WEYL COVARIANCE , HYDRODYNAMICS AND GRAVITY

A thesis submitted to
Tata Institute of Fundamental Research, Mumbai, India
for the degree of
Doctor of Philosophy
in
Physics

By
R.Loganayagam

Department of Theoretical Physics
Tata Institute of Fundamental Research
Homi Bhabha Rd, Mumbai 400005, India.

Jan 2015

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Shiraz Minwalla, at Tata Institute of Fundamental Research, Mumbai.

R. Loganayagam.

(R.Loganayagam)

In my capacity as the supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Shiraz Minwalla)

Date :

Prepared for submission to JHEP

Table of Contents

Contents

---

1 Synopsis

1.1 Motivation

Our understanding of the physical world relies greatly on the various pathbreaking experimental discoveries and the deep mathematical structures which have been uncovered in the quest to understand those observations. In this way, a variety of physical phenomena have received their common explanation through a minimal set of theoretical tools. It is especially exhilarating for a theoretical physicist to find apparently unrelated phenomena unified this way under common insights. A pertinent example in this regard is the way the theory of quantum fields has been used to explore a wide range of phenomena cutting across the conventional borders which separate the various fields of physics.

In a similar vein, it has been realised that string theory, a set of ideas that had originally originated in the experimental studies of gauge theories, also provides a natural framework to address an entirely different question altogether - that of constructing quantum theories of gravity. This realisation has over the past few decades led string theorists to a theoretical structure which seamlessly interweaves some of the central ideas of theoretical physics - from quantum mechanics to gauge theories, from general relativity to supersymmetry apart from opening up many new avenues in mathematical physics.

Notable in this regard is the way string theory incorporates and extends the idea of duality in the space of theories. Since duality will be a crucial theme of the work that is reported in this synopsis, it is essential that we briefly review the idea of duality at this point. Duality is a phenomenon in quantum theories whereby two theories which are classically distinct lead to the same theory after quantisation. The simplest and oft-used example of duality is the fact that a quantum theory of a many-particle system is identical to a theory of quantum fields despite the fact that a classical many particle system and a classical field theory are very different as physical systems.

Over the last century, many such dualities have been discovered and they have served as a rich source of insights whenever they are applicable. A non-exhaustive list of such field theoretical dualities include Kramers-Wannier duality in the Lattice Ising theory, the phenomenon of bosonisation in various $1+1$d systems, Seiberg dualities of the $\mathcal{N}=1$ superconformal fixed points, various $\mathcal{N}=2$ dualities inspired by the pioneering solution of Seiberg and Witten and the Montonen-Olive duality in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. String theory incorporates and extends various such dualities in its deep structure resulting in a rich web of string-string dualities connecting various string theories. More surprisingly, about a decade ago, it was realised that there are dualities which connect certain string theories on one side to field theories on the other side thus supporting the idea that the study of a particular corner in the space of all non-gravitational field theories leads one irrevocably into string theory and quantum gravity.

That one needed an apparently unrelated quest towards a quantum theory of gravity to lead us to such a statement about ordinary quantum field theories is itself revealing. More than anything else, it is humbling for a physicist to realise that our understanding of strongly coupled field theories is so rudimentary that some of the deepest statements about field theories in the last few decades has been discovered by pursuing an esoteric quest that according to naive intuition, seems far removed from such questions.

We live in very interesting times - despite the fact that physics has made enormous progress in understanding the many aspects of reality, there is still an enormous sea of challenging questions which are yet to be answered. These questions take various forms - what are the possible behaviours that are possible in a given system ? What is the most efficient way to understand such a behaviour ? What mathematical patterns underlie these behaviours ? It is fair to say that in many physical systems we are far away from answering these questions comprehensively. Our obstacles are many - there is no foolproof way to isolate the right degrees of freedom to describe a particular situation in physics except for symmetry considerations and prior experience. The most obvious variables are often the least useful because the phenomenon we are interested in arises out of strong interactions between the obvious degrees of freedom. Dualities, when present, give a rare window into these questions. In fact, it can be argued that almost everything that we analytically know about non-trivial quantum systems can be attributed to symmetries along with various exact and approximate dualities exhibited by the system (especially notable in this regard is the role of effective field theories which are essentially approximate dualities at the low-energy regime).

In this synopsis, we will describe a particular example of such a duality and its usefulness in answering questions about certain strongly coupled model field theories. These are questions regarding the transport phenomena in these systems which at the outset appear to be theoretically intractable, but we will take recourse to the symmetries and dualities to make progress. These dualities will lead us into the study of what naively appears to be an unrelated set of questions about black holes, black branes and their long-time evolution and at the end of it all, we will discover some very interesting transport phenomena which were discovered through the methods that are outlined in this synopsis.

1.2 Hydrodynamic description

How does a physicist proceed when confronted with a strongly coupled system ? Our first instinct is to ask for the basic symmetries that govern the system under study. Often strongly coupled field theories have spacetime symmetries of translations, rotations and boosts along with other global symmetries. Although exceptions to this statement (which involve spatially modulated and/or orientationally ordered phases of matter where these symmetries are spontaneously broken) are of great interest[1], we wish to focus on phases where these symmetries serve as good guides to the dynamics of the system[2]. In a relativistic theory, these symmetries

[1] After all, some of the most studied models in physics including solids, magnets and liquid crystals fall into this category.

[2] It would be an interesting exercise to generalise the methods detailed below to cover such ordered phases.There have been many attempts to apply holographic methods to spatially modulated phases but a general theory of the kind we describe below is yet to be developed.

guarantee the conservation of the energy-momentum tensor[3] $\nabla_\mu T^{\mu\nu} = 0$ and conservation of the charge current $\nabla_\mu J_i^\mu = 0$.

To begin with, we will look for a sector where the dynamics is essentially captured by the dynamics of the energy-momentum and charge flow. This in particular means that if there are other macroscopic degrees of freedom in the system, we are assuming that there is a sector where their dynamics can be decoupled and hence ignored. We do not know how generic the presence of such a sector is in strongly coupled systems[4] , but later on we will present examples where such a truncation to just the conserved charges can be justified at strong coupling.

Having said that, we should also add that the statics of many interacting systems have a thermodynamic limit where the conserved charges (along with entropy given as a function of these conserved charges) capture the essential physics. So, it is not entirely unnatural to expect that if we introduce a slow time-dependence with small spatial gradients such that local patches of the system are in thermodynamic equilibrium (and assuming we are able to introduce these spatial gradients without exciting the other macroscopic degrees of freedom), then the dynamics of the conserved quantities continues to be an effective way to describe the dynamics of the system.

Having convinced ourselves that this should be the case, we however face a conundrum - the energy momentum tensor $T^{\mu\nu}$ in d dimensions is a symmetric rank two tensor with $d(d+1)/2$ components and the d equations $\nabla_\mu T^{\mu\nu} = 0$ simply are not enough to determine the dynamics of all these components[5] ! Similarly, for the ith charge we have a single equation $\nabla_\mu J_i^\mu = 0$ which is not enough to determine the dynamics of d components of J_i^μ. This in turn implies that either we should give up on treating each and every component of $T^{\mu\nu}$ and J_i^μ as independent or we should add more dynamical equations to the set of conservation laws.

We will again be guided by our thermodynamic intuition - a static thermodynamic equilibrium state in a field theory often has a rest frame (specified by a unit time-like vector u^μ) which we can think of as a frame where there is no energy flow - more precisely, u^μ can be thought of as the time-like eigenvector of the energy momentum tensor T^μ_ν. Such a state is characterised by its rest frame energy density $\varepsilon \equiv T^{\mu\nu} u_\mu u_\nu$ and the rest frame charge density $n_i \equiv -J_i^\mu u_\mu$ (where we have crucially used our assumption that all other macroscopic degrees of freedom can be ignored). This fact combined with our physical intuition derived from various real systems leads us to the basic idea that resolves our problem - if we treat

[3]In a quantum field theory, a precise way to state the conservation laws is to invoke the Ward identities satisfied by arbitrary n-point functions of energy-momentum operator $\hat{T}^{\mu\nu}$ and the current operator $\hat{J}_i^\mu$. In our present discussion, we will focus mainly on the one-point functions $T^{\mu\nu}$ and J_i^μ ignoring admittedly interesting questions about the hydrodynamic fluctuations.

[4]What we do know is that phases with macroscopic degrees of freedom that are not conserved can exhibit qualitatively new features.

[5]Unless $d(d+1)/2 = d$ whose only non-trivial solution is $d = 1$. We may try to save ourselves specialising to conformal field theories where the trace degree of freedom T^μ_μ is removed, in which case we have enough equations only when $d(d+1)/2 - 1 = d$ or $d = 2$.

u^μ, ε and n_i as our basic variables in terms of which T^μ_ν and J^μ_i are specified, then we have as many variables as there are conservation laws resulting in a consistent dynamical system. When the macroscopic behaviour of a quatum field theory - strongly coupled or otherwise reduces to such a system, we say that we have a hydrodynamic description emerging out of that particular quantum field theory.

Given the argument above, one can ask whether there are actually calculable model quantum field theories in which such an emergence of hydrodynamic description can actually be exhibited and the corresponding T^μ_ν and J^μ_i be calculated as functions of u^μ, ε and n_i. This question leads us into the field of transport phenomena in various field theories where for weakly coupled field theories, there are by now various well-developed perturbative formalisms in non-equilibrium statistical physics to answer this question. In case of strongly coupled field theories, there is no general approach except numerical studies and studying transport in quantum field theories is still a computationally expensive proposition. So, it would appear that to exhibit the emergence of hydrodynamic behaviour in a strongly coupled system is for all practical purposes an intractable problem. While this pessimistic assertion may be justified in general, as we have mentioned before, there are strongly coupled systems which can be made tractable by using dualities and string theory with its cornucopia of dualities is an apt place to look for such model systems.

1.3 Gauge-Gravity duality and Supersymmetric Yang-Mills

In this synopsis, we will be focusing on a particular duality called gauge-gravity duality in string theory and ask what can it tell us about transport in strongly coupled field theories. For reasons of tractability, we will focus on a strongly coupled but a conformal quantum field theory which has no dimensionful quantity in its microscopic description [6]. Further, all the field theories we will consider have a parameter N characterising the number of degrees of freedom and we will be working in a large N limit. In higher dimensions, most known examples of strongly interacting conformal field theories happen to be supersymmetric, but supersymmetry will not play a direct role in our subsequent discussion.

A particularly good example of the kind of models under consideration is the $SU(N)$ supersymmetric gauge theory in 3+1 dimensions with a four-fold supersymmetry. Since we will later be discussing the transport properties in this model in some detail, we will briefly review the main features of this model. It has the following field content

- An $SU(N)$ gauge-field A^a_μ where $\mu = 0, 1, 2, 3$ denotes that it is a spacetime vector field and the index $a = 1, \ldots, N^2 - 1$ runs over the adjoint representation of the gauge group $SU(N)$.

[6] Note however that a conformal field theory at a finite temperature equilibrium state has a macroscopic scale - the temperature which is a characteristic of that state. Similar statements can be made about hydrodynamic states with a given energy density ε or charge density n.

