There are also some related developments in the study of Lifshitz and hyperscaling violating spacetimes, of interest for possible connections between AdS gravity and condensed matter physics, see [127].

2.2 Basic set-up and conventions

In this section we give a few details about the basic approach we will use; for more details see [24], and [26], [27].

We will consider the metric to be of the ADM form

$$ds^2 = -N^2dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt), \tag{2.1}$$

and work in the gauge

$$N = 1, N^i = 0. \tag{2.2}$$

The equations of motion obtained by varying N and N^i in the action must still be imposed. These equations will give rise to the constraints of spatial and time reparametrizations that play an important role in the subsequent discussion.

The background inflationary solution is a Friedmann-Robertson-Walker (FRW) spacetime with scale factor $a(t)$. Allowing for perturbations in the metric, we can write

$$h_{ij} \equiv a^2(t)\, g_{ij} = a^2(t)\,[\delta_{ij} + \gamma_{ij}], \tag{2.3}$$

with

$$\gamma_{ij} = 2\zeta\delta_{ij} + \widehat{\gamma}_{ij} \tag{2.4}$$

where $\widehat{\gamma}_{ij}$ is traceless.

A scalar field, the inflaton, ϕ, is also present in inflation (as mentioned in the introduction, we will restrict ourselves to the case with a single inflaton). It can be written as

$$\phi = \bar{\phi}(t) + \delta\phi(t, \boldsymbol{x}) \tag{2.5}$$

where $\bar{\phi}$ and $\delta\phi$ are the background value and the perturbation of the inflaton, respectively.

We will consider the wave function of the universe at late times, when the perturbations of interest have exited the horizon and stopped evolving in time. The wave function is actually a functional of the perturbations $\gamma_{ij}, \delta\phi$. Assuming the wave function is approximately Gaussian and that corrections are small, we can expand it in a Taylor series in the perturbations to get

$$\begin{aligned}\Psi[\delta\phi,\gamma_{ij}] = \exp\Bigg[\frac{M_{Pl}^2}{H^2}\Bigg(&-\frac{1}{2}\int d^3x\sqrt{g(\boldsymbol{x})}\,d^3y\sqrt{g(\boldsymbol{y})}\,\delta\phi(\boldsymbol{x})\,\delta\phi(\boldsymbol{y})\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle\\ &-\frac{1}{2}\int d^3x\sqrt{g(\boldsymbol{x})}\,d^3y\sqrt{g(\boldsymbol{y})}\,\gamma_{ij}(\boldsymbol{x})\gamma_{kl}(\boldsymbol{y})\langle T^{ij}(\boldsymbol{x})T^{kl}(\boldsymbol{y})\rangle\\ &+\frac{1}{3!}\int d^3x\sqrt{g(\boldsymbol{x})}\,d^3y\sqrt{g(\boldsymbol{y})}\,d^3z\sqrt{g(\boldsymbol{z})}\\ &\qquad\qquad\delta\phi(\boldsymbol{x})\,\delta\phi(\boldsymbol{y})\,\delta\phi(\boldsymbol{z})\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})\rangle\\ &+\frac{1}{4!}\int d^3x\sqrt{g(\boldsymbol{x})}\,d^3y\sqrt{g(\boldsymbol{y})}\,d^3z\sqrt{g(\boldsymbol{z})}\,d^3w\sqrt{g(\boldsymbol{w})}\\ &\qquad\delta\phi(\boldsymbol{x})\,\delta\phi(\boldsymbol{y})\,\delta\phi(\boldsymbol{z})\,\delta\phi(\boldsymbol{w})\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})O(\boldsymbol{w})\rangle+\ldots\Bigg)\Bigg].\end{aligned} \tag{2.6}$$

The ellipses denote additional terms which will not play an important role in this chapter.

The coefficient function for the quadratic term in $\delta\phi$ in eq.(2.6) is given by [4]

$$\langle O(\boldsymbol{k})O(\boldsymbol{k'})\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k'})\,k^3. \tag{2.7}$$

Let us also mention that in our conventions

$$\langle O(\boldsymbol{k})O(\boldsymbol{k'})\rangle = \int d^3x\,d^3y\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,e^{-i\boldsymbol{k'}\cdot\boldsymbol{y}}\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle. \tag{2.8}$$

We also note that the coefficient function for the quadratic term in γ_{ij} is given by [5]

$$\langle T^s(\boldsymbol{k_1})\,T^{s'}(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\delta^{s,s'}\,\frac{k_1^3}{2}, \tag{2.9}$$

where $T^s(\boldsymbol{k}) = T_{ij}(\boldsymbol{k})\,\epsilon^{s,ij}(-\boldsymbol{k})$, and the polarization tensor, $\epsilon^{s,ij}$, satisfies the normalization $\epsilon^{s,ij}\epsilon^{s'}_{ij} = 2\,\delta^{s,s'}$.

The wave function eq.(2.6) is obtained by doing a path integral with Bunch-Davies boundary conditions in the far past,

$$\Psi[\delta\phi,\gamma_{ij}] = \int[\mathcal{D}\delta\phi]\,[\mathcal{D}\gamma_{ij}]\,e^{i\,S[\delta\phi,\gamma_{ij}]}\,. \tag{2.10}$$

Our choice, eq.(2.2), does not fix the gauge completely. There is still the freedom to do spatial reparametrizations of the form

$$x^i \to x^i + \epsilon^i(\boldsymbol{x}), \tag{2.11}$$

[4] We follow the convention where bold face symbols, e.g. $\boldsymbol{k}$, stand for 3-vectors, and symbols without bold faces denote the magnitudes, e.g. $k \equiv |\boldsymbol{k}|$.

[5] The labels s, s' denote the two polarizations of the graviton.

and time reparametrization of the form

$$t \to t + \epsilon(\boldsymbol{x}),\ x^i \to x^i + v^i(t, \boldsymbol{x}), \tag{2.12}$$

where

$$v^i = \partial_i \epsilon \int \frac{1}{a^2(t)}\, dt. \tag{2.13}$$

Note that in de Sitter space eq.(2.13) becomes,

$$v^i = -\frac{1}{2H}\,(\partial_i \epsilon)\, e^{-2Ht}. \tag{2.14}$$

The wave function must be invariant under these coordinate transformations. In the classical limit, which we mainly consider here, the wave function is approximately

$$\Psi[\delta\phi, \gamma_{ij}] \sim e^{i\,S[\delta\phi, \gamma_{ij}]}, \tag{2.15}$$

and the invariance of the wave function arises from the invariance of the action with respect to the spatial and time reparametrizations. It is easy to see in the Hamilton-Jacobi formulation that for Einstein gravity, for example, the equation of motion obtained by varying N, N^i in the action, are exactly the equations which impose this invariance. More generally, the equations of motion can be complicated, but the ones obtained by varying N, N^i should, on general grounds, still impose this invariance.

The invariance of the wave function under eq.(2.11) and eq.(3.10) leads to conditions on the coefficient functions, introduced in eq.(2.6). In de Sitter space these constraints are exactly the same as Ward identities for conformal invariance in a conformal field theory, with the coefficient functions playing the role of correlation functions in the CFT. This is the essential reason why the study of the constraints imposed by conformal invariance on the wave function, and therefore expectation values, can be mapped to an analysis of constraints imposed on correlation functions in a CFT.

In de Sitter space the scale factor, eq.(2.3). is given by

$$a^2(t) = e^{2Ht} \tag{2.16}$$

where H, the Hubble parameter, is constant. More generally the Hubble parameter, defined by

$$\frac{\dot{a}}{a} \equiv H \tag{2.17}$$

will not be a constant.

Its variation gives two of the slow roll parameters which quantify the breaking of conformal invariance,

$$\epsilon_1 = -\frac{\dot{H}}{H^2}\,,\ \ \delta = \frac{\ddot{H}}{2H\dot{H}}\,. \tag{2.18}$$

Another parameter is given by

$$\frac{\dot{\phi}}{H}. \tag{2.19}$$

We often refer in this chapter to the "canonical model of slow roll inflation". By this we mean a theory with the action

$$S = \int d^4x \sqrt{-g}\, M_{Pl}^2 \left[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right], \tag{2.20}$$

where the potential is varying slowly enough to meet the conditions, eq.(2.22) and eq.(2.23). Note that in our normalization the scalar field is dimensionless, and V has dimensions of $[M]^2$. In this theory the Hubble parameter is given by

$$H^2 = \frac{1}{3}V. \tag{2.21}$$

In the slow roll approximation in this model, the conditions

$$\epsilon_1, \delta \ll 1, \tag{2.22}$$

and also

$$\frac{\dot{\phi}}{H} \ll 1, \tag{2.23}$$

are met.

The scalar field then approximately satisfies the equation

$$\dot{\phi} \simeq -\frac{1}{3H}V', \tag{2.24}$$

where a prime denotes derivative with respect to the scalar field. The slow roll parameters, ϵ_1 and δ, defined in eq.(2.18), are given by

$$\epsilon_1 = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \quad \text{and} \quad \delta = \epsilon_1 - \frac{V''}{V}, \tag{2.25}$$

and meeting the slow roll conditions, eq.(2.22), eq.(2.23) leads to the requirements,

$$\left(\frac{V'}{V}\right)^2 \ll 1, \tag{2.26}$$

and

$$\frac{V''}{V} \ll 1. \tag{2.27}$$

Also, in this model

$$\frac{\dot{\phi}}{H} = \sqrt{2\epsilon_1}. \tag{2.28}$$

As a result, from eq.(2.22) we see that

$$\frac{\dot{\phi}}{H} \gg \epsilon_1, \delta. \tag{2.29}$$

2.2.1 More general action and slow roll conditions

More generally, our analysis will allow for additional terms so that the full action could schematically take the form

$$S = \int d^4x \sqrt{-g}\, M_{Pl}^2 \left[\frac{1}{2}R - \frac{1}{2}(\partial\phi)^2 - V + \frac{c_1}{\Lambda^2}R^2 + \frac{c_2}{\Lambda^4}R^3 + \cdots\right], \tag{2.30}$$

where the additional terms, like the last two, also have additional derivatives. The R^2, R^3 terms above actually denote various terms with four derivative and six derivatives respectively. The coefficients c_1, c_2 are dimensionless, and in general could be functions of ϕ, while Λ denotes a higher energy cut-off scale, which could in string theory be the string scale, M_{st}, for example. The R^2, R^3 terms could be significant, for example, if the Hubble scale is of order the string scale in string theory. The ellipses stand for additional terms with higher derivatives on the metric, and also terms with additional derivatives on the inflaton. These would be suppressed by appropriate powers of Λ.

As was mentioned above, we are interested here in theories where the additional terms in eq.(2.30) give rise to an approximately conformally invariant dynamics for the perturbations. This can be ensured by taking both the Hubble parameter and the scalar to vary slowly, so that eq.(2.22) and eq.(2.23) are met. The background solution is then approximately de Sitter space with a constant scalar, which clearly preserves conformal invariance. And the perturbations about this background will then inherit this conformal symmetry. In the discussion which follows, it will be convenient for parameter counting to take

$$\epsilon_1 \sim \delta. \tag{2.31}$$

Corrections about the conformally invariant limit will then be suppressed by ϵ_1 and $\frac{\dot{\phi}}{H}$. With these features in mind we will take, in general, the conditions eq.(2.22) and eq.(2.23) to hold for approximate conformal invariance to arise.[6]

Once these conditions are met, it also follows from the field equations in the general case that eq.(2.29) is valid. As was mentioned above, we are assuming that there is an approximate de Sitter solution when $\frac{\dot{\phi}}{H}$ is small. The corrections to de Sitter space in such a solution arise because of extra contributions to the stress energy due to the non-vanishing value of $\dot{\phi}$. However, any such contribution must be of order $(\dot{\phi})^2$ or higher, since the scalar Lagrangian

[6] Strictly speaking, we have established that the conditions eq.(2.22), eq.(2.23) are sufficient, but perhaps not necessary. However, if they are violated the emergence of approximate conformal invariance for the dynamics of small perturbations would be something of an accident, which we view as being quite unlikely.

has at least two derivatives. Thus we learn that ϵ_1, δ can at most be of order

$$\epsilon_1, \delta \sim \left(\frac{\dot{\bar{\phi}}}{H}\right)^2, \tag{2.32}$$

and eq.(2.23) then leads to eq.(2.29). The equations, (2.22), (2.23) and (2.29) are what we will use in our derivation of the Ward identities.

We end with a few comments which are of relevance for the discussion in section 2.4.3, where we estimate the normalization of the homogeneous term S_h in the solution of the Ward identities. We begin by noting that when the higher derivative terms are important for the metric, H^2 will not be given in terms of V by eq.(4.8). Instead, the relation will be more complicated and have the form

$$H^2 f\left(\frac{H}{\Lambda}\right) = V, \tag{2.33}$$

where f is a function which depends on the higher derivative contributions. Now as long as the function $f \sim O(1)$, we get

$$H^2 \sim V. \tag{2.34}$$

Taking a time derivative then gives,

$$\frac{\dot{H}}{H^2} \sim \frac{V'}{H^2}\frac{\dot{\bar{\phi}}}{H}. \tag{2.35}$$

Using eq.(2.32) then leads to

$$\dot{\bar{\phi}} \sim \frac{V'}{H}. \tag{2.36}$$

It follows from eq.(2.34) and eq.(2.36) that the general slow roll case is in fact quite analogous to the canonical slow roll model. In particular, it follows from eq.(2.34), eq.(2.36) that

$$\frac{\dot{\bar{\phi}}}{H} \sim \sqrt{\epsilon_1}, \tag{2.37}$$

and also that in the slow roll expansion in general, an extra time derivative leads to a suppression by a factor of ϵ_1.

The function f in eq.(2.33) has the limiting behaviour $f \to 3$ when $\frac{H}{\Lambda} \to 0$. Eq. (2.34) is therefore a reasonable assumption if $f \sim O(1)$ also for $\frac{H}{\Lambda} \sim O(1)$, but it could be a bad approximation if f becomes big for $\frac{H}{\Lambda} \sim O(1)$.

2.3 The Ward identities

We now turn to a discussion of the Ward identities. It is convenient to first consider the case of pure de Sitter space, with no corrections, and then consider the inflationary spacetime.

2.3.1 de Sitter space

In de Sitter space the metric perturbations γ_{ij} and the scalar perturbation $\delta\phi$ both freeze out and become time independent at sufficiently late time, when their physical spatial momenta $\frac{|\mathbf{k}|}{\mathbf{a}}$ become much smaller than H.

The late time wave function is then a functional of these variables, as discussed in eq.(2.6). As was mentioned above in the comments after eq.(2.10), our choice eq.(2.2) does not fix the gauge completely. In the discussion below, it will be sometimes convenient to fix the remaining time reparametrization freedom, eq.(3.10), by setting the late time value of ζ to vanish,[7]

$$\zeta = 0. \tag{2.38}$$

It is possible to do this for a suitable choice of $\epsilon(\boldsymbol{x})$ because at late times, when v^i in eq.(2.13) vanishes, ζ transforms under

$$t \to t + \epsilon(\boldsymbol{x})$$

as

$$\zeta \to \zeta - H\epsilon(\boldsymbol{x}).$$

After this additional gauge fixing eq.(2.38), the Ward identities of special conformal transformations are then derived in this gauge by considering a combined spatial reparametrization and time reparametrization,

$$x^i \to x^i - 2(b_j x^j)x^i + b^i\left(\sum_j (x^j)^2 - \frac{e^{-2Ht}}{H^2}\right), \tag{2.39}$$

$$t \to t + 2\,\frac{b_j x^j}{H}, \tag{2.40}$$

which preserve the gauge condition eq.(2.38). Before proceeding, let us note that the special conformal transformations are specified by three parameters, $b^i, i = 1, \cdots 3$. Also, note that the last term in eq.(2.39), which goes like $b^i \frac{e^{-2Ht}}{H^2}$, can be dropped at late time.

The invariance of the wave function under the combined transformation, eq.(2.39), eq.(2.40), gives rise to constraints on the coefficient functions in eq.(2.6). In particular, for the coefficient function $\langle OOO\rangle$ in eq.(2.6), which is the coefficient of the term cubic in $\delta\phi$ in the wave function, this leads to the condition,

$$\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_1}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_2}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_3}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' = 0, \tag{2.41}$$

[7] This choice will be referred to as gauge A in section 2.3.2.

where $\mathcal{L}_{\boldsymbol{k}}^{\boldsymbol{b}}$ is the differential operator

$$\mathcal{L}_{\boldsymbol{k}}^{\boldsymbol{b}} = 2\left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right) - (\boldsymbol{b}\cdot\boldsymbol{k})\left(\frac{\partial}{\partial\boldsymbol{k}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right). \tag{2.42}$$

The prime symbols on the correlation functions in eq.(2.41) denote the correlation functions with the momentum conserving delta function stripped off:

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle = (2\pi)^3\,\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})\,\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle'. \tag{2.43}$$

We will follow a similar convention in this chapter for other correlation functions as well.

It is worth giving some more details leading to eq.(2.41). Since the asymptotic value of $\delta\phi$ is time independent, it only transforms under the spatial reparametrization, eq.(2.39),

$$\begin{aligned}\delta\phi &\to \delta\phi + \delta(\delta\phi(\boldsymbol{x})),\\ \delta(\delta\phi(\boldsymbol{x})) &= \left(2(\boldsymbol{b}\cdot\boldsymbol{x})x^i - \boldsymbol{x}^2 b^i\right)\partial_i(\delta\phi(\boldsymbol{x})).\end{aligned} \tag{2.44}$$

Requiring that the wave function is invariant gives rise to the condition

$$\Psi[\delta\phi] = \Psi[\delta\phi + \delta(\delta\phi)]. \tag{2.45}$$

For the coefficient $\langle OOO\rangle$ in position space this leads to the relation,

$$\langle(\delta O(\boldsymbol{x}))O(\boldsymbol{y})O(\boldsymbol{z})\rangle + \langle O(\boldsymbol{x})(\delta O(\boldsymbol{y}))O(\boldsymbol{z})\rangle + \langle O(\boldsymbol{x})O(\boldsymbol{y})(\delta O(\boldsymbol{z}))\rangle = 0, \tag{2.46}$$

where,

$$\delta O(\boldsymbol{x}) = \left(\boldsymbol{x}^2 b^i - 2(\boldsymbol{b}\cdot\boldsymbol{x})x^i\right)\partial_i O(\boldsymbol{x}) - 6(\boldsymbol{b}\cdot\boldsymbol{x})O(\boldsymbol{x}). \tag{2.47}$$

Eq.(2.47) becomes eq.(2.41) in momentum space. The wave function also depends on γ_{ij}, which transforms under eq.(2.39), eq.(2.40), but the resulting terms are not relevant for obtaining the identity eq.(2.46) and we omit them here.

The Ward identity for scale transformations can be derived in a similar way by requiring the invariance of the wave function under the coordinate transformation

$$t \to t+\lambda,\; x^i \to e^{-H\lambda}\,x^i \approx (1-H\lambda)x^i. \tag{2.48}$$

The scalar perturbation $\delta\phi$ transforms under this as

$$\begin{aligned}\delta\phi &\to \delta\phi\,+\,\delta(\delta\phi),\\ \delta(\delta\phi) &= H\lambda\, x^i\partial_i\delta\phi.\end{aligned} \tag{2.49}$$

For the coefficient function $\langle OOO\rangle$ this gives the relation

$$\langle(\delta O(\boldsymbol{x}))O(\boldsymbol{y})O(\boldsymbol{z})\rangle + \langle O(\boldsymbol{x})(\delta O(\boldsymbol{y}))O(\boldsymbol{z})\rangle + \langle O(\boldsymbol{x})O(\boldsymbol{y})(\delta O(\boldsymbol{z}))\rangle = 0\,, \tag{2.50}$$

where $\delta O(\boldsymbol{x})$ is now given by

$$\delta O(\boldsymbol{x}) = H\lambda \left(3 + x^i \partial_i\right) O(\boldsymbol{x}). \tag{2.51}$$

The first term on the RHS of eq.(2.51) arises as follows. Each factor of $\delta\phi(\boldsymbol{x})$ in the cubic term in the wave function, eq.(2.6), is accompanied by an integration measure, $\int d^3x \sqrt{g(\boldsymbol{x})}$. Since we are in the gauge $\zeta = 0$, $\sqrt{g} = 1$ and does not change under the transformation eq.(2.48). The change in the measure d^3x under eq.(2.48) then gives rise to this first term. We note that eq.(2.50) is what we would expect for an operator of dimension 3 in a CFT. In momentum space eq.(2.50) becomes

$$\left(\sum_{a=1}^{3} \boldsymbol{k_a} \cdot \frac{\partial}{\partial \boldsymbol{k_a}}\right) \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle = 0. \tag{2.52}$$

2.3.2 Inflationary spacetime

Now let us consider departures from the conformally invariant case which arise during inflation. In general, the metric begins to differ from the de Sitter case and this in turn affects the asymptotic behavior of the various perturbations. It turns out that for the limited purpose of deriving the Ward identities of interest, the departures of the metric from de Sitter space can be neglected. This is because these departures, which arise because H is no longer a constant, are proportional to ϵ_1, δ, eq.(2.18), whereas the Ward identity we seek arises at order $\frac{\dot{\phi}}{H}$. Since we have argued that the condition eq.(2.29), which is true in the canonical slow roll theory is also true more generally, it is consistent to take the background metric to be de Sitter space while keeping corrections of order $\frac{\dot{\phi}}{H}$.

This approximation leads to considerable simplification. The asymptotic behavior of perturbations continues to be that of de Sitter space. As a result, it is quite straightforward to connect with the analysis above in de Sitter space.

Choice of gauge

There is one subtlety in the inflationary case which needs to be kept in mind though. A variable which is often used to describe scalar perturbations in inflation is the variable $\mathcal{R}$, given by

$$\mathcal{R} = \zeta - \frac{H}{\dot{\phi}} \delta\phi\,. \tag{2.53}$$

The variable $\mathcal{R}$ has the advantage that it is invariant under linearized coordinate transformations, and is also constant outside the horizon. However, since $\dot{\phi}$ appears in the denominator on the RHS, taking the $\dot{\phi} \to 0$ limit, when the de Sitter description should become a good one, can sometimes be confusing when working directly in terms of $\mathcal{R}$.

The simplest way to deal with this complication is to use two different gauges. While the perturbations are inside the horizon and evolving, one can work in the gauge where eq.(2.38) is true. We refer to this as gauge A below. In this gauge the scalar perturbation is given by $\delta\phi$ which behaves in a smooth way, with a well defined Lagrangian for example, in the de Sitter limit. Once the perturbations leave the horizon, one can then go over to the gauge where

$$\delta\phi = 0 \tag{2.54}$$

is true. In this gauge the scalar perturbation is given by ζ and is a constant outside the horizon, so that the correlation functions in terms of ζ are time independent. We call this gauge B below. The required coordinate transformation is a time reparametrization eq.(3.10), with a suitably chosen time independent parameter $\epsilon(\mathbf{x})$. At the linearized level the variable ζ in gauge B is related to the variable $\delta\phi$ in gauge A by

$$\zeta = -\frac{H}{\dot{\phi}}\,\delta\phi\,. \tag{2.55}$$

Having calculated the correlation functions in gauge A it is a straightforward exercise, only involving a change of variables, to go over to gauge B.

This is in fact the procedure we will follow below. To begin, we will work in gauge A and construct the wave function in terms of $\delta\phi$ and the remaining degrees of freedom in the metric γ_{ij}. We can think of this wave function as being constructed in the epoch when the perturbations of interest are exiting the horizon. It will take the form given in eq.(2.6). We will then obtain relations between various coefficient functions of this wave function by demanding that it is invariant under suitable time and spatial reparametrizations. Then we will change the gauge and go to gauge B, and recast these relations now between correlation functions of ζ, which are conserved outside the horizon.

One more comment is in order before we proceed. Although the traceless component of the metric perturbation, $\widehat{\gamma}_{ij}$, eq.(2.4), will not play much of a role in the following discussion, we have in mind carrying out a spatial reparametrization eq.(2.11) so that at late time $\widehat{\gamma}_{ij}$ satisfies the condition,

$$\partial_i\widehat{\gamma}^{ij} = 0. \tag{2.56}$$

Indeed, only after this gauge fixing is $\mathcal{R}$ given by eq.(2.53).

The Ward identities

Setting $\zeta = 0$, eq.(2.38), to derive the Ward identity of special conformal transformations, we again choose the spatial and time reparametrizations, eq.(2.39), eq.(2.40), and demand that the wave function is invariant under them. The only new change is that since we are also keeping effects of order $\frac{\dot{\phi}}{H}$ now, the change in the scalar perturbation $\delta\phi$ has an extra term compared to eq.(2.44).

This extra term arises as follows. One wants the full inflaton field, eq.(2.5), to transform like a scalar under the coordinate transformation eq.(2.39), (2.40). That is, denoting a generic coordinate transformation as

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{2.57}$$

(where $\mu = 0, 1, 2, 3$), ϕ should transform as

$$\phi \to \phi - \epsilon^\mu \partial_\mu \phi. \tag{2.58}$$

It is easy to see that this gives rise to an extra term in the transformation for $\delta\phi$, so that, to this order

$$\delta\phi \to \delta\phi + \delta(\delta\phi) + \tilde{\delta}(\delta\phi), \tag{2.59}$$

where $\delta(\delta\phi)$ is the same as in eq.(2.44) and $\tilde{\delta}(\delta\phi)$, the extra contribution, is given by

$$\tilde{\delta}(\delta\phi(\boldsymbol{x})) = -2(\boldsymbol{b}\cdot\boldsymbol{x})\frac{\dot{\bar{\phi}}}{H}. \tag{2.60}$$

Now demanding that the wave function is invariant under the full change of $\delta\phi$ gives rise to a modified Ward identity, which takes the form

$$\begin{aligned} &\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_1}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_2}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_3}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' \\ &\qquad = 2\,\frac{\dot{\bar{\phi}}}{H}\left[\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_4}}\right]\left\{\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle'\Big|_{\boldsymbol{k_4}\to 0}\right\}, \end{aligned} \tag{2.61}$$

where $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is the same as defined in eq.(2.42).

Similarly, for the scaling transformation, eq.(2.48), we get the Ward identity

$$\left(\sum_{a=1}^{3}\boldsymbol{k_a}\cdot\frac{\partial}{\partial \boldsymbol{k_a}}\right)\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle = \frac{\dot{\bar{\phi}}}{H}\,\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle\Big|_{\boldsymbol{k_4}\to 0}. \tag{2.62}$$

Eq.(2.62) and especially eq.(2.61) are some of the main results of this chapter.

So far our discussion was in terms of the coefficient functions which appear in the wave function. It is useful to express the results in terms of correlation functions of perturbations. The expectation values of correlators involving $\delta\phi$ can be obtained from the wave function in the standard fashion. For example, the two point function is

$$\langle \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\rangle = \frac{\int [\mathcal{D}\delta\phi][\mathcal{D}\gamma_{ij}]\ |\Psi|^2\ \delta\phi(\boldsymbol{x})\,\delta\phi(\boldsymbol{y})}{\int [\mathcal{D}\delta\phi][\mathcal{D}\gamma_{ij}]\ |\Psi|^2}. \tag{2.63}$$

From eq.(2.6) we see that in momentum space this gives,

$$\langle\delta\phi(\boldsymbol{k})\delta\phi(\boldsymbol{k'})\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k'})\,\frac{1}{2}\frac{H^2}{M_{Pl}^2}\frac{1}{\langle O(\boldsymbol{k})O(\boldsymbol{k'})\rangle'} \tag{2.64}$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k'})\,\frac{H^2}{M_{Pl}^2}\frac{1}{2k^3}\,, \tag{2.65}$$

where we have used eq.(2.7).

Although it will not be very relevant for the present discussion, let us note that the RHS of eq.(2.63) is slightly imprecise. To make the sum over metrics well defined, the remaining gauge redundancy must also be removed. This is a general feature when calculating expectation values, [27]. While we are not being very explicit about this, we always have in mind fixing this redundancy by also taking $\hat{\gamma}_{ij}$ to be transverse, eq.(2.56). Note that ζ is already set to vanish in the gauge we are working with so far, eq.(2.38).

Once the correlation functions for $\delta\phi$ have been obtained, we can change gauge and go over to gauge B, eq.(2.54), as was discussed in subsection 2.3.2 above.

For the two point function, we see from eq.(2.65), eq.(2.38) and eq.(2.53) that the variable $\mathcal{R}$ has the two point function,

$$\langle\mathcal{R}(\boldsymbol{k})\mathcal{R}(\boldsymbol{k'})\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k'})\,\frac{H^2}{M_{Pl}^2}\frac{H^2}{\dot{\phi}^2}\frac{1}{2k^3}, \tag{2.66}$$

which is the standard result. In gauge B where eq.(2.54) is met,

$$\mathcal{R} = \zeta\,. \tag{2.67}$$

Thus, eq.(2.66) leads to,

$$\langle\zeta(\boldsymbol{k})\zeta(\boldsymbol{k'})\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k'})\,\frac{H^2}{M_{Pl}^2}\frac{H^2}{\dot{\phi}^2}\frac{1}{2k^3}. \tag{2.68}$$

For completeness, we also note that the graviton two-point function is given by

$$\langle\gamma_s(\boldsymbol{k_1})\gamma_{s'}(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\delta_{s,s'}\,\frac{H^2}{M_{Pl}^2}\frac{1}{k_1^3}, \tag{2.69}$$

where $\gamma_s = \frac{1}{2}\gamma_{ij}\,\epsilon_s^{ij}$.

At linear order the variable ζ in gauge B is related to $\delta\phi$ in gauge A by eq.(2.55). When we consider the three point function things get a little more complicated in going over to gauge B. Since the three point function is suppressed (due to the factor of $\dot{\phi}$ on the RHS of eq.(2.61)) the relation, eq.(2.55), is needed to second order. It turns out to be [8]

$$\zeta = -\frac{H}{\dot{\phi}}\,\delta\phi + \frac{1}{2}\frac{H}{\dot{\phi}}\left(\frac{\dot{H}}{H\dot{\phi}} - \frac{\ddot{\phi}}{\dot{\phi}^2}\right)\delta\phi^2. \tag{2.70}$$

It was shown in [24] that ζ in gauge B is in fact constant outside the horizon, and since we have gauge fixed completely, it is also a physical observable. This makes it a convenient variable to use. From eq.(2.70) and eq.(A.8) we get that [9]

$$\begin{aligned}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle =&\frac{1}{4}\frac{H^4}{M_{pl}^4}\frac{H^3}{\dot{\phi}^3}(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})\frac{1}{\prod_{a=1}^3 k_a^3}\\ &\left[-\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \left(\frac{\dot{H}}{H\dot{\phi}} - \frac{\ddot{\phi}}{\dot{\phi}^2}\right)\left(\sum_{a=1}^3 k_a^3\right)\right].\end{aligned} \tag{2.71}$$

Similarly, the four point function to leading order is given by

$$\begin{aligned}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle =&\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{CF}\\ &+\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{ET}\,.\end{aligned} \tag{2.72}$$

The two terms on the RHS of eq.(2.72) were calculated in [65] and [27], and are also given in eq.(A.19) and eq.(A.23) of appendix A.1.2. In particular, $\langle\zeta\zeta\zeta\zeta\rangle_{ET}$ is determined in terms of the $\langle OOT_{ij}\rangle$ correlator, and therefore completely fixed by conformal invariance, see [26].

By inverting eq.(2.71) and eq.(2.72), one can express $\langle OOO\rangle$ and $\langle OOOO\rangle$ in terms of the three point ζ correlator $\langle\zeta\zeta\zeta\rangle$, and $\langle\zeta\zeta\zeta\zeta\rangle_{CF}$ respectively, eq.(A.19). It turns out that the contribution of $\langle\zeta\zeta\zeta\zeta\rangle_{ET}$ to the RHS of the Ward identities vanishes. As a result, eq.(2.61) and (2.62) then become

$$\begin{aligned}&\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_1}}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' + \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_2}}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' + \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_3}}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle'\\ &\qquad= -4\frac{M_{Pl}^2}{H^2}\frac{\dot{\phi}^2}{H^2}\left[\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_4}}\right]\left\{k_4^3\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle'\Big|_{\boldsymbol{k_4}\to 0}\right\},\end{aligned} \tag{2.73}$$

and

$$\left[6 + \sum_{a=1}^3 \boldsymbol{k_a}\cdot\frac{\partial}{\partial\boldsymbol{k_a}}\right]\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = -2\frac{M_{Pl}^2}{H^2}\frac{\dot{\phi}^2}{H^2}k_4^3\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle'\Big|_{\boldsymbol{k_4}\to 0}, \tag{2.74}$$

with [10]

$$\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}} = \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}} + 6\left[\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right], \tag{2.75}$$

[8] It follows from inverting eq.(D.8) in [27] to obtain ζ in terms of $\delta\phi$.

[9] Note that the second term on the RHS of eq.(2.71) is of the same order as the first term, $\langle OOO\rangle'$. For instance, $\frac{\ddot{\phi}}{\dot{\phi}^2} = \left(\frac{\ddot{\phi}}{H\dot{\phi}}\right)\left(\frac{H}{\dot{\phi}}\right) = \frac{\delta}{\sqrt{2\epsilon_1}} \approx \sqrt{\epsilon_1}$.

[10] We remind the reader that a prime symbol on a correlator denotes that the momentum conserving delta function has been removed, see eq.(2.43).

and $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ as given in eq.(2.42).

In this way, we see that the Ward identities eq.(2.61) and eq.(2.62) derived above impose conditions on the physically observable three and four point correlators. Some of these Ward identities have been discussed in the literature before, e.g., setting $\boldsymbol{b} \propto \boldsymbol{k_4}$ in eq.(2.73) gives eq.(37) in [103].

2.4 Comments on the Ward identities

Let us comment on the Ward identities obtained above in more detail.

2.4.1 The canonical slow roll model as a check

The Ward identities obtained above can be checked in the canonical slow roll model, eq.(2.20), and shown to hold. For the slow roll model eq.(2.20), the three point function was obtained in [24]. The corresponding cubic coefficient function can be easily calculated, as discussed in appendix A.1.1, and is given by

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' = -\frac{3\epsilon_1 + 4\delta}{2\sqrt{2\epsilon_1}}\sum_a k_a^3 - \frac{1}{2}\sqrt{2\epsilon_1}\left(\frac{1}{2}\sum_{a\neq b} k_a k_b^2 + \frac{4}{k_t}\sum_{a>b} k_a^2 k_b^2\right), \tag{2.76}$$

where $k_a \equiv |\boldsymbol{k_a}|$, and $k_t = k_1 + k_2 + k_3$.

The four point function in this model was discussed in [65] and also in [27]. The corresponding coefficient function is given in eq.(4.33) of [27] (see appendix A.1.2 of this paper).

To check the Ward identity for scale invariance eq.(2.62), we note that since $\langle OOO\rangle'$ in eq.(2.76) is cubic in momenta, the LHS of eq.(2.62) vanishes. From eq.(6.21) and (6.22) of [27], it is easy to check that the RHS of eq.(2.62) also vanishes when $\boldsymbol{k_4} \to 0$. Thus the Ward identity eq.(2.62) holds.