- Four chiral fermions $\psi_a^{\alpha A}$ charged under the adjoint representation of $SU(N)$ where $\alpha = 1, 2$ denotes that it is a chiral fermion field and $A = 1, \ldots 4$ is the flavor index. The complex conjugate of $\psi_a^{\alpha A}$ is an anti-chiral spinor denoted as $\bar{\psi}^{\dot{\alpha}}_{Aa} = (\psi_a^{\alpha A})^*$

- Six real scalars $\Phi_a^{AB} = (\Phi_{aAB})^* = -\Phi_a^{BA}$ charged under the adjoint representation of $SU(N)$. It is convenient to define $\bar{\Phi}_{AB}$ via the relation

$$\Phi_a^{AB} \equiv \frac{1}{2}\epsilon^{ABCD}\bar{\Phi}^a_{CD}$$

The theory is defined by the Lagrangian density[1]

$$-\mathcal{L} = \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - (D_\mu\bar{\Phi}_{AB})_a(D^\mu\Phi^{AB})_a + i\psi_a^{\alpha A}\sigma^\mu_{\alpha\dot{\alpha}}(D_\mu\bar{\psi}^{\dot{\alpha}}_A)_a +$$
$$+ g^2 f^{abc}\Phi_b^{AB}\Phi_c^{CD}f^{ade}\bar{\Phi}^d_{AB}\bar{\Phi}^e_{CD} + g\sqrt{2}f^{abc}\left[\psi_a^{\alpha A}\bar{\Phi}^b_{AB}\psi_\alpha^{cB} + \bar{\psi}^a_{\dot{\alpha}A}\Phi_b^{AB}\bar{\psi}^{\dot{\alpha}}_{cB}\right]$$

where f^{abc} are the structure constant of the gauge group $SU(N)$, g is the gauge-coupling and D_μ is the covariant derivative appropriate to the gauge group. This theory is among the very few interacting conformal field theories known in $3 + 1$ dimensions. In particular, the coupling g in the above lagrangian does not run with scale and the classical scale-invariance of the above lagrangian is continued to be preserved by the quantum fluctuations.

This theory is manifestly invariant under a $SU(4)$ flavor symmetry (often called the R-symmetry of this theory) under which the different fields transform as

$$\psi_\alpha^A \to U^A{}_B\psi_\alpha^B \qquad \bar{\psi}_{\dot{\alpha}A} \to (U^*)_A{}^B\bar{\psi}_{\dot{\alpha}B}$$

and

$$\Phi^{AB} \to U^A_C\Phi^{CD}(U^T)_D^{\ B} \qquad \bar{\Phi}_{AB} \to (U^*)_A^{\ C}\bar{\Phi}_{CD}(U^\dagger)^D_{\ B}$$

where U is a 4×4 unitary matrix with ($UU^\dagger = 1$). This classical global symmetry is continued to be respected by the quantum fluctuations and hence we have quantum global $SU(4)$ symmetry in our theory with the corresponding Noether currents. For reasons that will become clearer later, it is especially convenient to focus on the transport of the Noether charge associated with a $U(1)$ subgroup of $SU(4)$ made up of Us of the form

$$U = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} \in SU(4)$$

One of the main results of the work presented here would be the claim(that we hope to substantiate by the end of this synopsis) that in an appropriate limit, using gauge-gravity duality, we can precisely argue for the existence of certain novel phenomena in the transport of this charge and compute the corresponding transport coefficients.

The study of this conformal field theory can be simplified further by taking the number of colors N in this theory to be large. Following ‘t Hooft, to get a sensible limit we should keep the ‘t Hooft coupling $\lambda \equiv g^2 N$ fixed as N is taken to infinity. Such a limit gives us an interacting theory whose interactions are governed by the parameter λ. This then is a precise model system, where the questions of the previous subsection can be asked - what are the transport properties of this model when λ is very large ? Are they even calculable ? Can one see the emergence of hydrodynamic behaviour in the macroscopic dynamics of this model ? The answer to these apparently intractable questions about an interacting theory is surprisingly in affirmative because of the gauge-gravity duality. A variety of transport phenomena in this strongly interacting theory using the gauge-gravity duality have been by now well-studied by various authors. In the course of this synopsis, we will describe only the studies to which I have contributed and briefly review those parts of other works which provide the necessary background.

1.4 Weyl Covariance and Hydrodynamics

Before applying gauge-gravity duality to derive the hydrodynamics emerging out of an underlying conformal theory, it is worthwhile to understand the structure of such a hydrodynamic description[7]. The fact that the microscopic theory is devoid of scales imposes strict constraints on the kind of transport that can occur in the macroscopic description. We will summarise in this subsection, a formalism developed in [2] to naturally incorporate these constraints into the hydrodynamic description.

A conformal field theory living on a d dimensional spacetime $\mathcal{M}$ with a metric $g_{\mu\nu}$ is by definition a theory which is covariant (upto quantum Weyl anomalies) under the Weyl transformation which replaces the old metric $g_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$ given by

$$g_{\mu\nu} = e^{2\phi(x)}\tilde{g}_{\mu\nu}; \qquad g^{\mu\nu} = e^{-2\phi(x)}\tilde{g}^{\mu\nu} \tag{1.1}$$

In some sense, these theories are naturally thought of (again upto Weyl anomalies) as living on a spacetime $\mathcal{M}$ which has a class of metrics $\mathcal{C}_g$ which are related to our original metric by a Weyl transformation. We will henceforth refer to the fluid which emerges out of such a conformal field theory at a finite temperature as a conformal fluid.

The conformal fluid which emerges out of a conformal field theory inherits Weyl-covariance from its microscopic theory. The question that we wish to answer is how this Weyl-covariance manifests itself at the level of hydrodynamics and how does one go about constructing Weyl-covariant quantities in hydrodynamics. Let u^μ be the unit time-like vector describing the fluid motion. Using $g_{\mu\nu}u^\mu u^\nu = \tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$, we get the Weyl transformation of the velocity $u^\mu = e^{-\phi}\tilde{u}^\mu$. Now, we can go ahead and construct various derivatives of this velocity field, provided we have a way of differentiating fields in a Weyl-covariant way.

[7] This subsection is based on the work that was done by the author and presented in the form of the paper [2].

To see how this might be possible, we begin with the Weyl-transformation of the Christoffel symbols

$$\Gamma_{\lambda\mu}{}^{\nu} = \widetilde{\Gamma}_{\lambda\mu}{}^{\nu} + \delta^{\nu}_{\lambda}\partial_{\mu}\phi + \delta^{\nu}_{\mu}\partial_{\lambda}\phi - \tilde{g}_{\lambda\mu}\tilde{g}^{\nu\sigma}\partial_{\sigma}\phi \tag{1.2}$$

from which it follows that

$$\mathcal{A}_{\nu} \equiv u^{\mu}\nabla_{\mu}u^{\nu} - \frac{\nabla_{\mu}u^{\mu}}{d-1}u_{\nu} = \tilde{\mathcal{A}}_{\nu} + \partial_{\nu}\phi. \tag{1.3}$$

This transformation means that hydrodynamics provides a natural 'gauge field' for Weyl transformations which can be used to construct a Weyl-covariant derivative. We define a Weyl covariant derivative $\mathcal{D}$ such that, if a tensorial quantity $Q^{\mu\ldots}_{\nu\ldots}$ obeys $Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$, then $\mathcal{D}_{\lambda}\, Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{\mathcal{D}}_{\lambda}\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$ where

$$\begin{aligned}\mathcal{D}_{\lambda}\, Q^{\mu\ldots}_{\nu\ldots} \equiv{}& \nabla_{\lambda}\, Q^{\mu\ldots}_{\nu\ldots} + w\, \mathcal{A}_{\lambda}Q^{\mu\ldots}_{\nu\ldots}\\ &+ \left[g_{\lambda\alpha}\mathcal{A}^{\mu} - \delta^{\mu}_{\lambda}\mathcal{A}_{\alpha} - \delta^{\mu}_{\alpha}\mathcal{A}_{\lambda}\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots\\ &- \left[g_{\lambda\nu}\mathcal{A}^{\alpha} - \delta^{\alpha}_{\lambda}\mathcal{A}_{\nu} - \delta^{\alpha}_{\nu}\mathcal{A}_{\lambda}\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots\end{aligned} \tag{1.4}$$

Note that the above covariant derivative is metric compatible ($\mathcal{D}_{\lambda}g_{\mu\nu} = 0$). In mathematical terms, what we have done is to use the additional mathematical structure provided by a fluid background (namely a unit time-like vector field with conformal weight $w = 1$) to define what is known as a *Weyl connection* over $(\mathcal{M}, \mathcal{C}_g)$ where $\mathcal{M}$ is the spacetime manifold with the conformal class of metrics $\mathcal{C}_g$.

A torsionless connection ∇^{weyl} is called a Weyl connection (see for example, [3] and references therein) if for every metric in the conformal class $\mathcal{C}_g$ there exists a one form $\mathcal{A}_{\mu}$ such that $\nabla^{weyl}_{\mu} g_{\nu\lambda} = 2\mathcal{A}_{\mu}g_{\nu\lambda}$. Having a fluid over the manifold provides us a natural one form $\mathcal{A}_{\mu}$ (see below), which can in turn be used to define a Weyl connection. The 'prolonged' covariant derivative $\mathcal{D}$ is related to this Weyl connection via the relation $\mathcal{D}_{\mu} = \nabla^{weyl}_{\mu} + w\mathcal{A}_{\mu}$. In terms of this covariant derivative, the condition for Weyl connection is just the statement of metric compatibility ($\mathcal{D}_{\lambda}g_{\mu\nu} = 0$) and the one-form $\mathcal{A}_{\mu}$ is uniquely determined by requiring that the covariant derivative of u^{μ} be transverse ($u^{\lambda}\mathcal{D}_{\lambda}u^{\mu} = 0$) and traceless ($\mathcal{D}_{\lambda}u^{\lambda} = 0$).

We can define a curvature associated with the Weyl-covariant derivative by the usual procedure of evaluating the commutator between two covariant derivatives. For a covariant vector field $V_{\mu} = e^{-w\phi}\widetilde{V}_{\mu}$, we get

$$\begin{aligned}[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]V_{\lambda} &= w\, \mathcal{F}_{\mu\nu}\, V_{\lambda} + \mathcal{R}_{\mu\nu\lambda}{}^{\alpha}\, V_{\alpha} \qquad \text{with}\\ \mathcal{F}_{\mu\nu} &= \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu}\\ \mathcal{R}_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} - \delta^{\alpha}_{[\mu}g_{\nu][\lambda}\delta^{\beta}_{\sigma]}\left(\nabla_{\alpha}\mathcal{A}_{\beta} + \mathcal{A}_{\alpha}\mathcal{A}_{\beta} - \frac{\mathcal{A}^2}{2}g_{\alpha\beta}\right) + \mathcal{F}_{\mu\nu}g_{\lambda\sigma}\end{aligned} \tag{1.5}$$

where we have introduced two new Weyl-invariant tensors $\mathcal{F}_{\mu\nu} = \widetilde{\mathcal{F}}_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} = \widetilde{\mathcal{R}}_{\mu\nu\lambda}{}^{\alpha}$ and $B_{[\mu\nu]} \equiv B_{\mu\nu} - B_{\nu\mu}$ indicates antisymmetrisation[8]. Since these Weyl-covariant counterparts of curvature tensors will play a useful role in the formulation of conformal hydrodynamics, we will briefly describe their properties[9].

We can write down similar expressions involving Ricci tensor, Ricci scalar and Einstein tensor.

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &\equiv \mathcal{R}_{\mu\alpha\nu}{}^{\alpha} = R_{\mu\nu} + (d-2)\left(\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \mathcal{A}^2 g_{\mu\nu}\right) + g_{\mu\nu}\nabla_\lambda \mathcal{A}^\lambda + \mathcal{F}_{\mu\nu} = \widetilde{\mathcal{R}}_{\mu\nu} \\
\mathcal{R} &\equiv \mathcal{R}_{\alpha}{}^{\alpha} = R + 2(d-1)\nabla_\lambda \mathcal{A}^\lambda - (d-2)(d-1)\mathcal{A}^2 = e^{-2\phi}\widetilde{\mathcal{R}} \\
\mathcal{G}_{\mu\nu} &\equiv \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu} = G_{\mu\nu} + (d-2)\left[\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \left(\nabla_\lambda \mathcal{A}^\lambda - \frac{d-3}{2}\mathcal{A}^2\right) g_{\mu\nu}\right] + \mathcal{F}_{\mu\nu}
\end{aligned} \tag{1.7}$$

These curvature tensors obey various Bianchi identities [10]

$$\begin{aligned}
\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} + \mathcal{R}_{\lambda[\mu\nu]}{}^{\alpha} &= 0 \\
\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_{[\mu}\mathcal{F}_{\nu]\lambda} &= 0 \\
\mathcal{D}_\lambda \mathcal{R}_{\mu\nu\alpha}{}^{\beta} + \mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda\alpha}{}^{\beta} &= 0
\end{aligned} \tag{1.8}$$

and various reduced Bianchi identities[11]

$$\begin{aligned}
\mathcal{R}_{[\mu\nu]} = \mathcal{R}_{\mu\nu\alpha}{}^{\alpha} &= d\ \mathcal{F}_{\mu\nu} \\
\mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda} + \mathcal{D}_\sigma \mathcal{R}_{\mu\nu\lambda}{}^{\sigma} &= 0 \\
\mathcal{D}_\lambda\left(\mathcal{G}^{\mu\lambda} - \mathcal{F}^{\mu\lambda}\right) &= 0
\end{aligned} \tag{1.9}$$

The tensor $\mathcal{R}_{\mu\nu\lambda\sigma}$ does not have the same symmetry properties as that of the usual Riemann tensor. For example,

$$\begin{aligned}
\mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu\sigma\lambda} &= 2\ \mathcal{F}_{\mu\nu} g_{\lambda\sigma} \\
\mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{R}_{\lambda\sigma\mu\nu} &= -\delta^{\alpha}_{[\mu} g_{\nu][\lambda}\delta^{\beta}_{\sigma]}\mathcal{F}_{\alpha\beta} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} - \mathcal{F}_{\lambda\sigma} g_{\mu\nu} \\
\mathcal{R}_{\mu\alpha\nu\beta} V^\alpha V^\beta - \mathcal{R}_{\nu\alpha\mu\beta} V^\alpha V^\beta &= \mathcal{F}_{\mu\nu}\ V^\alpha V_\alpha
\end{aligned} \tag{1.10}$$

The conformal tensors of the underlying spacetime manifold appear in the above formalism as a subset of conformal observeables in hydrodynamics. These conformal tensors are

---

[8] As is evident from the notation above, we use calligraphic alphabets to denote the Weyl-covariant counterparts of the usual curvature tensors. Our notation for the usual Riemann tensor is defined by the relation

$$[\nabla_\mu, \nabla_\nu] V_\lambda = R_{\mu\nu\lambda}{}^{\sigma} V_\sigma . \tag{1.6}$$

Note that the curvature tensors used in this synopsis are negative of those employed in [2].

[9] Note that these curvature tensors are essential even if one is in a flat spacetime, since most of these Weyl-covariant curvatures do not vanish for a general velocity configuration in flat spacetime.

[10] These identities can be derived from the Jacobi identity for the covariant derivative - $[\mathcal{D}_{[\mu}, [\mathcal{D}_{\nu]}, \mathcal{D}_\lambda] + [\mathcal{D}_\lambda, [\mathcal{D}_\mu, \mathcal{D}_\nu]] = 0$

[11] These identities are obtained from the Bianchi identities by contractions.

the Weyl-covariant tensors that are independent of the background fluid velocity. The Weyl curvature $C_{\mu\nu\lambda\sigma}$ is a well-known example of a conformal tensor. We have

$$\mathcal{C}_{\mu\nu\lambda\sigma} \equiv \mathcal{R}_{\mu\nu\lambda\sigma} + \delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \mathcal{S}_{\alpha\beta} = C_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} = e^{2\phi} \widetilde{\mathcal{C}}_{\mu\nu\lambda\sigma} \tag{1.11}$$

where the Schouten tensor $\mathcal{S}_{\mu\nu}$ is defined as

$$\mathcal{S}_{\mu\nu} \equiv \frac{1}{d-2}\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R} g_{\mu\nu}}{2(d-1)}\right) = S_{\mu\nu} + \left(\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \frac{\mathcal{A}^2}{2} g_{\mu\nu}\right) + \frac{\mathcal{F}_{\mu\nu}}{d-2} = \widetilde{\mathcal{S}}_{\mu\nu} \tag{1.12}$$

From equation (1.11), it is clear that $C_{\mu\nu\lambda\sigma} = \mathcal{C}_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu} g_{\lambda\sigma}$ is clearly a conformal tensor. Such an analysis can in principle be repeated for the other known conformal tensors in arbitrary dimensions.

The Weyl Tensor $C_{\mu\nu\lambda\sigma}$ has the same symmetry properties as that of Riemann Tensor $R_{\mu\nu\lambda\sigma}$.

$$\begin{gathered} C_{\mu\nu\lambda\sigma} = -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu} \\ \text{and} \quad C_{\mu\alpha\lambda}{}^{\alpha} = 0 \end{gathered} \tag{1.13}$$

From which it follows that $C_{\mu\alpha\nu\beta} u^\alpha u^\beta$ is a symmetric traceless and transverse tensor - a fact which will turn out to be important later in our discussion of conformal hydrodynamics.

Now, we turn to the study of how various quantities of relevance to hydrodynamics can be constructed in this formalism. The Weyl-covariant derivative of the velocity field naturally breaks up into a symmetric part and an antisymmetric part

$$\begin{aligned} P_{\mu\nu} &\equiv g^{\mu\nu} + u^\mu u^\nu \\ \mathcal{D}_\mu u^\nu &= \nabla_\mu u^\nu + u_\mu (u.\nabla) u^\nu - \frac{\nabla . u}{d-1} P_\mu{}^\nu = \sigma_\mu{}^\nu + \omega_\mu{}^\nu = e^{-\phi} \widetilde{\mathcal{D}}_\mu \tilde{u}^\nu, \\ \sigma^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda} \nabla_\lambda u^\nu + P^{\nu\lambda} \nabla_\lambda u^\mu\right) - \frac{\nabla . u}{d-1} P^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu + \mathcal{D}^\nu u^\mu\right) = e^{-3\phi} \tilde{\sigma}^{\mu\nu}, \\ \omega^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda} \nabla_\lambda u^\nu - P^{\nu\lambda} \nabla_\lambda u^\mu\right) = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu - \mathcal{D}^\nu u^\mu\right) = e^{-3\phi} \tilde{\omega}^{\mu\nu}. \end{aligned} \tag{1.14}$$

The shear strain rate $\sigma_{\mu\nu}$ is a symmetric traceless tensor which tells us the rate at which the fluid element around a point is sheared whereas the vorticity $\omega_{\mu\nu}$ is an antisymmetric tensor that roughly tells us how fast the fluid is swirled around a point. Other important quantities in hydrodynamics are the energy density ε, pressure p, entropy density s, temperature T, conserved charge density n_i and corresponding chemical potentials μ_i. The scaling properties of these thermodynamic quantities is directly determined by the naive dimensional analysis

$$\begin{gathered} \varepsilon = e^{-d\phi} \tilde{\varepsilon}\ , \quad p = e^{-d\phi} \tilde{p}\ , \quad s = e^{-(d-1)\phi} \tilde{s}\ , \\ T = e^{-\phi} \tilde{T}\ , \quad n_i = e^{-(d-1)\phi} \tilde{n}_i\ , \quad \mu_i = e^{-\phi} \tilde{\mu}_i\ . \end{gathered} \tag{1.15}$$

We now turn to the Weyl-transformation of the conserved currents $T_{\mu\nu}$ and J_i^μ. The scaling of J_i^μ is again fixed by naive dimensional analysis $J_i^\mu = e^{-d\phi} \tilde{J}_i^\mu$. It is easily shown that for such a J_i^μ, we have $\mathcal{D}_\mu J_i^\mu = \nabla_\mu J_i^\mu$ so that statement that it is conserved is consistent with Weyl-invariance.

We now turn to $T_{\mu\nu}$ - here we face a subtlety due to quantum anomalies in the Weyl transformation. If we ignore it, and use naive dimensional analysis to fix its scaling as $T^{\mu\nu}_{\text{classical}} = e^{-(d+2)\phi}\widetilde{T}^{\mu\nu}_{\text{classical}}$ then we get

$$\mathcal{D}_\mu T^{\mu\nu}_{\text{classical}} = \nabla_\mu T^{\mu\nu}_{\text{classical}} + \mathcal{A}^\nu T^{\mu}_{\mu\,\text{classical}}$$

so we recover the familiar statement that for the conservation of classical energy momentum to be consistent with the Weyl-covariance, we should also impose that the classical energy momentum tensor be traceless. Now, we turn to the actual quantum theory - the Weyl transformation of $T_{\mu\nu}$ in quantum theories is non-trivial because of the presence of Weyl anomaly - a quantum obstruction to the existence of a tensor which is both conserved and Weyl-covariant. This in turn means that $T^\lambda{}_\lambda = \mathcal{W}[g]$ where $\mathcal{W}[g]$ is the trace contribution due to the Weyl anomaly and it depends only on the microscopic field content and the ambient spacetime in which the conformal fluid lives and is independent of the state under consideration.

Often there exists another symmetric traceless tensor $T^{\mu\nu}_{\text{conf}}$ which is not conserved, but is Weyl-covariant.

$$T^{\mu\nu}_{\text{conf}} \equiv T^{\mu\nu} - T^{\mu\nu}_{anom}[g] = e^{-(d+2)\phi}\widetilde{T}^{\mu\nu}_{\text{conf}}$$

where $T^{\mu\nu}_{anom}[g]$ characterises the contribution due to Weyl anomaly which depends only on the background spacetime and the field content. We will assume from now on that this indeed the case for the models under consideration. In this case, one can show that the transformation of the energy-momentum tensor is such that the quantity

$$\mathcal{D}_\mu T^{\mu\nu} \equiv \nabla_\mu T^{\mu\nu} + \mathcal{A}^\nu(T^\mu{}_\mu - \mathcal{W}[g]) = e^{-(d+2)\phi}\tilde{\mathcal{D}}_\mu \widetilde{T}^{\mu\nu}$$

is Weyl-covariant. This in turn implies that the two statements of the conservation of $T_{\mu\nu}$ and freezing the trace of $T_{\mu\nu}$ to the Weyl-anomaly of the microscopic theory together can be packaged into a Weyl-covariant statement $\mathcal{D}_\mu T^{\mu\nu} = 0$. Since we have shown that the basic equations of conformal hydrodynamics are Weyl-covariant, the only constraint on the hydrodynamics coming from the underlying conformal field theory is that $T^{\mu\nu}_{\text{conf}}$ and J_i^μ be Weyl-covariant functionals of the basic hydrodynamic variables. This constraint is easily implemented by working within the manifestly Weyl-covariant formalism that we have just outlined. This then completes our programme to formulate hydrodynamics in a manifestly Weyl-covariant way.

We can now use this Weyl-covariant derivative to enumerate all the Weyl-covariant scalars, transverse vectors (i.e, vectors that are everywhere orthogonal to the fluid velocity field u^μ) and the transverse traceless tensors in the charged hydrodynamics that involve no more than second order derivatives. We will do this enumeration 'on-shell', i.e., we will enumerate those quantities which remain linearly independent even after the equations of motion are taken into account.

The basic fields in the charged hydrodynamics are the fluid velocity u^μ with weight unity, the fluid temperature T with with weight unity and the chemical potentials μ_i with weight unity. This implies that an arbitrary function of μ_i/T is Weyl-invariant and hence one could always multiply a Weyl-covariant tensor by such a function to get another Weyl-covariant tensor. Hence, in the following list only linearly independent fields appear. To make contact with the conventional literature on hydrodynamics we will work with the charge densities n_i (with weight $d-1$) rather than the chemical potentials μ_i. For simplicity, we will confine ourselves to the case where there is only one charge.