The check for the Ward identity of special conformal transformations, eq.(2.61), is more complicated because the four point coefficient function $\langle OOOO\rangle$ is an unwieldy large expression. Nevertheless, using Mathematica one can check that it is indeed valid. It is easy to see that the function k^3 satisfies the condition,

$$\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}\,(k^3) = 0, \tag{2.77}$$

where the operator $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is defined in eq.(2.42). The non-trivial contribution for the LHS of the Ward identity eq.(2.61) comes therefore from the second term in eq.(2.76). The $\langle OOOO\rangle$ coefficient function has two kinds of contributions, denoted by $\widehat{W}^S$ and $\widehat{R}^S$ (see eq.(A.11)). Of these, only the $\widehat{R}^S$ term contributes.

2.4.2 Constraint on the magnitude of the three point function

We see from eq.(2.76) that the cubic coefficient function $\langle OOO\rangle$ vanishes in the canonical slow roll model in the limit when the slow roll parameters vanish. This is well known and is responsible for the small magnitude of the non-Gaussianity in this model. One can argue more generally that the cubic coefficient $\langle OOO\rangle$ must vanish in the limit when all the slow roll parameters vanish. In the gravity calculation, this happens because in this limit $\delta\phi$ becomes a massless scalar field in de Sitter space with no potential, and therefore does not have a three point function. From the point of view of conformal invariance and the related CFT, in this limit the corresponding operator O is exactly marginal, and in a CFT it is well known that the three point function of an exactly marginal operator vanishes. This is analogous to what happens in 2 dimensional CFT, see for example section (15.8) of [128]. If this three point function would not vanish then $\langle O\rangle$ for example would have a log divergence at second order in perturbation theory, leading to a non-zero beta function for O. Thus, on general grounds, we know that the expectation value for the scalar three point function should be suppressed.

The Ward identity, eq.(2.61), allows us to estimate the magnitude of the three point function once non-vanishing values for the slow roll parameters are taken into account. Since the quartic coefficient function $\langle OOOO\rangle$ is not expected to vanish in the de Sitter limit, we see from eq.(2.61) that the RHS is of order $\frac{\dot{\phi}}{H}$. From this, it follows quite naturally that the $\langle OOO\rangle$ coefficient function will be of order $\frac{\dot{\phi}}{H}$. So we see that as long as conformal invariance is an approximate symmetry, the three point scalar correlator will be of order its value in the canonical slow roll model, eq.(2.71), and therefore be small. Although the functional form is not the same as in the standard f_{NL} parametrization, to get a rough idea, this magnitude corresponds to an $f_{NL} \sim O\big((\frac{\dot{\phi}}{H})^2\big)$. If observationally a scalar non-Gaussianity is observed in the near future, its magnitude would most likely be much bigger. Thus the considerations of this chapter show that such an observation would not only rule out the canonical slow roll model, but more generally any model which preserves approximate conformal invariance during inflation. Note that in our conventions, the scalar and tensor two point correlators are given in eq.(2.66) and eq.(2.69).

There is one important caveat to the above statement. As will be discussed in the next subsection, the Ward identity eq.(2.61) does not uniquely determine the coefficient function $\langle OOO\rangle$ and thus the scalar three point function $\langle\zeta\zeta\zeta\rangle$, in terms of $\langle OOOO\rangle$. The remaining freedom corresponds to the three point function of a dimension 3 primary scalar operator in a CFT, S_h, with an arbitrary overall normalization. However, as we argue there, with generic assumptions, in the slow roll approximation this normalization is expected to be small, making any such contribution to $\langle OOO\rangle$ even more suppressed than that which originates from the $\langle OOOO\rangle$ source term. In case these generic assumptions are somehow not met, and the normalization is bigger making S_h dominate, the functional form of the three point function will be fixed (upto a contact term) and this possibility can therefore also be checked observationally.

2.4.3 Solving the Ward identities to determine the three point function

In this subsection, we investigate the question of uniqueness: given a four point coefficient function $\langle OOOO\rangle$, to what extent do the Ward identities, eq.(2.61) and eq.(2.62), fix the three point coefficient function, $\langle OOO\rangle$. We find, not surprisingly, that there is very little freedom that remains. It corresponds to adding to the three point coefficient function a term whose form is the same as the three point function of a dimension 3 operator in a CFT, S_h. The momentum dependence of this additional function is completely fixed, and all that is left undetermined is its overall normalization.[11] Besides this normalization our conclusion is therefore that the three point function is completely fixed in terms of the four point function. This is an interesting result because unlike the three point function, the four point function, $\langle OOOO\rangle$, does not vanish in the conformally invariant case. By relating the two, we learn that the freedom allowed by the approximate conformal symmetry for the three point function is about the same as that in the four point function. Towards the end of this section we argue that the normalization constant for the additional term S_h should be suppressed generically in the slow roll approximation, so that even this remaining ambiguity is not important.

The Ward identities are in the form of linear differential equations for $\langle OOO\rangle'$, with $\langle OOOO\rangle'$ appearing on the RHS as a source or inhomogeneous term. Suppose there are two solutions for $\langle OOO\rangle'$ allowed by eq.(2.61), eq.(2.62). Let us denote their difference as

$$\langle OOO\rangle_1' - \langle OOO\rangle_2' = S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}). \tag{2.78}$$

It is clear that S_h solves the homogeneous equations,

$$\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\cdot\frac{\partial}{\partial\boldsymbol{k_a}}\Big)\,S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}) = 3\,S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}) \tag{2.79}$$

and

$$\big(\sum_{a=1}^{3}\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\big)\,S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}) = 0. \tag{2.80}$$

The RHS in eq.(2.79) arises because the delta function has been removed in defining $\langle OOO\rangle'$. By comparing with eq.(2.41) and eq.(2.52), we see that these are exactly the equations satisfied by the three point function of a dimension 3 operator in the CFT.

It is well known that the three point function of a dimension 3 primary in a CFT is fixed in position space upto overall normalization. We find a similar result on analyzing the two equations eq.(2.79) and eq.(2.80) in momentum space. Upto an additional constant, which

[11]There is also an additional constant associated with a contact term, see below.

affects only contact terms in position space, the only freedom in S_h allowed is the overall normalization. Details of this analysis are given in the appendix A.2.

Since $\langle OOO\rangle$ conserves overall momentum, it is easy to see that S_h can be taken to be a function of only the three scalars, k_a, $a = 1, \cdots 3$. Our analysis in appendix A.2 then gives,

$$S_h(k_1, k_2, k_3) = N\,\frac{1}{3}\Big[\ln(\lambda)\Big(\sum_{a=1}^{3} k_a^3\Big) + \ln(\sum_{a=1}^{3} k_a)\Big(\sum_{b=1}^{3} k_b^3\Big) - \sum_{a\neq b} k_a k_b^2 + k_1 k_2 k_3\Big], \quad (2.81)$$

where λ is a short distance cut-off which is introduced in obtaining the solution. As discussed in appendix A.2, in obtaining this final form for the solution we have also imposed conditions which arise from the operator product expansion. N is the overall undetermined normalization, and $\ln(\lambda)$ is the extra coefficient which multiplies the contact term $(\sum_a k_a^3)$. It is easy to see that $(\sum_a k_a^3)$ is a contact term because each component of $(\sum_a k_a^3)$ is independent and therefore analytic in at least one of the momenta.

We now give an argument for why N is likely to be suppressed in the slow roll limit, so that the contribution to $\langle OOO\rangle'$ which arises from S_h is sub-dominant compared to a solution of Ward identities with the $\langle OOOO\rangle$ source turned on, eq.(2.61), eq.(2.62).

To understand this point let us return to the canonical slow roll model. In this model, to leading order, no term of the form eq.(2.81) is present. One quick way to see this is to notice that in eq.(2.76) there is no term of the form $(\sum_a k_a^3)\ln(\sum_b k_b)$. At subleading order such a term does arise in this model, but it is suppressed with a coefficient of order $\epsilon_1^{3/2}$, as opposed to the leading terms in eq.(2.76), which are $O(\sqrt{\epsilon_1})$. Having understood this better in the canonical model below, we will then argue that it should be true more generally as well, leading to the suppression of the S_h contribution mentioned above.

In the canonical model, a term giving rise to a contribution of the form eq.(2.81) would arise from a contribution to the Lagrangian of the form

$$\int d^3x\, a^3\,(V'''\delta\phi^3). \quad (2.82)$$

Comparing with eq.(3.8) in [24], we see that such a contribution is in fact present (in the second line). However, it is not included in the final result for the three point function because it is suppressed. To keep the discussion simple we assume that eq.(2.31) is valid, and therefore that in the slow roll approximation every additional time derivative is suppressed with one factor of ϵ_1, as was discussed in section 3.2. It is then straightforward to see that, barring accidental cancellations, this requires every additional derivative of the potential to be suppressed by a factor of $\sqrt{\epsilon_1}$.

For example, from eq.(2.25) we see that

$$\frac{V'}{V} \sim \sqrt{\epsilon_1},\ \frac{V''}{V} \sim \epsilon_1, \quad (2.83)$$

so that

$$\frac{V''}{V'} \sim \sqrt{\epsilon_1}. \tag{2.84}$$

Similarly, since eq.(2.24) is valid, we have on taking two time derivatives

$$\partial_t^3 \bar{\phi} \sim \frac{V'''}{H} \dot{\bar{\phi}}^2. \tag{2.85}$$

Now

$$\partial_t^3 \bar{\phi} \sim \epsilon_1^2 H^2 \dot{\bar{\phi}}, \tag{2.86}$$

since the LHS has two additional time derivatives. This gives, on using eq.(2.28),

$$\frac{V'''}{H^2} \sim \epsilon_1^{3/2}. \tag{2.87}$$

So we see that V''' (in units of H^2) is smaller than the terms of order $\frac{\dot{\bar{\phi}}}{H} \sim \sqrt{\epsilon_1}$, retained in eq.(2.76).

In section 2.2.1 towards the end, we argued that quite generically eq.(2.34) and eq.(2.24) are expected to be valid for a general action of the form eq.(2.30) in the slow roll approximation. It then follows, as was mentioned there, that every additional time derivative will be suppressed by one additional power of ϵ_1, so that the argument above will go through, leading to eq.(2.87).

Let us end with some comments. First, if somehow due to say accidental cancellations, the normalization constant N is bigger than $O(\frac{\dot{\bar{\phi}}}{H})$, the three point function would be bigger in magnitude, making it more experimentally accessible. However, in this case if approximate conformal invariance is preserved, the functional form for $\langle OOO \rangle'$ must be as given by S_h, eq.(2.81), and is completely fixed, so this possibility can also be tested observationally. Second, by using the generalized Fourier transform discussed in appendix A.2, we can write down a formal solution for the three point function in terms of the four point function. For completeness, we present this result in appendix A.3. Finally, conformal perturbation theory is a standard way to study the consequences of small departures from conformal invariance. In this, one perturbs a conformally invariant theory by turning on a coupling constant that breaks conformal invariance, and then calculates correlators perturbatively in this coupling constant. Our approach above is different, and attempts to solve the Ward identities of scale and special conformal invariance after incorporating the effects of the breaking of these symmetries. This approach, which is akin to trying to solve the Callan-Symanzik equation for a small value of the beta function, can be more powerful in principle, although an explicit solution of the resulting Ward identities has not proved so easy in practice, as we see from appendix A.3.

2.5 Discussion

In this chapter we have studied the constraints imposed by approximate conformal invariance on the scalar three point function. This correlation function is of the greatest interest experimentally as a test of non-Gaussianity, and it is therefore important to understand how well it can be constrained in a model independent manner from symmetry considerations alone. In particular, we derived the Ward identities of scale and special conformal invariance and showed that these relate the three point function to the four point function in a particular limit, once the breaking of conformal invariance due to the non-zero values of slow roll parameters is taken into account.

We then investigated these Ward identities and found that they considerably constrain the three point function. We argued that as long as the dynamics is approximately conformally invariant, and the slow roll approximation is valid, the magnitude of the three point function should be suppressed, being of the same order as that found in the canonical slow roll model of inflation, eq.(2.20). Roughly, although the detailed functional form is different, this corresponds to $f_{NL} \sim O((\frac{\dot{\phi}}{H})^2)$. If an experimental discovery of non-Gaussianity is made in the near future it would almost certainly require a much bigger value for the three point correlator. Our analysis therefore says that such a discovery would not only rule out the canonical slow roll model of inflation, but in fact any model where conformal invariance is approximately valid, and the slow roll approximation is valid.

We also found that the Ward identities determine the three point function in terms of the four point function nearly completely. An additional function, S_h, is allowed, but its functional form is completely fixed, and corresponds to the three point function of a dimension 3 scalar primary operator in a CFT, only leaving the overall normalization and a coefficient of a contact term undetermined. We argued that generically the overall normalization should be suppressed in the slow roll approximation. If somehow this generic argument fails and the normalization is bigger leading to S_h dominating the three point function, the functional form of the three point function would still be completely fixed, allowing for an experimental test of this possibility as well.

Unlike the three point function, the four point function does not vanish in the leading slow roll approximation, and is conformally invariant. By relating the three point function to the four point function we therefore relate the three point function also to a conformally invariant correlator. Unfortunately, as is well known, the functional form of the four point function is not constrained very significantly by conformal invariance alone; as a result of the Ward identities this is also then true for the three point function. In the canonical slow roll model the four point function arises due to single graviton exchange. If the three point function is observed and found to deviate from its functional form in the canonical slow roll model, the four point function must also be different, suggesting perhaps that higher spin fields might be involved during inflation.

More generally, it would be worth extending the analysis in this chapter to include the breaking of conformal invariance to higher order in the slow roll expansion. The three point function, to leading non-vanishing order, only requires corrections of order $\frac{\dot{\phi}}{H}$ to be included, and these can be obtained without changing the background geometry, since corrections to the metric are of order the slow roll parameters, ϵ_1 and δ, eq.(2.18), and we have argued that these should be much smaller. But going beyond this order would require corrections in the de Sitter geometry also to be incorporated. This is an interesting question to pursue, both from the point of view of cosmology and also holography in approximately AdS spaces. Once the asymptotic behavior of the fields has been determined, the Ward identities should follow from the invariance of the wave function under time and spatial reparametrizations. We discuss this generalization to higher orders in slow roll in the next chapter.

Chapter 3

Symmetry constraints in inflation: higher orders in slow roll

3.1 Introduction

In this chapter, we explore the constraints imposed by the $O(1,4)$ symmetries on the perturbations produced during inflation to higher orders in slow roll. More specifically, we derive Ward identities arising due to the scale and special conformal transformations for the correlation functions of these perturbations. The Ward identities incorporate breaking of the $O(1,4)$ symmetries as well, and are valid to all orders in the slow roll expansion.

The analysis carried out is based on symmetries alone, and is independent of specific models. As a result, the Ward identities obtained can provide robust model independent checks of the central idea behind a large class of inflationary models, namely, that the inflationary dynamics (including the scalar sector) preserves approximate conformal invariance. These results should apply not only to slow roll models with different shapes of the inflationary potential, but also in situations where higher derivative corrections can become important, such as in string theory scenarios, with the Hubble scale during inflation being of order the string scale, which in turn is much smaller than the Planck scale.

The $O(1,4)$ symmetry algebra is also the symmetry algebra of a three dimensional Euclidean Conformal Field Theory (CFT), which is the motivation behind our calling it the conformal group. However, we should mention at the outset that we do not assume a dS/CFT type of correspondence in deriving the results. Rather the connection with a conformal field theory (with the breaking of conformal invariance also included) is only for the purpose of organizing the discussion of the symmetries.

This chapter is organized as follows. Section 3.2 contains the basic set-up. The central ideas and key results behind the derivation of the Ward identities are then discussed in section 3.3. For our analysis, it is useful to work with the late time wave function of the universe, when the modes of interest have exited the horizon. Constraints imposed by symmetries on the coefficient functions determining the wave function are discussed in section 3.4. The late time behaviour of the modes in the canonical slow roll model of inflation is discussed in subsection 3.6.1, and some aspects which arise when higher derivative corrections are incorporated are discussed in subsection 3.6.2. We end with a discussion in section 3.7. The three appendices B.1, B.2 and B.3 contain important supplementary material.

This chapter is largely based on [2]. The analysis we carry out is based on the seminal works [24] and [25]. It also develops ideas earlier reported in [26], [27] and [1]. There are many other references also of relevance. The use of conformal symmetry to constrain inflationary correlation functions has also been discussed in [84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 129, 130]. Approaches where the conformal symmetries are often thought of as being non-linearly realized include [96, 97, 98, 99, 100, 101, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 118, 131]. The idea of using time and spatial reparametrizations to derive Ward identities in the context of AdS was first discussed in [126].

Notation: Before proceeding, let us clarify the notation we will follow in this chapter. A dot above a quantity represents a time derivative, e.g. $\dot{\phi} \equiv d\phi/dt$. Spatial three vectors are written in boldface, e.g. $\boldsymbol{x}, \boldsymbol{k}$, etc. Also, k_a, k_b, etc. represent the magnitudes of the vectors $\boldsymbol{k_a}, \boldsymbol{k_b}$, whereas k_i, k_j, etc. represent the i^{th}, j^{th} components of $\boldsymbol{k}$. Unless otherwise stated, the spatial indices i, j, etc. will be raised and lowered using the Kronecker delta, δ_{ij}.

3.2 Basic set-up

In this section, we will outline the essential ideas behind the derivation of the Ward identities. Our discussion will be general and not tied to any specific model. In sections 3.6.1 and 3.6.2, we will discuss the concrete cases of the canonical model of slow roll inflation, and the presence of higher derivatives, respectively.

The dynamical degrees of freedom in the theories we consider will be the metric and a single scalar field.[1] We work with the ADM form of the metric,

$$ds^2 = -N^2 dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.1}$$

with N and N^i being the lapse and shift functions respectively. We choose the gauge

$$N = 1, N^i = 0. \tag{3.2}$$

This gauge is called the *synchronous* gauge.

[1] The discussion can be extended to include additional scalars. However, model independent observational predictions are not easy to make in such models.

The unperturbed background FRW solution is

$$ds^2 = -\,dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j. \tag{3.3}$$

The Hubble parameter is given by

$$H = \frac{\dot{a}}{a}. \tag{3.4}$$

Including the metric perturbations, denoted by γ_{ij}, gives

$$h_{ij} = a^2(t)\,[\delta_{ij} + \gamma_{ij}]. \tag{3.5}$$

Similarly, expanding the inflaton about the background value $\bar{\phi}(t)$ gives

$$\phi = \bar{\phi}(t) + \delta\phi. \tag{3.6}$$

The gauge choice, eq.(3.2), does not fix all the coordinate reparametrization invariance. There are two kinds of residual gauge transformations which can be carried out. These are spatial reparametrizations,

$$x^i \to x^i + v^i(\boldsymbol{x}), \tag{3.7}$$

under which

$$h_{ij} \to h_{ij} + \nabla_i v_j + \nabla_j v_i, \tag{3.8}$$

or equivalently

$$\gamma_{ij} \to \gamma_{ij} + \frac{1}{a^2(t)}\big(\nabla_i v_j + \nabla_j v_i\big). \tag{3.9}$$

We can also perform time reparametrizations

$$t \to t + \epsilon(\boldsymbol{x}), \tag{3.10}$$

along with accompanying spatial reparametrizations of the form

$$x^i \to x^i + w^i(t, \boldsymbol{x}) \tag{3.11}$$

with

$$w^i(t, \boldsymbol{x}) = \partial_i \epsilon(\boldsymbol{x}) \int^t dt'\,\frac{1}{a^2(t')}, \tag{3.12}$$

under which

$$\delta\phi \to \delta\phi + \dot{\bar{\phi}}(t)\,\epsilon(\boldsymbol{x}), \tag{3.13}$$

$$\gamma_{ij} \to \gamma_{ij} + 2\,\delta_{ij}\left(\frac{\dot{a}}{a}\right)\epsilon(\boldsymbol{x}) + \big(\partial_i w_j + \partial_j w_i\big). \tag{3.14}$$

Using the homogeneity of the background FRW solution, we can expand the perturbations in a basis of modes carrying fixed comoving momenta. Let ξ be a generic perturbation.

Then

$$\xi(t, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \xi(t, \boldsymbol{k}), \tag{3.15}$$

where the comoving momentum is $\boldsymbol{k}$. We will be interested in the behaviour of the perturbations at late times, when the modes of interest have left the horizon,

$$k^2/a^2 \ll H^2. \tag{3.16}$$

Using the time reparametrization symmetry at late times, we can set

$$\delta\phi = 0. \tag{3.17}$$

In this gauge, the perturbations freeze out once they exit the horizon, i.e., they become time independent, since their subsequent evolution becomes dominated by a frictional term proportional to the Hubble parameter. The remaining gauge invariance now corresponds to spatial reparametrizations, eq.(3.7). The choice of gauge eq.(3.17), and the freeze out of modes will be discussed in greater detail for the canonical slow roll model in section 3.6.1, and in the presence of higher derivative terms in section 3.6.2.

In the gauge eq.(3.17), all the remaining perturbations arise from the metric. We can decompose them as

$$\delta_{ij} + \gamma_{ij} = e^{2\zeta}[\delta_{ij} + \widehat{\gamma}_{ij}], \tag{3.18}$$

where $\widehat{\gamma}_{ij}$ is the traceless component. ζ determines the perturbations in the trace of the metric. To linear order in perturbations, we see from eq.(3.18) that

$$\gamma_{ij} = 2\zeta \, \delta_{ij} + \widehat{\gamma}_{ij}. \tag{3.19}$$

Going beyond the linear order, we will find that the definition given in eq.(3.18) leads to a simplification in our discussion of symmetries. To be more specific, it will turn out that the coefficient functions for the trace of the stress tensor will transform in a canonical way with this choice of variables.

It will be useful to carry out our symmetry based analysis in terms of the wave function of the universe. This wave function is actually a *functional* of the perturbations. Expanding at late times, when the perturbations become time independent, we get

$$\begin{aligned}\Psi[\gamma_{ij}] = \exp\Big[&- \frac{1}{2}\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle \\ &- \int d^3x\, d^3y\, \zeta(\boldsymbol{x})\, \widehat{\gamma}_{ij}(\boldsymbol{y})\, \langle T(\boldsymbol{x})\widehat{T}^{ij}(\boldsymbol{y})\rangle \\ &- \frac{1}{2}\int d^3x\, d^3y\, \widehat{\gamma}_{ij}(\boldsymbol{x})\widehat{\gamma}_{kl}(\boldsymbol{y})\, \langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle \\ &- \frac{1}{3!}\int d^3x\, d^3y\, d^3z\, \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\zeta(\boldsymbol{z})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})T(\boldsymbol{z})\rangle \\ &- \frac{1}{2}\int d^3x\, d^3y\, d^3z\, \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\widehat{\gamma}_{ij}(\boldsymbol{z})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\widehat{T}^{ij}(\boldsymbol{z})\rangle \\ &- \frac{1}{2}\int d^3x\, d^3y\, d^3z\, \zeta(\boldsymbol{x})\widehat{\gamma}_{ij}(\boldsymbol{y})\widehat{\gamma}_{kl}(\boldsymbol{z})\, \langle T(\boldsymbol{x})\widehat{T}^{ij}(\boldsymbol{y})\widehat{T}^{kl}(\boldsymbol{z})\rangle \\ &- \frac{1}{3!}\int d^3x\, d^3y\, d^3z\, \widehat{\gamma}_{ij}(\boldsymbol{x})\widehat{\gamma}_{kl}(\boldsymbol{y})\widehat{\gamma}_{mn}(\boldsymbol{z})\, \langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\widehat{T}^{mn}(\boldsymbol{z})\rangle \\ &+ \cdots\cdots \\ &- \frac{1}{m!\,n!}\int d^3x_1\cdots d^3x_{m+n}\, \zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_m})\, \widehat{\gamma}_{i_1j_1}(\boldsymbol{x_{m+1}})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{x_{m+n}})\times \\ &\qquad \Big\langle T(\boldsymbol{x_1})\cdots T(\boldsymbol{x_m})\widehat{T}^{i_1j_1}(\boldsymbol{x_{m+1}})\cdots\widehat{T}^{i_nj_n}(\boldsymbol{x_{m+n}})\Big\rangle + \cdots \Big].\end{aligned} \tag{3.20}$$

The quadratic terms in ζ and $\widehat{\gamma}_{ij}$ correspond to a Gaussian wave function; higher order terms give rise to non-Gaussianity.

Invariance with respect to the residual gauge invariance, namely with respect to the spatial reparametrization eq.(3.7), imposes constraints on the coefficient functions $\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle$, $\langle \hat{T}^{ij}(\boldsymbol{x})\hat{T}^{kl}(\boldsymbol{y})\rangle$ etc, which appear in this expansion. In fact, these coefficient functions have been written in a suggestive manner because the constraints take the form of Ward identities which are satisfied by correlation functions of the stress-energy tensor in a conformal field theory. This will be discussed further in section 3.4. Note that in eq.(3.20) we have also included a mixed term between ζ and $\hat{\gamma}_{ij}$ for generality, although such a term will vanish on further gauge fixing the spatial reparametrization invariance suitably, as we will see later. Let us also mention that as per our conventions, eq(3.19), T is related to the trace of the stress tensor T_{ij} by

$$T = 2T_{ii} \equiv 2\mathcal{T}, \tag{3.21}$$

so that the coefficient function for a general metric perturbation, γ_{ij}, is T^{ij}. Also, $\widehat{T}_{ij}$ is the traceless part of the stress-energy tensor T_{ij}.

The invariance with respect to spatial reparametrizations eq.(3.7) arises as follows. The wave function as a functional of the late time value for a generic perturbation ξ can be written as a path integral

$$\Psi[\xi] = \int_{initial}^{\xi} [\mathcal{D}\xi]\, e^{iS}, \tag{3.22}$$

where the initial conditions will be taken to be the Bunch-Davies vacuum. The action S has a pre-factor $1/G \sim M_{Pl}^2$. By suitably rescaling fields in terms of the Hubble parameter

H, we see that

$$S = \frac{M_{Pl}^2}{H^2}\,\tilde{S}, \tag{3.23}$$

where $\tilde{S}$ contains the rescaled fields which have been made dimensionless by the rescaling. Since no gravity waves have been detected so far, we know that [2]

$$\frac{H^2}{M_{Pl}^2} \leq 10^{-8}. \tag{3.24}$$

Thus the path integral on the RHS of eq.(3.22) can be evaluated in the semi-classical limit, by solving the equations of motion subject to the boundary conditions at late and early times. In particular, in the gauge eq.(3.2), the N, N^i equations must also be imposed. These equations give rise to the invariance of the wave function under spatial reparametrizations, eq.(3.7), after fixing the gauge, eq.(3.17), at late times.

3.3 The Ward identities

We are now ready to discuss the derivation of the Ward identities. We will be interested in the Ward identities which arise due to scale and special conformal transformations. It is useful to first consider the case of de Sitter space, with the background metric

$$ds^2 = -\,dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j. \tag{3.25}$$

This metric is well known to have an $O(1,4)$ symmetry with ten generators. Besides the three spatial translations, and three rotations along the spatial directions, this symmetry group includes scale transformations,

$$x^i \to \lambda x^i,\, t \to t - \frac{1}{H}\log(\lambda), \tag{3.26}$$

and three special conformal transformations,

$$\begin{aligned} &x^i \to x^i - 2(b_j x^j)x^i + b^i\left(\sum_j (x^j)^2 - \frac{1}{H^2}e^{-2Ht}\right), \\ &t \to t + \frac{2b_j x^j}{H}. \end{aligned} \tag{3.27}$$

The scale and special conformal symmetries give rise to Ward identities on the correlation functions of the perturbations. In de Sitter space these identities are met exactly; in inflationary backgrounds, there are corrections that arise due to the evolving inflaton which breaks these symmetries. We will derive the resulting identities for the correlation functions

[2] We take $M_{Pl} = \frac{1}{\sqrt{8\pi G}} \approx 10^{18}$GeV.

to all orders in the slow roll parameters

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \tag{3.28}$$

$$\delta = \frac{\ddot{H}}{2H\dot{H}}, \tag{3.29}$$

and

$$\epsilon = \frac{1}{2}\frac{\dot{\bar{\phi}}^2}{H^2}. \tag{3.30}$$

The identities for the scale transformations are the analogues of the Callan-Symanzik equations in field theory, which incorporate the running of the coupling constants. Similarly, we get identities for the special conformal transformations also incorporating the evolving inflaton.

Before proceeding, let us note that in the canonical slow roll model, discussed in section 3.6.1, ϵ and ϵ_1 are related as

$$\epsilon = \epsilon_1. \tag{3.31}$$

But more generally, when higher derivatives are included, they will not be related in this way. Also, for the slow roll conditions to hold,

$$\epsilon_1, \delta, \epsilon \ll 1. \tag{3.32}$$

The expectation values for the perturbations are obtained from the wave function in the standard manner. For example, for scalar perturbations ζ these are given by

$$\langle\zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\rangle = \frac{1}{\mathcal{N}}\int[\mathcal{D}\zeta]\,[\mathcal{D}\widehat{\gamma}_{ij}]\,|\Psi|^2\,\zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n}), \tag{3.33}$$

where $\mathcal{N}$ denotes the overall normalization factor in the path integral,

$$\mathcal{N} = \int[\mathcal{D}\zeta]\,[\mathcal{D}\widehat{\gamma}_{ij}]\,|\Psi|^2. \tag{3.34}$$

We will be interested in calculating these expectation values at late times, when the perturbations of interest have frozen out.

There is one important point which we must consider before we proceed. The sum over all metric perturbations $\widehat{\gamma}_{ij}$ on the RHS of eq.(3.33) is ill defined because we have not yet fixed the spatial reparametrization invariance symmetry. The integral on the RHS would diverge without fixing this symmetry. A conventional choice, which we will also make, is to take $\widehat{\gamma}_{ij}$ to be transverse,

$$\partial_i\widehat{\gamma}_{ij} = 0, \tag{3.35}$$

besides also being traceless. With this further gauge fixing, the path integral on the RHS of eq.(3.33) becomes finite. Note that since $\widehat{\gamma}_{ij}$ freezes out at late times, the additional gauge fixing required for eq.(C.2.2) can be achieved by a spatial reparametrization $x^i \to x^i + \epsilon^i(\boldsymbol{x})$ which preserves the synchronous gauge eq.(3.2).

Also note that after the additional gauge fixing, eq.(C.2.2), the resulting perturbations manifestly correspond to a scalar ζ with spin 0, and a tensor perturbation $\widehat{\gamma}_{ij}$ which has spin 2, with respect to the rotations along the spatial directions.

It is worth commenting here that the underlying reason for this further gauge fixing is that we are working with local correlation functions in a theory of quantum gravity. These correlations are well defined perturbatively about the inflationary background, but only after gauge fixing, as discussed above.

3.3.1 Ward identities for scale transformations

Under a scale transformation

$$x^i \to x^i + \lambda x^i,\ \lambda \ll 1, \tag{3.36}$$

ζ and $\widehat{\gamma}_{ij}$ transform as

$$\zeta \to \zeta + \lambda + \lambda\, x^i \partial_i \zeta\,, \tag{3.37}$$

$$\widehat{\gamma}_{ij} \to \widehat{\gamma}_{ij} + \lambda\, x^k \partial_k \widehat{\gamma}_{ij} \tag{3.38}$$

(see appendix B.1). Note that the transversality condition, eq.(C.2.2), is preserved by this transformation.

We can now consider changing variables in the path integral on the RHS of eq.(3.33), with ζ and $\widehat{\gamma}_{ij}$ transforming as given in eq.(3.37) and eq.(3.38), respectively. The measure in the path integral on the RHS of eq.(3.33) is invariant under spatial reparametrizations, and therefore under the change in eq.(3.36).

Naively, on the basis of what has been discussed so far, one might also conclude that the wave function Ψ is invariant under this transformation, leading to the condition

$$\left\langle \delta(\zeta(\boldsymbol{x_1}))\cdots\zeta(\boldsymbol{x_n})\right\rangle + \cdots + \left\langle \zeta(\boldsymbol{x_1})\cdots\delta(\zeta(\boldsymbol{x_n}))\right\rangle = 0, \tag{3.39}$$

where from eq.(3.37),

$$\delta\zeta = \lambda + \lambda x^i \partial_i \zeta. \tag{3.40}$$

This is incorrect.

Under the transformation eq.(3.36), $\widehat{\gamma}_{ij}$ transforms homogeneously, but ζ has a homogeneous and an inhomogeneous term in its transformation. The relations between the coefficient functions we obtain, as discussed in section 3.4, will ensure that terms in the wave function which are quadratic or higher order in the perturbations cancel amongst each other under this transformation. However, there is one term which arises from the leading term that is quadratic in ζ in eq.(3.20), to begin with, which needs to be handled with care and does not cancel, as is also discussed at the end of section 3.4.2. The quadratic terms in the wave function include

$$\Psi \sim \exp\Big(-\frac{1}{2}\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle \Big). \tag{3.41}$$

After the transformation eq.(3.37), we get a piece arising from the inhomogeneous term in the transformation of ζ,

$$\zeta \to \zeta + \lambda + \cdots\,, \tag{3.42}$$

which will now be linear in ζ,

$$\delta\Psi \sim \exp\Big(-\lambda\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle \Big). \tag{3.43}$$

This term will remain uncanceled. In contrast, the homogeneous term in the transformation of ζ,

$$\zeta \to \zeta + \lambda x^i\partial_i\zeta + \cdots\,, \tag{3.44}$$

will give rise to a term which is quadratic in ζ; this will cancel against a term coming from the piece of Ψ cubic in ζ.

Before proceeding, let us note that in eq.(3.20) there is another quadratic term,

$$\Psi \sim \exp\Big(-\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\widehat{\gamma}_{ij}(\boldsymbol{y})\, \langle T(\boldsymbol{x})\widehat{T}^{ij}(\boldsymbol{y})\rangle \Big), \tag{3.45}$$

involving both ζ and $\widehat{\gamma}_{ij}$, which could also have potentially contributed an additional piece. However, in the gauge eq.(C.2.2), this term in the wave function vanishes. This follows after noting that symmetries require the momentum space coefficient function $\langle \widehat{T}^{ij}(\boldsymbol{k_1})T(\boldsymbol{k_2})\rangle$ to be of the form

$$\langle \widehat{T}^{ij}(\boldsymbol{k_1})T(\boldsymbol{k_2})\rangle \sim (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\left(\frac{1}{3}\,\delta_{ij} - \frac{k_{1i}\,k_{1j}}{k_1^2}\right)\beta(k_1), \tag{3.46}$$

where $\beta(k_1)$ is a dimension 3 function of k_1.