At one derivative level,

- Weyl invariant scalars : None
- Weyl invariant transverse vector : $n^{-1}P_\mu^\nu \mathcal{D}_\nu n$.
  In d=4 , we also have $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha} u_\mu \nabla_\nu u_\lambda$.
- Weyl-invariant symmetric traceless transverse tensors : $T\sigma_{\mu\nu}$

At the two derivative level,

- Weyl invariant scalars :

$$\begin{aligned}&T^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad T^{-2}\omega_{\mu\nu}\omega^{\mu\nu}, \quad T^{-2}\mathcal{R},\\ &T^{-2}n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n \quad \text{and} \quad T^{-2}n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\mathcal{D}_\nu n\end{aligned} \tag{1.16}$$

  In d=4 ,

$$T^{-2}n^{-1}l^\mu \mathcal{D}_\mu n$$

- Weyl-invariant transverse vectors :

$$\begin{aligned}&T^{-1}P_\mu^\nu \mathcal{D}_\lambda {\sigma_\nu}^\lambda, \qquad T^{-1}P_\mu^\nu \mathcal{D}_\lambda {\omega_\nu}^\lambda,\\ &T^{-1}n^{-1}{\sigma_\mu}^\lambda \mathcal{D}_\lambda n \qquad \text{and} \qquad T^{-1}n^{-1}{\omega_\mu}^\lambda \mathcal{D}_\lambda n\end{aligned} \tag{1.17}$$

  In d=4 ,

$$T^{-1}\sigma_{\mu\nu}\, l^\nu$$

  .

- Weyl-invariant symmetric traceless transverse tensors : “

$$\begin{gathered}C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu},\\ {\omega_\mu}^\lambda\sigma_{\lambda\nu} + {\omega_\nu}^\lambda\sigma_{\lambda\mu}, \quad {\sigma_\mu}^\lambda\sigma_{\lambda\nu} - \frac{P_{\mu\nu}}{d-1}\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad {\omega_\mu}^\lambda\omega_{\lambda\nu} + \frac{P_{\mu\nu}}{d-1}\,\omega_{\alpha\beta}\omega^{\alpha\beta},\\ n^{-1}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha\mathcal{D}_\beta n, \quad n^{-2}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha n\,\mathcal{D}_\beta n.\end{gathered} \tag{1.18}$$

  In d=4,

$$\begin{aligned}&\frac{1}{4}\,{\epsilon^{\alpha\beta\lambda}}_\mu\,{\epsilon^{\gamma\theta\sigma}}_\nu C_{\alpha\beta\gamma\theta}\,u_\lambda u_\sigma, \qquad \frac{1}{2}\epsilon_{\alpha\beta\lambda(\mu}{C^{\alpha\beta}}_{\nu)\sigma}u^\lambda u^\sigma, \quad \mathcal{D}_{(\mu}l_{\nu)},\\ &n^{-1}\Pi_{\mu\nu}^{\alpha\beta}l_\alpha\mathcal{D}_\beta n, \quad n^{-1}{\epsilon^{\alpha\beta\lambda}}_{(\mu}\sigma_{\nu)\lambda}u_\alpha\mathcal{D}_\beta n.\end{aligned} \tag{1.19}$$

where we have introduced the projection tensor $\Pi^{\alpha\beta}_{\mu\nu}$ which projects out the transverse traceless symmetric part of second rank tensors

$$\Pi^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}\left[P^{\alpha}_{\mu}P^{\beta}_{\nu} + P^{\alpha}_{\nu}P^{\beta}_{\mu} - \frac{2}{d-1}P^{\alpha\beta}P_{\mu\nu}\right]$$

These invariants can now be used to write down the most general $T_{\mu\nu}$ and J^{μ}_{i} consistent with Weyl-covariance. The energy-momentum tensor and the charged currents of the fluid are usually divided into a zero-derivative part and a part involving at least one derivative

$$\begin{aligned} T^{\mu\nu}_{conf} =&\ p\,(g^{\mu\nu} + d\ u^{\mu}u^{\nu}) + \pi^{\mu\nu} \\ J^{\mu}_{i} =&\ \rho_i u^{\mu} + \nu^{\mu}_{i} \end{aligned} \tag{1.20}$$

where we can take the visco-elastic stress $\pi^{\mu\nu}$ to be transverse ($u_{\mu}\pi^{\mu\nu} = 0$) and traceless ($\pi^{\mu}{}_{\mu} = 0$) and the diffusion current ν^{μ}_{i} to be transverse ($u_{\lambda}\nu^{\lambda}_{i} = 0$). Hence, $\pi^{\mu\nu}$ and ν^{μ}_{i} are linear combination of transverse traceless Weyl-covariant tensors and transverse Weyl-covariant vectors of appropriate weight.

1.5 Derivative expansion in gauge-gravity duality

In order to obtain the explicit forms of $T_{\mu\nu}$ and J^{μ}_{i} , we have to 'solve' for the transport phenomenon in the strongly coupled field theory. As advertised before, this can be achieved by using gauge-gravity duality to reformulate this into a tractable problem. In this subsection, we will very quickly review the basic ideas behind this reformulation.

We begin with the interesting idea that large-N interacting field theories can often be rewritten in terms of a 'Master field' which is not really a single field in the same spacetime as the field theory but actually a collection of single-trace degrees of freedom that dominate the dynamics in the large-N limit. Often it is convenient to think of these degrees of freedom as living in an abstract space, their dynamics being governed in the Large-N limit by classical equations in that space. It is in this sense that large-N limit can be thought of as a classical limit. In practice however , with most field theories, it is almost impossible to guess the correct way to reformulate the large-N degrees of freedom and check whether this idea holds true for the model under consideration. This is one of the main reasons why large-N QCD remains an unsolved problem.

In a now celebrated work[4, 5], Juan Maldacena argued that for many quantum field theories, string theory gives a way out by providing us with a duality called gauge-gravity duality using which the large-N degrees of freedom and their dynamics are easily identified and tackled. In particular, he conjectured that the the $3+1$ dimensional $SU(N)$ gauge theory with fourfold supersymmetry is dual to a particular kind of a $9+1$ dimensional string theory called the Type IIB string theory quantised with an $\mathrm{AdS}_5\times\ \mathrm{S}^5$ boundary condition.

In this dual description, taking the large-N limit is same as taking the classical limit of the string theory and the relevant degrees of freedom and their dynamics in this limit are completely captured by the classical IIB string theory. Further taking the 't Hooft coupling large is equivalent in the dual description to making the typical mass of a massive string excitation very large so that only the massless degrees of freedom of the IIB string survive.

The massless degrees of freedom of IIB string theory contains a dynamical metric interacting with the other massless matter fields. Hence, we get the remarkable statement that under the gauge-gravity duality, large-N dynamics of the strongly coupled field theory gets mapped to a classical gravitational theory with an appropriate matter fields.

The fact that we get a metric in the gravity description indicates that the total energy-momentum survives as a good degree of freedom in the large-N dynamics of the strongly coupled field theory. In a similar way, the survival of the charge as a degree of freedom in the gauge theory side is encoded by the presence of a massless vector field on the gravity side. In particular, according to the gauge-gravity duality, the dynamics of these degrees of freedom are well-captured by the Einstein and Maxwell equations with the appropriate source terms. In a similar vein, one can now read off the other relevant degrees of freedom at strong coupling and their interactions directly from the low-energy lagrangian of the type IIB theory. Thus, Einstein equations in particular and gravitational dynamics in general emerge out of the large-N limit of an ordinary gauge theory and they serve as an effective way to encode the dynamics of the 'master field'.

Conversely, the classical gravitational dynamics in any gravity theory in $d+1$ dimensions with matter when studied with AdS_{d+1} boundary conditions mimics the dynamics of the 'master field' for a large-N conformal field theory in d dimensions. Given this fact, we will in the rest of the synopsis (except for the last few subsections) work in a general gravity theory in $d+1$ dimensions with the understanding that putting $d = 3+1 = 4$ will give us the answers for the $3+1$ dimensional $SU(N)$ gauge theory with the fourfold supersymmetry.

It is a fact in classical gravity that any two derivative gravity theory having AdS as a solution and having no particles with spin> 2 has a consistent truncation to a sector with just gravity and a negative cosmological constant. Such a sector has an action

$$S = \frac{1}{16\pi G_{AdS}} \int d^{d+1}x \sqrt{g_{d+1}} \left[R + d(d-1)\right]$$

This implies that at least within the class of strongly coupled field theories whose large-N description takes the form of a gravitational theory, it is quite generic to have a sector where we can focus on the dynamics of energy and momentum alone without worrying about exciting other modes. This statement justifies our assumption at the beginning of the synopsis where we chose to focus on a sector with only conserved charges excited.

There are no results of similar generality once we try to truncate to a sector keeping only the metric and a massless vector field in the bulk. Generically, there exists other excitations which do not decouple, but in special cases, we can achieve a consistent truncation. To be concrete we will look at the IIB gravity compactified on an S^5. In particular, there exists a consistent truncation consisting of just the metric and a $U(1)$ gauge field - this happens in the sector which is equally charged under all the three cartans of $SO(6)$. We can write down a truncated action of the form

$$\begin{aligned} S &= \frac{1}{16\pi G_{\rm AdS}} \int \left[\sqrt{-g_5}(R+12) - \frac{1}{2}\mathbf{F}\wedge *_5\mathbf{F} + \frac{2\kappa}{3}\mathbf{A}\wedge\mathbf{F}\wedge\mathbf{F} \right] \\ &= \frac{1}{16\pi G_{\rm AdS}} \int \sqrt{-g_5} \left[R + 12 - \frac{1}{4}F_{AB}F^{AB} + \frac{\kappa}{6}\epsilon^{PQRST}A_P F_{QR} F_{ST} \right] \end{aligned} \tag{1.21}$$

with $\kappa = (2\sqrt{3})^{-1}$ for the IIB gravity. The $SO(6)$ massless vector fields in the gravity theory encode the $SU(4) \cong SO(6)$ charge current degrees of freedom and working through the duality in detail, one concludes that there exists a sector in the hydrodynamics of the gauge theory where we can decouple all excitations except the energy-momentum and a $U(1)$ charge associated with the subgroup

$$U = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} \in SU(4)$$

that we had introduced before. As we will see by the end of this synopsis, this particular sector exhibits some very interesting transport phenomena.

We will now go on to describe how we can use the duality to study the non-equilibrium dynamics of the field theory under consideration. We begin with the equilibrium configurations in the field theory at a given temperature and chemical potentials which are dual to equilibrium black-brane configurations in gravity at the same temperature and chemical potentials. These black-brane configurations are gravitational configurations with a planar horizon and by various semiclassical arguments due to Bekenstein and Hawking, they are known to exhibit thermodynamic properties like entropy and temperature.

For example, when all the chemical potentials are turned off, the black-brane solution at rest takes the form

$$ds^2 = 2dvdr - r^2[1-(br)^{-d}]dv^2 + r^2 dx_{d-1}^2 \tag{1.22}$$

in the ingoing Eddington-Finkelstein co-ordinates. No matter fields other than the metric are excited in these solutions. In general, a uniformly moving black-brane solution with a constant velocity u^μ is given by

$$ds^2 = -2u_\mu dx^\mu dr + r^2\eta_{\mu\nu}dx^\mu dx^\nu + r^2(br)^{-d}u_\mu u_\nu dx^\mu dx^\nu \tag{1.23}$$

The thermodynamic properties of this solution are

$$\begin{gathered} \varepsilon = \frac{(d-1)}{16\pi G_{AdS} b^d}\ , \quad p = \frac{1}{16\pi G_{AdS} b^d}\ , \quad s = \frac{1}{4 G_{AdS} b^d}\ , \quad T = \frac{d}{4\pi b}\ , \\ n_i = 0\ , \qquad \mu_i = 0\ . \end{gathered} \tag{1.24}$$

Hence, this solution that we have written down describes thermal state in the corresponding conformal field theory. In the same vein, one can construct thermal states with chemical potentials turned on by looking for charged black-brane solutions solving Einstein-Maxwell equations with the appropriate sources.

Having understood the equilibrium configurations, we can then set up a systematic expansion where we allow the the thermodynamic quantities to slowly vary and find the corresponding time-dependent gravitational configurations. We will just review the setting up of this expansion in the zero chemical potential case - this method has been generalised to the finite chemical potential case, which while technically intricate is nevertheless straightforward. The logic behind - and the method of implementation of - this perturbative procedure have been described in detail in [6]. Further it was described in [6], how this perturbative procedure establishes a map between solutions of fluid dynamics and regular long wavelength solutions of Einstein gravity with a negative cosmological constant.

We start with the ansatz

$$g_{MN} = g^{(0)}_{MN} + \epsilon g^{(1)}_{MN} + \epsilon^2 g^{(2)}_{MN} + \ldots$$

Here $g^{(0)}_{MN}$ is given by (1.23), ϵ is the small parameter of the derivative expansion, and $g^{(k)}_{MN}$ are the corrections to the bulk metric that we will determine with the aid of the bulk Einstein equation. In implementing our perturbative procedure we adopt a choice of gauge. As in all the metrics described above, we use the coordinates r, x^μ for our bulk spaces. We use x^μ as coordinates that parameterise the boundary and r is a radial coordinate. In order to give precise meaning to our coordinates we need to adopt a choice of gauge-we choose the gauge $g_{rr} = 0$ together with $g_{r\mu} = -u_\mu$.

The Bulk Einstein equations decompose into 'constraints' on the boundary hydrodynamic data and 'dynamical equations' for the bulk metric along the tubes which are solved order by order in the derivative expansion. The dynamical equations determine the corrections that should be added to our initial metric to make it a solution of the Einstein equations. At each order, we get inhomogeneous linear equations -but, with the same homogeneous parts. These inhomogeneous linear equations obtained from Einstein equations can be solved order by order by imposing regularity at the zeroth order future horizon and appropriate asymptotic fall off at the boundary.These boundary conditions - together with a clear definition of velocity, which fixes the ambiguity of adding zero modes - give a unique solution for the metric, as a function of the original boundary velocity and temperature profile inputted into the metric $g^{(0)}_{MN}$ - order by order in the boundary derivative expansion.

Now, we turn to the 'constraints'. The 'constraints' on the boundary data can be shown to be equivalent to the requirement of the conservation of the boundary stress tensor. Recall that we have already used the dynamical Einstein equations to determine the full bulk metric - and hence the boundary stress tensor - as a function of the input velocity and temperature fields. It follows that the constraint Einstein equations reduce simply to the equations of fluid dynamics, i.e. the requirement of a conserved stress tensor which, in turn, is a given function of temperature and velocity fields. In the next few subsections, we will detail the results obtained by performing such a boundary derivative expansion.

1.6 Hydrodynamics from gravity in arbitrary dimensions

Before presenting the metric that is dual to a given hydrodynamic state, we will pause to ask how the Weyl covariance of the hydrodynamic state gets encoded in the metric[12]. Let us begin by performing the boundary Weyl-transformation in the boosted black-brane metric (1.23)

$$u_\mu = e^{\phi}\tilde{u}_\mu, \quad \eta_{\mu\nu} = e^{2\phi}\tilde{g}_{\mu\nu} \quad \text{and} \quad b = e^{\phi}\tilde{b}$$

where the transformation of b can be fixed by its relation to the thermodynamic quantities. Further, since for a constant velocity u^μ, the Weyl connection $\mathcal{A}_\mu = 0$ we also have $0 = \tilde{\mathcal{A}}_\mu + \partial_\mu\phi$

Under such a transformation, the metric becomes

$$ds^2 = -2\tilde{u}_\mu dx^\mu(d\tilde{r} + \tilde{r}\tilde{\mathcal{A}}_\nu dx^\nu) + \tilde{r}^2\tilde{g}_{\mu\nu}dx^\mu dx^\nu + \tilde{r}^2(\tilde{b}\tilde{r})^{-d}\tilde{u}_\mu\tilde{u}_\nu dx^\mu dx^\nu \tag{1.25}$$

where we have defined $\tilde{r}$ via $r = e^{-\phi}\tilde{r}$. We note that written in this form the black-brane metric is form-invariant under the boundary Weyl-transformations ! This is a particular example of the general principle that the boundary Weyl-transformations get realised in the gravity theory as a specific set of diffeomorphisms that redefine the radial slicing. The fact that different slicings of the same gravity solution may in fact be interpreted as states of the same theory in distinct though Weyl equivalent background metrics, reflects the Weyl invariance of the dual field theory.

To make this more precise, any metric that obeys that gauge choice $g_r^{(d+1)}r = 0$ and $g_r^{(d+1)}\mu = -u\mu$ can be put in the form

$$ds^2 = -2u_\mu(x)dx^\mu(dr + \mathcal{V}_\nu(r,x)dx^\nu) + \mathfrak{G}_{\mu\nu}(r,x)dx^\mu dx^\nu \tag{1.26}$$

where $\mathfrak{G}_{\mu\nu}$ is transverse, i.e., $u^\mu\mathfrak{G}_{\mu\nu} = 0$. [13]

Consider now a bulk-diffeomorphism of the form $r = e^{-\phi}\tilde{r}$ along with a scaling in the temperature of the form $b = e^{\phi}\tilde{b}$ where we assume that $\phi = \phi(x)$ is a function only of the boundary co-ordinates. The metric components transform as

$$\mathcal{V}_\mu = e^{-\phi}\left[\tilde{\mathcal{V}}_\mu + \tilde{r}\ \partial_\mu\phi\right], \qquad u_\mu = e^{\phi}\tilde{u}_\mu \quad \text{and} \quad \mathfrak{G}_{\mu\nu} = \tilde{\mathfrak{G}}_{\mu\nu} \tag{1.28}$$

Recall however that, within our procedure, the quantities $\mathfrak{G}_{\mu\nu}$ and $\mathcal{V}_\mu$ are each functions of u^μ and b. Now u^μ and b each pick up a factor of e^ϕ under the same diffeomorphism. We

[12] In this subsection, we summarise the work that was first presented in the paper[7]

[13] All the Greek indices are raised and lowered using the boundary metric $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = \lim_{r\to\infty} r^{-2}\left[\mathfrak{G}_{\mu\nu} - u_{(\mu}\mathcal{V}_{\nu)}\right] \tag{1.27}$$

and u_μ is the unit time-like velocity field in the boundary, i.e., $g^{\mu\nu}u_\mu u_\nu = -1$.

conclude that consistency demands that $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ are functions of b and u^μ that transform like a connection and remain invariant under boundary Weyl transformation respectively. It follows immediately that, for instance $\mathfrak{G}_{\mu\nu}$ is a linear sum of the Weyl invariant forms listed before, with coefficients that are arbitrary functions of br. Similarly, $\mathcal{V}_\mu - r\mathcal{A}_\mu$ is a linear sum of Weyl-covariant vectors (both transverse and non-transverse) with weight unity. Symmetry requirements do not constrain the form of these coefficients, which have to be determined via direct calculation.

Using a Weyl-covariant form of the procedure outlined in [6], we find that the final metric for the zero chemical potential case in arbitrary dimensions can be written in a Weyl-covariant form (with only terms involving no more than two derivatives shown)

$$\begin{aligned}
ds^2 = & -2u_\mu dx^\mu \left(dr + r\ A_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right] dx^\mu dx^\nu \\
& + \frac{1}{(br)^d}(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})u_\mu u_\nu dx^\mu dx^\nu + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right] dx^\mu dx^\nu \\
& - 2(br)^2\left[K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + K_2(br)\frac{u_\mu u_\nu}{(br)^d}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{2(d-1)} - \frac{L(br)}{(br)^d}u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\alpha\sigma^\alpha{}_\lambda\right] dx^\mu dx^\nu \\
& - 2(br)^2 H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] dx^\mu dx^\nu \\
& + 2(br)^2 H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] dx^\mu dx^\nu + \ldots
\end{aligned} \tag{1.29}$$

The various functions appearing in the metric are defined by the integrals

$$\begin{aligned}
F(br) &\equiv \int_{br}^\infty \frac{y^{d-1}-1}{y(y^d-1)}dy \\
H_1(br) &\equiv \int_{br}^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy \\
H_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi y^{d-3}dy\left[1 + (d-1)yF(y) + 2y^2F'(y)\right] \\
&= \frac{1}{2}F(br)^2 - \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi \frac{y^{d-2}-1}{y(y^d-1)}dy \\
K_1(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\int_\xi^\infty dy\ y^2F'(y)^2 \\
K_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\Big[1 - \xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1} \\
&\qquad + \Big(2(d-1)\xi^d - (d-2)\Big)\int_\xi^\infty dy\ y^2F'(y)^2\Big] \\
L(br) &\equiv \int_{br}^\infty \xi^{d-1}d\xi\int_\xi^\infty dy\ \frac{y-1}{y^3(y^d-1)}
\end{aligned}$$

We have checked using Mathematica upto $d = 10$ that the above metric solves Einstein equations with negative cosmological constant provided the velocity u^μ satisfies the hydrody-

Value of τ_ω/b for various dimensions		
d	Value of $\tau_\omega/b = \int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy$	τ_ω/b (Numerical)
3	$\frac{1}{2}\left(\text{Log }3 - \frac{\pi}{3\sqrt{3}}\right)$	0.247006...
4	$\frac{1}{2}\text{Log }2$	0.346574...
5	$\frac{1}{4}\left(\text{Log }5 + \frac{2\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}} - \frac{2}{\sqrt{5}}\text{ArcCoth }\sqrt{5}\right)$	0.396834...
6	$\frac{1}{4}\left(\text{Log }3 + \frac{\pi}{3\sqrt{3}}\right)$	0.425803...

namic equations. The above formula hence gives a map from the solutions of hydrodynamic equations to the solutions of Einstein equations thus connecting two widely different fields of classical physics !

The dual stress tensor corresponding to this metric is given by

$$\begin{aligned} T^{\text{conf}}_{\mu\nu} &= p\left(g_{\mu\nu} + du_\mu u_\nu\right) - 2\eta\sigma_{\mu\nu} \\ &- 2\eta\tau_\omega\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] \\ &+ 2\eta b\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] \end{aligned} \tag{1.30}$$

with

$$p = \frac{1}{16\pi G_{\text{AdS}} b^d}\ , \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{\text{AdS}} b^{d-1}} \quad \text{and} \quad \tau_\omega = b\int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy \tag{1.31}$$

This result is a generalisation to the fluid dynamical stress tensor on an arbitrary curved manifold in general dimension d reported in [2, 6, 8–11] for special values of d and most recently by [12] for flat space in arbitrary dimensions. The values of τ_ω for some of the lower dimensions is shown[14] in the table 5.3.2. The way the various Weyl-covariant terms combine together in the energy-momentum tensor is intriguing - in fact, it can be shown[14] that there are universal relations between the second-order transport coefficients that come out of gravity.

These perturbative solutions of Einstein equations can be compared against a class of exact solutions of Einstein's equations. This class of solutions is the set of rotating black holes in the global AdS spaces. The dual boundary stress tensor to these solutions varies on the length scale unity (if we choose our boundary sphere to be of unit radius). On the other hand the temperature of these black holes may be taken to be arbitrarily large. It

[14] More generally, the integral appearing in the expression for τ_ω can be evaluated in terms of the derivative of the Gamma function or more directly in terms of 'the harmonic number function' with the fractional argument (as was noted in [13])

$$\tau_\omega = -\frac{b}{d}\left[\gamma_E + \frac{d}{dz}\text{Log }\Gamma(z)\right]_{z=2/d} = -\frac{b}{d}\text{Harmonic}[2/d-1]$$

For large d, τ_ω has an expansion of the form $\tau_\omega/b = 1/2 - \pi^2/(3d^2) + \ldots$.

follows that, in the large temperature limit, these black holes are dual to 'slowly varying' field theory configurations that should be well described by fluid dynamics. All of these remarks, together with nontrivial evidence for this expectation was described in [15]. In fact, we can go further[7] and complete the programme initiated in [15] for uncharged blackholes by demonstrating that the full bulk metric of these high temperature rotating black holes agrees in detail with the 2nd order bulk metric determined by our analysis earlier . This exercise was already carried out in [11] for the special case $d = 4$.