Keeping this uncanceled linear term, eq.(3.43), gives us then the correct Ward identity

$$\begin{aligned} &\big\langle \delta(\zeta(\boldsymbol{x_1}))\cdots\zeta(\boldsymbol{x_n})\big\rangle + \cdots + \big\langle \zeta(\boldsymbol{x_1})\cdots\delta(\zeta(\boldsymbol{x_n}))\big\rangle \\ &\qquad = 2\lambda\int d^3x\, d^3y\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\, \langle \zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\zeta(\boldsymbol{x})\rangle. \end{aligned} \tag{3.47}$$

We will be interested in the expectation values for ζ with non-zero momentum. Since λ is a constant, we can drop the piece linear in λ on the LHS of eq.(3.47), leading to the Ward identity

$$\left(\sum_{\mathbf{a}=1}^{n}\boldsymbol{x_a}\cdot\frac{\partial}{\partial \boldsymbol{x_a}}\right)\langle\zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\rangle = 2\int d^3x\, d^3y\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\, \langle \zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\zeta(\boldsymbol{x})\rangle. \tag{3.48}$$

Expressing this in momentum space gives [3]

$$\left(3(n-1)+\sum_{a=1}^{n} k_a \frac{\partial}{\partial k_a}\right)\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_n})\rangle' = \\ -\frac{1}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_{n+1}})\rangle'\Big|_{\boldsymbol{k_{n+1}}\to 0}. \tag{3.49}$$

Similarly, for correlation functions of tensor perturbations $\widehat{\gamma}_{ij}$ we get

$$\left(\sum_{\mathbf{a}=1}^{n} \boldsymbol{x_a}\cdot\frac{\partial}{\partial \boldsymbol{x_a}}\right)\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{x_1})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{x_n})\rangle = \\ 2\int d^3x\, d^3y\,\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\,\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{x_1})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{x_n})\zeta(\boldsymbol{x})\rangle, \tag{3.50}$$

which in momentum space takes the form

$$\left(3(n-1)+\sum_{a=1}^{n} k_a \frac{\partial}{\partial k_a}\right)\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{k_n})\rangle' = \\ -\frac{1}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{k_n})\zeta(\boldsymbol{k_{n+1}})\rangle'\Big|_{\boldsymbol{k_{n+1}}\to 0}. \tag{3.51}$$

Mixed identities involving both tensor and scalar perturbations can also be similarly obtained. These are given by

$$\left(3(n-1)+\sum_{a=1}^{n} k_a \frac{\partial}{\partial k_a}\right)\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\zeta(\boldsymbol{k_{m+1}})\cdots\zeta(\boldsymbol{k_n})\rangle' \\ = -\frac{1}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\zeta(\boldsymbol{k_{m+1}})\cdots\zeta(\boldsymbol{k_{n+1}})\rangle'\Big|_{\boldsymbol{k_{n+1}}\to 0}. \tag{3.52}$$

Equations (3.49) and (3.51) are examples of Maldacena consistency conditions in the literature [24]. These are exact to all orders in the slow roll expansion.

The physical picture behind these relations is easy to state. The LHS of eq.(3.48) is the change of the n-point correlator under an overall change of scale. Exactly such a transformation is generated by a scalar perturbation $\zeta(\boldsymbol{k_{n+1}})$ in the limit of very long wavelength, $k_{n+1}\to 0$, leading to the identity eq.(3.49).

Comments: The reader will note that the scale transformation eq.(3.36) is different from the isometry in de Sitter space, eq.(3.26). In de Sitter space, metric perturbations and also perturbations for test scalars freeze out at late times and become time independent. Thus, in effect, the scale transformation becomes eq.(3.36). In the inflationary case, once

[3] A prime $'$ symbol on a correlation function denotes the suppression of the overall momentum conserving delta function. For e.g.

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle'.$$

we choose the gauge where eq.(3.17) is met, we cannot make any time reparametrization, eq.(3.10). Thus the only symmetries available are spatial reparametrizations.

Similarly, the special conformal transformations, which we will consider next, eq.(3.53), are different from the corresponding isometries in de Sitter space, eq.(3.27). However, again at late times, their action on time independent fields will be the same as eq.(3.53).

It is also worth emphasizing that our derivation of the Ward identities for scale invariance obtained here is quite general. As mentioned above, it is valid to all orders in the slow roll expansion, and thus should hold even when the slow roll conditions are not valid. The assumptions one has used are that one can go to the gauge eq.(3.17), and that the remaining metric perturbations in this gauge then freeze out due to the cosmological expansion. The residual spatial reparametrizations are then enough to give rise to the Ward identities above. A similar comment will also apply to the Ward identities of special conformal invariance we derive next.

3.3.2 Ward identities for special conformal transformations

We next turn to the special conformal transformations,

$$x^i \to x^i + \alpha^i(\boldsymbol{x}),\ \alpha^i(\boldsymbol{x}) = -2(\boldsymbol{b}\cdot\boldsymbol{x})x^i + b^i\boldsymbol{x}^2. \tag{3.53}$$

Here there is an important extra subtlety. Consider the transformation of ζ and $\widehat{\gamma}_{ij}$ under eq.(3.53) (see appendix B.1),

$$\zeta(\boldsymbol{x}) \to \zeta(\boldsymbol{x}) - 2(\boldsymbol{b}\cdot\boldsymbol{x}) + \alpha^i\partial_i\zeta(\boldsymbol{x})\,, \tag{3.54}$$

$$\widehat{\gamma}_{ij}(\boldsymbol{x}) \to \widehat{\gamma}_{ij}(\boldsymbol{x}) + \alpha^m\partial_m\widehat{\gamma}_{ij}(\boldsymbol{x}) + 2\,\mathcal{M}^{\boldsymbol{b}}_{im}(\boldsymbol{x})\widehat{\gamma}_{jm}(\boldsymbol{x}) + 2\,\mathcal{M}^{\boldsymbol{b}}_{jm}(\boldsymbol{x})\widehat{\gamma}_{im}(\boldsymbol{x}), \tag{3.55}$$

where $\mathcal{M}^{\boldsymbol{b}}_{ij}(\boldsymbol{x})$ is given by

$$\mathcal{M}^{\boldsymbol{b}}_{ij}(\boldsymbol{x}) = x_i b_j - x_j b_i. \tag{3.56}$$

It is easy to see that the transformation eq.(3.55) does not preserve the transverse gauge condition eq.(C.2.2) we have chosen for $\widehat{\gamma}_{ij}$. We must therefore carry out a compensating coordinate transformation

$$\begin{aligned} x^i &\to x^i + v^i(\boldsymbol{x}),\\ v^i(\boldsymbol{x}) &= -\frac{6b^m\widehat{\gamma}_{im}(\boldsymbol{x})}{\partial^2}, \end{aligned} \tag{3.57}$$

which then restores the transversality condition on $\widehat{\gamma}_{ij}$. Under this compensating transformation, ζ and $\widehat{\gamma}_{ij}$ transform as

$$\zeta(\boldsymbol{x}) \to \zeta(\boldsymbol{x}) - \frac{6b^m\widehat{\gamma}_{km}(\boldsymbol{x})}{\partial^2}\,\partial_k\zeta(\boldsymbol{x}) - \frac{2b^m\partial_i\widehat{\gamma}_{jm}(\boldsymbol{x})}{\partial^2}\,\widehat{\gamma}_{ij}(\boldsymbol{x}), \tag{3.58}$$

and

$$
\begin{aligned}
\widehat{\gamma}_{ij}(\boldsymbol{x}) \to \widehat{\gamma}_{ij}(\boldsymbol{x}) - 6b^m &\left[\partial_i\left(\frac{\widehat{\gamma}_{jm}(\boldsymbol{x})}{\partial^2}\right) + \partial_j\left(\frac{\widehat{\gamma}_{im}(\boldsymbol{x})}{\partial^2}\right)\right] \\
&- 6b^m\left[\widehat{\gamma}_{ik}(\boldsymbol{x})\,\partial_j\left(\frac{\widehat{\gamma}_{km}(\boldsymbol{x})}{\partial^2}\right) + \widehat{\gamma}_{jk}(\boldsymbol{x})\,\partial_i\left(\frac{\widehat{\gamma}_{km}(\boldsymbol{x})}{\partial^2}\right)\right] \\
&- 6b^m\partial_k\widehat{\gamma}_{ij}(\boldsymbol{x})\left(\frac{\widehat{\gamma}_{km}(\boldsymbol{x})}{\partial^2}\right) \\
&+ 4b^m\,\widehat{\gamma}_{ab}(\boldsymbol{x})\,\partial_a\left(\frac{\widehat{\gamma}_{bm}(\boldsymbol{x})}{\partial^2}\right)\left(\delta_{ij} + \widehat{\gamma}_{ij}(\boldsymbol{x})\right).
\end{aligned}
\tag{3.59}
$$

The Ward identities then arise because of the combined transformations eq.(3.54) and eq.(3.58) for the transformation of ζ, and eq.(3.55) and eq.(3.59) for the transformation of $\widehat{\gamma}_{ij}$. Note that the compensating transformation parameter v^i itself depends on $\widehat{\gamma}_{ij}$. As a result, the compensating transformation becomes non-linear in the perturbations.

Once this subtlety requiring a compensating coordinate transformation is taken care of, the rest of the analysis follows along similar lines to that for the scale transformation case. The wave function Ψ is invariant under spatial reparametrizations, and therefore under the combined transformations eq.(3.53) and eq.(3.57). More correctly, this is true for all terms in the wave function which are quadratic or higher order in the perturbations. However, the inhomogeneous term in eq.(3.54), $-2(\boldsymbol{b}\cdot\boldsymbol{x})$, gives rise to a term in the change of the wave function which is linear in ζ. This term does not cancel. As a result we get a Ward identity for scalar perturbations of the form

$$
\begin{aligned}
4\int d^3x\, d^3y\, (\boldsymbol{b}\cdot\boldsymbol{x})\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\langle\zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\zeta(\boldsymbol{y})\rangle& \\
+ \langle\delta^C\zeta(\boldsymbol{x_1})\cdots\zeta(\boldsymbol{x_n})\rangle + \cdots + \langle\zeta(\boldsymbol{x_1})\cdots\delta^C\zeta(\boldsymbol{x_n})\rangle &= 0,
\end{aligned}
\tag{3.60}
$$

where $\delta^C\zeta$ denotes the complete homogeneous change in ζ under eq.(3.54) and eq.(3.58),

$$
\delta^C\zeta(\boldsymbol{x}) = \big(-2(\boldsymbol{b}\cdot\boldsymbol{x})x^k + b^k\boldsymbol{x}^2\big)\,\partial_k\zeta(\boldsymbol{x}) - \frac{6b^m\widehat{\gamma}_{km}(\boldsymbol{x})}{\partial^2}\,\partial_k\zeta(\boldsymbol{x}) - \frac{2b^m\partial_i\widehat{\gamma}_{jm}(\boldsymbol{x})}{\partial^2}\,\widehat{\gamma}_{ij}(\boldsymbol{x}). \tag{3.61}
$$

In momentum space this takes the form

$$
\begin{aligned}
\big\langle\delta(\zeta(\boldsymbol{k_1}))\cdots\zeta(\boldsymbol{k_n})\big\rangle + \cdots + \big\langle\zeta(\boldsymbol{k_1})\cdots\delta(\zeta(\boldsymbol{k_n}))\big\rangle &= \\
-2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_{n+1}}}\right)&\frac{\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_{n+1}})\rangle}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\bigg|_{\boldsymbol{k_{n+1}}\to 0},
\end{aligned}
\tag{3.62}
$$

where $\delta(\zeta(\boldsymbol{k}))$ is given by

$$
\begin{aligned}
\delta(\zeta(\boldsymbol{k})) = \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}}\,\zeta(\boldsymbol{k}) + 6\,b^m k^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\zeta(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})& \\
+ 2\,b^m k^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\widehat{\gamma}_{ij}(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,\widehat{\gamma}_{jm}(\tilde{\boldsymbol{k}}),&
\end{aligned}
\tag{3.63}
$$

and the operator $\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is given by

$$\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}} = 2\left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right) - (\boldsymbol{b}\cdot\boldsymbol{k})\left(\frac{\partial}{\partial\boldsymbol{k}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right) + 6\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right). \tag{3.64}$$

Similarly, for the tensor perturbations we get the Ward identity

$$\begin{aligned}&\left\langle\delta\big(\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\big)\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{k_n})\right\rangle + \cdots + \left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\delta\big(\widehat{\gamma}_{i_nj_n}(\boldsymbol{k_n})\big)\right\rangle\\ &\qquad = -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_{n+1}}}\right)\frac{\left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_nj_n}(\boldsymbol{k_n})\zeta(\boldsymbol{k_{n+1}})\right\rangle}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\Bigg|_{\boldsymbol{k_{n+1}}\to 0},\end{aligned} \tag{3.65}$$

where $\delta(\widehat{\gamma}_{ij})$ is the complete change in $\widehat{\gamma}_{ij}$ in momentum space, given by

$$\begin{aligned}\delta\big(\widehat{\gamma}_{ij}(\boldsymbol{k})\big) =& \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}}\,\widehat{\gamma}_{ij}(\boldsymbol{k}) + 2\,\tilde{\mathcal{M}}^{\boldsymbol{b}}_{im}(\boldsymbol{k})\,\widehat{\gamma}_{jm}(\boldsymbol{k}) + 2\,\tilde{\mathcal{M}}^{\boldsymbol{b}}_{jm}(\boldsymbol{k})\,\widehat{\gamma}_{im}(\boldsymbol{k})\\ &+ 6b^m\,\frac{1}{k^2}\left(k_i\,\widehat{\gamma}_{jm}(\boldsymbol{k}) + k_j\,\widehat{\gamma}_{im}(\boldsymbol{k})\right)\\ &+ 6b^m\int\frac{d^3k'}{(2\pi)^3}\,\frac{\widehat{\gamma}_{km}(\boldsymbol{k'})}{(k')^2}\left(k'_i\,\widehat{\gamma}_{jk}(\boldsymbol{k}-\boldsymbol{k'}) + k'_j\,\widehat{\gamma}_{ik}(\boldsymbol{k}-\boldsymbol{k'})\right)\\ &+ 6b^m\int\frac{d^3k'}{(2\pi)^3}\,\frac{k'_l}{|\boldsymbol{k}-\boldsymbol{k'}|^2}\,\widehat{\gamma}_{ij}(\boldsymbol{k'})\,\widehat{\gamma}_{lm}(\boldsymbol{k}-\boldsymbol{k'})\\ &- 4b^m\delta_{ij}\int\frac{d^3k'}{(2\pi)^3}\,\frac{k_a}{|\boldsymbol{k}-\boldsymbol{k'}|^2}\,\widehat{\gamma}_{ab}(\boldsymbol{k'})\,\widehat{\gamma}_{bm}(\boldsymbol{k}-\boldsymbol{k'})\\ &- 4b^m\int\frac{d^3k'}{(2\pi)^3}\frac{d^3k''}{(2\pi)^3}\,\frac{(k_a-k'_a)}{|\boldsymbol{k}-\boldsymbol{k'}-\boldsymbol{k''}|^2}\,\widehat{\gamma}_{ij}(\boldsymbol{k'})\,\widehat{\gamma}_{ab}(\boldsymbol{k''})\,\widehat{\gamma}_{bm}(\boldsymbol{k}-\boldsymbol{k'}-\boldsymbol{k''}),\end{aligned} \tag{3.66}$$

where

$$\tilde{\mathcal{M}}^{\boldsymbol{b}}_{ij}(\boldsymbol{k}) = b_j\,\frac{\partial}{\partial k^i} - b_i\,\frac{\partial}{\partial k^j}\,. \tag{3.67}$$

Finally, we can write the general Ward identity for the variation of a mixed correlator involving m tensor perturbations $\widehat{\gamma}_{ij}(\boldsymbol{k})$ and $(n-m)$ scalar perturbations $\zeta(\boldsymbol{k})$ as

$$\begin{aligned}&\left\langle\delta\big(\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\big)\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\,\zeta(\boldsymbol{k_{m+1}})\cdots\zeta(\boldsymbol{k_n})\right\rangle + \cdots\\ &+ \left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\delta\big(\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\big)\,\zeta(\boldsymbol{k_{m+1}})\cdots\zeta(\boldsymbol{k_n})\right\rangle\\ &+ \left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\,\delta(\zeta(\boldsymbol{k_{m+1}}))\cdots\zeta(\boldsymbol{k_n})\right\rangle + \cdots\\ &+ \left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\,\zeta(\boldsymbol{k_{m+1}})\cdots\delta(\zeta(\boldsymbol{k_n}))\right\rangle\\ &= -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_{n+1}}}\right)\frac{\left\langle\widehat{\gamma}_{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{\gamma}_{i_mj_m}(\boldsymbol{k_m})\zeta(\boldsymbol{k_{m+1}})\cdots\zeta(\boldsymbol{k_{n+1}})\right\rangle}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\Bigg|_{\boldsymbol{k_{n+1}}\to 0},\end{aligned} \tag{3.68}$$

where $\delta(\zeta(\boldsymbol{k}))$ is given by eq.(4.22) and $\delta(\widehat{\gamma}_{ij}(\boldsymbol{k}))$ is given by eq.(3.66).

These identities are again exact, like the ones for scale transformations derived in section 3.3.1. They are valid to all orders in the slow roll expansion, and are one of the key results of this chapter. Note that due to the non-linear nature of the transformations eq.(3.58) and eq.(3.59), the resulting identities are in fact quite complicated. We will see in section 3.5, following [27], that the identity for the four point scalar perturbations in de Sitter space is indeed met.

It is important to note that both connected and disconnected contributions to the correlation functions may be important in the Ward identities. As mentioned in eq.(3.23), after suitable rescaling the action has a factor of M_{Pl}^2/H^2 in front of it. Since this ratio is large, eq.(3.24), the situation in cosmology is analogous to the large N limit in AdS/CFT, with M_{Pl}^2/H^2 playing the role of N^2. Disconnected components in the Ward identities can then often dominate over connected ones. More accurately, the non-linear nature of the transformation in the compensating spatial reparametrization means that different number of fields will be present in correlation functions involved in the LHS of the Ward identities eq.(3.62), eq.(3.65). The suppression at large N due to additional fields can be compensated for by including additional disconnected components. For more details and an explicit example see section 3.5.

Let us also note that in the Ward identity for scalar perturbations eq.(3.62), for the cases $n = 2, 3$, the extra terms in $\delta\zeta(\boldsymbol{k})$ due to the compensating spatial reparametrization can be neglected to the leading order in H^2/M_{Pl}^2. The extra terms are the last two terms on the RHS of eq.(4.22). This will become clearer from the discussion in section 3.5. As a result, for the cases $n = 2, 3$, the Ward identity eq.(3.62) takes the form,

$$\left(\sum_{a=1}^{2} \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right) \langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_3}}\right) \frac{\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle}{\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\Bigg|_{\boldsymbol{k_3}\to 0}, \tag{3.69}$$

and

$$\left(\sum_{a=1}^{3} \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right) \langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle = -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_4}}\right) \frac{\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle}{\langle\zeta(\boldsymbol{k_4})\zeta(-\boldsymbol{k_4})\rangle'}\Bigg|_{\boldsymbol{k_4}\to 0}. \tag{3.70}$$

For the special case $\boldsymbol{b} \propto \boldsymbol{k_3}$ in eq.(3.69) and $\boldsymbol{b} \propto \boldsymbol{k_4}$ in eq.(3.70), we obtain the conformal consistency relations derived in eq.(37) of [103]. However, as we will see in section 3.5, for the case $n \geq 4$ in eq.(3.62) the extra terms in $\delta\zeta(\boldsymbol{k})$ due to the compensating spatial reparametrization cannot be neglected. Thus, the Ward identities obtained from eq.(3.62) for $\boldsymbol{b} \propto \boldsymbol{k_{n+1}}$, $n \geq 4$, have additional terms as compared to eq.(37) of [103].

This concludes our summary of some of the main results of this chapter.

3.4 Conditions on the coefficient functions

In this section, we will show how the invariance of the wave function under the scale and special conformal transformations, eq. (3.36) and eq.(3.53), lead to conditions on the coefficient functions which are analogous to Ward identities in a conformal field theory. These Ward identities also incorporate the breaking of conformal invariance in the inflationary background.

We start with the Ward identities of spatial reparametrizations in general. These will give rise to conditions analogous to Ward identities of stress-energy conservation in a field theory. We then obtain the identities for scale and special conformal transformations.

In this section, it will be convenient to work with a general metric perturbation $\widehat{\gamma}_{ij}$ without further imposing the transversality condition eq.(C.2.2). Once the constraints on the coefficient functions have been obtained for these general perturbations, they will lead to constraints on expectation values which can be calculated only after further gauge fixing, as explained in section 3.3 above.

3.4.1 Spatial reparametrizations

Under a spatial reparametrization eq.(3.7), the perturbations ζ and $\widehat{\gamma}_{ij}$ transforms as

$$\zeta \to \zeta + \frac{1}{3}\,\partial_i v_i + v^i \partial_i \zeta + \frac{1}{3}\,\partial_i v_j\, \widehat{\gamma}_{ij}, \tag{3.71}$$

and

$$\begin{aligned} \widehat{\gamma}_{ij} \to \widehat{\gamma}_{ij} + \Big(\partial_i v_j + \partial_j v_i - \frac{2}{3}\,\partial_a v_a\, \delta_{ij}\Big) + \Big(&\widehat{\gamma}_{ik}\, \partial_j v^k + \widehat{\gamma}_{jk}\, \partial_i v^k \\ &+ v^k \partial_k \widehat{\gamma}_{ij} - \frac{2}{3}\,\partial_a v_a\, \widehat{\gamma}_{ij} - \frac{2}{3}\,\partial_a v_b\, \widehat{\gamma}_{ab}\,(\delta_{ij} + \widehat{\gamma}_{ij})\Big). \end{aligned} \tag{3.72}$$

See appendix B.1 for some details of the derivation.

Now, for the invariance of the wave function under spatial reparametrizations, the terms proportional to the transformation parameter v_i that get generated because of the transformations eq.(3.71), eq.(3.72), must cancel with one another. Consider first such terms which are linear in ζ. These terms are produced by the first and second terms of the wave function, eq.(3.20). These are

$$\begin{aligned} &\delta\Big(-\frac{1}{2}\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\, \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\Big) \\ &\qquad = -\frac{1}{6}\int d^3x\, d^3y \Big[\partial_i v_i(\boldsymbol{x})\, \zeta(\boldsymbol{y}) + \zeta(\boldsymbol{x})\, \partial_{y^i} v_i(\boldsymbol{y})\Big] \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} &\delta\Big(-\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\, \widehat{\gamma}_{ij}(\boldsymbol{y})\, \langle T(\boldsymbol{x})\widehat{T}^{ij}(\boldsymbol{y})\rangle\Big) \\ &\qquad\qquad = -\,2\int d^3x\, d^3y\, \zeta(\boldsymbol{x})\, \partial_{y^i} v_j(\boldsymbol{y})\, \langle T(\boldsymbol{x})\widehat{T}^{ij}(\boldsymbol{y})\rangle. \end{aligned} \tag{3.74}$$

Mutual cancellation of the terms in eq.(3.73) and eq.(3.74) produces the Ward identity [4]

$$\partial_i\langle\widehat{T}^{ij}(\boldsymbol{x})T(\boldsymbol{y})\rangle+\frac{1}{6}\,\partial_j\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle=0. \tag{3.75}$$

Using

$$T^{ij}=\widehat{T}^{ij}+\frac{1}{3}\delta_{ij}\mathcal{T}, \tag{3.76}$$

and eq.(3.21), this can also be written as

$$\partial_i\langle T^{ij}(\boldsymbol{x})\mathcal{T}(\boldsymbol{y})\rangle=0. \tag{3.77}$$

Similarly, canceling the extra terms in the wave function which are linear in $\widehat{\gamma}_{ij}$ gives

$$\partial_i\langle T^{ij}(\boldsymbol{x})T^{kl}(\boldsymbol{y})\rangle=0. \tag{3.78}$$

Proceeding in a similar manner, and canceling terms linear in v_i and quadratic in ζ gives

$$\begin{aligned}\partial_i\langle\widehat{T}^{ij}(\boldsymbol{x})T(\boldsymbol{y})T(\boldsymbol{z})\rangle=&\,\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{y})\right]\langle T(\boldsymbol{x})T(\boldsymbol{z})\rangle+\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle\\&-\frac{1}{6}\,\partial_j\langle T(\boldsymbol{x})\,T(\boldsymbol{y})T(\boldsymbol{z})\rangle.\end{aligned} \tag{3.79}$$

Eq.(3.79) can be rewritten in terms of the complete stress-energy tensor T^{ij} and its trace $\mathcal{T}$ as

$$\partial_i\langle T^{ij}(\boldsymbol{x})\mathcal{T}(\boldsymbol{y})\mathcal{T}(\boldsymbol{z})\rangle=\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{y})\right]\langle\mathcal{T}(\boldsymbol{x})\mathcal{T}(\boldsymbol{z})\rangle+\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle\mathcal{T}(\boldsymbol{x})\mathcal{T}(\boldsymbol{y})\rangle. \tag{3.80}$$

Similarly, canceling terms proportional to $\zeta\,\widehat{\gamma}_{ij}$ produces the Ward identity

$$\begin{aligned}\partial_i\langle T^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})T(\boldsymbol{z})\rangle=&\,\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{y})\right]\langle\widehat{T}^{kl}(\boldsymbol{x})T(\boldsymbol{z})\rangle\\&+\frac{1}{2}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle T(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle\\&+\frac{1}{3}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{y})\right]\langle\widehat{T}^{kl}(\boldsymbol{y})T(\boldsymbol{z})\rangle\\&-\delta_{jl}\,\partial_i\left[\delta^3(\boldsymbol{x}-\boldsymbol{y})\right]\langle T^{ik}(\boldsymbol{y})T(\boldsymbol{z})\rangle,\end{aligned} \tag{3.81}$$

and proportional to $\widehat{\gamma}_{ij}\widehat{\gamma}_{kl}$ gives

$$\begin{aligned}\partial_i\langle T^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\widehat{T}^{mn}(\boldsymbol{z})\rangle=&\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle\widehat{T}^{mn}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle\\&+\frac{2}{3}\,\partial_j\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle\widehat{T}^{kl}(\boldsymbol{y})\widehat{T}^{mn}(\boldsymbol{z})\rangle\\&-2\,\delta_{jn}\,\partial_i\left[\delta^3(\boldsymbol{x}-\boldsymbol{z})\right]\langle\widehat{T}^{kl}(\boldsymbol{y})T^{im}(\boldsymbol{z})\rangle.\end{aligned} \tag{3.82}$$

[4] Note that unless otherwise stated, ∂_i stands for the derivative with respect to x^i, i.e. $\partial/\partial x^i$.

3.4.2 Scale transformations

We now turn to deriving conditions on the coefficient functions for the invariance of the wave function under scale transformations, eq.(3.36). The change in ζ and $\widehat{\gamma}_{ij}$ under these transformations is given by eq.(3.37) and eq.(3.38) respectively.

The procedure we follow is the same as outlined above. Canceling the terms linear in the transformation parameter λ and quadratic in ζ gives

$$\int d^3z \, \langle T(\boldsymbol{x})T(\boldsymbol{y})T(\boldsymbol{z})\rangle = \left[x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}\right] \langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle + 6\,\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle, \tag{3.83}$$

which in momentum space takes the form

$$\lim_{\boldsymbol{k_3}\to 0} \langle T(\boldsymbol{k_1})T(\boldsymbol{k_2})T(\boldsymbol{k_3})\rangle = -\left(\sum_{a=1}^{2} k_a \frac{\partial}{\partial k_a}\right) \langle T(\boldsymbol{k_1})T(\boldsymbol{k_2})\rangle, \tag{3.84}$$

where $k_a \equiv |\boldsymbol{k_a}|$.

In general, the n-point correlation function of the T operators will be related under scaling to the $(n-1)$-point correlation function through the relation

$$\lim_{\boldsymbol{k_n}\to 0} \langle T(\boldsymbol{k_1})\cdots T(\boldsymbol{k_n})\rangle = -\left(\sum_{a=1}^{n-1} k_a \frac{\partial}{\partial k_a}\right) \langle T(\boldsymbol{k_1})\cdots T(\boldsymbol{k_{n-1}})\rangle. \tag{3.85}$$

Requiring the cancellation of the extra quadratic terms in $\widehat{\gamma}_{ij}$ gives us the Ward identity

$$\int d^3z \, \langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})T(\boldsymbol{z})\rangle = \left[x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}\right] \langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle + 6\,\langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle, \tag{3.86}$$

which translates in momentum space to

$$\lim_{\boldsymbol{k_3}\to 0} \langle \widehat{T}^{ij}(\boldsymbol{k_1})\widehat{T}^{kl}(\boldsymbol{k_2})T(\boldsymbol{k_3})\rangle = -\left[\sum_{a=1}^{2} k_a \frac{\partial}{\partial k_a}\right] \langle \widehat{T}^{ij}(\boldsymbol{k_1})\widehat{T}^{kl}(\boldsymbol{k_2})\rangle. \tag{3.87}$$

The general form of the scaling Ward identity relating the n-point correlation function of $(n-1)$ $\widehat{T}^{ij}$ operators and one insertion of T, to the $(n-1)$-point correlation function of the $\widehat{T}^{ij}$ operators is

$$\begin{aligned} &\lim_{\boldsymbol{k_n}\to 0} \langle \widehat{T}^{i_1 j_1}(\boldsymbol{k_1})\cdots \widehat{T}^{i_{n-1} j_{n-1}}(\boldsymbol{k_{n-1}})T(\boldsymbol{k_n})\rangle = \\ &\qquad - \left[\sum_{a=1}^{n-1} k_a \frac{\partial}{\partial k_a}\right] \langle \widehat{T}^{i_1 j_1}(\boldsymbol{k_1})\cdots \widehat{T}^{i_{n-1} j_{n-1}}(\boldsymbol{k_{n-1}})\rangle. \end{aligned} \tag{3.88}$$

One can also write the general scaling Ward identity relating the $(n+1)$-point correlation function involving m insertions of $\widehat{T}^{ij}$ and $(n+1-m)$ insertions of T, with the n-point correlation function of m insertions of $\widehat{T}^{ij}$ and $(n-m)$ of T, as

$$\begin{aligned}\lim_{\boldsymbol{k_{n+1}}\to 0}\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_mj_m}(\boldsymbol{k_m})T(\boldsymbol{k_{m+1}})\cdots T(\boldsymbol{k_{n+1}})\rangle = \\ -\left[\sum_{a=1}^{n} k_a\frac{\partial}{\partial k_a}\right]\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_mj_m}(\boldsymbol{k_m})T(\boldsymbol{k_{m+1}})\cdots T(\boldsymbol{k_n})\rangle.\end{aligned} \tag{3.89}$$

One final comment. We began this subsection by considering terms which are quadratic in ζ, eq.(3.83). There is also a term which is linear in both ζ and the transformation parameter λ. Since λ is spatially constant, this term has support only at zero momentum in the wave function and we neglect it here. However, in deriving expectation values, this term which is uncanceled plays a crucial role, as was discussed in section 3.3.1 above.

3.4.3 Special conformal transformations

We will now derive Ward identities for the invariance of the wave function under special conformal transformations, eq.(3.53). The change in ζ and $\widehat{\gamma}_{ij}$ under this is given by eq.(3.54) and eq.(3.55). Note that, as was mentioned at the beginning of this section, we are considering a general graviton perturbation here and have not fixed it to be transverse; as a result we do not have to worry about the fact that a special conformal transformation leads to the gauge eq.(C.2.2) not being preserved.

We start again with terms which are linear in b_i and quadratic in ζ. Invariance of Ψ then gives

$$\begin{aligned}3\,\boldsymbol{b}\cdot(\boldsymbol{x}+\boldsymbol{y})\,\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle - \frac{1}{2}\left[\alpha^i(\boldsymbol{x})\partial_{x^i}+\alpha^i(\boldsymbol{y})\partial_{y^i}\right]\langle T(\boldsymbol{x})T(\boldsymbol{y})\rangle \\ = \int d^3z\,(\boldsymbol{b}\cdot\boldsymbol{z})\langle T(\boldsymbol{x})T(\boldsymbol{y})T(\boldsymbol{z})\rangle,\end{aligned} \tag{3.90}$$

which in momentum space has the form

$$\frac{1}{2}\left[\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_1}}+\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_2}}\right]\langle T(\boldsymbol{k_1})T(\boldsymbol{k_2})\rangle = -\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_3}}\right)\langle T(\boldsymbol{k_1})T(\boldsymbol{k_2})T(\boldsymbol{k_3})\rangle\bigg|_{\boldsymbol{k_3}\to 0}, \tag{3.91}$$

where $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is the operator

$$\begin{aligned}\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}} &= 2\left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)-(\boldsymbol{b}\cdot\boldsymbol{k})\left(\frac{\partial}{\partial\boldsymbol{k}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\\ &= (\boldsymbol{b}\cdot\boldsymbol{k})\left(-\frac{2}{k}\frac{\partial}{\partial k}+\frac{\partial^2}{\partial k^2}\right).\end{aligned} \tag{3.92}$$

In general, the n-point correlation function of T operators will be related to the $(n-1)$-point function under a special conformal transformation as

$$\frac{1}{2}\left(\sum_{a=1}^{n-1}\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right)\langle T(\boldsymbol{k_1})\cdots T(\boldsymbol{k_{n-1}})\rangle = -\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_n}}\right)\langle T(\boldsymbol{k_1})\cdots T(\boldsymbol{k_n})\rangle\Big|_{\boldsymbol{k_n}\to 0}. \tag{3.93}$$

Similarly, canceling the extra terms quadratic in $\widehat{\gamma}_{ij}$ gives

$$\left[\mathcal{D}^{\boldsymbol{b}}_{\boldsymbol{x}}+\mathcal{D}^{\boldsymbol{b}}_{\boldsymbol{y}}\right]\langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle = -\int d^3z\,(\boldsymbol{b}\cdot\boldsymbol{z})\langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})T(\boldsymbol{z})\rangle, \tag{3.94}$$

where the action of the operator $\mathcal{D}^{\boldsymbol{b}}_{\boldsymbol{x}}$ is defined by

$$\begin{aligned}\mathcal{D}^{\boldsymbol{b}}_{\boldsymbol{x}}\,\widehat{T}^{ij}(\boldsymbol{x}) = -3\,(\boldsymbol{b}\cdot\boldsymbol{x})\,\widehat{T}^{ij}(\boldsymbol{x}) + \frac{1}{2}\,\alpha^m(\boldsymbol{x})\,\partial_m\widehat{T}^{ij}(\boldsymbol{x})&\\ -\mathcal{M}^{\boldsymbol{b}}_{mi}(\boldsymbol{x})\widehat{T}^{mj}(\boldsymbol{x}) - \mathcal{M}^{\boldsymbol{b}}_{mj}(\boldsymbol{x})\widehat{T}^{im}(\boldsymbol{x}),&\end{aligned} \tag{3.95}$$

with $\mathcal{M}^{\boldsymbol{b}}_{ij}(\boldsymbol{x})$ as defined in eq.(3.56). The Ward identity eq. (3.94) can be expressed in the momentum space as

$$\left[\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k_1}}+\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k_2}}\right]\langle \widehat{T}^{ij}(\boldsymbol{k_1})\widehat{T}^{kl}(\boldsymbol{k_2})\rangle = -\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_3}}\right)\langle \widehat{T}^{ij}(\boldsymbol{k_1})\widehat{T}^{kl}(\boldsymbol{k_2})T(\boldsymbol{k_3})\rangle\Big|_{\boldsymbol{k_3}\to 0}, \tag{3.96}$$

where the momentum space operator $\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is defined by

$$\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k}}\,\widehat{T}^{ij}(\boldsymbol{k}) = \frac{1}{2}\,\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}\,\widehat{T}^{ij}(\boldsymbol{k}) - \tilde{\mathcal{M}}^{\boldsymbol{b}}_{mi}(\boldsymbol{k})\widehat{T}^{mj}(\boldsymbol{k}) - \tilde{\mathcal{M}}^{\boldsymbol{b}}_{mj}(\boldsymbol{k})\widehat{T}^{im}(\boldsymbol{k}), \tag{3.97}$$

with the operator $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ as given in eq.(3.92), and $\tilde{\mathcal{M}}^{\boldsymbol{b}}_{ij}(\boldsymbol{k})$ as given by eq.(3.67).