Consider the AdS-Kerr BHs in arbitrary dimensions - exact solution for the rotating blackholes in general AdS_{d+1} in different coordinates is derived in reference [16]. Following [16], we begin by defining two integers n and ϵ via $d = 2n + \epsilon$ with $\epsilon = d \bmod 2$. We can then parametrise the $d + 1$ dimensional AdS Kerr solution by a radial co-ordinate r, a time co-ordinate $\hat{t}$ along with $d - 1 = 2n + \epsilon - 1$ spheroidal co-ordinates on S^{d-1}. We will choose these spheroidal co-ordinates to be $n+\epsilon$ number of direction cosines $\hat{\mu}_i$ (obeying $\sum_{k=1}^{n+\epsilon} \hat{\mu}_k^2 = 1$) and $n + \epsilon$ azimuthal angles $\hat{\varphi}_i$ with $\hat{\varphi}_{n+1} = 0$ identically. The angular velocities along the different $\hat{\varphi}_i$s are denoted by a_i ($a_{n+1}$is taken to be zero identically).

In this 'altered' Boyer-Lindquist co-ordinates, AdS Kerr metric assumes the form (See equation (E.3) of the [16])

$$\begin{aligned} ds^2 = &-W(1+r^2)d\hat{t}^2 + \frac{\mathfrak{F}dr^2}{1-2M/V} + \frac{2M}{V\mathfrak{F}}\left(W d\hat{t} - \sum_{i=1}^{n} \frac{a_i \hat{\mu}_i^2 d\hat{\varphi}_i}{1-a_i^2}\right)^2 \\ &+ \sum_{i=1}^{n+\epsilon} \frac{r^2+a_i^2}{1-a_i^2}\left[d\hat{\mu}_i^2 + \hat{\mu}_i^2 d\hat{\varphi}_i^2\right] - \frac{1}{W(1+r^2)}\left(\sum_{i=1}^{n+\epsilon} \frac{r^2+a_i^2}{1-a_i^2}\hat{\mu}_i d\hat{\mu}_i\right)^2 \end{aligned} \tag{1.32}$$

where

$$W \equiv \sum_{i=1}^{n+\epsilon} \frac{\hat{\mu}_i^2}{1-a_i^2} \quad ; \quad V \equiv r^d \, (1+\frac{1}{r^2}) \prod_{i=1}^{n} (1+\frac{a_i^2}{r^2}) \quad \text{and} \quad \mathfrak{F} \equiv \frac{1}{1+r^2} \sum_{i=1}^{n+\epsilon} \frac{r^2 \hat{\mu}_i^2}{r^2+a_i^2} \tag{1.33}$$

This expression can be further simplified[7] and the AdS Kerr metric in arbitrary dimensions can be rewritten in the form

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu \left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda \omega_{\lambda\nu}\right] dx^\mu dx^\nu \\ &+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu \end{aligned} \tag{1.34}$$

where the determinant of a tensor M^λ_σ is defined by

$$\epsilon_{\mu\nu\ldots} M^\mu_\alpha M^\nu_\beta \ldots = \det\left[M^\lambda_\sigma\right] \epsilon_{\alpha\beta\ldots}$$

We have checked this form explicitly using Mathematica upto $d = 8$. It is easily checked that this metric agrees (upto second order in boundary derivative expansion) with the metric presented in (5.23) upon inserting the velocity and temperature fields as

$$
\begin{aligned}
u^\mu \partial_\mu \equiv \partial_t + a_i \partial_{\varphi_i} \quad , \quad \mathcal{A}_\mu = 0 \quad , \quad b \equiv (2M)^{-1/d} \\
g_{\mu\nu} \equiv W \left[-dt^2 + \sum_i \left(d\mu_i^2 + \mu_i^2 d\varphi_i^2 \right) \right]
\end{aligned}
\tag{1.35}
$$

The exact energy momentum tensor for the AdS Kerr Black Hole described can be computed

$$
T^{\mu\nu}_{conf} = p(g^{\mu\nu} + du^\mu u^\nu) \quad \text{with} \quad p = \frac{1}{16\pi G_{AdS} b^d} \tag{1.36}
$$

which is consistent with (5.26) if we take into account the fact that $\sigma_{\mu\nu} = 0$ in these configurations.

1.7 Global Charge transport in $\mathcal{N} = 4$ SYM

In the previous subsection, we presented our calculation of various transport coefficients in the simplest sector of hydrodynamics where the entire dynamics is that of the energy-momentum transport. In this subsection, we will turn to a slightly more complicated sector where there is an interplay between charge transport and energy transport. This in the gravity language translates to the study of Einstein-Maxwell system with appropriate sources.

As we had emphasised before, given a generic gravitational theory describing the large N behaviour of a strongly coupled CFT, we are not guaranteed to find a truncation to just the Einstein-Maxwell system. In the field theory language, it implies that at least in these class of systems it is very difficult to disentangle the charge dynamics from the dynamics of the other long wavelength modes. We had already mentioned that there do exist theories where such a truncation is achievable and we gave an example of a sector in the supersymmetric gauge theory where this happens- this is IIB gravity compactified on an S^5 with a consistent truncation consisting of just the metric and a $U(1)$ gauge field in the sector which is equally charged under all the three cartans of $SO(6)$.

We can repeat the derivative expansion to solve the Einstein-Maxwell system with a Chern-Simons term order by order for the metric and the gauge field. This was done in [17] and the expressions that we finally get for the metric and the gauge field are complicated and we will refer the reader to [17] for the full expressions. In turn, the energy momentum and charge transport upto second order in the boundary derivative expansion could be computed from that metric and gauge field and this gives us many new transport coefficients in the supersymmetric gauge theory. We again refer the reader to [17] (and [18] which appeared independently with similar results) for the detailed expressions . But in this synopsis, we will try to focus on an interesting result that comes out of these expressions.

To see this very clearly, it is useful to focus our attention on a known exact solution of the Einstein-Maxwell-Chern-Simons system which describes a charged rotating blackhole.

We begin with a truncated action of the form

$$\begin{aligned} S &= \frac{1}{16\pi G_{\rm AdS}} \int \left[\sqrt{-g_5}(R+12) - \frac{1}{2}\mathbf{F}\wedge *_5\mathbf{F} + \frac{2\kappa}{3}\mathbf{A}\wedge\mathbf{F}\wedge\mathbf{F}\right] \\ &= \frac{1}{16\pi G_{\rm AdS}} \int \sqrt{-g_5}\left[R+12 - \frac{1}{4}F_{AB}F^{AB} + \frac{\kappa}{6}\epsilon^{PQRST}A_P F_{QR}F_{ST}\right] \end{aligned} \tag{1.37}$$

with $\kappa = (2\sqrt{3})^{-1}$ for the IIB gravity.

This action has many known blackhole solutions. The general blackhole solutions which solves the equations coming out of this action was found in [19]. Their solution is given by[15]

$$\begin{aligned} ds^2 = &-\frac{(r^2+1)\,\Delta_\Theta dt_1{}^2}{(1-\omega_1{}^2)(1-\omega_2{}^2)} + \frac{2(m-q\omega_1\omega_2)}{\rho^2} - \frac{q^2}{\rho^4} \\ &+ \frac{(\mathrm{d}\psi_1+dt_1\omega_2)^2\,(r^2+\omega_2{}^2)\cos^2\Theta}{1-\omega_2{}^2} + \frac{(\mathrm{d}\phi_1+dt_1\omega_1)^2\,(r^2+\omega_1{}^2)\sin^2\Theta}{1-\omega_1{}^2} \\ &+ \frac{\rho^2\mathrm{d}r^2 r^2}{q^2-2\omega_1\omega_2 q-2mr^2+(r^2+1)(r^2+\omega_1{}^2)(r^2+\omega_2{}^2)} \\ &+ \frac{\rho^2\mathrm{d}\Theta^2}{\Delta_\Theta} + \frac{2q}{\rho^2}\left(\omega_1(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta + (\mathrm{d}\phi_1+dt_1\omega_1)\omega_2\sin^2\Theta\right) \\ &\times\left[\frac{\Delta_\Theta dt_1}{(1-\omega_1{}^2)(1-\omega_2{}^2)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-\omega_2{}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-\omega_1{}^2}\right] \\ \mathbf{A} = &-\frac{\sqrt{3}q}{\rho^2}\left[\frac{\Delta_\Theta dt_1}{(1-\omega_1{}^2)(1-\omega_2{}^2)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-\omega_2{}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-\omega_1{}^2}\right] \end{aligned} \tag{1.38}$$

where we use the definitions

$$\begin{aligned} \rho^2 &\equiv r^2 + \omega_1{}^2\cos^2\Theta + \omega_2{}^2\sin^2\Theta \\ \Delta_\Theta &\equiv 1 - \omega_1{}^2\cos^2\Theta - \omega_2{}^2\sin^2\Theta \end{aligned} \tag{1.39}$$

After some manipulations (which closely follow the methods outlined in [7, 11]) , we find that the final metric and the gauge field can be written in a manifestly Weyl-covariant form

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu\,(dr + r\,\mathcal{A}_\nu dx^\nu) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right]dx^\mu dx^\nu \\ &\quad + \left[\left(\frac{2m}{\rho^2} - \frac{q^2}{\rho^4}\right)u_\mu u_\nu + \frac{q}{2\rho^2}u_{(\mu}l_{\nu)}\right]dx^\mu dx^\nu \\ \mathbf{A} &= \frac{\sqrt{3}q}{\rho^2}u_\mu dx^\mu \quad ; \quad \rho^2 \equiv r^2 + \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \quad ; \quad l_\mu \equiv \epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma} \end{aligned} \tag{1.40}$$

with

$$\begin{aligned} T_{\mu\nu} &= p(g_{\mu\nu} + 4u_\mu u_\nu) + 2\kappa l_{(\mu}J_{\nu)} + \frac{1}{64\pi G_{\rm AdS}}\left(R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{R^2}{12}g_{\mu\nu}\right) \\ J_\mu &= nu_\mu \qquad \text{where} \quad l_\mu \equiv \epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma} \quad ; \quad p \equiv \frac{m}{8\pi G_{\rm AdS}} \quad \text{and} \qquad n \equiv \frac{\sqrt{3}q}{8\pi G_{\rm AdS}} \end{aligned} \tag{1.41}$$

[15]Note that the parameter q here is the negative of the one used in [19].

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the energy current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid made of $\mathcal{N} = 4$ SYM matter. This new non-dissipative transport can be traced back to the Chern-Simons term in the gravity theory which according to the gauge-gravity duality, encodes the information about the global anomalies in the field theory. This suggests that this transport is closely related to the $U(1)^3$ global anomaly in the field theory.

It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation. However, an indirect explanation was provided by the authors of [20], where they give a clever entropic argument which relates this coefficient to the anomaly. This suggests the possibility that such a transport is universal,i.e., it is present in any field theory which has global anomalies and it would be useful to explicitly check whether this is the case by calculating this transport coefficient in a calculable model - say a spin model.

The presence of such an effect was indirectly observed by the authors of [15] where they noted a discrepancy between the thermodynamics of charged rotating AdS blackholes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N} = 4$ SYM fluid. In fact, one could go further and compare the first order metric obtained in [17, 18] with the rotating blackhole metric written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.

1.8 Conclusion

In this synopsis, we have tried to summarise some of our work on the hydrodynamics that emerges out of an underlying strongly coupled conformal field theory at a finite temperature and chemical potentials. The hydrodynamic transport in such theories can be studied using the methods of gauge-gravity duality which reformulate the questions about transport into questions in a classical gravity theory interacting with matter. Further, we have seen that the underlying Weyl covariance of the conformal field theory finds a natural expression both in hydrodynamics and in gravity.

Our study has thrown up very interesting questions about the hydrodynamic transport - we have already remarked on the apparent universality of second order transport coefficients from gravity which is puzzling. We can also enquire as to whether there is a clever way by which such universal relations can be derived at all orders in derivative expansion.

A more intriguing phenomenon is the novel kind of non-dissipative transport in the hydrodynamics which seems to be driven by an underlying anomaly. As we had already remarked‘, it will be worthwhile look for a solvable model where this kind of transport is present. Given the possibility that it is a universal effect, there is a slim hope that there might be experimental systems where such a transport can be measured.

We have in this synopsis derived an expression which provides a metric solution to Einstein equations for a give hydrodynamic state. This immediately raises the possibility that a transition to turbulence in the hydrodynamics is accompanied by a turbulence like phenomenon in gravity. It would be interesting to see whether we can develop a detailed phenomenology of such a transition in gravity.

2 List of Publications

2.1 Publications by the author summarised in this synopsis/thesis

[P1] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812**, 116 (2008), [arXiv:0809.4272 [hep-th]].

[P2] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," JHEP **1101**, 094 (2011), [arXiv:0809.2596 [hep-th]].

[P3] R. Loganayagam,"Entropy Current in Conformal Hydrodynamics," JHEP **0805**, 087 (2008), [arXiv:0801.3701 [hep-th]].

[P4] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, "Large rotating AdS black holes from fluid mechanics," JHEP **0809**, 054 (2008), [arXiv:0708.1770 [hep-th]].

2.2 Publications by the author during the course of Graduate study at TIFR

[A1] P. Basu, J. Bhattacharya, S. Bhattacharyya, R. Loganayagam, S. Minwalla and V. Umesh "Small Hairy Black Holes in Global AdS Spacetime" , JHEP **1010**, 045 (2010) [arXiv:1003.3232 [hep-th]].

[A2] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812**, 116 (2008), [arXiv:0809.4272 [hep-th]].

[A3] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," JHEP **1101**, 094 (2011), [arXiv:0809.2596 [hep-th]].

[A4] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi and S. R. Wadia, "Forced Fluid Dynamics from Gravity," JHEP **0902**, 018 (2009), arXiv:0806.0006 [hep-th].

[A5] M. P. Heller, P. Surowka, R. Loganayagam, M. Spalinski and S. E. Vazquez, "On a consistent AdS/CFT description of boost-invariant plasma," arXiv:0805.3774 [hep-th]. Published under the title "Consistent Holographic Description Of Boost-Invariant Plasma" Phys. Rev. Lett. **102**, 041601 (2009).

[A6] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani and H. S. Reall, "Local Fluid Dynamical Entropy from Gravity," JHEP **0806**, 055 (2008), [arXiv:0803.2526 [hep-th]].

[A7] R. Loganayagam,"Entropy Current in Conformal Hydrodynamics," JHEP **0805**, 087 (2008), [arXiv:0801.3701 [hep-th]].

[A8] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, "Large rotating AdS black holes from fluid mechanics," JHEP **0809**, 054 (2008), [arXiv:0708.1770 [hep-th]].

3 Scale Invariance in Hydrodynamics

We now turn to a more detailed description of various topics discussed above. We will begin by reviewing the Weyl-covariant formalism introduced by the author in [2] which would be then used throughout the rest of the thesis.

Before applying gauge-gravity duality to derive the hydrodynamics emerging out of an underlying conformal theory, it is worthwhile to understand the structure of such a hydrodynamic description[16]. The fact that the microscopic theory is devoid of scales imposes strict constraints on the kind of transport that can occur in the macroscopic description. We will summarise in this subsection, a formalism developed in [2] to naturally incorporate these constraints into the hydrodynamic description.

The plan of this chapter is as follows - In §3.1, we introduce a manifestly Weyl-covariant derivative especially suited to the study of conformal fluids and list the various conformal observables that occur in fluid mechanics. Since, we are interested in conformal fluids on arbitrary spacetimes, we describe in some detail the various curvature related observables that occur in conformal hydrodynamics. This is followed by the section §3.2, where the equations of fluid mechanics are formulated in a conformally covariant way. We end §3.2 by writing down the derivative expansion for a conformal fluid exact up to second derivative terms.

Next, we proceed in section §3.3 to find a derivative expansion of the local entropy current for a conformal fluid which obeys the second law of thermodynamics. We make a proposal for the entropy current of a conformal fluid living in arbitrary spacetimes (with $d > 3$). Next,in section §3.4, we turn to the specific case of $\mathcal{N} = 4$ SYM and find the corresponding expression for the entropy flux.

This is followed by the section §3.5 where we compare the method adopted in this chapter with the existing theories of relativistic hydrodynamics. In the final section, we discuss future directions and conclude. In appendix (3.6.1) , we prove some very useful identities that were used in the body of the paper. This is followed by appendix (3.6.2) where we discuss the various terms that can in principle occur in the energy-momentum tensor of a conformal fluid.

3.1 Conformal Tensors In Hydrodynamics

A conformal field theory living on a d dimensional spacetime $\mathcal{M}$ with a metric $g_{\mu\nu}$ is by definition a theory which is covariant (upto quantum Weyl anomalies) under the Weyl transformation which replaces the old metric $g_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$ given by

$$g_{\mu\nu} = e^{2\phi(x)}\tilde{g}_{\mu\nu}; \qquad g^{\mu\nu} = e^{-2\phi(x)}\tilde{g}^{\mu\nu} \tag{3.1}$$

In some sense, these theories are naturally thought of (again upto Weyl anomalies) as living on a spacetime $\mathcal{M}$ which has a class of metrics $\mathcal{C}_g$ which are related to our original metric

[16] This subsection is based on the work that was done by the author and presented in the form of the paper [2].

by a Weyl transformation. We will henceforth refer to the fluid which emerges out of such a conformal field theory at a finite temperature as a conformal fluid.

The conformal fluid which emerges out of a conformal field theory inherits Weyl-covariance from its microscopic theory. The question that we wish to answer is how this Weyl-covariance manifests itself at the level of hydrodynamics and how does one go about constructing Weyl-covariant quantities in hydrodynamics. Let u^μ be the unit time-like vector describing the fluid motion. Using $g_{\mu\nu}u^\mu u^\nu = \tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$, we get the Weyl transformation of the velocity $u^\mu = e^{-\phi}\tilde{u}^\mu$. Now, we can go ahead and construct various derivatives of this velocity field, provided we have a way of differentiating fields in a Weyl-covariant way.

To see how this might be possible, we begin with the Weyl-transformation of the Christoffel symbols

$$\Gamma_{\lambda\mu}{}^{\nu} = \widetilde{\Gamma}_{\lambda\mu}{}^{\nu} + \delta^\nu_\lambda\partial_\mu\phi + \delta^\nu_\mu\partial_\lambda\phi - \tilde{g}_{\lambda\mu}\tilde{g}^{\nu\sigma}\partial_\sigma\phi \tag{3.2}$$

from which it follows that

$$\mathcal{A}_\nu \equiv u^\mu\nabla_\mu u^\nu - \frac{\nabla_\mu u^\mu}{d-1}u_\nu = \tilde{\mathcal{A}}_\nu + \partial_\nu\phi. \tag{3.3}$$

This transformation means that hydrodynamics provides a natural 'gauge field' for Weyl transformations which can be used to construct a Weyl-covariant derivative. We define a Weyl covariant derivative $\mathcal{D}$ such that, if a tensorial quantity $Q^{\mu\ldots}_{\nu\ldots}$ obeys $Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$, then $\mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{\mathcal{D}}_\lambda\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$ where

$$\begin{aligned}\mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} \equiv\ & \nabla_\lambda\, Q^{\mu\ldots}_{\nu\ldots} + w\ \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots}\\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots\\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots\end{aligned} \tag{3.4}$$

Note that the above covariant derivative is metric compatible ($\mathcal{D}_\lambda g_{\mu\nu} = 0$). In mathematical terms, what we have done is to use the additional mathematical structure provided by a fluid background (namely a unit time-like vector field with conformal weight $w = 1$) to define what is known as a *Weyl connection* over $(\mathcal{M}, \mathcal{C}_g)$ where $\mathcal{M}$ is the spacetime manifold with the conformal class of metrics $\mathcal{C}_g$.

A torsionless connection ∇^{weyl} is called a Weyl connection (see for example, [3] and references therein) if for every metric in the conformal class $\mathcal{C}_g$ there exists a one form $\mathcal{A}_\mu$ such that $\nabla^{weyl}_\mu g_{\nu\lambda} = 2\mathcal{A}_\mu g_{\nu\lambda}$. Having a fluid over the manifold provides us a natural one form $\mathcal{A}_\mu$ (see below), which can in turn be used to define a Weyl connection. The 'prolonged' covariant derivative $\mathcal{D}$ is related to this Weyl connection via the relation $\mathcal{D}_\mu = \nabla^{weyl}_\mu + w\mathcal{A}_\mu$. In terms of this covariant derivative, the condition for Weyl connection is just the statement of metric compatibility ($\mathcal{D}_\lambda g_{\mu\nu} = 0$) and the one-form $\mathcal{A}_\mu$ is uniquely determined by requiring that the covariant derivative of u^μ be transverse ($u^\lambda\mathcal{D}_\lambda u^\mu = 0$) and traceless ($\mathcal{D}_\lambda u^\lambda = 0$).

We can define a curvature associated with the Weyl-covariant derivative by the usual procedure of evaluating the commutator between two covariant derivatives. For a covariant vector field $V_\mu = e^{-w\phi}\widetilde{V}_\mu$, we get

$$
\begin{split}
[\mathcal{D}_\mu, \mathcal{D}_\nu]V_\lambda &= w\ \mathcal{F}_{\mu\nu}\ V_\lambda + \mathcal{R}_{\mu\nu\lambda}{}^{\alpha}\ V_\alpha \qquad \text{with}\\
\mathcal{F}_{\mu\nu} &= \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu\\
\mathcal{R}_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} - \delta^{\alpha}_{[\mu} g_{\nu][\lambda}\delta^{\beta}_{\sigma]}\left(\nabla_\alpha \mathcal{A}_\beta + \mathcal{A}_\alpha\mathcal{A}_\beta - \frac{\mathcal{A}^2}{2}g_{\alpha\beta}\right) + \mathcal{F}_{\mu\nu}g_{\lambda\sigma}
\end{split}
\tag{3.5}
$$

where we have introduced two new Weyl-invariant tensors $\mathcal{F}_{\mu\nu} = \widetilde{\mathcal{F}}_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} = \widetilde{\mathcal{R}}_{\mu\nu\lambda}{}^{\alpha}$ and $B_{[\mu\nu]} \equiv B_{\mu\nu} - B_{\nu\mu}$ indicates antisymmetrisation[17]. Since these Weyl-covariant counterparts of curvature tensors will play a useful role in the formulation of conformal hydrodynamics, we will briefly describe their properties[18].