Following a similar procedure as outlined above, we get the Ward identity for special conformal transformations relating the n-point correlation function with $(n-1)$ insertions of $\widehat{T}^{ij}$ and one insertion of T, to the $(n-1)$ point correlation function of $\widehat{T}^{ij}$ to be

$$\begin{aligned}\left(\sum_{a=1}^{n-1}\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right)\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_{n-1}j_{n-1}}(\boldsymbol{k_{n-1}})\rangle =&\\ -\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_n}}\right)\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_{n-1}j_{n-1}}(\boldsymbol{k_{n-1}})T(\boldsymbol{k_n})\rangle\Big|_{\boldsymbol{k_n}\to 0}&.\end{aligned} \tag{3.98}$$

In general, the Ward identity of special conformal invariance relating the n-point correlation function with m insertions of $\widehat{T}^{ij}$ and $(n-m)$ insertions of T, to the $(n+1)$-point correlation function with m insertions of $\widehat{T}^{ij}$ and $(n+1-m)$ insertions of T is given by

$$\begin{aligned}&\left[\left(\sum_{a=1}^{m}\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right)+\frac{1}{2}\left(\sum_{r=m+1}^{n}\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_r}}\right)\right]\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_mj_m}(\boldsymbol{k_m})T(\boldsymbol{k_{m+1}})\cdots T(\boldsymbol{k_n})\rangle\\ &= -\left(\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{k_{n+1}}}\right)\langle \widehat{T}^{i_1j_1}(\boldsymbol{k_1})\cdots\widehat{T}^{i_mj_m}(\boldsymbol{k_m})T(\boldsymbol{k_{m+1}})\cdots T(\boldsymbol{k_{n+1}})\rangle\Big|_{\boldsymbol{k_{n+1}}\to 0},\end{aligned} \tag{3.99}$$

where the operators $\tilde{\mathcal{D}}^{\boldsymbol{b}}_{\boldsymbol{k}}$ and $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ are given in eq.(3.97) and eq.(3.92) respectively.

Finally, as in the case of the scale transformations, there is one remaining term linear in $\widehat{\gamma}_{ij}$ and b_i, with support at zero momentum, which is uncanceled. In calculating expectation values, section 3.3, this term will vanish once we choose the transverse gauge, eq.(C.2.2).

3.5 Explicit checks of the special conformal Ward identities

In this section, we present a few checks of the Ward identities of special conformal transformations, eq.(3.62) and eq.(3.65). Our analysis above has been to the leading order in H^2/M_{Pl}^2, and we will verify the Ward identities to this order below.

3.5.1 Basic checks

Consider the Ward identity eq.(3.62) for the case of $n = 2$. We have

$$\begin{aligned}&\left[\sum_{a=1}^{2}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle+\Bigg\{\Bigg(6\,b^m k_1^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\langle\zeta(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k_2})\,\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle\\&+2\,b^m k_1^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\,\widehat{\gamma}_{jm}(\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k_2})\rangle\Bigg)+\Big(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_2}\Big)\Bigg\}\\&=-2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_3}}\right)\frac{1}{\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle\Big|_{\boldsymbol{k_3}\to 0}.\end{aligned}\tag{3.100}$$

We will first consider the identity eq.(3.100) in the de Sitter limit. We substitute

$$\zeta=-\frac{H}{\dot{\bar{\phi}}}\,\delta\phi,\tag{3.101}$$

and take the limit $\dot{\bar{\phi}}\to 0$. Keeping only the leading terms, we get

$$\begin{aligned}\left[\sum_{a=1}^{2}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle=-\Bigg\{\Bigg(6\,b^m k_1^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\langle\delta\phi(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\,\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle\Bigg)\\+\Big(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_2}\Big)\Bigg\}.\end{aligned}\tag{3.102}$$

Next, we introduce suitable factors of H/M_{Pl}. The wave function eq.(3.20) arises from the action eq.(3.162), which has a factor of $1/G\sim M_{Pl}^2$ in front of it. Thus, after suitably rescaling by powers of the Hubble parameter, Ψ will go like

$$\begin{aligned}\Psi \sim \exp\bigg[-\frac{M_{Pl}^2}{H^2}\bigg\{\frac{1}{2}\int d^3x\, d^3y\, \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\, \langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle \\ +\frac{1}{2}\int d^3x\, d^3y\, \widehat{\gamma}_{ij}(\boldsymbol{x})\widehat{\gamma}_{kl}(\boldsymbol{y})\, \langle \widehat{T}^{ij}(\boldsymbol{x})\widehat{T}^{kl}(\boldsymbol{y})\rangle \\ +\frac{1}{3!}\int d^3x\, d^3y\, d^3z\, \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\delta\phi(\boldsymbol{z})\, \langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})\rangle \\ +\frac{1}{3!}\int d^3x\, d^3y\, d^3z\, \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\widehat{\gamma}_{ij}(\boldsymbol{z})\, \langle O(\boldsymbol{x})O(\boldsymbol{y})\widehat{T}^{ij}(\boldsymbol{z})\rangle + \cdots\bigg\}\bigg].\end{aligned} \tag{3.103}$$

As a result, we see that the propagators $\langle\delta\phi\,\delta\phi\rangle$ or $\langle\widehat{\gamma}_{ij}\widehat{\gamma}_{kl}\rangle$ behave like H^2/M_{Pl}^2, while each vertex, e.g., the three point vertices $\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})\rangle$ and $\langle O(\boldsymbol{x})O(\boldsymbol{y})\widehat{T}^{ij}(\boldsymbol{z})\rangle$ in eq.(3.103), go like M_{Pl}^2/H^2. With this, one can argue that

$$\langle\delta\phi\,\delta\phi\,\widehat{\gamma}_{ij}\rangle \sim \frac{H^4}{M_{Pl}^4}. \tag{3.104}$$

From eq.(3.104), it becomes clear that to leading order in H^2/M_{Pl}^2, the correlation function $\langle\delta\phi(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\,\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle$ in the RHS of eq.(3.102) is suppressed compared to $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ in the LHS and eq.(3.102) reduces to

$$\left[\sum_{a=1}^{2}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle = 0. \tag{3.105}$$

This condition is the statement of conformal invariance of the two point function $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle$ in de Sitter space, and it is easy to verify that it is met.

Next, let us consider the $n=2$ scalar Ward identity, eq.(3.100), to the first non-trivial order in the slow roll approximation (but still to the leading order in H^2/M_{Pl}^2). The terms proportional to $\langle\zeta\zeta\widehat{\gamma}\rangle$ and $\langle\widehat{\gamma}\widehat{\gamma}\zeta\rangle$ in eq.(3.100) scale as H^4/M_{Pl}^4, and can be dropped compared to other terms, which scale as H^2/M_{Pl}^2, in the limit $H \ll M_{Pl}$. Eq.(3.100) then reduces to

$$\left[\sum_{a=1}^{2}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_3}}\right)\frac{1}{\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle\bigg|_{\boldsymbol{k_3}\to 0}. \tag{3.106}$$

In the canonical slow-roll model we have

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\frac{H^2}{M_{Pl}^2}\,\frac{1}{4\epsilon}\,k_1^{-3+n_S}, \tag{3.107}$$

and [5]

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle = (2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\frac{H^4}{M_{Pl}^4}\frac{1}{4\epsilon^2}\frac{1}{\prod_a(2k_a)^3} \\ \left[(3\epsilon-2\eta)\sum_{a=1}^{3}k_a^3+2\epsilon\left(\frac{1}{2}\sum_{a\neq b}k_ak_b^2+\frac{4}{k_t}\sum_{a>b}k_a^2k_b^2\right)\right]. \tag{3.108}$$

Working to the leading order in the slow roll parameters ϵ, η as well as the scalar tilt n_S, one finds that eq.(3.106) implies

$$n_S = 2(\eta-3\epsilon), \tag{3.109}$$

which is the correct result, [24].

Note that the case $n = 3$ for the Ward identity eq.(3.62) in the slow roll approximation was discussed in detail in [1].

Next, we turn to the graviton two-point correlator and consider the Ward identity eq.(3.65). Again dropping terms which are subleading in H^2/M_{Pl}^2, and working in the de Sitter limit $\dot{\bar{\phi}} \to 0$, we get,

$$\begin{aligned}\left[\sum_{a=1}^{2}\widehat{\mathcal{L}}_{\boldsymbol{k_a}}^{\boldsymbol{b}}\right]&\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1})\widehat{\gamma}_{kl}(\boldsymbol{k_2})\rangle \\ &= -\Big(2\,\tilde{\mathcal{M}}_{im}^{\boldsymbol{b}}(\boldsymbol{k_1})\langle\widehat{\gamma}_{jm}(\boldsymbol{k_1})\widehat{\gamma}_{kl}(\boldsymbol{k_2})\rangle+2\,\tilde{\mathcal{M}}_{jm}^{\boldsymbol{b}}(\boldsymbol{k_1})\langle\widehat{\gamma}_{im}(\boldsymbol{k_1})\widehat{\gamma}_{kl}(\boldsymbol{k_2})\rangle \\ &\quad+2\,\tilde{\mathcal{M}}_{km}^{\boldsymbol{b}}(\boldsymbol{k_2})\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1})\widehat{\gamma}_{ml}(\boldsymbol{k_2})\rangle+2\,\tilde{\mathcal{M}}_{lm}^{\boldsymbol{b}}(\boldsymbol{k_2})\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1})\widehat{\gamma}_{km}(\boldsymbol{k_2})\rangle \\ &\quad+\frac{6b^m}{k_1^2}\left(k_{1i}\,\langle\widehat{\gamma}_{jm}(\boldsymbol{k_1})\widehat{\gamma}_{kl}(\boldsymbol{k_2})\rangle+k_{1j}\,\langle\widehat{\gamma}_{im}(\boldsymbol{k_1})\widehat{\gamma}_{kl}(\boldsymbol{k_2})\rangle\right) \\ &\quad+\frac{6b^m}{k_2^2}\left(k_{2k}\,\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1})\widehat{\gamma}_{lm}(\boldsymbol{k_2})\rangle+k_{2l}\,\langle\widehat{\gamma}_{ij}(\boldsymbol{k_1})\widehat{\gamma}_{km}(\boldsymbol{k_2})\rangle\right)\Big).\end{aligned} \tag{3.110}$$

The two-point graviton correlator is

$$\langle\widehat{\gamma}_{ij}(\boldsymbol{k})\widehat{\gamma}_{kl}(-\boldsymbol{k})\rangle' = \frac{P_{ijkl}(\boldsymbol{k})}{k^3}, \tag{3.111}$$

where $P_{ijkl}(\boldsymbol{k})$ is given in eq.(5.2) of [27]. An explicit calculation then shows that eq.(3.110) is indeed met. Note that the last two terms on the RHS of eq.(3.110) come from the compensating spatial reparametrization which maintains the transverse gauge for $\widehat{\gamma}_{ij}$.

3.5.2 The scalar four point function

In this subsection, we consider the Ward identity eq.(3.62) for the case $n = 4$. We will work to the leading order in H^2/M_{Pl}^2, and to the leading order in $\dot{\bar{\phi}}/H$, i.e. in the de Sitter limit.

[5]The result eq.(3.108) is from [24], with $k_t = k_1 + k_2 + k_3$.

Eq.(3.62) for the case of $n = 4$ then gives

$$\begin{aligned}
&\left[\sum_{a=1}^{4} \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right] \langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle + \Bigg\{ \Bigg(2\, b^m k_1^i \int \frac{d^3\tilde{k}}{(2\pi)^3}\, \frac{1}{\tilde{k}^2} \times \\
&\Big[3\, \langle \zeta(\boldsymbol{k_1} - \tilde{\boldsymbol{k}})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle + \langle \widehat{\gamma}_{ij}(\boldsymbol{k_1} - \tilde{\boldsymbol{k}})\, \widehat{\gamma}_{jm}(\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle \Big] \Bigg) \\
&+ \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_2}\big) + \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_3}\big) + \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_4}\big) \Bigg\} \\
&= -\, 2 \left(\boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{k_5}} \right) \frac{1}{\langle \zeta(\boldsymbol{k_5})\zeta(-\boldsymbol{k_5})\rangle'}\, \langle \zeta(\boldsymbol{k_1}) \cdots \zeta(\boldsymbol{k_5})\rangle \Big|_{\boldsymbol{k_5} \to 0} .
\end{aligned} \tag{3.112}$$

We next write eq.(3.112) in terms of $\delta\phi$ by using eq.(3.101), and take the de Sitter limit $\dot{\bar{\phi}} \to 0$. In this limit, the terms in eq.(3.112) that survive are

$$\begin{aligned}
&\left[\sum_{a=1}^{4} \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right] \langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle \\
&= -\Bigg\{ \Bigg(6\, b^m k_1^i \int \frac{d^3\tilde{k}}{(2\pi)^3}\, \frac{1}{\tilde{k}^2}\, \langle \delta\phi(\boldsymbol{k_1} - \tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle \Bigg) \\
&+ \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_2}\big) + \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_3}\big) + \big(\boldsymbol{k_1} \leftrightarrow \boldsymbol{k_4}\big) \Bigg\}.
\end{aligned} \tag{3.113}$$

Introducing suitable factors of H/M_{Pl} by rescaling the wave function, eq.(3.103), we see that for connected correlators,

$$\langle \delta\phi\delta\phi\delta\phi\delta\phi\rangle \sim \frac{H^6}{M_{Pl}^6}\,; \quad \langle \delta\phi\widehat{\gamma}_{ij}\delta\phi\delta\phi\delta\phi\rangle \sim \frac{H^8}{M_{Pl}^8}. \tag{3.114}$$

From eq.(3.114), it seems that for the Hubble scale being much small compared to the Planck scale, $H \ll M_{Pl}$, the correlation function $\langle \delta\phi(\boldsymbol{k_1} - \tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle$ in the RHS of eq.(3.113) is suppressed compared to $\langle \delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle$ in the LHS. However, one should also consider disconnected contributions to the RHS of eq.(3.113) which may contribute to the same order of H/M_{Pl} as the LHS. In particular, there is a disconnected contribution to the five point function $\langle \delta\phi\delta\phi\delta\phi\delta\phi\widehat{\gamma}_{ij}\rangle$ in eq.(3.113), which goes as

$$\langle \delta\phi\delta\phi\rangle\, \langle \delta\phi\delta\phi\widehat{\gamma}_{ij}\rangle \sim \frac{H^6}{M_{Pl}^6}, \tag{3.115}$$

and which is of the same order as $\langle \delta\phi\delta\phi\delta\phi\delta\phi\rangle$. With these considerations, eq.(3.113) in the limit $H \ll M_{Pl}$ becomes

$$
\begin{aligned}
&\left[\sum_{a=1}^{4}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle\\
&=-\Bigg\{\Bigg(6\,b^m k_1^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\frac{1}{\tilde{k}^2}\langle\delta\phi(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\rangle\langle\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle\\
&+\big(\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_3}\big)+\big(\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_4}\big)\Bigg)+\big(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_2}\big)+\big(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_3}\big)+\big(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_4}\big)\Bigg\}.
\end{aligned}
\tag{3.116}
$$

It is important to note that there are other possible disconnected contributions to the five point correlator $\langle\delta\phi(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\rangle$, such as

$$
\langle\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle\langle\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_1}-\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_4})\rangle. \tag{3.117}
$$

However, this requires $\boldsymbol{k_2}+\boldsymbol{k_3}=\boldsymbol{k_1}+\boldsymbol{k_4}=0$, and will not contribute in general.

Eq.(3.116) gives the change in the four point correlator $\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle$ under a special conformal transformation in the exact de Sitter limit. Using the relations

$$
\begin{aligned}
\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle&=(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle'\\
&=(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\frac{H^2}{M_{Pl}^2}\frac{1}{2k_1^3},
\end{aligned}
\tag{3.118}
$$

and

$$
\begin{aligned}
\langle\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle=&-2\,\langle\delta\phi(\boldsymbol{k_3})\delta\phi(-\boldsymbol{k_3})\rangle'\langle\delta\phi(\boldsymbol{k_4})\delta\phi(-\boldsymbol{k_4})\rangle'\langle\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\widehat{\gamma}_{kl}(-\tilde{\boldsymbol{k}})\rangle'\\
&\langle\widehat{T}_{kl}(\tilde{\boldsymbol{k}})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle\\
=&-2\,\frac{H^4}{M_{Pl}^4}\frac{\langle\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}})\widehat{\gamma}_{kl}(-\tilde{\boldsymbol{k}})\rangle'\,\langle\widehat{T}_{kl}(\tilde{\boldsymbol{k}})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle}{(2k_3^3)(2k_4^3)},
\end{aligned}
\tag{3.119}
$$

we can write eq.(3.116) as

$$
\begin{aligned}
&\left[\sum_{a=1}^{4}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle=\\
&\quad\Bigg[\frac{H^6}{M_{Pl}^6}\frac{6b^m P_{imkl}(\boldsymbol{k_1}+\boldsymbol{k_2})}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^5}\Bigg\{\left(\frac{k_{1i}}{k_2^3}+\frac{k_{2i}}{k_1^3}\right)\frac{\langle\widehat{T}_{kl}(\boldsymbol{k_1}+\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle}{(2k_3^3)(2k_4^3)}\\
&\qquad+\left(\frac{k_{3i}}{k_4^3}+\frac{k_{4i}}{k_3^3}\right)\frac{\langle\widehat{T}_{kl}(\boldsymbol{k_3}+\boldsymbol{k_4})O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle}{(2k_1^3)(2k_2^3)}\Bigg\}\Bigg]\\
&\qquad+\Big[\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_3}\Big]+\Big[\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_4}\Big],
\end{aligned}
\tag{3.120}
$$

where we have used the eq.(3.111).

Eq.(3.120) gives us the change in the scalar four point function under a special conformal transformation. We can verify this by performing an explicit check. The four point function

$\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle$ was calculated in [27]. It is given by

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle = \int [\mathcal{D}\delta\phi]\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})P[\delta\phi], \tag{3.121}$$

where $P[\delta\phi]$ is the probability distribution function

$$P[\delta\phi] = \int [\mathcal{D}\widehat{\gamma}_{ij}]\,\left|\Psi[\delta\phi,\widehat{\gamma}_{ij}]\right|^2. \tag{3.122}$$

An explicit expression for $P[\delta\phi]$ was obtained starting from the wave function, eq.(3.103), in eq.(5.3) of [27]. It is given by [6]

$$\begin{aligned}P[\delta\phi] = \exp\Bigg[\frac{M_{Pl}^2}{H^2}\Bigg(&-\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\,\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle\\
&+\int\prod_{J=1}^{4}\left\{\frac{d^3k_J}{(2\pi)^3}\delta\phi(\boldsymbol{k}_J)\right\}\Bigg\{-\frac{1}{12}\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})O(-\boldsymbol{k_3})O(-\boldsymbol{k_4})\rangle\\
&+\frac{1}{2}\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\widehat{T}_{ij}(\boldsymbol{k_1}+\boldsymbol{k_2})\rangle'\,\langle O(-\boldsymbol{k_3})O(-\boldsymbol{k_4})\widehat{T}_{kl}(\boldsymbol{k_3}+\boldsymbol{k_4})\rangle'\\
&(2\pi)^3\delta^3\left(\sum_{J=1}^{4}\boldsymbol{k}_J\right)P_{ijkl}(\boldsymbol{k_1}+\boldsymbol{k_2})\frac{1}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^3}\Bigg\}\Bigg)\Bigg].\end{aligned} \tag{3.123}$$

From eq.(3.121) and eq.(3.123), we see that the four point function has two types of contributions,

$$\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle = \langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{CF} + \langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{ET}. \tag{3.124}$$

Here, $\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{CF}$ is the term proportional to $\langle OOOO\rangle$,

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle_{CF} = -\frac{1}{8}\frac{H^6}{M_{Pl}^6}\frac{1}{\prod_{a=1}^4 k_a^3}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle, \tag{3.125}$$

and $\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{ET}$ is the term proportional to $\langle OO\widehat{T}_{ij}\rangle'\langle OO\widehat{T}_{kl}\rangle'$,

$$\begin{aligned}\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle_{ET} = \frac{1}{4}\frac{H^6}{M_{Pl}^6}\frac{1}{\prod_{a=1}^4 k_a^3}\Big[&I(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4})\\
+I(\boldsymbol{k_1},\boldsymbol{k_3},\boldsymbol{k_2},\boldsymbol{k_4})+I(\boldsymbol{k_1},\boldsymbol{k_4},\boldsymbol{k_3},\boldsymbol{k_2})&+I(\boldsymbol{k_3},\boldsymbol{k_2},\boldsymbol{k_1},\boldsymbol{k_4})\\
+I(\boldsymbol{k_4},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_1})&+I(\boldsymbol{k_3},\boldsymbol{k_4},\boldsymbol{k_1},\boldsymbol{k_2})\Big],\end{aligned} \tag{3.126}$$

[6] In [27], the term $\langle\widehat{T}_{kl}OO\rangle$ in the wave function appeared with a coefficient $1/4$ (see eq(2.36) of [27]). But in the present chapter we choose to have a $1/2$, which means we need to consistently replace

$$\langle\widehat{T}_{kl}OO\rangle_{there}\rightarrow 2\langle\widehat{T}_{kl}OO\rangle_{here}$$

while using expressions from [27].

where $I(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4})$ is given in eq.(E.13) of [27],

$$I(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4}) = \int \frac{d^3k_5}{(2\pi)^3}\frac{d^3k_6}{(2\pi)^3}\frac{\langle \widehat{T}_{ij}(\boldsymbol{k}_5)\widehat{T}_{kl}(\boldsymbol{k}_6)\rangle}{k_5^3\ k_6^3} \langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\widehat{T}_{ij}(\boldsymbol{k}_5)\rangle\langle O(\boldsymbol{k_3})O(\boldsymbol{k_4})\widehat{T}_{kl}(\boldsymbol{k}_6)\rangle. \tag{3.127}$$

Now, the term $\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{CF}$ is invariant under a special conformal transformation, whereas the term $\langle\delta\phi\delta\phi\delta\phi\delta\phi\rangle_{ET}$ does change. We therefore have

$$\left[\sum_{a=1}^{4}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle = \left[\sum_{a=1}^{4}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle_{ET}. \tag{3.128}$$

As discussed in appendix (E.2) of [27], we have

$$\begin{aligned}\left[\sum_{a=1}^{4}\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right]\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle_{ET} &= \frac{1}{4}\frac{H^6}{M_{Pl}^6}\frac{1}{\prod_{a=1}^4 k_a^3}\Big[\delta^C I(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4})\\ &+\delta^C I(\boldsymbol{k_1}, \boldsymbol{k_3}, \boldsymbol{k_2}, \boldsymbol{k_4}) + \delta^C I(\boldsymbol{k_1}, \boldsymbol{k_4}, \boldsymbol{k_3}, \boldsymbol{k_2}) + \delta^C I(\boldsymbol{k_3}, \boldsymbol{k_2}, \boldsymbol{k_1}, \boldsymbol{k_4})\\ &+ \delta^C I(\boldsymbol{k_4}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_1}) + \delta^C I(\boldsymbol{k_3}, \boldsymbol{k_4}, \boldsymbol{k_1}, \boldsymbol{k_2})\Big],\end{aligned} \tag{3.129}$$

with $\delta^C I(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4})$ given in eq.(E.23) of [27],

$$\delta^C I(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4}) = 12 b_m \int \frac{d^3k}{(2\pi)^3}\frac{P_{imkl}(\boldsymbol{k})}{k^5}\, k_j\,\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\widehat{T}_{ij}(\boldsymbol{k})\rangle \langle O(\boldsymbol{k_3})O(\boldsymbol{k_4})\widehat{T}_{kl}(-\boldsymbol{k})\rangle. \tag{3.130}$$

By using the Ward identity eq.(3.8) of [27] expressed in momentum space,

$$k_j\langle\widehat{T}_{ij}(\boldsymbol{k})O(\boldsymbol{k'_1})O(\boldsymbol{k'_2})\rangle = \frac{1}{2}\Big(k'_{2i}\,\langle O(\boldsymbol{k'_1}+\boldsymbol{k})O(\boldsymbol{k'_2})\rangle + k'_{1i}\,\langle O(\boldsymbol{k'_2}+\boldsymbol{k})O(\boldsymbol{k'_1})\rangle\Big), \tag{3.131}$$

we can calculate the RHS of eq.(3.129). This gives the result eq.(3.120) for the change in the four point function, and completes the check.

3.6 Late time behaviour of modes

In this section, we elaborate on the late time behaviour of modes in the canonical model of slow roll inflation and also after including higher derivative terms in the action.

3.6.1 The canonical model of slow roll inflation

We have discussed in section 3.2 that one can use the residual time reparametrization invariance, eq.(3.10), in the gauge eq.(3.2), to set $\delta\phi = 0$, and that the remaining perturbations all become time independent in this gauge, at late times. Here we demonstrate this behaviour explicitly in the canonical slow roll model of inflation. The behaviour in the presence of higher derivative terms is discussed in section 3.6.2.

The action for the canonical model of slow roll inflation is given by

$$S = M_{Pl}^2 \int d^4x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right). \tag{3.132}$$

In the canonical model, the Hubble parameter eq.(4.8) is given by

$$H^2 = \frac{V}{3}. \tag{3.133}$$

The background $\bar{\phi}(t)$ satisfies the equation of motion

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0, \tag{3.134}$$

which in the slow roll approximation reduces to

$$\dot{\bar{\phi}} \approx -\frac{V'}{3H}, \tag{3.135}$$

where a $'$ denotes a derivative with respect to the scalar field. Using eq.(3.133) and eq.(3.135), the slow roll parameters ϵ_1, δ and ϵ, defined in eq.(3.28), eq.(3.29) and eq.(3.30), can be expressed as

$$\epsilon_1 = \epsilon = \frac{1}{2} \left(\frac{V'}{V} \right)^2, \; \delta = \epsilon_1 - \frac{V''}{V}. \tag{3.136}$$

The slow roll conditions, eq.(3.32), are

$$\left(\frac{V'}{V} \right)^2 \ll 1, \; \frac{V''}{V} \ll 1. \tag{3.137}$$

For the purpose of convenience in calculations, it is helpful to further decompose the metric perturbation γ_{ij}, eq.(3.5), as follows [29]

$$\gamma_{ij} = [A\,\delta_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i + D_{ij}], \tag{3.138}$$

where A, B transform as scalars, C_i transforms like a 3-vector and D_{ij} transforms as a rank-2 tensor under spatial rotations. Note that the perturbation C_i is divergence-less, and D_{ij} is transverse and traceless,

$$\partial_i C_i = 0, \; \partial_i D_{ij} = 0, \;\; D_{ii} = 0. \tag{3.139}$$

The Einstein equations to linear order in the perturbations about the FRW inflationary background are given by (see appendix B.3) [7]

$$-\,V'(\bar{\phi})\,\delta\phi = \frac{1}{2a^2}\nabla^2 A - \frac{1}{2}\ddot{A} - 3\left(\frac{\dot{a}}{a}\right)\dot{A} - \frac{1}{2}\left(\frac{\dot{a}}{a}\right)\nabla^2\dot{B}, \tag{3.140}$$

$$0 = A - a^2\ddot{B} - 3a\dot{a}\dot{B}, \tag{3.141}$$

$$-\,\dot{\bar{\phi}}\,\delta\phi = \dot{A}, \tag{3.142}$$

$$-\left(2\,\dot{\bar{\phi}}\,\delta\dot{\phi} - V'(\bar{\phi})\,\delta\phi\right) = \frac{3}{2}\ddot{A} + 3\left(\frac{\dot{a}}{a}\right)\dot{A} + \frac{1}{2}\nabla^2\ddot{B} + \left(\frac{\dot{a}}{a}\right)\nabla^2\dot{B}, \tag{3.143}$$

for the scalar perturbations $A, B, \delta\phi$. The vector perturbations C_i satisfy the equation

$$\nabla^2\dot{C}_i = 0. \tag{3.144}$$

For the tensor perturbations D_{ij} we get

$$\ddot{D}_{ij} + 3\left(\frac{\dot{a}}{a}\right)\dot{D}_{ij} - \frac{1}{a^2}\nabla^2 D_{ij} = 0. \tag{3.145}$$

The equation of motion for $\delta\phi$ is

$$\delta\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\delta\dot{\phi} + V''(\bar{\phi})\,\delta\phi - \frac{1}{a^2}\nabla^2\delta\phi = -\frac{1}{2}\,\dot{\bar{\phi}}\left(3\dot{A} + \nabla^2\dot{B}\right). \tag{3.146}$$

Eq.(3.141) can be used to solve for B in terms of A,

$$B(t,\boldsymbol{x}) = \int^t dt'\,\frac{1}{a(t')^3}\left(G_1(\boldsymbol{x}) + \int^{t'} dt''\,a(t'')A(t'',\boldsymbol{x})\right) + G_2(\boldsymbol{x}), \tag{3.147}$$

where G_1, G_2 are arbitrary functions of $\boldsymbol{x}$.

The late time behaviour of these equations can be obtained by dropping all spatial derivatives of the form ∇^2/a^2 in eqs.(3.140), (3.145) and (3.146). In addition, due to the $1/a^3$ pre-factor in eq.(3.147), we get that

$$B(t,\boldsymbol{x}) \approx G_2(\boldsymbol{x}) \text{ for t} \to \infty, \tag{3.148}$$

so that all time derivatives of B vanish at late times. Equations (3.140), (3.143) and (3.145) then simplify to

$$V'(\bar{\phi})\,\delta\phi = \frac{1}{2}\ddot{A} + 3\left(\frac{\dot{a}}{a}\right)\dot{A}, \tag{3.149}$$

$$-\left(2\,\dot{\bar{\phi}}\,\delta\dot{\phi} - V'(\bar{\phi})\,\delta\phi\right) = \frac{3}{2}\ddot{A} + 3\left(\frac{\dot{a}}{a}\right)\dot{A}, \tag{3.150}$$

[7] For brevity, we present the equations in units with $M_{Pl}^2 = \frac{1}{8\pi G} = 1$.

$$\ddot{D}_{ij} + 3\left(\frac{\dot{a}}{a}\right)\dot{D}_{ij} = 0, \tag{3.151}$$

and eq.(3.146) becomes

$$\delta\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\delta\dot{\phi} + V''(\bar{\phi})\,\delta\phi + \frac{3}{2}\,\dot{\bar{\phi}}\dot{A} = 0. \tag{3.152}$$

As discussed in appendix B.3, the late time behaviour for $A, \delta\phi$ is

$$A(t,\boldsymbol{x}) = P_1(\boldsymbol{x}) - 2\left(\frac{\dot{a}}{a}\right)P_2(\boldsymbol{x}), \tag{3.153}$$

and

$$\delta\phi(t,\boldsymbol{x}) = -\,\dot{\bar{\phi}}(t)\,P_2(\boldsymbol{x}), \tag{3.154}$$

where P_1, P_2 are time independent functions of $\boldsymbol{x}$. Also, the perturbations C_i, D_{ij} become time independent.

We can now carry out a time reparametrization

$$t \to t + P_2(\boldsymbol{x}), \tag{3.155}$$

along with the accompanying spatial reparametrization, eq.(3.11), which maintains the gauge choice eq.(3.2). Note that under the time reparametrization eq.(3.10), and the accompanying spatial reparametrization eq.(3.11), the perturbations change as

$$\delta A = 2\left(\frac{\dot{a}}{a}\right)\epsilon(\boldsymbol{x}), \tag{3.156}$$

$$\delta B = 2\,\epsilon(\boldsymbol{x})\int^t dt'\,\frac{1}{a^2(t')}, \tag{3.157}$$

$$\delta C_i = 0, \tag{3.158}$$

$$\delta D_{ij} = 0, \tag{3.159}$$

$$\delta(\delta\phi) = \dot{\bar{\phi}}\,\epsilon(\boldsymbol{x}). \tag{3.160}$$

We see from eq.(3.154) and eq.(3.160) that the change eq.(3.155) sets the late time value of $\delta\phi$ to vanish. In addition, using eq.(3.156) we see that the value of A is given by

$$A \to A' = P_1(\boldsymbol{x}), \tag{3.161}$$

while C, D_{ij} are unchanged and therefore continue to be time independent. B is changed by the the time reparametrization eq.(3.155), see eq.(3.157), but this change vanishes at late times, and thus B too continues to be time independent. Thus, we see that in the gauge $\delta\phi = 0$ all the perturbations freeze out at late times.[8]

3.6.2 Higher derivative corrections

In the discussion above, we have considered the canonical model of slow roll inflation, with the action in eq.(3.132). The action for this model involves two-derivative terms. One of the main motivations of our work is to be able to use symmetry considerations in more complicated situations where explicit computations or models may be unavailable. An example is the possibility that the Hubble scale H during inflation is of order the string scale M_{st}, so that higher derivative corrections to eq.(3.132) would be important. Given our limited knowledge of string theory in time dependent situations, explicit models or calculations for such a scenario are not possible today. But a symmetry based analysis should still be possible, as we discuss further in this section.

The more general situation we have in mind is the one with an effective action having higher order terms of the schematic form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R + (\partial\phi)^2 - 2V + \frac{R^2}{\Lambda^2} + \frac{R^3}{\Lambda^4} + \frac{(\partial\phi)^4}{\Lambda^2} + \cdots\right]. \tag{3.162}$$

The higher derivative terms are important because $H \sim O(\Lambda)$. In eq.(3.162), Λ is the underlying cutoff scale, which would be of order the string scale M_{st} in string theory. The term R^2/Λ^2 schematically denotes four derivative terms, and so on. Also, the coefficients of each of the higher order terms can in general be a function of ϕ. Let us note that in the background solution the contribution from terms like $(\partial\phi)^4$ will be small, since the inflaton will be evolving slowly. However, these terms will be important in determining the behaviour of the perturbations, since the perturbations will start out with physical wavelengths $\lambda \ll H^{-1}$, and then freeze out at a time when $\lambda \simeq H^{-1}$.

Let us note that $H \sim O(\Lambda)$ is consistent with the bounds on the tensor perturbations, since Λ can be much smaller than M_{pl}, as indeed happens in weakly coupled string theory. The condition $\Lambda \ll M_{Pl}$ also ensures that all quantum loop effects are small, and it is only tree level effects involving the higher derivative corrections which are important in the kind of scenario we have in mind.

In fact, considerations of the last few sections can be extended in a straightforward way to situations of this type. The crucial point is that even with the higher derivative terms present, one can argue that solutions with the same asymptotic behaviour as in the two-derivative case continue to exist. The underlying reason for this is that the asymptotic behaviour in the two-derivative case follows from gauge invariance. We will discuss this in

[8]The perturbations $\zeta, \widehat{\gamma}_{ij}$, which appear in section 3.2 and the discussion thereafter, are given by

$$\zeta = \frac{1}{2}A + \frac{1}{6}\nabla^2 B\,, \quad \text{and} \quad \widehat{\gamma}_{ij} = D_{ij} + \partial_i C_j + \partial_j C_i + \partial_i\partial_j B - \frac{1}{3}\delta_{ij}\nabla^2 B.$$

more detail in the next subsection. Given this fact, the arguments leading to the Ward identities can be easily seen to apply in cases with higher derivative corrections as well. A further change of coordinates allows us to set $\delta\phi$ to vanish, as discussed in section 3.2, and the invariance of the wave function under the residual spatial reparametrizations in the synchronous gauge then leads to the Ward identities of interest.

Freezing of the perturbations

We start with a discussion of the spin-2 component D_{ij}, eq.(3.138), which corresponds to gravity waves. In the two-derivative theory it satisfies the eq.(3.145). At late times, when k^2/a^2 becomes sufficiently small, this becomes eq.(3.151), which has the general solution eq.(B.38). In particular, D_{ij} becomes time independent, satisfying eq.(B.39), since the additional solution proportional to K_{ij} in eq.(B.38) dies out as $t \to \infty$. Higher derivative terms would result in contributions to the equation of motion with either additional spatial derivatives, and/or additional time derivatives. All terms with spatial derivatives will become small, since the physical wavelength λ for fixed $\boldsymbol{k}$ becomes large at late times. Thus the only terms which survive will have additional time derivatives. It is then clear that the solution eq.(B.39) will continue to hold even when higher derivative corrections are included.

However, in the presence of higher derivative terms there could be additional solutions which do not die out at large t. We will assume that the correct boundary conditions in the far past are such that any such solution is not "turned on" in the far future, leading to eq.(B.39).

Exponentially growing solutions would signify an instability. Our assumption that they are absent is consistent with the background inflationary solution being stable. There could be additional oscillatory solutions though, which are non-decaying. We cannot rule these out except by appealing to the initial conditions. However, the following possibility is worth mentioning in this context. The additional oscillatory solutions might be present if the higher derivative corrections in eq.(3.162) arise in the first place by integrating out massive particles with a mass $\sim O(\Lambda)$. This could happen in an underlying theory where all particles, the massive ones and the graviton, satisfy second order equations of motion, leading to a well posed initial value problem. In this case, the graviton will indeed have the solution discussed above, eq.(B.39), with a second solution which decays, eq.(B.38). If $\Lambda \sim H$, these additional particles would also be produced during inflation, with a suitable Boltzmann suppression. However, the Ward identities we derive in section 3.3 will continue to hold in this case as well. The wave function in the presence of these fields will continue to be invariant under spatial reparametrizations, and thus after integrating these heavy fields out, eq.(3.33), the same Ward identities will follow for ζ and $\widehat{\gamma}_{ij}$.

The discussion for spin-1 is even more straightforward. The solution found in the two-derivative case is pure gauge, since there are no physical degrees of freedom with spin 1.

Starting from the unperturbed solution of the form eq.(3.3), and carrying out a spatial reparametrization

$$x^i \to x^i + \epsilon^i(\boldsymbol{x}), \tag{3.163}$$

one gets

$$C_i = \epsilon_i - \partial_i \partial^{-2}(\partial \cdot \epsilon), \tag{3.164}$$

so that the most general time independent C_i can be turned on with a suitable choice of $\epsilon^i(\boldsymbol{x})$. This makes it clear that a solution of the form eq.(B.37) must continue to exist in the presence of higher derivative terms too.

Finally we come to the scalar perturbations. In the two-derivative theory, the late time behaviour for solutions was found to be eq.(3.153), eq.(3.148) and eq.(3.154), for A, B and $\delta\phi$ respectively. We now argue, in analogy with the case of spin-1 above, that the existence of solutions exhibiting this behaviour follows from spatial and time reparametrizations which preserve the synchronous gauge eq.(3.2). Starting from eq.(3.3) and doing the transformation eq.(3.163) gives

$$B = 2\,\partial^{-2}(\partial \cdot \epsilon), \tag{3.165}$$

so that the most general time independent B can be turned on. Also, starting from eq.(3.3) and carrying out a transformation

$$x^i \to x^i(1+\epsilon), \tag{3.166}$$

where ϵ is a constant, gives

$$A = 2\,\epsilon. \tag{3.167}$$

The late time behaviour of A with higher derivative terms will still be determined by an equation where all spatial derivatives can be dropped. The solution eq.(3.167) then means that actually

$$A \to P_1(\boldsymbol{x}) \tag{3.168}$$

will be a solution to the small perturbation equations. Finally, doing the time reparametrization eq.(3.10) gives rise to the solution

$$A = 2H\epsilon(\boldsymbol{x}),\ \delta\phi = \dot{\bar{\phi}}\,\epsilon(\boldsymbol{x}). \tag{3.169}$$

Putting all these solutions together, we get the general late time behaviour seen in eq.(3.153), eq.(3.148) and eq.(3.154).

Since the solutions in the spin-0 case arise just from gauge invariance, they stay valid even in the presence of the higher derivative terms. As in the case of the spin-2 mode, there could as well be additional solutions which do not decay, but we will assume that they are either not turned on due to the initial conditions, or are of oscillatory type arising due to additional massive particles, which do not invalidate the arguments for the Ward identities.

3.7 Discussion

In this chapter, we have derived the Ward identities that arise from scale and special conformal transformations for single field inflation. Our results are given in eq.(3.49) and eq.(3.62) for the scalar perturbations, and eq.(3.51) and eq.(3.65) for the tensor perturbations. Similar results for mixed correlators can also be easily obtained, see eq.(3.52) and eq.(3.68).

The Ward identities for the special conformal transformations also involve a contribution due to a compensating spatial reparametrization, as explained in section 3.3.2. The underlying reason for this is that we are working with local correlators in a quantum theory of gravity. Such correlators can be defined in perturbation theory after suitable gauge fixing, but a compensating spatial reparametrization must then be carried out to preserve the gauge, for deriving the Ward identities of special conformal transformations [27]. The Ward identities for scale invariance do not require such a compensating transformation and are well known in the literature already, [24], and called the Maldacena consistency conditions.

The Ward identities we obtain also incorporate the breaking of the $O(1,4)$ symmetry. In fact, this breaking is incorporated to all orders in the slow roll parameters. The resulting relations can be thought of as being the analogues of the Callan-Symanzik equation, but now for both scale and special conformal transformations.

When the slow roll conditions are approximately valid, the Ward identities impose useful constraints on the correlation functions. The coefficient functions which appear in the wave function, and which transform in a manner analogous to correlation functions in a conformal field theory, can then be constrained order by order in the slow roll approximation, and the resulting constraints on the expectation values, in agreement with the Ward identities, can then be obtained. For the scalar three point function, which is observationally the most important one for non-Gaussianity, this was discussed in [1].

We work in a theory where the degrees of freedom are the metric and a scalar field. However, it is worth emphasizing that our results are also valid in situations where there are extra massive fields present during inflation, with masses of order the Hubble scale, or even higher. The wave function in the presence of such fields must still meet the equations of motion imposed by varying the shift and lapse functions, and thus must be invariant under the spatial reparametrizations discussed in section 3.4.1, in the gauge eq.(3.17) at late times. As a result, after the heavy fields are integrated out in deriving the expectation values for ζ and $\widehat{\gamma}_{ij}$, in the step analogous to eq.(3.33), the same Ward identities as before, eq.(3.49), eq.(3.51), eq.(3.62) and eq.(3.65) are obtained.

In contrast, our results are not valid when there are additional scalar fields which are much lighter than the Hubble scale, as in multi-field models of inflation. In this case, it is well known that the results are model dependent, and do not follow just from the underlying symmetries.[9]

In the context of AdS physics, space-times where conformal symmetry is broken are important in the study of condensed matter physics and QCD. Examples include Lifshitz and hyperscaling violating geometries. The Ward identities for the stress tensor etc. can be obtained in such situations in a way completely analogous to what we have used above. Some discussion of the identities in such situations can be found in [127].

The scalar three point function: Since the scalar three point function is of the greatest interest as a test of non-Gaussianity, let us end by commenting on it in some more detail. The Ward identities of interest here are eq.(3.49) and eq.(3.62) for $n = 3$, and relate the scalar 3 and 4 point correlators. These Ward identities were studied to the leading order in the slow roll expansion in [1]. The resulting relations in terms of coefficient functions are given in eq.(3.24) and eq.(3.25) of [1],

$$\left(\sum_{a=1}^{3} \boldsymbol{k_a} \cdot \frac{\partial}{\partial \boldsymbol{k_a}}\right) \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle = \frac{\dot{\bar{\phi}}}{H} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3}) O(\boldsymbol{k_4})\rangle\Big|_{\boldsymbol{k_4}\to 0}, \tag{3.170}$$

and

$$\begin{aligned} \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_1}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_2}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k_3}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' \\ = 2\,\frac{\dot{\bar{\phi}}}{H}\left[\boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{k_4}}\right] \left\{ \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3}) O(\boldsymbol{k_4})\rangle'\Big|_{\boldsymbol{k_4}\to 0} \right\}, \end{aligned} \tag{3.171}$$

with $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ defined in eq.(3.92). Note that in the leading slow roll approximation, the four point coefficient function $\langle OOOO\rangle$ can be calculated in the conformally invariant limit [27]. As a result of the factor of $\dot{\bar{\phi}}$ on the RHS of eq.(3.170) and eq.(3.171), the three point function $\langle OOO\rangle$ will be suppressed. Converting to expectation values, one gets that

$$\frac{\langle \zeta\zeta\zeta\rangle}{\langle \zeta\zeta\rangle^2} \sim \epsilon, \tag{3.172}$$

where the slow roll parameter ϵ is given in eq.(3.30). Although the functional form one will get in general is different, this roughly corresponds to

$$f_{NL} \sim \epsilon. \tag{3.173}$$

It is well known that the parameter r which measures the ratio of the power in the tensor

[9] See [132] for interesting progress in this direction.

to scalar perturbations is given by [10]

$$r \equiv \frac{P_t(k)}{P_\zeta(k)} = 16\epsilon. \tag{3.174}$$

Note that in theories which are not of the type described by a canonical model of inflation, eq.(3.132), e.g., those involving higher derivative corrections, ϵ_1 and ϵ as defined in eq.(3.28) and eq.(3.30) need not be the same. In these theories also, eq.(3.172) with the definition of ϵ given in eq.(3.30) is still valid to leading order in the slow roll parameters. We see from eq.(3.173) and eq.(3.174) that there is therefore an interesting tie-in between the ratio of power in the scalar and tensor perturbations, and the non-Gaussianity. This connection is well known in the canonical slow roll models, but we see here that it is more general, since eq.(3.173) follows from symmetry considerations alone.

The estimate in eq.(3.172) should actually be thought of as a lower bound. A contribution due to an intermediate graviton (or the stress energy tensor running as an intermediate in the $\langle OOOO\rangle$ correlator) will give a contribution of this order to the non-Gaussianity. However, as has been emphasized in [66], if there are additional particles of mass of order the Hubble scale which couple more strongly than the graviton, the contribution can be even bigger.[11]

Keeping the above considerations in mind we can phrase this tie-in between the two scales as follows. If tensor perturbations are observed in the future, so that ϵ is known, we would have a firm prediction on a lower bound on non-Gaussianity that follows only from conformal invariance. On the other hand, if the non-Gaussianity is observed and found to be of a bigger magnitude than the bound on ϵ that arises from constraints on the tensor perturbations, eq.(3.174), then it would rule out the scenario of approximate conformal invariance. More correctly, it will rule out this scenario together with the assumption that particles which appear as intermediate states, and contribute to the non-Gaussianity, couple to the inflaton only with gravitational strength.

[10] The tensor and scalar power spectra $P_t(k), P_\zeta(k)$ are

$$P_t(k) = \frac{H^2}{M_{Pl}^2}\frac{4}{k^3}, \text{ and } P_\zeta(k) = \frac{H^2}{M_{Pl}^2}\frac{1}{\epsilon}\frac{1}{4k^3}.$$

[11] Similarly, in theories where conformal invariance is violated to a significant extent, the non-Gaussianity can be bigger, e.g. in DBI inflation.

Chapter 4

Applications and consequences of the Ward identities

4.1 Introduction

In this chapter, we continue to explore the symmetry properties of the correlation functions of the perturbations produced during inflation. In particular, we discuss some applications and consequences of the Ward identities derived in the previous chapters. The present chapter is largely based on [3], and is organized as follows. Section 4.2 provides a quick recap of some properties of de Sitter space relevant for the present chapter, and helps set-up the notation. In section 4.3, we consider a class of models which are not of the standard slow roll type. Instead, in these models, called generalized single field models, the inflaton can roll quickly in units of the Hubble parameter, H, while the spacetime is still approximately de Sitter space. Using the earlier calculations of the three and four point correlation functions of scalar perturbations in these models, [133] and [134], we explicitly check that the Ward identities derived and discussed in the previous chapters are valid for this class of models as well.

In section 4.4 we discuss the scalar three point function in slow roll inflation in some detail. This correlation function, which is observationally most significant in the study of non-Gaussianity, was first calculated in [24] using the in-in formalism. The Ward identities suggest a somewhat different way to calculate this correlation function. These identities relate the three point function to the scalar four point function in a particular limit, with the coefficient of the four point function being suppressed by a power of the slow roll parameter $\dot{\bar{\phi}}/H$, [1, 2]. This suggests that the leading slow roll result for the three point function can be calculated from the four point function in the de Sitter approximation (where the slow roll parameters can be set to vanish). We make this explicit in section 4.4 by carrying out the calculation along these lines. We show that replacing one of the legs in the four point function calculation in de Sitter space with a factor of the slow roll parameter $\dot{\bar{\phi}}/H$ does give the correct result for the three point function. This way of thinking about the three and higher point correlators, motivated by the AdS/CFT correspondence [72, 73, 74], and the resulting discussion of the Ward identities was implicit in some of the earlier literature, [27], and has also played an important role in the recent discussions in [66]. See also [67, 70, 135].

We end with a discussion in section 4.5. Appendices C.1 and C.2 provide additional details.

Notation: The Planck mass is given by $M_{Pl} = 1/\sqrt{8\pi G}$. We denote the conformal time coordinate by η. Spatial three vectors are denoted by boldface letters, e.g. $\boldsymbol{x}, \boldsymbol{k}$ etc. $\boldsymbol{k_a}, a = 1, 2, \ldots$ denotes the momentum vectors $\boldsymbol{k_1}, \boldsymbol{k_2}, \ldots$ etc, whereas $k^i, i = 1, 2, 3$ denotes the components of $\boldsymbol{k}$. The magnitude of a vector is denoted by the corresponding ordinary letter, e.g. $x \equiv |\boldsymbol{x}|$. A dot above a quantity denotes ordinary time derivative, e.g. $\dot{f} \equiv df/dt$.

4.2 Some properties of de Sitter space

We start by presenting some key properties of de Sitter space, which we will refer to throughout the rest of the chapter. Four dimensional de Sitter space in planar coordinates is given by the line element

$$ds^2 = -dt^2 + e^{2Ht} dx^i dx_i, \tag{4.1}$$

where $-\infty < t, x^i < \infty$. In our calculations, we will make use of the conformal time coordinate η, given by

$$\eta = -\frac{1}{H} e^{-Ht}, \tag{4.2}$$

where $-\infty < \eta \leq 0$. The line element in eq.(4.1) then takes the form

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^i dx_i). \tag{4.3}$$

Note that the coordinates $(t, \boldsymbol{x})$ or $(\eta, \boldsymbol{x})$ cover only half of de Sitter space.

Four dimensional de Sitter space has the following isometries,

(i) Translations: $x^i \to x^i + \epsilon^i$; (4.4a)

(ii) Rotations: $x^i \to x^i + \omega^i_{\ j} x^j$, $\omega_{ij} = -\omega_{ji}$; (4.4b)

(iii) Dilatations: $\eta \to (1+\epsilon)\eta$, $x^i \to (1+\epsilon)x^i$; (4.4c)

(iv) Special Conformal Transformations: $\eta \to (1 + 2\boldsymbol{b}\cdot\boldsymbol{x})\eta$,
$$x^i \to x^i + 2\boldsymbol{b}\cdot\boldsymbol{x}\, x^i + b^i(\eta^2 - \boldsymbol{x}^2). \tag{4.4d}$$

Here, the parameters $\epsilon^i, \omega_{ij}, \epsilon$ and b^i are all infinitesimal. These isometries impose important constraints on the correlation functions of inflationary perturbations.

4.3 Conformal invariance and general single field models of inflation

The canonical single field slow roll models of inflation are characterized by the action

$$S = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}\,\Big(R - g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - 2V(\phi)\Big), \tag{4.5}$$

where $V(\phi)$ is the potential for the inflaton ϕ. For inflation to occur, the potential is assumed to be approximately flat over a range of values for ϕ. The inflaton evolves slowly along this flat part during inflation, leading to exponential expansion. In the homogeneous limit, the inflaton is purely a function of time, $\phi \equiv \bar\phi(t)$, and the metric is the unperturbed FRW metric,

$$ds^2 = -\,dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{4.6}$$

where $a(t)$ is the scale factor of the universe. The metric can equivalently be expressed in terms of the conformal time coordinate η as[1]

$$ds^2 = a(\eta)^2\left(-\,d\eta^2 + \delta_{ij}dx^i dx^j\right). \tag{4.7}$$

Also, the Hubble parameter is given by

$$H = \frac{\dot a}{a}. \tag{4.8}$$

The homogeneous field $\bar\phi$ satisfies the equations

$$\begin{aligned} 3H^2 &= \frac{1}{2}\dot{\bar\phi}^2 + V(\bar\phi),\\ \dot H &= -\frac{1}{2}\dot{\bar\phi}^2,\\ 0 &= \ddot{\bar\phi} + 3H\dot{\bar\phi} + \frac{dV(\bar\phi)}{d\bar\phi}. \end{aligned} \tag{4.9}$$

The slow roll conditions are imposed by setting the slow roll parameters ϵ_1, δ_1 to be much less than unity, where

$$\epsilon_1 = -\frac{\dot H}{H^2}, \tag{4.10}$$

$$\delta_1 = \frac{\ddot H}{2H\dot H}. \tag{4.11}$$

The slow roll criterion $\epsilon_1, \delta_1 \ll 1$ ensures that the universe remains approximately de Sitter during the inflationary phase. The slow roll conditions can also be expressed in terms of

[1] The general relation between the ordinary time t and the conformal time η is

$$d\eta = \frac{dt}{a(t)}.$$

the slow roll parameters ϵ, δ, where

$$\epsilon = \frac{1}{2}\frac{\dot{\bar{\phi}}^2}{H^2}, \tag{4.12}$$

$$\delta = \frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}}. \tag{4.13}$$

Note that $\epsilon = \epsilon_1$ and $\delta = \delta_1$ in the canonical slow roll model due to the background equations eq.(4.9). Another set of slow roll parameters are the potential slow roll parameters, defined by

$$\begin{aligned} \epsilon_{\rm v} &= \frac{1}{2}\left(\frac{V'}{V}\right)^2, \\ \eta_{\rm v} &= \frac{V''}{V}. \end{aligned} \tag{4.14}$$

In the slow roll approximation, these are also related to the parameters ϵ_1, δ due to the background equations eq.(4.9), via $\epsilon_{\rm v} = \epsilon_1$ and $\eta_{\rm v} = \epsilon_1 - \delta$.

Perturbations to the homogeneous situation discussed above are introduced in the ADM formalism. The metric in the ADM formalism takes the form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{4.15}$$

where h_{ij} is the induced metric on the spatial three surface labeled by time t, and N, N^i are the lapse and shift functions, respectively. One needs to make a choice of gauge to fix the diffeomorphism invariance of the theory. A convenient choice is the *synchronous gauge*, defined by imposing the conditions

$$N = 1, N^i = 0. \tag{4.16}$$

The perturbed metric in this gauge has the form

$$\begin{aligned} h_{ij} &= a^2\left[(1+2\zeta)\delta_{ij} + \widehat{\gamma}_{ij}\right], \\ \widehat{\gamma}_{ii} &= 0. \end{aligned} \tag{4.17}$$

where $\zeta, \widehat{\gamma}_{ij}$ are the scalar and tensor perturbations in the metric, respectively, with $\widehat{\gamma}_{ij}$ being traceless. The perturbed inflaton is given by

$$\phi = \bar{\phi}(t) + \delta\phi(t, \boldsymbol{x}). \tag{4.18}$$

Note that in the ADM formalism ϕ, h_{ij} are the dynamical variables, whereas N, N^i are Lagrange multipliers. One thus needs to impose the equations of motion of N, N^i as constraints in the gauge eq.(4.16). In the wave function of the universe approach, the equations of motion of N, N^i correspond to time and spatial reparametrization invariance of the wave function. In [2], as discussed in the previous chapters, general Ward identities were derived for single field models of inflation as a consequence of these reparametrization invariance constraints. These Ward identities are satisfied by the correlation functions of the curvature perturbation ζ, and the transverse and traceless tensor perturbations $\widehat{\gamma}_{ij}$.[2] For instance, for a scaling transformation, we have

$$\begin{aligned}\left(3(n-1)+\sum_{a=1}^{n} k_a \frac{\partial}{\partial k_a}\right)\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_n})\rangle' = \\ -\frac{1}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_{n+1}})\rangle'\Big|_{\boldsymbol{k_{n+1}}\to 0},\end{aligned} \tag{4.19}$$

where a $'$ on a correlation function denotes the suppression of the overall momentum conserving δ-function; for e.g.

$$\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_n})\rangle = (2\pi)^3\delta^3\Big(\sum_{a=1}^{n}\boldsymbol{k_a}\Big)\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_n})\rangle'. \tag{4.20}$$

Similarly, for special conformal transformations, we have the Ward identity

$$\begin{aligned}\big\langle\delta(\zeta(\boldsymbol{k_1}))\cdots\zeta(\boldsymbol{k_n})\big\rangle+\cdots+\big\langle\zeta(\boldsymbol{k_1})\cdots\delta(\zeta(\boldsymbol{k_n}))\big\rangle = \\ -2\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k_{n+1}}}\right)\frac{\langle\zeta(\boldsymbol{k_1})\cdots\zeta(\boldsymbol{k_{n+1}})\rangle}{\langle\zeta(\boldsymbol{k_{n+1}})\zeta(-\boldsymbol{k_{n+1}})\rangle'}\Big|_{\boldsymbol{k_{n+1}}\to 0},\end{aligned} \tag{4.21}$$

where $\delta(\zeta(\boldsymbol{k}))$ is given by

$$\begin{aligned}\delta(\zeta(\boldsymbol{k})) = \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}}\,\zeta(\boldsymbol{k}) + 6\,b^m k^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\zeta(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,\widehat{\gamma}_{im}(\tilde{\boldsymbol{k}}) \\ + 2\,b^m k^i\int\frac{d^3\tilde{k}}{(2\pi)^3}\,\frac{1}{\tilde{k}^2}\,\widehat{\gamma}_{ij}(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,\widehat{\gamma}_{jm}(\tilde{\boldsymbol{k}}),\end{aligned} \tag{4.22}$$

and the operator $\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}}$ is given by

$$\widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k}} = 2\left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right) - (\boldsymbol{b}\cdot\boldsymbol{k})\left(\frac{\partial}{\partial\boldsymbol{k}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right) + 6\left(\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right). \tag{4.23}$$

Here, we have reproduced the Ward identities satisfied by the correlation functions of ζ. Similar Ward identities are satisfied by the correlation functions of the tensor perturbation $\widehat{\gamma}_{ij}$ as well. For details on the derivation of the Ward identities see the previous chapters.

We would now like to check the validity of these Ward identities for more general single field models of inflation, where the matter part of the Lagrangian density is an arbitrary function of the scalar field ϕ and its first derivatives. These models of inflation follow from

[2] In the gauge eq.(4.16), the tensor perturbations $\widehat{\gamma}_{ij}$ can be made transverse, $\partial_i\widehat{\gamma}_{ij}=0$, at late times, using the spatial reparametrization $x^i \to x^i + v^i(\boldsymbol{x})$.

the action (see [136, 137])

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{Pl}^2 R + 2P(X,\phi)\right], \tag{4.24}$$

where

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{4.25}$$

and $P(X,\phi)$ is an arbitrary function of X,ϕ. The speed of sound parameter c_s which characterizes these general single field models of inflation is defined as

$$c_s^2 \equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}. \tag{4.26}$$

Clearly, for the canonical slow roll model of inflation, where $P(X,\phi) = X - V(\phi)$, the speed of sound is $c_s = 1$. For models with more general form of the matter Lagrangian $P(X,\phi)$ than the canonical slow roll model, we have $c_s \neq 1$.

In these general models of single field inflation, one defines three "slow variation parameters," given by ϵ_1, eq.(4.10), and

$$\eta_1 = \frac{\dot{\epsilon}_1}{\epsilon_1 H}, \tag{4.27}$$

$$s = \frac{\dot{c}_s}{c_s H}. \tag{4.28}$$

For inflation to occur, the three slow variation parameters must be small,

$$\epsilon_1, \eta_1, s \ll 1. \tag{4.29}$$

However, the parameter ϵ, eq.(4.12), which is small in the canonical slow roll model of inflation, need not be small for the more general models.[3]

[3]Consider, for instance, the DBI model of inflation [81, 82]. In this model, one has

$$P(X,\phi) = -\frac{1}{f(\phi)}\sqrt{1-2Xf(\phi)} + \frac{1}{f(\phi)} - V(\phi),$$

where the inflaton ϕ is the position of a D3 brane moving in a warped throat, and $f(\phi)$ is the warping factor. The energy density and pressure for this model are given by

$$\rho = \frac{1}{f(\phi)\sqrt{1-2Xf(\phi)}} - \frac{1}{f(\phi)} + V(\phi),$$

and

$$p = -\frac{1}{f(\phi)}\sqrt{1-2Xf(\phi)} + \frac{1}{f(\phi)} - V(\phi).$$

The speed of sound can be calculated using eq.(4.26), and is given by

$$c_s = \sqrt{1-2Xf(\phi)}.$$

Working in the homogeneous limit, the inflaton becomes purely a function of time, $\phi \equiv \phi(t)$. We then have $X = \dot{\phi}^2/2$. The speed of sound then becomes

$$c_s = \sqrt{1-\dot{\phi}^2 f(\phi)}.$$

The two, three and four point functions for the curvature perturbation ζ in these models of inflation have been computed explicitly. For our purpose, we follow the references [133, 134]. We have reproduced their results in appendix C.1 for completeness.

For the case of $n = 2$, the scaling Ward identity eq.(4.19) becomes,

$$\left(3 + \sum_{a=1}^{2} k_a \frac{\partial}{\partial k_a}\right) \langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle' = - \frac{\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle'}{\langle \zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\bigg|_{\boldsymbol{k_3}\to 0}, \tag{4.30}$$

which is also known as the Maldacena consistency condition. An explicit check for its validity in the $P(X, \phi)$ models of inflation was performed in [133], and it was found to hold true.

We now check the Ward identities eqs.(4.19) and (4.21) for the case of $n = 3$. Their explicit form is

$$\left(6 + \sum_{a=1}^{3} k_a \frac{\partial}{\partial k_a}\right) \langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = - \frac{\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle'}{\langle \zeta(\boldsymbol{k_4})\zeta(-\boldsymbol{k_4})\rangle'}\bigg|_{\boldsymbol{k_4}\to 0}, \tag{4.31}$$

for the scaling transformation, and for the special conformal transformation we have

$$\left(\sum_{a=1}^{3} \widehat{\mathcal{L}}^{\boldsymbol{b}}_{\boldsymbol{k_a}}\right) \langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle = -2\left(\boldsymbol{b}\cdot \frac{\partial}{\partial \boldsymbol{k_4}}\right) \frac{\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle}{\langle \zeta(\boldsymbol{k_4})\zeta(-\boldsymbol{k_4})\rangle'}\bigg|_{\boldsymbol{k_4}\to 0}. \tag{4.32}$$

Using the expressions for the three and four point functions given in appendix C.1, we performed a check of these Ward identities on Mathematica. We find that the scaling Ward identity is met, since the LHS and RHS of eq.(4.31) vanish individually. Similarly, we find

In this model, the inflaton evolves relativistically, and the parameter ϵ defined in eq.(4.12) is not small. Consequently, the speed of sound is very small, $c_s \ll 1$. This gives an approximate expression for $\dot{\phi}$,

$$|\dot{\phi}| \approx \frac{1}{\sqrt{f(\phi)}}.$$

Also, the Friedmann and continuity equations have the form

$$3M_{Pl}^2 H^2 = \rho,$$
$$\dot{\rho} = -3H(\rho + p).$$

Using these equations, one finds that the expression for the slow variation parameter ϵ_1, defined in eq.(4.10), is given by

$$\epsilon_1 = \frac{3\dot{\phi}^2}{2\left(\frac{1}{f}(1-c_s) + c_s V\right)} \approx \frac{3}{2}\frac{1}{1 + c_s f V},$$

where the approximate expression follows from the condition that $c_s \ll 1$. To get a de Sitter like phase of exponential expansion, one must have $\epsilon_1 \ll 1$. Thus, the potential must satisfy the condition

$$2c_s f V \gg 1.$$

that the special conformal Ward identity, eq.(4.32), is also met. As discussed in appendix C.1, the four point function has two parts, one coming from a contact interaction term and the other from an intermediate scalar exchange. An interesting point to note is that the dominant contribution to the RHS of eq.(4.32) in the limit $\boldsymbol{k_4} \to 0$ comes from the four point contact interaction term, whereas the intermediate scalar exchange contribution is subleading.

4.4 Bulk calculation of the scalar three point function in inflation

The three point function for the scalar perturbation ζ, eq.(4.17), in inflation was computed in [24] using the in-in formalism. In this section, we present an alternate approach for computing the same. This approach arises as a consequence of the scaling and special conformal Ward identities relating the three and four point functions at the leading order in the slow roll approximation, see [1]. The Ward identities suggest that the three point function must follow from a computation of the four point function, with one of the external legs replaced by the time derivative of the homogeneous background, $\dot{\bar{\phi}}$. Working in the Bunch-Davies vacuum, we will show that this is indeed the case, providing another check for the validity of the Ward identities. Our method follows the approach utilized in [27] for computing the inflationary four point function of ζ. The discussion here is also related to [66] and [67], where related ideas are used to examine the effect of higher spin fields on non-Gaussianity.

The present technique for computing the inflationary three point function relies on an important analogy between calculations in dS and AdS spaces. We first calculate the wave function, in terms of the late time values for the perturbations, and then the correlation functions can be computed from the wave function. To compute the inflationary three point function for ζ, we need to evaluate the wave function $\Psi[\delta\phi]$, where $\delta\phi$ is the perturbation to the inflaton, eq.(4.18), and is related to ζ by a change of gauge. The wave function $\Psi[\delta\phi]$ has the schematic form

$$\Psi[\delta\phi] = \exp\left[\frac{M_{Pl}^2}{H^2}\left(-\frac{1}{2}\int \delta\phi\delta\phi\,\langle OO\rangle + \frac{1}{3!}\int \delta\phi\delta\phi\delta\phi\,\langle OOO\rangle + \cdots\right)\right], \tag{4.33}$$

where

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,k_1^3, \tag{4.34}$$

and $\langle OOO\rangle, \ldots$ are the coefficient functions. Once we have the wave function $\Psi[\delta\phi]$, in particular the cubic coefficient $\langle OOO\rangle$, we can get the three point function for $\delta\phi$ by using

$$\langle\delta\phi\delta\phi\delta\phi\rangle = \frac{\int[\mathcal{D}\delta\phi]\,\delta\phi\delta\phi\delta\phi\,\big|\Psi[\delta\phi]\big|^2}{\int[\mathcal{D}\delta\phi]\,\big|\Psi[\delta\phi]\big|^2}, \tag{4.35}$$

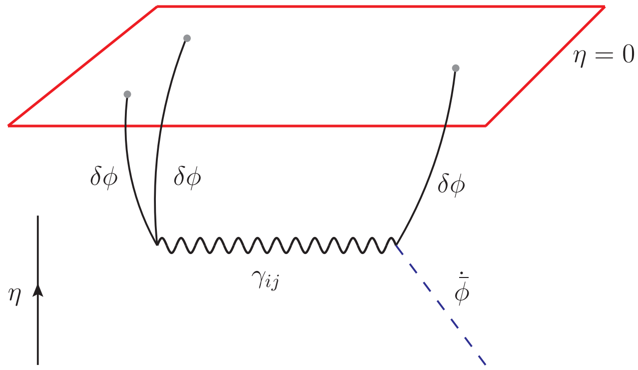


Figure 4.1: The three point function in de Sitter space, with one leg being the time-derivative of the background $\bar{\phi}$.

which is the standard quantum mechanical prescription to compute expectation values. Knowing the three point function for $\delta\phi$, we can get the three point function for ζ by simply performing a change of gauge. Thus, the whole computation boils down to computing the cubic term $\langle OOO \rangle$ in the wave function $\Psi[\delta\phi]$.

The analogy with AdS space also suggests that this cubic term can be calculated using the analogue of Feynman-Witten bulk-to-boundary propagators. In fact, as already mentioned, the three point function gets related to the four point function with one leg replaced by the background value of the inflaton, $\dot{\bar{\phi}}$, as shown in figure 4.1, see [1, 66]. The propagators in figure 4.1 are bulk-to-boundary propagators in de Sitter space, and the interaction vertex is given in figure 4.2, which arises by expanding the action about the inflationary background as will be explained shortly. In fact, since de Sitter space can be analytically continued to Euclidean AdS (EAdS) space, the whole calculation can be done conveniently by first working in EAdS space and then continuing the result back to de Sitter space. Note that the Bunch-Davies vacuum in de Sitter space corresponds to choosing the boundary condition that deep in the interior of EAdS space all perturbations become regular. This is the procedure we follow in the computation below.

To be more specific, consider the metric of four dimensional EAdS space in Poincare coordinates,

$$ds^2 = \frac{\mathrm{R}_{\mathrm{AdS}}^2}{z^2}\left(dz^2 + \sum_{i=1}^{3} dx_i\, dx^i\right), \tag{4.36}$$

where $\mathrm{R}_{\mathrm{AdS}}$ is the EAdS radius, and the coordinate $z \in [0, \infty)$. Under the analytic continuation

$$z = -i\eta, \tag{4.37}$$

and

$$\mathrm{R}_{\mathrm{AdS}} = \frac{i}{H}, \tag{4.38}$$

the metric eq.(4.36) goes to the metric of four dimensional de Sitter space, eq.(4.3). Also, the partition function in EAdS space is related to the wave function in de Sitter space via the same analytic continuation. In the semiclassical approximation, where one can replace the path integral involved in the calculation of the partition function or the wave function by its saddle point value, one can write

$$Z_{\text{EAdS}}[\Phi(x)] \equiv \mathrm{e}^{-S^{\text{EAdS}}_{on-shell}[\Phi(x)]} \xleftrightarrow[\mathrm{R}_{\text{AdS}}=\frac{i}{H}]{z=-i\eta} \Psi[\Phi(x)] \equiv \mathrm{e}^{iS^{\text{dS}}_{on-shell}[\Phi(x)]}, \tag{4.39}$$

where the EAdS partition function Z_{EAdS} is a functional of the boundary value of the field $\Phi(x)$ as $z \to 0$, whereas the wave function Ψ in de Sitter space is a functional of the late time value of the field $\Phi(x)$ as $\eta \to 0$. The other boundary condition imposed while computing the on-shell action is to demand regularity of the solution deep in the interior, $z \to \infty$, of EAdS; as mentioned above, this corresponds to the choice of Bunch-Davies vacuum in the far past, $\eta \to -\infty$, in de Sitter space.

4.4.1 Computing the coefficient function $\langle OOO \rangle$

In the EAdS space, with the metric given by eq.(4.36), we start with the action

$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{g} \Big(R - 2\Lambda - (\nabla\phi)^2 - 2V(\phi) \Big), \tag{4.40}$$

where $\phi(z, \boldsymbol{x}) = \bar{\phi}(z) + \delta\phi(z, \boldsymbol{x})$ is the inflaton written in AdS coordinates, and Λ is the cosmological constant, which is related to R_{AdS} by

$$\Lambda = -\frac{3}{\mathrm{R}^2_{\text{AdS}}}. \tag{4.41}$$

We expand the metric perturbatively as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{4.42}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed background metric given in eq.(4.36), and $\delta g_{\mu\nu}$ is the perturbation. Substituting eq.(4.42) into the action eq.(4.40) and expanding, we get

$$S = S_0 + S^{(2)}_{\text{grav}} - \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \partial_\mu(\delta\phi)\partial_\nu(\delta\phi) + S_{int}, \tag{4.43}$$

where S_0 is the action for the unperturbed background, $S^{(2)}_{\text{grav}}$ is the part of the action which is quadratic in the metric perturbation $\delta g_{\mu\nu}$, and S_{int} is the interaction term, given by

$$S_{int} = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{\bar{g}}\, \delta g_{\mu\nu} T^{\mu\nu}, \tag{4.44}$$

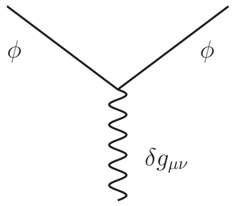


Figure 4.2: The qualitative three point bulk interaction vertex between two scalars and a graviton.

where the energy-momentum tensor $T_{\mu\nu}$ for the scalar field ϕ is given by

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\,\bar{g}_{\mu\nu}\Big(\bar{g}^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + 2V(\phi)\Big). \tag{4.45}$$

The interaction term eq.(4.44) gives rise to an interaction vertex between two scalars and a graviton, depicted qualitatively in figure 4.2.

From $S^{(2)}_{\rm grav}$, we can compute the propagator for the graviton. For doing so, we choose the gauge [4]

$$\delta g_{zz} = 0,\ \delta g_{zi} = 0, \tag{4.46}$$

with $i = 1, 2, 3$. The graviton propagator in this gauge is given by [138, 139]

$$\begin{aligned}&\mathcal{G}_{ij,kl}(z_1, \boldsymbol{x_1}; z_2, \boldsymbol{x_2})\\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x_1}-\boldsymbol{x_2})}\int_0^\infty \frac{dp^2}{2}\left[\frac{J_{\frac{3}{2}}(pz_1)J_{\frac{3}{2}}(pz_2)}{\sqrt{z_1z_2}(\boldsymbol{k}^2+p^2)}\frac{1}{2}(\mathcal{T}_{ik}\mathcal{T}_{jl}+\mathcal{T}_{il}\mathcal{T}_{jk}-\mathcal{T}_{ij}\mathcal{T}_{kl})\right],\end{aligned} \tag{4.47}$$

where

$$\mathcal{T}_{ij} = \delta_{ij} + \frac{k_ik_j}{p^2}. \tag{4.48}$$

Note that the graviton propagator in eq.(4.47) is not transverse. We can however decompose it into a transverse part and a longitudinal part. The transverse graviton propagator is given by

$$\begin{aligned}&\tilde{\mathcal{G}}_{ij,kl}(z_1, \boldsymbol{x_1}; z_2, \boldsymbol{x_2})\\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x_1}-\boldsymbol{x_2})}\int_0^\infty \frac{dp^2}{2}\left[\frac{J_{\frac{3}{2}}(pz_1)J_{\frac{3}{2}}(pz_2)}{\sqrt{z_1z_2}(\boldsymbol{k}^2+p^2)}\frac{1}{2}(\tilde{\mathcal{T}}_{ik}\tilde{\mathcal{T}}_{jl}+\tilde{\mathcal{T}}_{il}\tilde{\mathcal{T}}_{jk}-\tilde{\mathcal{T}}_{ij}\tilde{\mathcal{T}}_{kl})\right],\end{aligned} \tag{4.49}$$

where

$$\tilde{\mathcal{T}}_{ij} = \delta_{ij} - \frac{k_ik_j}{k^2}. \tag{4.50}$$

The longitudinal part is then essentially the difference between the full propagator, eq.(4.47), and the transverse piece, eq.(4.49).

[4]This gauge naturally goes over to the gauge $N = 1, N^i = 0$ under analytic continuation, which is used in our inflationary calculations.

From eq.(4.43), we see that the scalar field $\delta\phi$ behaves essentially like a free scalar field in EAdS space, with only gravitational interactions. Thus for a particular momentum mode carrying momentum $\boldsymbol{k}$, we have

$$\delta\phi_{\boldsymbol{k}}(x, \boldsymbol{z}) = \phi_0(\boldsymbol{k})(1 + kz)\,\mathrm{e}^{-kz}\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4.51}$$

where we have chosen the solution for $\delta\phi$ which is regular as $z \to \infty$.

To calculate the EAdS partition function, we need to compute the on-shell action. In particular, to extract the unknown coefficient $\langle OOO \rangle$, we need to evaluate the contribution from the Feynman-Witten diagrams of figure 4.3. This contribution is given by

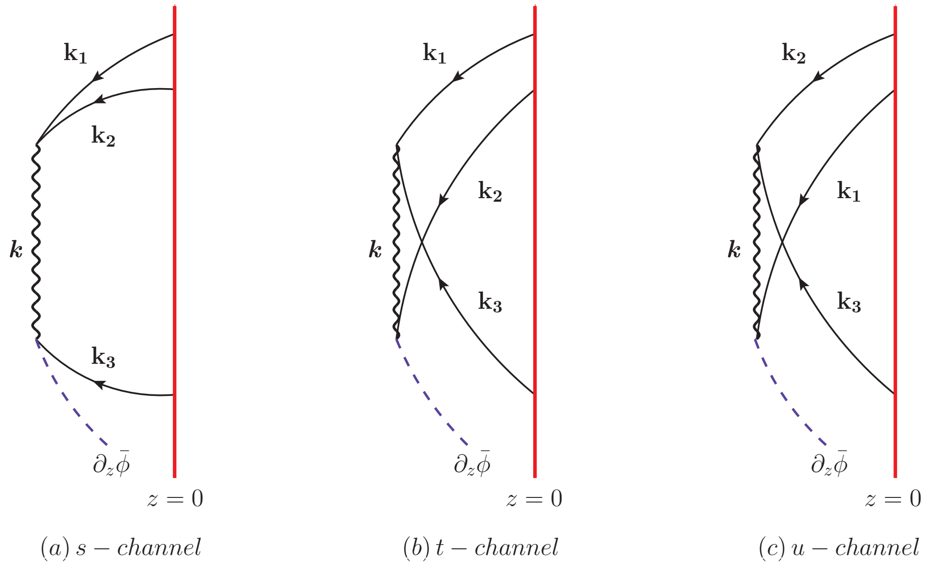


Figure 4.3: Feynman-Witten diagrams for the s, t, and u channel processes contributing to the calculation of the EAdS partition function. $z = 0$ is the EAdS boundary. The wavy line denotes the bulk-to-bulk graviton propagator, the solid lines represent the bulk-to-boundary propagators for the scalar field $\delta\phi$, and the dashed line represents the background. The exchanged graviton carries momentum $\boldsymbol{k}$.

$$\begin{aligned} S^{EAdS}_{on-shell} = \frac{1}{2}\, M^2_{\mathrm{Pl}}\, \mathrm{R}^2_{\mathrm{AdS}} \int \frac{dz_1}{z_1^4}\frac{dz_2}{z_2^4}\, d^3x_1 d^3x_2\, \bar{g}^{i_1 i_2}\bar{g}^{j_1 j_2} T_{i_1 j_1}(z_1, \boldsymbol{x_1})\times \\ \mathcal{G}_{i_2 j_2, k_2 l_2}(z_1, \boldsymbol{x_1}; z_2, \boldsymbol{x_2})\bar{g}^{k_1 k_2}\bar{g}^{l_1 l_2} T_{k_1 l_1}(z_2, \boldsymbol{x_2}). \end{aligned} \tag{4.52}$$

As already discussed above, we can write the graviton propagator as a sum of a transverse and a longitudinal part. This gives us

$$S^{EAdS}_{on-shell} = \frac{1}{2}\, M^2_{\mathrm{Pl}}\, \mathrm{R}^2_{\mathrm{AdS}}\, (\mathcal{W} + 2\mathcal{R}), \tag{4.53}$$

where $\mathcal{W}$ is the transverse graviton contribution,

$$\mathcal{W} = \int dz_1 dz_2\, d^3x_1 d^3x_2\, T_{ij}(z_1, \boldsymbol{x_1}) \tilde{\mathcal{G}}_{ij,kl}(z_1, \boldsymbol{x_1}; z_2, \boldsymbol{x_2}) T_{kl}(z_2, \boldsymbol{x_2}), \tag{4.54}$$

where we have used $\bar{g}^{ij} = z^2 \delta^{ij}$. The contribution to the on-shell action from the longitudinal part of the exchanged graviton is written in the form of a "remainder" term $\mathcal{R}$, which has the form

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \tag{4.55}$$

with $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ given by

$$\begin{aligned}
\mathcal{R}_1 &= -\int \frac{dz}{z^2}\, d^3x\, T_{zj}(z, \boldsymbol{x})\, \frac{1}{\partial^2}\, T_{zj}(z, \boldsymbol{x}), \\
\mathcal{R}_2 &= -\frac{1}{2} \int \frac{dz}{z}\, d^3x\, \partial_j T_{zj}(z, \boldsymbol{x})\, \frac{1}{\partial^2}\, T_{zz}(z, \boldsymbol{x}), \\
\mathcal{R}_3 &= -\frac{1}{4} \int \frac{dz}{z^2}\, d^3x\, \partial_j T_{zj}(z, \boldsymbol{x}) \left(\frac{1}{\partial^2}\right)^2 \partial_i T_{zi}(z, \boldsymbol{x}).
\end{aligned} \tag{4.56}$$

We can now perform the computation of the on-shell action. Some details of the calculation are given in appendix C.2. The contribution from the transverse part of the graviton exchanged vanishes, see appendix C.2.2. The contribution from the longitudinal part is calculated in appendix C.2.1, and is given by

$$\begin{aligned}
\mathcal{R} = \frac{1}{2} \frac{\dot{\phi}}{H} (2\pi)^3 \delta^3 \Big(\sum_{a=1}^{3} \boldsymbol{k_a}\Big) \Big(\prod_{a=1}^{3} \phi_0(\boldsymbol{k_a})\Big) \times \\
\Bigg[-\frac{1}{2} \sum_{a=1}^{3} k_a^3 + \frac{1}{2} \sum_{a \neq b} k_a k_b^2 + \frac{4}{K} \sum_{a<b} k_a^2 k_b^2 \Bigg].
\end{aligned} \tag{4.57}$$

Using eq.(4.53), we see that the EAdS on-shell action is

$$\begin{aligned}
S^{EAdS}_{on-shell} = \frac{1}{2} M_{\mathrm{Pl}}^2 \mathrm{R}_{\mathrm{AdS}}^2 \frac{\dot{\phi}}{H} (2\pi)^3 \delta^3 \Big(\sum_{a=1}^{3} \boldsymbol{k_a}\Big) \Big(\prod_{a=1}^{3} \phi_0(\boldsymbol{k_a})\Big) \times \\
\Bigg[-\frac{1}{2} \sum_{a=1}^{3} k_a^3 + \frac{1}{2} \sum_{a \neq b} k_a k_b^2 + \frac{4}{K} \sum_{a<b} k_a^2 k_b^2 \Bigg].
\end{aligned} \tag{4.58}$$

From eq.(4.58), by taking derivatives with respect to the boundary value ϕ_0 of the field $\delta\phi$, we obtain the three point coefficient function $\langle OOO \rangle$,

$$\begin{aligned}
\langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3}) \rangle = -\frac{1}{2} \frac{\dot{\phi}}{H} (2\pi)^3 \delta^3 \Big(\sum_{a=1}^{3} \boldsymbol{k_a}\Big) \times \\
\Bigg[-\frac{1}{2} \sum_{a=1}^{3} k_a^3 + \frac{1}{2} \sum_{a \neq b} k_a k_b^2 + \frac{4}{K} \sum_{a<b} k_a^2 k_b^2 \Bigg],
\end{aligned} \tag{4.59}$$

where we have made use of the analytic continuation eq.(4.38).

4.4.2 The three point function $\langle\zeta\zeta\zeta\rangle$

We now proceed to compute the inflationary three point function $\langle\zeta\zeta\zeta\rangle$. From the wave function eq.(4.33), one finds that

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle = \frac{1}{4}\frac{H^4}{M_{\rm Pl}^4}\frac{\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle}{\prod_{a=1}^{3}\langle O(\boldsymbol{k_a})O(-\boldsymbol{k_a})\rangle'}. \tag{4.60}$$

Substituting the result eq.(4.59) in eq.(4.60), and using eq.(4.34), we get

$$\begin{aligned}\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle = -\frac{1}{8}\frac{H^4}{M_{\rm Pl}^4}\frac{\dot\phi}{H}(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\frac{1}{k_a^3}\Big)\times \\ \Big[-\frac{1}{2}\sum_{a=1}^{3}k_a^3+\frac{1}{2}\sum_{a\neq b}k_ak_b^2+\frac{4}{K}\sum_{a<b}k_a^2k_b^2\Big].\end{aligned} \tag{4.61}$$

Now, to obtain the three point function for the perturbation ζ, we need to perform a change of gauge from $\delta\phi$ to ζ in eq.(4.61). The second order change of gauge relating $\delta\phi$ and ζ is given by (see [24] for details)

$$\zeta = -\frac{H}{\dot\phi}\delta\phi + \frac{1}{2}\Big(\frac{1}{2}+\frac{\ddot\phi H}{\dot\phi^3}\Big)\delta\phi^2. \tag{4.62}$$

Performing the change of gauge eq.(4.62) in eq.(4.61), we get the final result

$$\begin{aligned}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle = (2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\frac{1}{(2k_a^3)}\Big)\frac{H^4}{M_{\rm Pl}^4}\frac{H^2}{\dot\phi^2}\times \\ \Big[\Big(\frac{1}{2}+\frac{2H\ddot\phi}{\dot\phi^3}\Big)\sum_{a=1}^{3}k_a^3+\frac{1}{2}\sum_{a\neq b}k_ak_b^2+\frac{4}{K}\sum_{a<b}k_a^2k_b^2\Big],\end{aligned} \tag{4.63}$$

which is indeed the expression for the three point function of ζ computed in [24].

The present method of calculation also provides us with an understanding of which region in the bulk makes a significant contribution to the late time $\eta\to 0$ correlation functions. E.g. for the three point function, when $k_3\to 0$ keeping k_1, k_2 fixed, the corresponding bulk-to-boundary propagators for the scalars, shown in figure 4.1, go deep inside the bulk. Thus the behaviour in the far past becomes important. This is a version of the UV-IR connection in de Sitter or inflationary space times, and has also been discussed in [66].

4.5 Discussion

We have studied the Ward identities for scale and special conformal transformations in the context of inflation and de Sitter space in this chapter. It was argued earlier in chapters 2, 3 that these Ward identities follow from the coordinate reparametrization symmetries of the system. The coordinate reparametrization invariance can be used to set the perturbation in the inflaton to vanish, $\delta\phi = 0$, at late times. The resulting perturbations in single field models then correspond to scalar perturbations ζ, and tensor perturbations $\widehat{\gamma}_{ij}$ in the metric. The residual spatial reparametrization symmetries present give rise to Ward identities for the correlation functions of these perturbations.

For generalized models of single field inflation, it was shown here that the Ward identities are indeed valid, as would be expected from the general nature of the arguments leading to these identities. We should mention that some of these Ward identities were checked in an earlier work [103].

Finally, we described an alternate calculation for the three point function for scalar perturbations in standard slow roll inflation in the Bunch-Davies vacuum. This calculation is motivated by techniques drawn from the AdS/CFT correspondence and is related to other recent papers, including [27, 66, 67, 70], and could be useful in thinking about the implications of additional fields during inflation, including those with higher spin. See also [68, 71, 135, 140].

The additional checks presented in this chapter put the Ward identities on a very solid footing. They thus indeed provide robust model independent constraints on single field models of inflation.

Chapter 5

Conclusions

The aim of the thesis was to study the constraints imposed by underlying symmetries on correlation functions of perturbations in single field models of inflation. Inflation is the dominant paradigm to explain the approximate isotropy and homogeneity of the universe. Inflation also gives rise to quantum perturbations which lead to the observed anisotropy in the Cosmic Microwave Background, and which seed the formation of large scale structure in the universe.

During inflation, the spacetime is well approximated by four dimensional de Sitter space, which is a maximally symmetric FRW cosmology, with the symmetry group $O(1,4)$. The time evolution of the inflaton and its back-reaction on the metric breaks these symmetries, but this breaking is small if the slow roll conditions are satisfied. The symmetry algebra of $O(1,4)$ is the same as the symmetry algebra of a three dimensional Euclidean Conformal Field Theory. Because of this, we referred to the $O(1,4)$ symmetry of de Sitter space as the conformal symmetry group. It includes translations and rotations along the spatial directions, as well as a scale transformation, and three special conformal transformations.

In chapter 2, single field slow roll inflation was studied at the leading order, and it was shown that the symmetry constraints on the correlation functions of scalar and tensor perturbations can be expressed in terms of the Ward identities of conformal invariance. More precisely, in the de Sitter limit, where the slow roll parameters can be set to vanish, we obtained Ward identities for exact conformal invariance. Incorporating departures from exact conformal invariance by taking into account the non-vanishing of the slow roll parameter $\dot{\bar{\phi}}/H$ then gave rise to Ward identities which included the breaking of the de Sitter symmetries. The scaling Ward identities gave rise to the Maldacena consistency condition, and additional similar constraints arise due to the special conformal transformations.

Further study, presented in chapter 3, showed that the Ward identities follow from the constraints of reparametrization invariance and should be more generally valid. This allows the breaking of conformal invariance during inflation, due to the evolution of inflaton, to be incorporated systematically even beyond the leading order in the slow roll parameters. After appropriate gauge fixing, we argued that the leftover spatial reparametrization invariance can be used to derive Ward identities for scale and special conformal transformations. The Ward identities so obtained are valid to all orders in slow roll expansion. The derivation of the Ward identities for special conformal transformations required an additional field-dependent compensating spatial reparametrization to maintain the gauge choice.

The situation is analogous to what happens in a field theory which is not scale invariant. The correlations in such a theory still satisfy the Callan-Symanzik equation, which now involves contributions due to the non-vanishing of the beta functions. In the conformally invariant limit, the beta functions vanish and the Ward identities simplify and constrain the correlators in a more powerful way. In the near conformal limit, where the beta functions are small, there can still be significant constraints from the Ward identities. In the same way, for inflation the Ward identities are generally valid since they arise from the constraints of spatial reparametrization invariance, which is a gauge symmetry of general relativity, and must hold very generally. In the slow roll limit, where there is approximate conformal invariance, these conditions can impose significant constraints on the correlation functions for the scalar and tensor perturbations.

We relied on the late time wave function of the universe for the derivation of the Ward identities. The late time limit was relevant because the inflationary perturbations freeze out after horizon crossing in the appropriate gauge. The invariance of the wave function under the spatial reparametrizations meant that the coefficient functions appearing in a semiclassical expansion of the wave function transformed in a way analogous to the correlation functions of a marginal primary operator in a three dimensional conformal field theory.

In chapter 4, we studied the generalized single field models of inflation in some detail. In these models of inflation the parameter $\dot{\bar{\phi}}/H$ is not necessarily small. We found that the Ward identities are valid for this class of models as well. Another interesting consequence of the Ward identities is a method to compute the scalar three point function in slow roll inflation from the scalar four point function calculation, by setting one of the external legs in the Feynman-Witten diagrams contributing to the four point function to be $\dot{\bar{\phi}}$. We explicitly computed the three point function using this method, providing another check for the validity of the Ward identities.

The key aspect of the symmetry based analysis carried out in this thesis is that it is model independent. Constraints which arise, for example, in the approximately conformally invariant limit, probe basic features of the inflationary model in a model independent way. These constraints can have significant observational consequences, and can therefore give rise to model independent tests for the inflationary paradigm.

Appendix A

Appendices for Chapter 2

A.1 More on $\langle OOO \rangle$ and $\langle OOOO \rangle$ in the canonical model of slow roll inflation

In this appendix, we discuss in some more detail the coefficient functions $\langle OOO \rangle$ and $\langle OOOO \rangle$ in the canonical model of slow roll inflation. We divide this appendix into two subsections, one for each of them.

A.1.1 The three point coefficient function $\langle OOO \rangle$

The three point scalar correlator $\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3}) \rangle$ in the canonical slow roll model, eq.(2.20), was computed in [24],

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3}) \rangle = (2\pi)^3 \delta^3\big(\boldsymbol{k_1} + \boldsymbol{k_2} + \boldsymbol{k_3}\big) \frac{H^4}{\dot{\phi}^4} \frac{H^4}{M_{Pl}^4} \frac{1}{\prod_a (2k_a^3)} A, \tag{A.1}$$

with

$$A = \left(\frac{2\ddot{\phi}}{H\dot{\phi}} + \frac{\dot{\phi}^2}{2H^2} \right) \big(\sum_a k_a^3 \big) + \frac{\dot{\phi}^2}{H^2} \left[\frac{1}{2} \sum_{a \neq b} k_a k_b^2 + \frac{4}{k_t} \sum_{a > b} k_a^2 k_b^2 \right]. \tag{A.2}$$

Here, $k_a = |\boldsymbol{k_a}|$ and $k_t = k_1 + k_2 + k_3$. Using the definitions of the slow-roll parameters, ϵ_1 and δ, eq.(2.18), and the eq.(2.28), in eq.(A.1) and eq.(A.2) above, we can obtain the expression for $\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3}) \rangle$ in terms of ϵ_1, δ as

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3}) \rangle = (2\pi)^3 \delta^3\big(\boldsymbol{k_1} + \boldsymbol{k_2} + \boldsymbol{k_3}\big) \frac{1}{4\epsilon_1^2} \frac{H^4}{M_{Pl}^4} \frac{1}{\prod_a (2k_a^3)} A, \tag{A.3}$$

with

$$A = (\epsilon_1 + 2\delta) \big(\sum_a k_a^3 \big) + 2\,\epsilon_1 \left[\frac{1}{2} \sum_{a \neq b} k_a k_b^2 + \frac{4}{k_t} \sum_{a > b} k_a^2 k_b^2 \right]. \tag{A.4}$$

We can also express the relation between ζ and $\delta\phi$, as given in eq.(2.70), in terms of the parameters ϵ_1 and δ as

$$\zeta = -\frac{1}{\sqrt{2\epsilon_1}}\,\delta\phi - \left(\frac{\epsilon_1+\delta}{4\epsilon_1}\right)\delta\phi^2. \tag{A.5}$$

Then from eq.(A.3) and eq.(A.4), we get

$$\begin{aligned}\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle = &-(2\pi)^3\delta^3\left(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3}\right)\frac{H^4}{M_{Pl}^4}\frac{1}{\prod_a(2k_a^3)}\times\\ &\left[\left(\frac{3\epsilon_1+4\delta}{\sqrt{2\epsilon_1}}\right)\sum_a k_a^3+\sqrt{2\epsilon_1}\left(\frac{1}{2}\sum_{a\neq b}k_a k_b^2+\frac{4}{k_t}\sum_{a>b}k_a^2k_b^2\right)\right].\end{aligned} \tag{A.6}$$

Now, to obtain a relationship between $\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle$ and $\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle$, we use the momentum space expression for the wave function eq.(2.6), given by

$$\begin{aligned}\psi[\delta\phi] = \exp\Bigg[\frac{M_{Pl}^2}{H^2}\Bigg(&-\frac{1}{2!}\int\frac{d^3\boldsymbol{k_1}}{(2\pi)^3}\frac{d^3\boldsymbol{k_2}}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\,\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})\rangle\\ &+\frac{1}{3!}\int\frac{d^3\boldsymbol{k_1}}{(2\pi)^3}\frac{d^3\boldsymbol{k_2}}{(2\pi)^3}\frac{d^3\boldsymbol{k_3}}{(2\pi)^3}\,\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\times\\ &\langle O(-\boldsymbol{k_1})O(-\boldsymbol{k_2})O(-\boldsymbol{k_3})\rangle\Bigg)\Bigg],\end{aligned} \tag{A.7}$$

where we have kept only the relevant terms. This gives

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\rangle = \frac{1}{4}\frac{H^4}{M_{Pl}^4}\frac{\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle}{\prod_{a=1}^3\langle O(\boldsymbol{k_a})O(-\boldsymbol{k_a})\rangle'}\,. \tag{A.8}$$

Using the expression for $\langle O(\boldsymbol{k_a})O(-\boldsymbol{k_a})\rangle'$, eq.(2.7), in eq.(A.8), and using eq.(A.6) we obtain the relation

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' = -\frac{3\epsilon_1+4\delta}{2\sqrt{2\epsilon_1}}\sum_a k_a^3-\frac{1}{2}\sqrt{2\epsilon_1}\left(\frac{1}{2}\sum_{a\neq b}k_ak_b^2+\frac{4}{k_t}\sum_{a>b}k_a^2k_b^2\right), \tag{A.9}$$

which is same as the expression in eq.(2.76).

A.1.2 The four point coefficient function $\langle OOOO\rangle$

The scalar four point coefficient function $\langle OOOO\rangle$ in the canonical slow roll model was calculated in [65] and [27]. It is given, see eq.(4.32) of [27], as

$$\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)O(\boldsymbol{x}_4)\rangle = \int\prod_{a=1}^{4}\frac{d^3k_a}{(2\pi)^3}e^{i\boldsymbol{k_a}\cdot\boldsymbol{x_a}}\,\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle\,, \tag{A.10}$$

where

$$
\begin{aligned}
\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle = -4\,(2\pi)^3\delta^3(\sum_{a=1}^{4}\boldsymbol{k_a})\Bigg[\frac{1}{2}\Bigg\{&\widehat{W}^S(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) \\
+\widehat{W}^S(\boldsymbol{k_1},\boldsymbol{k_3},\boldsymbol{k_2},\boldsymbol{k_4})+\widehat{W}^S(\boldsymbol{k_1},\boldsymbol{k_4},\boldsymbol{k_3},\boldsymbol{k_2})\Bigg\}&+\widehat{R}^S(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) \\
+\widehat{R}^S(\boldsymbol{k_1},\boldsymbol{k_3},\boldsymbol{k_2},\boldsymbol{k_4})+\widehat{R}^S(\boldsymbol{k_1},\boldsymbol{k_4},\boldsymbol{k_3},\boldsymbol{k_2})\Bigg]&,
\end{aligned}
\tag{A.11}
$$

with $\widehat{W}^S$ being the contribution from the transverse component of the graviton exchanged, given by

$$
\begin{aligned}
&\widehat{W}^S(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) = -2\Bigg[\Bigg\{\boldsymbol{k_1}.\boldsymbol{k_3}+\frac{\{(\boldsymbol{k_2}+\boldsymbol{k_1}).\boldsymbol{k_1}\}\{(\boldsymbol{k_4}+\boldsymbol{k_3}).\boldsymbol{k_3}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\} \\
&\Bigg\{\boldsymbol{k_2}.\boldsymbol{k_4}+\frac{\{(\boldsymbol{k_1}+\boldsymbol{k_2}).\boldsymbol{k_2}\}\{(\boldsymbol{k_3}+\boldsymbol{k_4}).\boldsymbol{k_4}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\}+\Bigg\{\boldsymbol{k_1}.\boldsymbol{k_4}+\frac{\{(\boldsymbol{k_2}+\boldsymbol{k_1}).\boldsymbol{k_1}\}\{(\boldsymbol{k_4}+\boldsymbol{k_3}).\boldsymbol{k_4}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\} \\
&\Bigg\{\boldsymbol{k_2}.\boldsymbol{k_3}+\frac{\{(\boldsymbol{k_2}+\boldsymbol{k_1}).\boldsymbol{k_2}\}\{(\boldsymbol{k_4}+\boldsymbol{k_3}).\boldsymbol{k_3}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\}-\Bigg\{\boldsymbol{k_1}.\boldsymbol{k_2}-\frac{\{(\boldsymbol{k_2}+\boldsymbol{k_1}).\boldsymbol{k_1}\}\{(\boldsymbol{k_1}+\boldsymbol{k_2}).\boldsymbol{k_2}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\} \\
&\Bigg\{\boldsymbol{k_3}.\boldsymbol{k_4}-\frac{\{(\boldsymbol{k_3}+\boldsymbol{k_4}).\boldsymbol{k_4}\}\{(\boldsymbol{k_4}+\boldsymbol{k_3}).\boldsymbol{k_3}\}}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Bigg\}\Bigg]\times \\
&\Bigg[\Bigg\{\frac{k_1k_2(k_1+k_2)^2\left((k_1+k_2)^2-k_3^2-k_4^2-4k_3k_4\right)}{(k_1+k_2-k_3-k_4)^2(k_1+k_2+k_3+k_4)^2(k_1+k_2-|\boldsymbol{k_1}+\boldsymbol{k_2}|)(k_1+k_2+|\boldsymbol{k_1}+\boldsymbol{k_2}|)} \\
&\Big(-\frac{k_1+k_2}{2k_1k_2}-\frac{k_1+k_2}{-(k_1+k_2)^2+k_3^2+k_4^2+4k_3k_4}+\frac{k_1+k_2}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2-(k_1+k_2)^2} \\
&+\frac{1}{-k_1-k_2+k_3+k_4}-\frac{1}{k_1+k_2+k_3+k_4}+\frac{3}{2(k_1+k_2)}\Big)+(1,2\leftrightarrow 3,4)\Bigg\} \\
&-\frac{|\boldsymbol{k_1}+\boldsymbol{k_2}|^3\left(-k_1^2-4k_2k_1-k_2^2+|\boldsymbol{k_1}+\boldsymbol{k_2}|^2\right)\left(-k_3^2-4k_4k_3-k_4^2+|\boldsymbol{k_1}+\boldsymbol{k_2}|^2\right)}{2\left(-k_1^2-2k_2k_1-k_2^2+|\boldsymbol{k_1}+\boldsymbol{k_2}|^2\right)^2\left(-k_3^2-2k_4k_3-k_4^2+|\boldsymbol{k_1}+\boldsymbol{k_2}|^2\right)^2}\Bigg].
\end{aligned}
\tag{A.12}
$$

The longitudinal contribution from the graviton is denoted by $\widehat{R}^S$, and is given by

$$
\widehat{R}^S(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) = \frac{A_1(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4})}{(k_1+k_2+k_3+k_4)}+\frac{A_2(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4})}{(k_1+k_2+k_3+k_4)^2}+\frac{A_3(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4})}{(k_1+k_2+k_3+k_4)^3} \tag{A.13}
$$

with

$$
\begin{aligned}
A_1(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) =& \left[\frac{\boldsymbol{k_3}\cdot\boldsymbol{k_4}\left(\boldsymbol{k_1}\cdot\boldsymbol{k_2}\left(k_1^2+k_2^2\right)+2k_1^2k_2^2\right)}{8|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}+\{1,2\Leftrightarrow 3,4\}\right] \\
&-\frac{k_1^2\boldsymbol{k_2}\cdot\boldsymbol{k_3}k_4^2+k_1^2\boldsymbol{k_2}\cdot\boldsymbol{k_4}k_3^2+\boldsymbol{k_1}\cdot\boldsymbol{k_3}k_2^2k_4^2+\boldsymbol{k_1}\cdot\boldsymbol{k_4}k_2^2k_3^2}{2|\boldsymbol{k_1}+\boldsymbol{k_2}|^2} \\
&-\frac{\left(\boldsymbol{k_1}\cdot\boldsymbol{k_2}\left(k_1^2+k_2^2\right)+2k_1^2k_2^2\right)\left(\boldsymbol{k_3}\cdot\boldsymbol{k_4}\left(k_3^2+k_4^2\right)+2k_3^2k_4^2\right)}{8|\boldsymbol{k_1}+\boldsymbol{k_2}|^4},
\end{aligned}
\tag{A.14}
$$

$$A_2(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) = -\frac{1}{8|\boldsymbol{k_1}+\boldsymbol{k_2}|^4}\Big[k_3k_4(k_3+k_4)\left(\boldsymbol{k_1}\cdot\boldsymbol{k_2}\left(k_1^2+k_2^2\right)+2k_1^2k_2^2\right)$$
$$(k_3k_4+\boldsymbol{k_3}\cdot\boldsymbol{k_4})+k_1k_2(k_1+k_2)(k_1k_2+\boldsymbol{k_1}\cdot\boldsymbol{k_2})\left(\boldsymbol{k_3}\cdot\boldsymbol{k_4}\left(k_3^2+k_4^2\right)+2k_3^2k_4^2\right)\Big]$$
$$-\frac{1}{2|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Big[k_1^2\boldsymbol{k_2}\cdot\boldsymbol{k_3}k_4^2(k_2+k_3)+k_1^2\boldsymbol{k_2}\cdot\boldsymbol{k_4}k_3^2(k_2+k_4)$$
$$+\boldsymbol{k_1}\cdot\boldsymbol{k_3}k_2^2k_4^2(k_1+k_3)+\boldsymbol{k_1}\cdot\boldsymbol{k_4}k_2^2k_3^2(k_1+k_4)\Big]$$
$$+\Big[\frac{\boldsymbol{k_1}\cdot\boldsymbol{k_2}}{8|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\big((k_1+k_2)\left(\boldsymbol{k_3}\cdot\boldsymbol{k_4}\left(k_3^2+k_4^2\right)+2k_3^2k_4^2\right)$$
$$+k_3k_4(k_3+k_4)(k_3k_4+\boldsymbol{k_3}\cdot\boldsymbol{k_4})\big)+\{1,2\Leftrightarrow 3,4\}\Big], \tag{A.15}$$

$$A_3(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4}) = -\frac{k_1k_2k_3k_4(k_1+k_2)(k_3+k_4)(k_1k_2+\boldsymbol{k_1}\cdot\boldsymbol{k_2})(k_3k_4+\boldsymbol{k_3}\cdot\boldsymbol{k_4})}{4|\boldsymbol{k_1}+\boldsymbol{k_2}|^4}$$
$$-\frac{k_1k_2k_3k_4(k_1\boldsymbol{k_2}\cdot\boldsymbol{k_3}k_4+k_1\boldsymbol{k_2}\cdot\boldsymbol{k_4}k_3+\boldsymbol{k_1}\cdot\boldsymbol{k_3}k_2k_4+\boldsymbol{k_1}\cdot\boldsymbol{k_4}k_2k_3)}{|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}$$
$$+\frac{1}{4|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}\Big[k_1k_2(k_1k_2+\boldsymbol{k_1}\cdot\boldsymbol{k_2})\left(\boldsymbol{k_3}\cdot\boldsymbol{k_4}\left(k_3^2+k_4^2\right)+2k_3^2k_4^2\right)$$
$$+\boldsymbol{k_1}\cdot\boldsymbol{k_2}k_3k_4(k_1+k_2)(k_3+k_4)(k_3k_4+\boldsymbol{k_3}\cdot\boldsymbol{k_4})+\{1,2\Leftrightarrow 3,4\}\Big]$$
$$+\frac{3k_1k_2k_3k_4(k_1k_2+\boldsymbol{k_1}\cdot\boldsymbol{k_2})(k_3k_4+\boldsymbol{k_3}\cdot\boldsymbol{k_4})}{4|\boldsymbol{k_1}+\boldsymbol{k_2}|^2}. \tag{A.16}$$

From eq.(2.72), we can see that $\langle\zeta\zeta\zeta\zeta\rangle$ is made up of two parts. Among them, $\langle\zeta\zeta\zeta\zeta\rangle_{CF}$ gets contribution from the four point coefficient function $\langle OOOO\rangle$. Similar to eq.(A.8), one can derive a relation between $\langle OOOO\rangle$ and $\langle\delta\phi\,\delta\phi\,\delta\phi\,\delta\phi\rangle_{CF}$ using the momentum space wave function. The relation is given by

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\delta\phi(\boldsymbol{k_3})\delta\phi(\boldsymbol{k_4})\rangle_{CF} = \frac{1}{8}\frac{H^6}{M_{Pl}^6}\frac{\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle}{\prod_{a=1}^4\langle O(\boldsymbol{k_a})O(-\boldsymbol{k_a})\rangle'}. \tag{A.17}$$

Inverting eq.(2.70), we obtain $\delta\phi$ in terms of ζ. Working upto linear order in $\delta\phi$, we get

$$\delta\phi = -\frac{\dot{\phi}}{H}\,\zeta\,. \tag{A.18}$$

Using eq.(A.18) in eq.(A.17), we obtain

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{CF} = \frac{1}{8}\frac{H^6}{M_{Pl}^6}\frac{H^4}{\dot{\phi}^4}\frac{\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle}{\prod_{a=1}^4\langle O(\boldsymbol{k_a})O(-\boldsymbol{k_a})\rangle'} \tag{A.19}$$

Similarly, the other contribution in $\langle\zeta\zeta\zeta\zeta\rangle$, i.e. $\langle\zeta\zeta\zeta\zeta\rangle_{ET}$, comes from integrating out a boundary graviton. The corresponding $\langle\delta\phi\,\delta\phi\,\delta\phi\,\delta\phi\rangle_{ET}$ was computed in eq.(5.6) of [27],

$$\begin{aligned}\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle_{ET} = 4(2\pi)^3\delta^3\Big(\sum_{a=1}^{4}\boldsymbol{k_a}\Big)\frac{H^6}{M_{Pl}^6}\frac{1}{\prod_{a=1}^{4}(2k_a^3)} \\ \Big[\widehat{G}^S(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)+\widehat{G}^S(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_4)+\widehat{G}^S(\mathbf{k}_1,\mathbf{k}_4,\mathbf{k}_3,\mathbf{k}_2)\Big],\end{aligned} \tag{A.20}$$

with $\widehat{G}^S$ being given by (eq.(5.7) of [27])

$$\begin{aligned}\widehat{G}^S(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \frac{S(\widetilde{\mathbf{k}},\mathbf{k}_1,\mathbf{k}_2)S(\widetilde{\mathbf{k}},\mathbf{k}_3,\mathbf{k}_4)}{|\mathbf{k}_1+\mathbf{k}_2|^3}\Bigg[\Bigg\{\mathbf{k}_1.\mathbf{k}_3+\frac{\{(\mathbf{k}_2+\mathbf{k}_1).\mathbf{k}_1\}\{(\mathbf{k}_4+\mathbf{k}_3).\mathbf{k}_3\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\} \\ \Bigg\{\mathbf{k}_2.\mathbf{k}_4+\frac{\{(\mathbf{k}_1+\mathbf{k}_2).\mathbf{k}_2\}\{(\mathbf{k}_3+\mathbf{k}_4).\mathbf{k}_4\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\}+\Bigg\{\mathbf{k}_1.\mathbf{k}_4+\frac{\{(\mathbf{k}_2+\mathbf{k}_1).\mathbf{k}_1\}\{(\mathbf{k}_4+\mathbf{k}_3).\mathbf{k}_4\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\} \\ \Bigg\{\mathbf{k}_2.\mathbf{k}_3+\frac{\{(\mathbf{k}_2+\mathbf{k}_1).\mathbf{k}_2\}\{(\mathbf{k}_4+\mathbf{k}_3).\mathbf{k}_3\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\}-\Bigg\{\mathbf{k}_1.\mathbf{k}_2-\frac{\{(\mathbf{k}_2+\mathbf{k}_1).\mathbf{k}_1\}\{(\mathbf{k}_1+\mathbf{k}_2).\mathbf{k}_2\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\} \\ \Bigg\{\mathbf{k}_3.\mathbf{k}_4-\frac{\{(\mathbf{k}_3+\mathbf{k}_4).\mathbf{k}_4\}\{(\mathbf{k}_4+\mathbf{k}_3).\mathbf{k}_3\}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\}\Bigg],\end{aligned} \tag{A.21}$$

with

$$S(\widetilde{\mathbf{k}},\mathbf{k}_1,\mathbf{k}_2) = (k_1+k_2+k_3)-\frac{\sum_{i>j}k_ik_j}{(k_1+k_2+k_3)}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\Bigg|_{\mathbf{k}_3=\widetilde{\mathbf{k}}=-(\mathbf{k}_1+\mathbf{k}_2)}. \tag{A.22}$$

In eq.(A.20), one can use eq.(A.18) to obtain

$$\begin{aligned}\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{ET} = 4\,(2\pi)^3\delta^3\Big(\sum_{a=1}^{4}\boldsymbol{k_a}\Big)\frac{H^4}{\dot{\phi}^4}\frac{H^6}{M_{Pl}^6}\frac{1}{\prod_{a=1}^{4}(2k_a^3)}\times \\ \Big[\widehat{G}^S(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3},\boldsymbol{k_4})+\widehat{G}^S(\boldsymbol{k_1},\boldsymbol{k_3},\boldsymbol{k_2},\boldsymbol{k_4})+\widehat{G}^S(\boldsymbol{k_1},\boldsymbol{k_4},\boldsymbol{k_3},\boldsymbol{k_2})\Big].\end{aligned} \tag{A.23}$$

Thus, $\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{CF}$, in eq.(A.19), and $\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle_{ET}$, in eq.(A.23), give the two contributions mentioned on the RHS. of eq.(2.72).

A.2 Solving the homogeneous equation for $\langle OOO\rangle$

In this appendix, we calculate the homogeneous contribution to the three point function $\langle OOO\rangle'$, denoted by $S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3})$, eq.(2.78). For this, we need to solve the equations eq.(2.79) and eq.(2.80). We start by rewriting eq.(2.80) in a slightly different manner which is more suited for the purpose of our calculation. Note that the function $S_h(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3})$ is a function only of the magnitudes k_1,k_2 and k_3. Thus it will be beneficial for us if we express the derivative operators in eq.(2.42) in terms of the magnitudes k_1,k_2 and k_3, rather then in terms of the components of $\boldsymbol{k_1}, \boldsymbol{k_2}$ and $\boldsymbol{k_3}$. Using

$$\frac{\partial}{\partial k_i} = \frac{k_i}{k}\frac{\partial}{\partial k}\,, \tag{A.24}$$

where k is the magnitude and k_i is the i^{th} component of a generic vector $\boldsymbol{k}$, we can re-express the derivative operator $\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}}$ as

$$\mathcal{L}^{\boldsymbol{b}}_{\boldsymbol{k}} = (\boldsymbol{b}\cdot\boldsymbol{k})\,\Theta(k) \tag{A.25}$$

with

$$\Theta(k) = -\frac{2}{k}\frac{\partial}{\partial k} + \frac{\partial^2}{\partial k^2}\,. \tag{A.26}$$

Eq.(2.80) can then be written as

$$\left[(\boldsymbol{b}\cdot\boldsymbol{k_1})\Theta(k_1) + (\boldsymbol{b}\cdot\boldsymbol{k_2})\Theta(k_2) + (\boldsymbol{b}\cdot\boldsymbol{k_3})\Theta(k_3)\right] S_h(k_1,k_2,k_3) = 0. \tag{A.27}$$

With the choice for the parameter of the special conformal transformation, $\boldsymbol{b}$, to be perpendicular to $\boldsymbol{k_3}$, *i.e.* $\boldsymbol{b}\perp\boldsymbol{k_3}$, eq.(A.27) becomes

$$(\Theta(k_1) - \Theta(k_2))\,S_h(k_1,k_2,k_3) = 0. \tag{A.28}$$

Similarly, we can make another independent choice for the parameter $\boldsymbol{b}$, $\boldsymbol{b}\perp\boldsymbol{k_2}$, and obtain

$$(\Theta(k_1) - \Theta(k_3))\,S_h(k_1,k_2,k_3) = 0. \tag{A.29}$$

The other possible independent choice, $\boldsymbol{b}\perp\boldsymbol{k_1}$, gives an equation that is a linear combination of eq.(A.28) and eq.(A.29).

We will now analyze solutions to these equations. Our analysis is related to that carried out in [26]. Let us consider a complete set of functions $f_z(k)$ defined in the range $z\in(-\infty,\infty)$, given by

$$f_z(k) = (1+ikz)\,e^{-ikz}. \tag{A.30}$$

Any general function, say $H(k)$, can be expanded in terms of $f_z(k)$ in a souped-up Fourier transform as

$$H(k) = \int_{-\infty}^{\infty} dz\, f_z(k)\,\tilde{H}(z). \tag{A.31}$$

The functions $f_z(k)$ are actually eigenfunctions of the operators $\Theta(k)$, satisfying

$$\Theta(k) f_z(k) = -z^2 f_z(k). \tag{A.32}$$

It is also important to note that the inverse of the transformation in eq.(A.31) is given by,

$$\tilde{H}(z) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(k\, e^{ikz} \int^{k} \frac{H(q)}{q^2} dq \right). \tag{A.33}$$

Using eq.(A.31), we can expand the function $S_h(k_1, k_2, k_3)$ as

$$S_h(k_1, k_2, k_3) = \int_{-\infty}^{\infty} dz_1\, dz_2\, dz_3\, f_{z_1}(k_1) f_{z_2}(k_2) f_{z_3}(k_3)\, \mathcal{M}(z_1, z_2, z_3). \tag{A.34}$$

Substituting $S_h(k_1, k_2, k_3)$ from eq.(A.34) into eq.(A.28) and eq.(A.29), we obtain

$$z_1^2 = z_2^2 = z_3^2, \tag{A.35}$$

which in turn allows us to write $S_h(k_1, k_2, k_3)$ as

$$S_h(k_1, k_2, k_3) = \sum_{n_1, n_2, n_3 = \pm 1} \int_0^{\infty} dz\, \mathcal{F}_{n_1 n_2 n_3}(k_1, k_2, k_3, z)\, \mathcal{M}_{n_1 n_2 n_3}(z), \tag{A.36}$$

where $\mathcal{M}_{n_1 n_2 n_3}(z)$ are a set of 8 functions corresponding to the 8 possible choices of the set $\{n_1, n_2, n_3\}$, and $\mathcal{F}_{n_1 n_2 n_3}(k_1, k_2, k_3, z)$ is given by

$$\mathcal{F}_{n_1 n_2 n_3}(k_1, k_2, k_3, z) = (1 + i n_1 k_1 z)\,(1 + i n_2 k_2 z)\,(1 + i n_3 k_3 z)\, e^{-i(n_1 k_1 + n_2 k_2 + n_3 k_3) z}. \tag{A.37}$$

Using eq.(A.24), we can also rewrite eq.(2.79) as

$$\left(k_1 \frac{\partial}{\partial k_1} + k_2 \frac{\partial}{\partial k_2} + k_3 \frac{\partial}{\partial k_3} \right) S_h(k_1, k_2, k_3) = 3\, S_h(k_1, k_2, k_3). \tag{A.38}$$

Using eq.(A.36) and eq.(A.37) in eq.(A.38) we get

$$\begin{aligned} \left[\sum_{a=1}^{3} k_a \frac{\partial}{\partial k_a} \right] S_h(k_1, k_2, k_3) = - \sum_{n_1, n_2, n_3 = \pm 1} \int_0^{\infty} dz\, \mathcal{F}_{n_1 n_2 n_3}(k_1, k_2, k_3, z) \times \\ \frac{\partial}{\partial z} \left[z\, \mathcal{M}_{n_1 n_2 n_3}(z) \right]. \end{aligned} \tag{A.39}$$

Combining eq.(A.38) and eq.(A.39) we obtain

$$\frac{\partial}{\partial z} \left[z\, \mathcal{M}_{n_1 n_2 n_3}(z) \right] + 3\, \mathcal{M}_{n_1 n_2 n_3}(z) = 0. \tag{A.40}$$

This has the general solution

$$\mathcal{M}_{n_1 n_2 n_3}(z) = \frac{m_{n_1 n_2 n_3}}{z^4}, \tag{A.41}$$

where $m_{n_1 n_2 n_3}$ is a z independent constant. Thus, eq.(A.41) fixes the functional dependence of $\mathcal{M}$ on z. Using eq.(A.41) in eq.(A.36) we see that

$$S_h(k_1, k_2, k_3) = \sum_{n_1,n_2,n_3=\pm 1} m_{n_1 n_2 n_3} \int_0^\infty \frac{dz}{z^4} \mathcal{F}_{n_1 n_2 n_3}(k_1, k_2, k_3, z). \tag{A.42}$$

To make the integration in eq.(A.42) well defined as $z \to \infty$, we add a small imaginary component to k_a. The integral is also divergent as $z \to 0$. We regularize it by putting a small cut-off at $z = \lambda$. On carrying out the integral we get

$$\begin{aligned} S_h(k_1, k_2, k_3) = \sum_{n_1,n_2,n_3=\pm 1} m_{n_1 n_2 n_3} \Bigg\{ & \frac{1}{3\lambda^3} + \frac{1}{2\lambda} \sum_{a=1}^{3} (n_a k_a)^2 + \\ & + i\Big(-\frac{4}{9} \sum_{a=1}^{3} (n_a k_a)^3 - \frac{1}{3} \sum_{a \neq b} n_a k_a (n_b k_b)^2 + \frac{1}{3} \prod_{a=1}^{3} (n_a k_a) \Big) \\ & - \frac{i}{3} \Big(\sum_{a=1}^{3} (n_a k_a)^3 \Big) \Big(\int_\lambda^\infty \frac{dz}{z} e^{-i(n_1 k_1 + n_2 k_2 + n_3 k_3) z} \Big) \Bigg\}. \end{aligned} \tag{A.43}$$

This gives us the solution to the homogeneous equations eq.(2.79) and eq.(2.80). At this stage, it consists of a sum of eight distinct functions, corresponding to the eight distinct choices for the set (n_1, n_2, n_3). We will now take various limits of the answer in eq.(A.43) and find a unique solution.

First of all, we remove the first two terms in the solution eq.(A.43) which go like powers of $1/\lambda$, since their presence would violate conformal invariance. We next consider the last term involving the integral. We can explicitly evaluate this integral to get

$$\int_\lambda^\infty \frac{dz}{z} e^{-i\,(\sum_a n_a k_a)\,z} = \Gamma[0, i(\sum_a n_a k_a)\lambda] = -\gamma - \frac{i\pi}{2} - ln\big[\lambda\big(\sum_a n_a k_a\big)\big] + O(\lambda). \tag{A.44}$$

Here, γ is the Euler-Mascheroni constant and ln denotes the natural logarithm. The $O(\lambda)$ terms appearing in eq.(A.44) vanish in the limit $\lambda \to 0$. Thus, our answer becomes

$$\begin{aligned} S_h(k_1, k_2, k_3) = \sum_{n_1,n_2,n_3=\pm 1} m_{n_1 n_2 n_3} \Bigg\{ & \frac{i}{3} \Big(\sum_{a=1}^{3} (n_a k_a)^3 \Big) \Big(\gamma + \frac{i\pi}{2} + ln\big[\lambda\big(\sum_a n_a k_a\big)\big] \Big) \\ & + i\Big(-\frac{4}{9} \sum_{a=1}^{3} (n_a k_a)^3 - \frac{1}{3} \sum_{a \neq b} n_a k_a (n_b k_b)^2 + \frac{1}{3} \prod_{a=1}^{3} (n_a k_a) \Big) \Bigg\}. \end{aligned} \tag{A.45}$$

We will now consider the behavior of eq.(A.45) in the limit $k_1 \approx k_2 \gg k_3$. We know that the momentum space three point function is related to the position space expression by the standard Fourier transform. Thus

$$\begin{aligned}
\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle &= \int d^3x_1\, d^3x_2\, d^3x_3\, e^{-i(\sum_a \boldsymbol{k_a}\cdot\boldsymbol{x_a})}\langle O(\boldsymbol{x_1})O(\boldsymbol{x_2})O(\boldsymbol{x_3})\rangle \\
&= \int d^3x_1\, d^3x_2\, d^3x_3\, e^{-i\{(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})\cdot\boldsymbol{x_1}+\boldsymbol{k_2}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})+\boldsymbol{k_3}\cdot(\boldsymbol{x_3}-\boldsymbol{x_1})\}} \\
&\qquad\qquad\qquad \langle O(0)O(\boldsymbol{x_2}-\boldsymbol{x_1})O(\boldsymbol{x_3}-\boldsymbol{x_1})\rangle \\
&= \int d^3x_1\, d^3x\, d^3y\, e^{-i\{(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})\cdot\boldsymbol{x_1}+\boldsymbol{k_2}\cdot\boldsymbol{x}+\boldsymbol{k_3}\cdot\boldsymbol{y}\}}\langle O(0)O(\boldsymbol{x})O(\boldsymbol{y})\rangle \\
&= (2\pi)^3\,\delta^3\big(\sum_{a=1}^{3}\boldsymbol{k_a}\big)\int d^3x\, d^3y\, e^{-i(\boldsymbol{k_2}\cdot\boldsymbol{x}+\boldsymbol{k_3}\cdot\boldsymbol{y})}\langle O(0)O(\boldsymbol{x})O(\boldsymbol{y})\rangle, \qquad \text{(A.46)}
\end{aligned}$$

where we have used the notation $\boldsymbol{x_2}-\boldsymbol{x_1}=\boldsymbol{x}$ and $\boldsymbol{x_3}-\boldsymbol{x_1}=\boldsymbol{y}$. Now, as we are interested in the limit $k_2\to\infty\Rightarrow x\to 0$ (where $x\equiv|\boldsymbol{x}|$), we can use the Operator Product Expansion (OPE)

$$O(0)O(\boldsymbol{x}) = \frac{A}{x^3}\,O(\boldsymbol{x})+\ldots \qquad \text{(A.47)}$$

where A is a constant. Substituting eq.(A.47) into eq.(A.46) then gives us

$$\begin{aligned}
\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle &\approx (2\pi)^3\,\delta^3\big(\sum_{a=1}^{3}\boldsymbol{k_a}\big)\int d^3x\, d^3y\, e^{-i(\boldsymbol{k_2}\cdot\boldsymbol{x}+\boldsymbol{k_3}\cdot\boldsymbol{y})}\,\frac{1}{x^3}\,\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle \\
&= (2\pi)^3\,\delta^3\big(\sum_{a=1}^{3}\boldsymbol{k_a}\big)\int d^3x\, d^3y\, e^{-i(\boldsymbol{k_2}\cdot\boldsymbol{x}+\boldsymbol{k_3}\cdot\boldsymbol{y})}\,\frac{1}{x^3}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^6} \qquad \text{(A.48)} \\
&\approx (2\pi)^3\,\delta^3\big(\sum_{a=1}^{3}\boldsymbol{k_a}\big)\int d^3x\, d^3y\, e^{-i(\boldsymbol{k_2}\cdot\boldsymbol{x}+\boldsymbol{k_3}\cdot\boldsymbol{y})}\,\frac{1}{x^3}\,\frac{1}{y^6}\,,
\end{aligned}$$

where we have used the fact that $k_2\gg k_3\Rightarrow x\ll y$. The leading k_2 dependence in this limit is thus given by the integral

$$\int d^3x\,\frac{e^{-i\boldsymbol{k_2}\cdot\boldsymbol{x}}}{x^3}\sim ln(\lambda k_2),\ \lambda\to 0. \qquad \text{(A.49)}$$

Using dimensional analysis to fix the k_3 dependence in eq.(A.48), we find that the three point function in this limit is of the form

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle\sim(2\pi)^3\,\delta^3\big(\sum_{a=1}^{3}\boldsymbol{k_a}\big)\,k_3^3\,ln(\lambda k_2). \qquad \text{(A.50)}$$

From eq.(2.78), eq.(A.45) and eq.(A.50), we see that only two terms, $(n_1,n_2)=(1,1)$ or $(-1,-1)$ are consistent with this behaviour. Now, by taking the similar limit $k_1\ll k_2\approx k_3$ and following the steps outlined above, we can see that the signs of k_2 and k_3 should also be identical: $(n_2,n_3)=(1,1)$ or $(-1,-1)$. Combining these two results, we see that out of the eight possibilities in eq.(A.45) for (n_1,n_2,n_3), only two survive: $(n_1,n_2,n_3)=(1,1,1)$ and $(n_1,n_2,n_3)=(-1,-1,-1)$.

Note that the choice $(n_1, n_2, n_3) = (-1, -1, -1)$ differs from $(n_1, n_2, n_3) = (1, 1, 1)$ only by an overall sign, which can be absorbed into the coefficient. By suitably redefining λ and the normalization N to absorb some constants, we then get S_h to be given by eq.(2.81).

A.3 A prescription to calculate $\langle OOO \rangle$ from $\langle OOOO \rangle$

In this appendix, we will argue that for a given scalar four point coefficient function $\langle OOOO \rangle$ in general, not necessarily for the canonical slow roll model, the Ward identity in eq.(2.61) can be solved, in principle, to get the three point coefficient function $\langle OOO \rangle$. We start by decomposing $\langle OOO \rangle'$ into two parts

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' = S_h(k_1, k_2, k_3) + S_i(k_1, k_2, k_3)\,, \tag{A.51}$$

where $S_h(k_1, k_2, k_3)$ is the homogeneous piece eq.(2.81), and $S_i(k_1, k_2, k_3)$ is a particular solution to the inhomogeneous Ward identity eq.(2.61). To calculate the particular solution $S_i(k_1, k_2, k_3)$, we rewrite the eq.(2.61) as,

$$\begin{aligned}\mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_1}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_2}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_3}}\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})\rangle'& \\ = b^j\, f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}).&\end{aligned} \tag{A.52}$$

Here, $f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3})$ is assumed to be an arbitrary vector function of the three momenta $\boldsymbol{k_a}$. Comparing with eq.(2.61), we can see that

$$f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}) = 2\,\frac{\dot{\bar{\phi}}}{H}\,\frac{\partial}{\partial k_4^j}\,\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})O(\boldsymbol{k_3})O(\boldsymbol{k_4})\rangle'\Big|_{\boldsymbol{k_4}\to 0}. \tag{A.53}$$

Note that from eq.(A.52), $f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3})$ is symmetric under the permutations of its arguments. We can write the most general vector function $f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3})$ with the above property as

$$f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}) = k_{1j}\,F(k_1, k_2, k_3) + k_{2j}\,F(k_2, k_3, k_1) + k_{3j}\,F(k_3, k_1, k_2)\,, \tag{A.54}$$

such that $F(k_1, k_2, k_3)$ is an arbitrary function and is symmetric under the exchange of its last two arguments.

Next, we make a choice for $\boldsymbol{b}$, the parameter of special conformal transformation, to be perpendicular to $\boldsymbol{k_3}$,

$$\boldsymbol{b} = \boldsymbol{k_2} - \frac{\boldsymbol{k_2}\cdot\boldsymbol{k_3}}{k_3^2}\boldsymbol{k_3}. \tag{A.55}$$

Using eq.(A.54) and eq.(A.55), the RHS of eq.(A.52) becomes

$$b^j\, f_j(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}) = \left(k_2^2 - \frac{(\boldsymbol{k_2}\cdot\boldsymbol{k_3})^2}{k_3^2}\right) g(k_1, k_2, k_3), \tag{A.56}$$

with the definition,

$$g(k_1, k_2, k_3) = F(k_2, k_1, k_3) - F(k_1, k_2, k_3). \tag{A.57}$$

It is obvious from the definition that $g(k_1, k_2, k_3)$ is antisymmetric under the exchange of its first two arguments. Also, using eq.(A.25), eq.(A.26) and eq.(A.55), we can write the LHS of eq.(A.52) as,

$$\begin{aligned}\mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_1}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_2}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' + \mathcal{L}^{\mathbf{b}}_{\boldsymbol{k_3}} \langle O(\boldsymbol{k_1}) O(\boldsymbol{k_2}) O(\boldsymbol{k_3})\rangle' \\ = \left(k_2^2 - \frac{(\boldsymbol{k_2}\cdot\boldsymbol{k_3})^2}{k_3^2}\right) \left(\Theta(k_2) - \Theta(k_1)\right) S_i(k_1, k_2, k_3).\end{aligned} \tag{A.58}$$

From eq.(A.56) and eq.(A.58), we see that the Ward identity eq.(A.52) becomes,

$$\left(\Theta(k_2) - \Theta(k_1)\right) S_i(k_1, k_2, k_3) = g(k_1, k_2, k_3). \tag{A.59}$$

Next we expand both $S_i(k_1, k_2, k_3)$ and $g(k_1, k_2, k_3)$ in terms of the functions $f_z(k)$, eq.(A.30),

$$S_i(k_1, k_2, k_3) = \int_{-\infty}^{\infty} dz_1\, dz_2\, dz_3\, \mathcal{F}(k_1, k_2, k_3, z_1, z_2, z_3)\, \mathcal{M}(z_1, z_2, z_3), \tag{A.60}$$

$$g(k_1, k_2, k_3) = \int_{-\infty}^{\infty} dz_1\, dz_2\, dz_3\, \mathcal{F}(k_1, k_2, k_3, z_1, z_2, z_3)\, \mathcal{N}(z_1, z_2, z_3), \tag{A.61}$$

with

$$\mathcal{F}(k_1, k_2, k_3, z_1, z_2, z_3) = (1 + ik_1z_1)\,(1 + ik_2z_2)\,(1 + ik_3z_3)\, e^{-i(k_1z_1 + k_2z_2 + k_3z_3)}. \tag{A.62}$$

Substituting eq.(A.60) and eq.(A.61) into eq.(A.59) gives us,

$$\mathcal{M}(z_1, z_2, z_3) = \frac{\mathcal{N}(z_1, z_2, z_3)}{z_1^2 - z_2^2}. \tag{A.63}$$

Using the definition of the inverse transformation in eq.(A.33), we can invert eq.(A.61) to obtain $\mathcal{N}(z_1, z_2, z_3)$ in terms of $g(k_1, k_2, k_3)$ as

$$\begin{aligned}\mathcal{N}(z_1, z_2, z_3) = -\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\, k_1\, k_2\, k_3\, e^{i(k_1z_1 + k_2z_2 + k_3z_3)} \\ \left(\int^{k_1}\int^{k_2}\int^{k_3} \frac{g(q_1, q_2, q_3)}{q_1^2q_2^2q_3^2}\, dq_1dq_2dq_3\right).\end{aligned} \tag{A.64}$$

Using eq.(A.64) and eq.(A.63) in eq.(A.60), we finally obtain

$$\begin{aligned}S_i(k_1, k_2, k_3) = -\int_{-\infty}^{\infty} dz_1dz_2dz_3\, \frac{\mathcal{F}(k_1, k_2, k_3, z_1, z_2, z_3)}{(z_1^2 - z_2^2)} \left[\int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\frac{dp_3}{2\pi}\, p_1p_2p_3 \right. \\ \left. e^{i(p_1z_1 + p_2z_2 + p_3z_3)} \left(\int^{p_1}\int^{p_2}\int^{p_3} \frac{g(q_1, q_2, q_3)}{q_1^2q_2^2q_3^2}\, dq_1dq_2dq_3\right)\right].\end{aligned} \tag{A.65}$$

Thus, given a four point coefficient function $\langle OOOO \rangle$, we should first calculate the function $g(q_1, q_2, q_3)$, eq.(A.57). Knowing g, we can evaluate the integral in eq.(A.65) to obtain the function S_i. Eq. (A.51) then gives us the three point coefficient function $\langle OOO \rangle$, as desired. Note that the expression above is a formal one. In particular, we know that the solution to the Ward identities is not unique, with an ambiguity of the form given by S_h, eq.(2.81). This ambiguity should be related to an ambiguity in how to carry out the integrals in eq.(A.65).

Appendix B

Appendices for Chapter 3

B.1 Transformation of perturbations under spatial reparametrizations

In this appendix, we would like to give some details about the transformation properties of the perturbations under spatial reparametrizations. We consider the perturbed line element in the gauge eq.(3.2),

$$ds^2 = -\,dt^2 + h_{ij}(t,\boldsymbol{x})\,dx^i dx^j, \tag{B.1}$$

with

$$h_{ij} \equiv a^2(t)\,g_{ij} = a^2(t)\,e^{2\zeta}\,[\delta_{ij} + \widehat{\gamma}_{ij}], \tag{B.2}$$

where

$$\widehat{\gamma}_{ii} = 0. \tag{B.3}$$

Consider now a spatial reparametrization of the form eq.(3.7). The change in h_{ij} under this transformation is

$$\delta h_{ij} = \nabla_i v_j + \nabla_j v_i\,. \tag{B.4}$$

Eq.(B.4) implies

$$\begin{aligned}
\delta g_{ij} &= \frac{1}{a^2(t)}\left[\partial_i v_j + \partial_j v_i - 2\,\Gamma^a_{ij}\,v_a\right] \\
&= \frac{1}{a^2(t)}\left[\partial_i\left(h_{jk}v^k\right) + \partial_j\left(h_{ik}v^k\right) - v_a\,h^{ab}\big(\partial_i h_{jb} + \partial_j h_{ib} - \partial_b h_{ij}\big)\right] \\
&= g_{jk}\,\partial_i v^k + g_{ik}\,\partial_j v^k + v^k\partial_k g_{ij}\,,
\end{aligned} \tag{B.5}$$

where indices will now be raised and lowered by δ_{ij}. Eq.(B.5) gives us

$$\delta g_{ii} = 2g_{ik}\,\partial_i v^k + v^k\partial_k g_{ii}. \tag{B.6}$$

Putting g_{ij} from eq.(B.2) in eq.(B.6) gives the change in ζ under spatial reparametrizations, eq.(3.7), as

$$\delta\zeta = \frac{1}{3}\,\partial_i v_i + v^k\partial_k\zeta + \frac{1}{3}\,\partial_i v_j\,\widehat{\gamma}_{ij}\,, \tag{B.7}$$

which is the result quoted in eq.(3.71). Once we have calculated $\delta\zeta$, we can insert the full g_{ij} in eq.(B.5) to get the change in $\widehat{\gamma}_{ij}$ as

$$\begin{aligned}\delta\widehat{\gamma}_{ij} = \left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\,\partial_a v_a\,\delta_{ij}\right) + \Big(\widehat{\gamma}_{ik}\,\partial_j v^k + \widehat{\gamma}_{jk}\,\partial_i v^k + \\ + v^k\partial_k\widehat{\gamma}_{ij} - \frac{2}{3}\,\partial_a v_a\,\widehat{\gamma}_{ij} - \frac{2}{3}\,\partial_a v_b\,\widehat{\gamma}_{ab}\,(\delta_{ij}+\widehat{\gamma}_{ij})\Big).\end{aligned} \tag{B.8}$$

For simplicity, we call the terms in eqs.(B.7) and (B.8) which are proportional to the perturbations as the homogeneous pieces of the transformation, and the parts independent of the perturbations as the inhomogeneous pieces of the transformation.

Having obtained eq.(B.7) and eq.(B.8), we can calculate the changes $\delta\zeta$ and $\delta\widehat{\gamma}_{ij}$ for the specific cases of scale transformations, eq.(3.36), special conformal transformations, eq.(3.53), and the compensating spatial reparametrization, eq.(3.57). For scale transformations, the change in ζ and $\widehat{\gamma}_{ij}$ is given by eq.(3.37) and eq.(3.38) respectively. Similarly, for the special conformal transformations, the changes are given by eq.(3.54) and eq.(3.55), and for the compensating spatial reparametrization, the changes are eq.(3.58) and eq.(3.59).

B.2 The scalar and tensor spectral tilts

As a simple check on the Ward identities, we consider here the 2-point correlators. For scalar perturbations, the scaling Ward identity in eq.(3.49) relates the 2-point expectation value to the 3-point expectation value,

$$\left(3+\sum_{a=1}^{2} k_a\frac{\partial}{\partial k_a}\right)\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle' = -\,\frac{1}{\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\,\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle'\Big|_{\boldsymbol{k_3}\to 0}. \tag{B.9}$$

The expression for $\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle$ is

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\frac{H^2}{M_{Pl}^2}\,\frac{1}{4\epsilon}\,k_1^{-3+n_S}, \tag{B.10}$$

where n_S is the scalar tilt. Thus, we get

$$\left[\sum_{a=1}^{2} k_a\,\frac{\partial}{\partial k_a}\right]\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle' = (-3+n_S)\langle\zeta(\boldsymbol{k_1})\zeta(-\boldsymbol{k_1})\rangle', \tag{B.11}$$

which on substituting back into the eq.(B.9) gives the well known Maldacena consistency condition

$$\lim_{\boldsymbol{k_3}\to 0} \langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = -\, n_S \,\langle\zeta(\boldsymbol{k_1})\zeta(-\boldsymbol{k_1})\rangle' \,\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'. \tag{B.12}$$

By using the expression for $\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle$ from [24], we get [1]

$$\lim_{\boldsymbol{k_3}\to 0} \langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = (6\epsilon - 2\eta)\,\langle\zeta(\boldsymbol{k_1})\zeta(-\boldsymbol{k_1})\rangle' \,\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'. \tag{B.13}$$

Putting the limit from eq.(B.13) into eq.(B.12) gives the expression for the scalar tilt as

$$n_S = 2\eta - 6\epsilon\,, \tag{B.14}$$

which is indeed the correct expression, [24]. For a pedagogical discussion see [31, 32].

Similarly, consider the tensor Ward identity in eq.(3.51), with $n = 2$. This has the form

$$\left(3 + \sum_{a=1}^{2} k_a \frac{\partial}{\partial k_a}\right)\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\rangle' = -\,\frac{1}{\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'}\,\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle'\Big|_{\boldsymbol{k_3}\to 0}. \tag{B.15}$$

In writing eq.(B.15), we have introduced the two polarization tensors for the graviton, e^s_{ij}, through the relation

$$\widehat{\gamma}_{ij}(\boldsymbol{k}) = \sum_{s=1}^{2} e^s_{ij}(\boldsymbol{k})\,\widehat{\gamma}_s(\boldsymbol{k}). \tag{B.16}$$

Now, the expression for $\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\rangle$ has the form

$$\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\delta_{s,s'}\,\frac{H^2}{M_{Pl}^2}\,k_1^{-3+n_T}, \tag{B.17}$$

where n_T is the tensor tilt. By using the expression eq.(B.17) in eq.(B.15), we get

$$\lim_{\boldsymbol{k_3}\to 0} \langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = -\, n_T\,\delta_{s,s'}\,\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_s(-\boldsymbol{k_1})\rangle'\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'. \tag{B.18}$$

We can calculate the limit on the left side of eq.(B.18) by using the expression for the correlator $\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle$ from [24]. This gives

$$\lim_{\boldsymbol{k_3}\to 0} \langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_{s'}(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle' = 2\epsilon\,\delta_{s,s'}\langle\widehat{\gamma}_s(\boldsymbol{k_1})\widehat{\gamma}_s(-\boldsymbol{k_1})\rangle'\langle\zeta(\boldsymbol{k_3})\zeta(-\boldsymbol{k_3})\rangle'. \tag{B.19}$$

Then by comparing eq.(B.18) and eq.(B.19), we get

$$n_T = -\,2\epsilon\,, \tag{B.20}$$

which is the correct expression for the tensor tilt.

[1] The slow roll parameter η is given by $\eta \equiv \epsilon - \delta = \frac{V''}{V}$.

B.3 The behaviour of perturbations in canonical slow roll

In this appendix, we provide some details of the analysis given in section 3.6.1. We follow [29] for our calculations. Our gauge choice, eq.(3.2), is same as the synchronous gauge of [29] (see section **5.3 (B)**). The relevant equations are eqs.(5.3.28)-(5.3.33) for scalar perturbations, eq.(5.1.51) for vector perturbations, and eq.(5.1.53) for tensor perturbations.

The energy-momentum tensor for the inflaton can be calculated by varying the matter part of the action (3.132) with respect to the metric. It is given by

$$T^{\mu\nu} = -g^{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) + g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi. \tag{B.21}$$

This has the form of the energy-momentum tensor for a perfect fluid,

$$T^{\mu\nu} = (\rho + P)\,u^\mu u^\nu + P g^{\mu\nu}, \tag{B.22}$$

with the energy density ρ, pressure P, and the four-velocity u^μ given by

$$\rho = -\frac{1}{2}(\nabla\phi)^2 + V(\phi), \tag{B.23}$$

$$P = -\frac{1}{2}(\nabla\phi)^2 - V(\phi), \tag{B.24}$$

$$u^\mu = -[-(\nabla\phi)^2]^{-1/2}\,g^{\mu\nu}\partial_\nu\phi. \tag{B.25}$$

For our purpose, we specialize to the case of single field slow roll inflation. We then have

$$\bar{\rho} = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}), \tag{B.26}$$

$$\bar{P} = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}), \tag{B.27}$$

$$\bar{u}^0 = 1,\ \bar{u}^i = 0, \tag{B.28}$$

for the homogeneous background $\bar{\phi}(t)$. By expanding eqs.(B.23)-(B.25) to linear order in the perturbation $\delta\phi$, we get

$$\delta\rho = \dot{\bar{\phi}}\,\delta\dot{\phi} + V'(\bar{\phi})\,\delta\phi, \tag{B.29}$$

$$\delta P = \dot{\bar{\phi}}\,\delta\dot{\phi} - V'(\bar{\phi})\,\delta\phi, \tag{B.30}$$

$$\delta u = -\frac{\delta\phi}{\dot{\bar{\phi}}}, \tag{B.31}$$

where δu is defined through $\delta u_i \equiv \partial_i \delta u + \delta u_i^V$, and $\delta u_i^V = 0$ for single field inflation. Also, for single field inflation, the anisotropic stresses in the perturbed energy-momentum tensor vanish,

$$\pi^S = 0,\ \pi_i^V = 0,\ \pi_{ij}^T = 0. \tag{B.32}$$

By using the eqs.(B.26)-(B.32) above in the eqs.(5.3.28)-(5.3.33), eq.(5.1.51) and eq.(5.1.53) of [29], we obtain the perturbed Einstein equations eqs.(3.140)-(3.146) given in section 3.6.1 for the scalar, vector and tensor perturbations, along with the equation of motion for the background $\bar{\phi}(t)$, eq.(3.134). Note that the perturbations G_j in eq.(5.1.51) of [29] vanish due to our gauge choice eq.(3.2).

We now provide some details for calculating the late time behaviour of the perturbations. To solve for the perturbation A, we consider eq.(3.149). Inserting $\delta\phi$ from eq.(3.142) into eq.(3.149), we get an equation purely for the perturbation A,

$$\frac{\ddot{A}}{2} + \left[3\left(\frac{\dot{a}}{a}\right) + \frac{V'(\bar{\phi})}{\dot{\bar{\phi}}}\right]\dot{A} = 0. \tag{B.33}$$

By using the background eq.(3.134) in eq.(B.33), we get

$$\ddot{A} - 2\left(\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}}\right)\dot{A} = 0. \tag{B.34}$$

The general solution to eq.(B.34) is

$$A(t, \boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) \int^t dt' \, \dot{\bar{\phi}}^2(t'), \tag{B.35}$$

where $P_1(\boldsymbol{x}), P_2(\boldsymbol{x})$ are two arbitrary functions of $\boldsymbol{x}$. Eq.(B.35) on using the background equation

$$\dot{H} \equiv \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = -\frac{1}{2}\dot{\bar{\phi}}^2 \tag{B.36}$$

becomes

$$A(t, \boldsymbol{x}) = P_1(\boldsymbol{x}) - 2\left(\frac{\dot{a}}{a}\right) P_2(\boldsymbol{x}),$$

which is the solution quoted in eq.(3.153).

Once we have obtained the solution for A, it is straight forward to obtain the solution for the perturbation $\delta\phi$. From eq.(3.142) and eq.(3.153), it follows that

$$\delta\phi(t, \boldsymbol{x}) = -\dot{\bar{\phi}}(t)\, P_2(\boldsymbol{x}),$$

as given in eq.(3.154). One can check explicitly that the solutions eq.(3.153), eq.(3.154) satisfy the other equations, namely eq.(3.150) and eq.(3.152).

The equation for the perturbation C_i, eq.(3.144), has the general solution

$$C_i(t, \boldsymbol{x}) = \partial^{-2} Q_i(\boldsymbol{x}), \tag{B.37}$$

which shows that the perturbation C_i is frozen for non-zero momentum modes, which are the ones of interest to us.

Finally, we consider eq.(3.151) for the tensor perturbations. The general solution is

$$D_{ij}(t,\boldsymbol{x}) = \tilde{D}_{ij}(\boldsymbol{x}) + K_{ij}(\boldsymbol{x}) \int^{t} dt' \exp\left[-3\int^{t'} dt'' \left(\frac{\dot{a}}{a}\right)\right], \tag{B.38}$$

which in the late time limit also gets frozen,

$$D_{ij}(t,\boldsymbol{x}) \approx \tilde{D}_{ij}(\boldsymbol{x}) \text{ for t} \to \infty. \tag{B.39}$$

Appendix C

Appendices for Chapter 4

C.1 Correlation functions of ζ for the $P(X, \phi)$ models of inflation

In this appendix, to make the thesis self-contained, we present the results for the two, three and four point functions of the curvature perturbation ζ for the general single field models of inflation introduced in section 4.3. These results are taken from [133, 134].

The two point function is given by

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\,\frac{H^2}{M_{Pl}^2}\,\frac{1}{c_s}\,\frac{1}{4\epsilon k_1^3}. \tag{C.1}$$

The three point function to the leading order in the "slow variation" parameter ϵ is given by

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\rangle = (2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\,\frac{H^4}{M_{Pl}^4}\,\frac{2}{c_s^2\epsilon^2}\prod_{a=1}^{3}\Big(\frac{1}{2k_a^3}\Big)\,\mathcal{A}, \tag{C.2}$$

where [1]

$$\begin{aligned}\mathcal{A} =& \Big(\frac{1}{c_s^2}-1-\frac{2\lambda}{\Sigma}\Big)\frac{3k_1^2k_2^2k_3^2}{2K^3}\\ &+\Big(\frac{1}{c_s^2}-1\Big)\Big(-\frac{1}{K}\sum_{a<b}k_a^2k_b^2+\frac{1}{2K^2}\sum_{a\neq b}k_a^2k_b^3+\frac{1}{8}\sum_{a=1}^{3}k_a^3\Big),\end{aligned} \tag{C.3}$$

with λ and Σ being defined as

$$\lambda = X^2P_{,XX}+\frac{2}{3}X^3P_{,XXX}\,, \tag{C.4}$$

[1]The term $\mathcal{A}$ can have subleading corrections which are $O(\epsilon)$. For details, see [133].

and

$$\Sigma = XP_{,X} + 2X^2 P_{,XX}. \tag{C.5}$$

Note that $K = k_1 + k_2 + k_3$. Another important quantity which we will need later is

$$\mu = \frac{1}{2}X^2 P_{,XX} + 2X^3 P_{,XXX} + \frac{2}{3}X^4 P_{,XXXX}. \tag{C.6}$$

The four point function receives contributions from a contact interaction term, as well as from an intermediate scalar exchange. The complete expression for the four point function to the leading order is given by

$$\langle \zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\zeta(\boldsymbol{k_3})\zeta(\boldsymbol{k_4})\rangle = (2\pi)^3 \delta^3 \Big(\sum_{a=1}^{4} \boldsymbol{k_a} \Big) \frac{H^6}{M_{Pl}^6} \frac{2}{c_s^3 \epsilon^3} \prod_{a=1}^{4} \left(\frac{1}{2k_a^3} \right) \mathcal{T}, \tag{C.7}$$

where $\mathcal{T}$ is given by

$$\begin{aligned} \mathcal{T} =& \left(\frac{\lambda}{\Sigma} \right)^2 \mathcal{T}_{s1} + \frac{\lambda}{\Sigma} \left(\frac{1}{c_s^2} - 1 \right) \mathcal{T}_{s2} + \left(\frac{1}{c_s^2} - 1 \right)^2 \mathcal{T}_{s3} \\ &+ \left(\frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right) \mathcal{T}_{c1} + \left(\frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right) \mathcal{T}_{c2} + \left(\frac{1}{c_s^2} - 1 \right) \mathcal{T}_{c3}. \end{aligned} \tag{C.8}$$

Here, $\mathcal{T}_{s1}, \mathcal{T}_{s2}, \mathcal{T}_{s3}$ are contributions coming from an intermediate scalar exchange, and are given by

$$\begin{aligned} \mathcal{T}_{s1} =& \bigg[\frac{9}{8} k_1^2 k_2^2 k_3^2 k_4^2 k_{12} \frac{1}{(k_1 + k_2 + k_{12})^3 M^3} \\ &+ \frac{9}{4} k_1^2 k_2^2 k_3^2 k_4^2 k_{12} \frac{1}{M^3} \left(\frac{6M^2}{\tilde{K}^5} + \frac{3M}{\tilde{K}^4} + \frac{1}{\tilde{K}^3} \right) \bigg] + 23 \text{ Permutations}, \end{aligned} \tag{C.9}$$

$$\begin{aligned} \mathcal{T}_{s2} = \bigg[&- \frac{3}{32} (\boldsymbol{k_3} \cdot \boldsymbol{k_4})\, k_{12} k_1^2 k_2^2 \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_3, k_4, M) \\ &- \frac{3}{16} (\boldsymbol{k_{12}} \cdot \boldsymbol{k_4}) \frac{k_1^2 k_2^2 k_3^2}{k_{12}} \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_{12}, k_4, M) \\ &- \frac{3}{16} (\boldsymbol{k_3} \cdot \boldsymbol{k_4})\, k_{12} k_1^2 k_2^2\, G_{ab}(k_3, k_4) - \frac{3}{8} (\boldsymbol{k_{12}} \cdot \boldsymbol{k_4}) \frac{k_1^2 k_2^2 k_3^2}{k_{12}} G_{ab}(k_{12}, k_4) \\ &- \frac{3}{16} (\boldsymbol{k_1} \cdot \boldsymbol{k_2})\, k_{12} k_3^2 k_4^2\, G_{ba}(k_1, k_2) + \frac{3}{8} (\boldsymbol{k_{12}} \cdot \boldsymbol{k_2}) \frac{k_1^2 k_3^2 k_4^2}{k_{12}} G_{ba}(-k_{12}, k_2) \bigg] \\ &+ 23 \text{ Permutations}, \end{aligned} \tag{C.10}$$

$$
\begin{aligned}
\mathcal{T}_{s3} = \Bigg[& \frac{1}{128}(\boldsymbol{k_1}\cdot\boldsymbol{k_2})(\boldsymbol{k_3}\cdot\boldsymbol{k_4})k_{12}\,F(k_1,k_2,k_1+k_2+k_{12})\,F(k_3,k_4,M) \\
& + \frac{1}{32}(\boldsymbol{k_1}\cdot\boldsymbol{k_2})(\boldsymbol{k_{12}}\cdot\boldsymbol{k_4})\frac{k_3^2}{k_{12}}\,F(k_1,k_2,k_1+k_2+k_{12})\,F(k_{12},k_4,M) \\
& - \frac{1}{32}(\boldsymbol{k_{12}}\cdot\boldsymbol{k_2})(\boldsymbol{k_{12}}\cdot\boldsymbol{k_4})\frac{k_1^2k_3^2}{k_{12}^3}\,F(k_{12},k_2,k_1+k_2+k_{12})\,F(k_{12},k_4,M) \\
& + \frac{1}{64}(\boldsymbol{k_1}\cdot\boldsymbol{k_2})(\boldsymbol{k_3}\cdot\boldsymbol{k_4})k_{12}\,G_{bb}(k_1,k_2,k_3,k_4) \\
& + \frac{1}{32}(\boldsymbol{k_1}\cdot\boldsymbol{k_2})(\boldsymbol{k_{12}}\cdot\boldsymbol{k_4})\frac{k_3^2}{k_{12}}\,G_{bb}(k_1,k_2,k_{12},k_4) \\
& - \frac{1}{32}(\boldsymbol{k_{12}}\cdot\boldsymbol{k_2})(\boldsymbol{k_3}\cdot\boldsymbol{k_4})\frac{k_1^2}{k_{12}}\,G_{bb}(-k_{12},k_2,k_3,k_4) \\
& - \frac{1}{16}(\boldsymbol{k_{12}}\cdot\boldsymbol{k_2})(\boldsymbol{k_{12}}\cdot\boldsymbol{k_4})\frac{k_1^2k_3^2}{k_{12}^3}\,G_{bb}(-k_{12},k_2,k_{12},k_4)\Bigg] + 23\,\text{Permutations},
\end{aligned}
\tag{C.11}
$$

where we have used the notation

$$
\begin{aligned}
&\tilde{K} = k_1+k_2+k_3+k_4, \\
&\boldsymbol{k_{12}} = \boldsymbol{k_1}+\boldsymbol{k_2}, \\
&k_{12} = |\boldsymbol{k_{12}}|, \\
&M = k_3+k_4+k_{12}.
\end{aligned}
\tag{C.12}
$$

Also, the functions F, G_{ab}, G_{ba} and G_{bb} used in eqs.(C.10) and (C.11) above are defined as

$$
F(u,v,m) \equiv \frac{1}{m^3}\big[2uv+(u+v)m+m^2\big],
\tag{C.13}
$$

$$
\begin{aligned}
G_{ab}(u,v) \equiv{}& \frac{1}{M^3\tilde{K}^3}\big[2uv+(u+v)M+M^2\big] \\
&+ \frac{3}{M^2\tilde{K}^4}\big[2uv+(u+v)M\big] + \frac{12}{M\tilde{K}^5}uv,
\end{aligned}
\tag{C.14}
$$

$$
\begin{aligned}
G_{ba}(u,v) \equiv{}& \frac{1}{M^3\tilde{K}} + \frac{1}{M^3\tilde{K}^2}(u+v+M) \\
&+ \frac{1}{M^3\tilde{K}^3}\big[2uv+2(u+v)M+M^2\big] + \frac{3}{M^2\tilde{K}^4}\big[2uv+(u+v)M\big] \\
&+ \frac{12}{M\tilde{K}^5}uv,
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
G_{bb}(u,v,x,y) \equiv{}& \frac{1}{M^3\tilde{K}}\big[2xy+(x+y)M+M^2\big] \\
&+ \frac{1}{M^3\tilde{K}^2}\big[2xy(u+v)+\big(2xy+(u+v)(x+y)\big)M+(u+v+x+y)M^2\big] \\
&+ \frac{2}{M^3\tilde{K}^3}\big[2uvxy+\big(2xy(u+v)+uv(x+y)\big)M \\
&\qquad + (uv+ux+uy+vx+vy+xy)M^2\big] \\
&+ \frac{6}{M^2\tilde{K}^4}uvxy\left(2+M\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{x}+\frac{1}{y}\right)\right) + \frac{24}{M\tilde{K}^5}uvxy.
\end{aligned}
\tag{C.16}
$$

Also, $\mathcal{T}_{c1}, \mathcal{T}_{c2}, \mathcal{T}_{c3}$ in eq.(C.8) are contributions coming from the four point contact interaction, and are given by

$$\mathcal{T}_{c1} = 36\, \frac{k_1^2 k_2^2 k_3^2 k_4^2}{\tilde{K}^5}, \tag{C.17}$$

$$\mathcal{T}_{c2} = -\frac{1}{8} \frac{k_1^2 k_2^2 (\boldsymbol{k_3} \cdot \boldsymbol{k_4})}{\tilde{K}^3} \left(1 + \frac{3(k_3 + k_4)}{\tilde{K}} + \frac{12 k_3 k_4}{\tilde{K}^2}\right) + 23\,\text{Permutations}, \tag{C.18}$$

and

$$\begin{aligned} \mathcal{T}_{c3} = \frac{1}{32} \frac{(\boldsymbol{k_1} \cdot \boldsymbol{k_2})(\boldsymbol{k_3} \cdot \boldsymbol{k_4})}{\tilde{K}} \Bigg(1 + \frac{\sum_{a<b} k_a k_b}{\tilde{K}^2} + \frac{3 k_1 k_2 k_3 k_4}{\tilde{K}^3} \sum_{a=1}^{4} \frac{1}{k_a} \\ + \frac{12 k_1 k_2 k_3 k_4}{\tilde{K}^4}\Bigg) + 23\,\text{Permutations}. \end{aligned} \tag{C.19}$$

C.2 Calculating the Euclidean AdS on-shell action

In this appendix, we provide some details for calculating the EAdS on-shell action to determine the unknown coefficient $\langle OOO \rangle$.

C.2.1 The longitudinal graviton contribution

We first compute the contribution to the EAdS on-shell action coming from the longitudinal part of the exchanged graviton, which we have denoted by $\mathcal{R}$; see eq.(4.55). For the purpose of calculations, it will be convenient to express $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ given in eq.(4.56) in momentum space. We get

$$\begin{aligned} \mathcal{R}_1 &= \int \frac{dz}{z^2} \frac{d^3k}{(2\pi)^3}\, T_{zj}(z, -\boldsymbol{k})\, \frac{1}{k^2}\, T_{zj}(z, \boldsymbol{k}), \\ \mathcal{R}_2 &= -\frac{i}{2} \int \frac{dz}{z} \frac{d^3k}{(2\pi)^3}\, k_j T_{zj}(z, -\boldsymbol{k})\, \frac{1}{k^2}\, T_{zz}(z, \boldsymbol{k}), \\ \mathcal{R}_3 &= -\frac{1}{4} \int \frac{dz}{z^2} \frac{d^3k}{(2\pi)^3}\, k_j T_{zj}(z, -\boldsymbol{k})\, \frac{1}{k^4}\, k_i T_{zi}(z, \boldsymbol{k}). \end{aligned} \tag{C.20}$$

Now, to compute the remainder terms, we need to find the quantities $T_{zj}(z, \boldsymbol{k})$ and $T_{zz}(z, \boldsymbol{k})$, with the energy-momentum tensor given in eq.(4.45). Note that to get the coefficient $\langle OOO \rangle$, out of the two factors of $T_{\mu\nu}$ in each of the remainder terms, one must give a contribution proportional to $\delta\phi\delta\phi$, and the other must give a contribution proportional to $\bar{\phi}\delta\phi$; see figure 4.3. Consider first the case when the energy-momentum tensor contributes a factor of $\delta\phi\delta\phi$. For this case

$$T_{zj}(z, \boldsymbol{x}) = \partial_z \delta\phi\, \partial_j \delta\phi, \tag{C.21}$$

which on substituting the expression for $\delta\phi$ from eq.(4.51) and converting to momentum space gives [2]

$$T_{zj}(z,\boldsymbol{k}) = -iz\,(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\,\phi_0(\boldsymbol{p})\phi_0(\boldsymbol{q})\,p^2 q_j\,(1+qz)\,\mathrm{e}^{-(p+q)z}, \tag{C.22}$$

where $\boldsymbol{p}, \boldsymbol{q}$ are the momentum labels carried by the two external $\delta\phi$ legs. Similarly, we also have

$$T_{zz}(z,\boldsymbol{x}) = \frac{1}{2}\Big((\partial_z\delta\phi)^2 - (\partial_i\delta\phi)^2\Big), \tag{C.23}$$

which in momentum space takes the form

$$\begin{aligned} T_{zz}(z,\boldsymbol{k}) = \frac{1}{2}\,(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\,\phi_0(\boldsymbol{p})\phi_0(\boldsymbol{q})\times \\ \big[p^2q^2z^2 + (\boldsymbol{p}\cdot\boldsymbol{q})(1+pz)(1+qz)\big]\,\mathrm{e}^{-(p+q)z}. \end{aligned} \tag{C.24}$$

Consider now the case when the energy-momentum tensor contributes one factor of $\bar{\phi}$ and one factor of $\delta\phi$. For this case, we have

$$T_{zj}(z,\boldsymbol{x}) = \partial_z\bar{\phi}\,\partial_j\delta\phi, \tag{C.25}$$

which in momentum space is given by

$$T_{zj}(z,\boldsymbol{k}) = i\,(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})\,\partial_z\bar{\phi}\,\phi_0(\boldsymbol{p})\,p_j(1+pz)\,\mathrm{e}^{-pz}, \tag{C.26}$$

where $\boldsymbol{p}$ is the momentum carried by the external leg $\delta\phi$. We also have [3]

$$T_{zz}(z,\boldsymbol{x}) = \partial_z\bar{\phi}\,\partial_z\delta\phi - \frac{\mathrm{R}^2_{\mathrm{AdS}}}{z^2}\,V'(\bar{\phi})\,\delta\phi, \tag{C.27}$$

implying that

$$T_{zz}(z,\boldsymbol{k}) = -(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})\left[p^2z\partial_z\bar{\phi} + \frac{\mathrm{R}^2_{\mathrm{AdS}}}{z^2}\,V'(\bar{\phi})(1+pz)\right]\phi_0(\boldsymbol{p})\,\mathrm{e}^{-pz}. \tag{C.28}$$

We can now jump into the computation of the remainder terms. Consider for instance the s-channel process shown in figure 4.3. The contribution from this process to $\mathcal{R}_1$, defined in

[2]For $\delta\phi$ given in eq.(4.51), we have

$$\begin{aligned} \partial_z\delta\phi_{\boldsymbol{k}}(z,\boldsymbol{x}) &= -k^2z\,\phi_0(\boldsymbol{k})\mathrm{e}^{-kz}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \\ \partial_j\delta\phi_{\boldsymbol{k}}(z,\boldsymbol{x}) &= ik_j\,\phi_0(\boldsymbol{k})(1+kz)\,\mathrm{e}^{-kz}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \end{aligned}$$

[3]We use the notation $V'(\bar{\phi}) \equiv dV(\bar{\phi})/d\bar{\phi}$. Also note that we have an additional factor of $\mathrm{R}^2_{\mathrm{AdS}}$ multiplying the $V'(\bar{\phi})$ term in eq. (C.27) as compared to eq.(4.45). It is because in writing the EAdS on-shell action eq.(4.52), we extracted out an overall factor of $\mathrm{R}^2_{\mathrm{AdS}}$.

eq.(C.20), is

$$\mathcal{R}_1^s = 2\int \frac{dz}{z^2}\frac{d^3k}{(2\pi)^3}\bigg[-iz\,(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k})\,\phi_0(\boldsymbol{k_1})\phi_0(\boldsymbol{k_2})\times \\ \mathrm{e}^{-(k_1+k_2)z}\Big\{k_1^2k_{2j}(1+k_2z)+k_2^2k_{1j}(1+k_1z)\Big\}\bigg] \\ \times\frac{1}{k^2}\Big[i\,(2\pi)^3\delta^3(\boldsymbol{k_3}-\boldsymbol{k})\,\partial_z\bar{\phi}\,\phi_0(\boldsymbol{k_3})\,k_{3j}(1+k_3z)\,\mathrm{e}^{-k_3z}\Big], \tag{C.29}$$

which on simplification yields

$$\mathcal{R}_1^s = -2\,\frac{\dot{\bar{\phi}}}{H}\,(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\phi_0(\boldsymbol{k_a})\Big)\frac{1}{k_3^2}\times \\ \int_0^\infty\frac{dz}{z^2}\mathrm{e}^{-Kz}\bigg[\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})+k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\ +z\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(k_2+k_3)+k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})(k_1+k_3)\Big\} \\ +z^2\Big\{k_1^2k_2k_3(\boldsymbol{k_2}\cdot\boldsymbol{k_3})+k_2^2k_1k_3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\}\bigg], \tag{C.30}$$

where we have used the fact that under the analytic continuation given in eq.(4.37), $z\partial_z\bar{\phi}$ goes over to $-\dot{\bar{\phi}}/H$, and since we are working to the leading order in slow roll, we can consider this factor to be a constant and pull it out of the z-integral. Eq.(C.30) gives

$$\mathcal{R}_1^s = -\,2\,\frac{\dot{\bar{\phi}}}{H}\,(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\phi_0(\boldsymbol{k_a})\Big)\frac{1}{k_3^2}\times \\ \bigg[\frac{1}{K}\Big\{k_1^2k_2k_3(\boldsymbol{k_2}\cdot\boldsymbol{k_3})+k_2^2k_1k_3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\ +\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(k_2+k_3)+k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})(k_1+k_3)\Big\}\Big(\int_0^\infty\frac{dz}{z}\,\mathrm{e}^{-Kz}\Big) \\ +\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})+k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\}\Big(\int_0^\infty\frac{dz}{z^2}\,\mathrm{e}^{-Kz}\Big)\bigg]. \tag{C.31}$$

Now, the remaining two integrals in eq.(C.31) are divergent. Let us replace the lower limit of the remaining integrals by ρ, where $\rho\to 0$. Then

$$\int_\rho^\infty\frac{dz}{z}\,\mathrm{e}^{-Kz} = \Gamma\big[0,K\rho\big], \tag{C.32}$$

and

$$\int_\rho^\infty\frac{dz}{z^2}\,\mathrm{e}^{-Kz} = \frac{1}{\rho}-K-K\,\Gamma\big[0,K\rho\big]. \tag{C.33}$$

Using the results of eqs.(C.32) and (C.33) in eq.(C.31), we get

$$
\begin{aligned}
\mathcal{R}_1^s = -2\,\frac{\dot{\bar{\phi}}}{H}\,(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\phi_0(\boldsymbol{k_a})\Big)\frac{1}{k_3^2}\times \\
\Bigg[\frac{1}{K}\Big\{k_1^2k_2k_3(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) + k_2^2k_1k_3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\
-\Big(K + K\,\Gamma\big[0,K\rho\big] - \frac{1}{\rho}\Big)\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) + k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\
+\,\Gamma\big[0,K\rho\big]\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(k_2+k_3) + k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})(k_1+k_3)\Big\}\Bigg].
\end{aligned}
\tag{C.34}
$$

This is our final answer for $\mathcal{R}_1^s$. From figure 4.3, it is clear that the t and u channel contributions to $\mathcal{R}_1$ can be easily obtained from $\mathcal{R}_1^s$ by performing momentum interchanges. In particular,

$$
\mathcal{R}_1^t = \mathcal{R}_1^s(\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_3}),
\tag{C.35}
$$

and

$$
\mathcal{R}_1^u = \mathcal{R}_1^s(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_3}).
\tag{C.36}
$$

The complete expression for $\mathcal{R}_1$ is then

$$
\mathcal{R}_1 = \mathcal{R}_1^s + \mathcal{R}_1^t + \mathcal{R}_1^u.
\tag{C.37}
$$

Following exactly the same procedure as above, we can now compute $\mathcal{R}_2$, eq.(C.20). We get[4]

$$
\begin{aligned}
\mathcal{R}_2^s = -\frac{1}{2}\,\frac{\dot{\bar{\phi}}}{H}\,(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\phi_0(\boldsymbol{k_a})\Big)\frac{1}{k_3^2}\times \\
\Bigg[\frac{1}{K^2}\Big\{k_1^2k_2k_3^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) + k_1k_2^2k_3^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3}) - k_1k_2k_3^3(\boldsymbol{k_1}\cdot\boldsymbol{k_2}) - k_1^2k_2^2k_3^3\Big\} \\
+\frac{1}{K}\Big\{k_1^2k_3^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) + k_2^2k_3^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3}) - k_1k_2k_3^2(\boldsymbol{k_1}\cdot\boldsymbol{k_2}) - k_1^2k_2^2k_3^2 \\
- k_3^3(\boldsymbol{k_1}\cdot\boldsymbol{k_2})(k_1+k_2) - 3k_1^2k_2k_3(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) - 3k_1k_2^2k_3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\
+ k_3^2\Big(K - \frac{1}{\rho}\Big)(\boldsymbol{k_1}\cdot\boldsymbol{k_2}) + 3\Big(K + K\,\Gamma\big[0,K\rho\big] - \frac{1}{\rho}\Big)\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3}) + k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\Big\} \\
- 3\Gamma\big[0,K\rho\big]\Big\{k_1^2(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(k_2+k_3) + k_2^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})(k_1+k_3)\Big\}\Bigg].
\end{aligned}
\tag{C.38}
$$

As before, we have

$$
\mathcal{R}_2^t = \mathcal{R}_2^s(\boldsymbol{k_2}\leftrightarrow\boldsymbol{k_3}),\ \ \mathcal{R}_2^u = \mathcal{R}_2^s(\boldsymbol{k_1}\leftrightarrow\boldsymbol{k_3}),
\tag{C.39}
$$

[4]In calculating $\mathcal{R}_2$, we have used the slow roll approximation $V'(\bar{\phi}) \approx -3H\dot{\bar{\phi}}$.

and the complete expression is

$$\mathcal{R}_2 = \mathcal{R}_2^s + \mathcal{R}_2^t + \mathcal{R}_2^u. \tag{C.40}$$

Finally, evaluating $\mathcal{R}_3$, we find that

$$\mathcal{R}_3 = -\frac{1}{4}\mathcal{R}_1. \tag{C.41}$$

Combining the expressions for $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$, we get an expression for $\mathcal{R}$, eq.(4.55), given by

$$\begin{aligned}\mathcal{R} = \frac{1}{2}\frac{\dot{\bar{\phi}}}{H}(2\pi)^3\delta^3\Big(\sum_{a=1}^{3}\boldsymbol{k_a}\Big)\Big(\prod_{a=1}^{3}\phi_0(\boldsymbol{k_a})\Big)\times \\ \Big[-\frac{1}{2}\sum_{a=1}^{3}k_a^3 + \frac{1}{2}\sum_{a\neq b}k_a k_b^2 + \frac{4}{K}\sum_{a<b}k_a^2 k_b^2 - \frac{1}{2\rho}\sum_{a=1}^{3}k_a^2\Big].\end{aligned} \tag{C.42}$$

The cut-off dependent term in eq.(C.42) has to be removed by the addition of a suitable local counter-term. The final expression for the remainder term is then given by eq.(4.57).

C.2.2 The transverse graviton contribution

We now proceed to calculate the contribution to the EAdS on-shell action coming from the transverse part of the exchanged graviton. This contribution is denoted by $\mathcal{W}$, and is defined in eq.(4.54). Out of the two T_{ij} terms in $\mathcal{W}$, one must contribute a piece proportional to $\delta\phi\delta\phi$, and the other must contribute a piece proportional to $\partial_z\bar{\phi}\,\delta\phi$. Using the expression eq.(4.45) for the energy-momentum tensor, we get

$$T_{ij}(z,\boldsymbol{x}) = -\Big(\partial_z\bar{\phi}\,\partial_z\delta\phi + \frac{\mathrm{R}_{\mathrm{AdS}}^2}{z^2}V'(\bar{\phi})\,\delta\phi\Big)\delta_{ij}. \tag{C.43}$$

The key point to note in eq.(C.43) is that T_{ij} is proportional to δ_{ij}. Thus, while calculating $\mathcal{W}$, the contraction of T_{ij} with the transverse graviton propagator $\tilde{\mathcal{G}}_{ij,kl}$ gives the trace $\tilde{\mathcal{G}}_{ii,kl}$, which vanishes. Therefore, there is no contribution from the transverse graviton exchange to the EAdS on-shell action, i.e.

$$\mathcal{W} = 0. \tag{C.44}$$

Therefore, the only contribution to the EAdS on-shell action comes from the longitudinal part of the exchanged graviton, eq.(4.57).

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