We can write down similar expressions involving Ricci tensor, Ricci scalar and Einstein tensor.

$$
\begin{split}
\mathcal{R}_{\mu\nu} \equiv \mathcal{R}_{\mu\alpha\nu}{}^{\alpha} &= R_{\mu\nu} + (d-2)\left(\nabla_\mu\mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu - \mathcal{A}^2 g_{\mu\nu}\right) + g_{\mu\nu}\nabla_\lambda\mathcal{A}^\lambda + \mathcal{F}_{\mu\nu} = \widetilde{\mathcal{R}}_{\mu\nu}\\
\mathcal{R} \equiv \mathcal{R}_{\alpha}{}^{\alpha} &= R + 2(d-1)\nabla_\lambda\mathcal{A}^\lambda - (d-2)(d-1)\mathcal{A}^2 = e^{-2\phi}\widetilde{\mathcal{R}}\\
\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}g_{\mu\nu} &= G_{\mu\nu} + (d-2)\left[\nabla_\mu\mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu - \left(\nabla_\lambda\mathcal{A}^\lambda - \frac{d-3}{2}\mathcal{A}^2\right)g_{\mu\nu}\right] + \mathcal{F}_{\mu\nu}
\end{split}
\tag{3.7}
$$

These curvature tensors obey various Bianchi identities [19]

$$
\begin{split}
\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} + \mathcal{R}_{\lambda[\mu\nu]}{}^{\alpha} &= 0\\
\mathcal{D}_\lambda\mathcal{F}_{\mu\nu} + \mathcal{D}_{[\mu}\mathcal{F}_{\nu]\lambda} &= 0\\
\mathcal{D}_\lambda\mathcal{R}_{\mu\nu\alpha}{}^{\beta} + \mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda\alpha}{}^{\beta} &= 0
\end{split}
\tag{3.8}
$$

and various reduced Bianchi identities[20]

$$
\begin{split}
\mathcal{R}_{[\mu\nu]} = \mathcal{R}_{\mu\nu\alpha}{}^{\alpha} &= d\ \mathcal{F}_{\mu\nu}\\
\mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda} + \mathcal{D}_\sigma\mathcal{R}_{\mu\nu\lambda}{}^{\sigma} &= 0\\
\mathcal{D}_\lambda\left(\mathcal{G}^{\mu\lambda} - \mathcal{F}^{\mu\lambda}\right) &= 0
\end{split}
\tag{3.9}
$$

[17] As is evident from the notation above, we use calligraphic alphabets to denote the Weyl-covariant counterparts of the usual curvature tensors. Our notation for the usual Riemann tensor is defined by the relation

$$
[\nabla_\mu, \nabla_\nu]V_\lambda = R_{\mu\nu\lambda}{}^{\sigma}V_\sigma. \tag{3.6}
$$

Note that the curvature tensors used in this synopsis are negative of those employed in [2].

[18] Note that these curvature tensors are essential even if one is in a flat spacetime, since most of these Weyl-covariant curvatures do not vanish for a general velocity configuration in flat spacetime.

[19] These identities can be derived from the Jacobi identity for the covariant derivative - $[\mathcal{D}_{[\mu}, [\mathcal{D}_{\nu]}, \mathcal{D}_\lambda] + [\mathcal{D}_\lambda, [\mathcal{D}_\mu, \mathcal{D}_\nu]] = 0$

[20] These identities are obtained from the Bianchi identities by contractions.

The tensor $\mathcal{R}_{\mu\nu\lambda\sigma}$ does not have the same symmetry properties as that of the usual Riemann tensor. For example,

$$\begin{aligned}
\mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu\sigma\lambda} &= 2\ \mathcal{F}_{\mu\nu} g_{\lambda\sigma} \\
\mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{R}_{\lambda\sigma\mu\nu} &= -\delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \mathcal{F}_{\alpha\beta} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} - \mathcal{F}_{\lambda\sigma} g_{\mu\nu} \\
\mathcal{R}_{\mu\alpha\nu\beta} V^{\alpha} V^{\beta} - \mathcal{R}_{\nu\alpha\mu\beta} V^{\alpha} V^{\beta} &= \mathcal{F}_{\mu\nu}\ V^{\alpha} V_{\alpha}
\end{aligned} \tag{3.10}$$

The conformal tensors of the underlying spacetime manifold appear in the above formalism as a subset of conformal observeables in hydrodynamics. These conformal tensors are the Weyl-covariant tensors that are independent of the background fluid velocity. The Weyl curvature $C_{\mu\nu\lambda\sigma}$ is a well-known example of a conformal tensor. We have

$$\mathcal{C}_{\mu\nu\lambda\sigma} \equiv \mathcal{R}_{\mu\nu\lambda\sigma} + \delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \mathcal{S}_{\alpha\beta} = C_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} = e^{2\phi} \widetilde{\mathcal{C}}_{\mu\nu\lambda\sigma} \tag{3.11}$$

where the Schouten tensor $\mathcal{S}_{\mu\nu}$ is defined as

$$\mathcal{S}_{\mu\nu} \equiv \frac{1}{d-2}\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R} g_{\mu\nu}}{2(d-1)}\right) = S_{\mu\nu} + \left(\nabla_{\mu}\mathcal{A}_{\nu} + \mathcal{A}_{\mu}\mathcal{A}_{\nu} - \frac{\mathcal{A}^2}{2} g_{\mu\nu}\right) + \frac{\mathcal{F}_{\mu\nu}}{d-2} = \widetilde{\mathcal{S}}_{\mu\nu} \tag{3.12}$$

Often in the study of conformal tensors , it is useful to rewrite other curvature tensors in terms of the Schouten and the Weyl curvature tensors-

$$\begin{aligned}
\mathcal{R}_{\mu\nu\lambda\sigma} &= \mathcal{C}_{\mu\nu\lambda\sigma} - \delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \mathcal{S}_{\alpha\beta}, \qquad \mathcal{R} = 2(d-1)\mathcal{S}_{\lambda}{}^{\lambda} \\
\mathcal{R}_{\mu\nu} &= (d-2)\mathcal{S}_{\mu\nu} + \mathcal{S}_{\lambda}{}^{\lambda} g_{\mu\nu}, \qquad \mathcal{G}_{\mu\nu} = (d-2)(\mathcal{S}_{\mu\nu} - \mathcal{S}_{\lambda}{}^{\lambda} g_{\mu\nu})
\end{aligned} \tag{3.13}$$

From equation (3.11), it is clear that $C_{\mu\nu\lambda\sigma} = \mathcal{C}_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu} g_{\lambda\sigma}$ is clearly a conformal tensor. Such an analysis can in principle be repeated for the other known conformal tensors in arbitrary dimensions.

The Weyl Tensor $C_{\mu\nu\lambda\sigma}$ has the same symmetry properties as that of Riemann Tensor $R_{\mu\nu\lambda\sigma}$.

$$\begin{aligned}
C_{\mu\nu\lambda\sigma} &= -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu} \\
&\text{and} \quad C_{\mu\alpha\lambda}{}^{\alpha} = 0
\end{aligned} \tag{3.14}$$

From which it follows that $C_{\mu\alpha\nu\beta} u^{\alpha} u^{\beta}$ is a symmetric traceless and transverse tensor - a fact which will turn out to be important later in our discussion of conformal hydrodynamics.

Now, we turn to the study of how various quantities of relevance to hydrodynamics can be constructed in this formalism. The Weyl-covariant derivative of the velocity field naturally breaks up into a symmetric part and an antisymmetric part

$$\begin{aligned}
P_{\mu\nu} &\equiv g^{\mu\nu} + u^{\mu} u^{\nu} \\
\mathcal{D}_{\mu} u^{\nu} &= \nabla_{\mu} u^{\nu} + u_{\mu}(u.\nabla)u^{\nu} - \frac{\nabla.u}{d-1} P_{\mu}{}^{\nu} = \sigma_{\mu}{}^{\nu} + \omega_{\mu}{}^{\nu} = e^{-\phi}\widetilde{\mathcal{D}}_{\mu}\tilde{u}^{\nu}, \\
\sigma^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda}\nabla_{\lambda}u^{\nu} + P^{\nu\lambda}\nabla_{\lambda}u^{\mu}\right) - \frac{\nabla.u}{d-1} P^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}^{\mu}u^{\nu} + \mathcal{D}^{\nu}u^{\mu}\right) = e^{-3\phi}\tilde{\sigma}^{\mu\nu}, \\
\omega^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda}\nabla_{\lambda}u^{\nu} - P^{\nu\lambda}\nabla_{\lambda}u^{\mu}\right) = \frac{1}{2}\left(\mathcal{D}^{\mu}u^{\nu} - \mathcal{D}^{\nu}u^{\mu}\right) = e^{-3\phi}\widetilde{\omega}^{\mu\nu}.
\end{aligned} \tag{3.15}$$

The shear strain rate $\sigma_{\mu\nu}$ is a symmetric traceless tensor which tells us the rate at which the fluid element around a point is sheared whereas the vorticity $\omega_{\mu\nu}$ is an antisymmetric tensor that roughly tells us how fast the fluid is swirled around a point. Other important quantities in hydrodynamics are the energy density ε, pressure p, entropy density s, temperature T, conserved charge density n_i and corresponding chemical potentials μ_i. The scaling properties of these thermodynamic quantities is directly determined by the naive dimensional analysis

$$\begin{aligned}\varepsilon = e^{-d\phi}\tilde{\varepsilon}\ , \quad p = e^{-d\phi}\tilde{p}\ , \quad s = e^{-(d-1)\phi}\tilde{s}\ , \\ T = e^{-\phi}\tilde{T}\ , \quad n_i = e^{-(d-1)\phi}\tilde{n}_i\ , \quad \mu_i = e^{-\phi}\tilde{\mu}_i\ .\end{aligned} \tag{3.16}$$

3.2 Conformal hydrodynamics

In this section, we reformulate the fundamental equations of fluid mechanics in a Weyl-covariant form. The basic equations of fluid mechanics are the conservation of energy-momentum and various other charges -

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \text{and} \qquad \nabla_\mu J^\mu = 0 \tag{3.17}$$

But, these equations are not manifestly Weyl-covariant. To cast them into a manifestly Weyl-covariant form, we need the transformation of the stress tensor and the currents - $T^{\mu\nu} = e^{-(d+2)\phi}\widetilde{T}^{\mu\nu} + \ldots$ and $J^\mu = e^{-w\phi}\tilde{J}^\mu$ respectively (where $\ldots$ denotes the contributions due to the Weyl anomaly $T^\lambda{}_\lambda = \mathcal{W}$. The Weyl Anomaly $\mathcal{W}$ only on the microscopic field content and the ambient spacetime in which the conformal fluid lives.). Then, we can impose a manifestly Weyl covariant[21] set of equations

$$\begin{aligned}\mathcal{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} + \mathcal{A}^\nu(T^\mu{}_\mu - \mathcal{W}) = 0 \\ \mathcal{D}_\mu J^\mu = \nabla_\mu J^\mu + (w-d)\mathcal{A}_\mu J^\mu = 0\end{aligned} \tag{3.18}$$

These equations coincide with (3.17) provided $T^{\mu\nu}$ is a traceless tensor of conformal weight $d+2$ apart from the anomalous contribution and the conformal weight w of the conserved current is equal to the number of dimensions of the spacetime. The second condition is same as requiring that the charge associated with the charge currents be a dimensionless scalar.

The entropy current J^μ_S of the fluid also has a conformal weight equal to the spacetime dimensions. This means that we can write the statement of the second law in a manifestly conformal way as

$$\mathcal{D}_\mu J^\mu_S = \nabla_\mu J^\mu_S \geq 0 \tag{3.19}$$

[21] The Weyl transformation of the stress tensor in quantum theories is non-trivial because of the presence of Weyl anomaly . The situation is simplified if we assume that there exists a symmetric tensor $T^{\mu\nu}_{\text{conf}} = T^{\mu\nu} - \mathcal{W}^{\mu\nu}[g] = e^{-(d+2)\phi}\widetilde{T}^{\mu\nu}_{\text{conf}}$ where $\mathcal{W}^{\mu\nu}[g]$ characterizes the contribution due to Weyl anomaly which depends only on the background spacetime and the field content. In that case, though $T^{\mu\nu}$ does not transform homogeneously under the Weyl transformations, one can show that $\mathcal{D}_\mu T^{\mu\nu} = e^{-(d+2)\phi}\widetilde{\mathcal{D}}_\mu\tilde{T}^{\mu\nu}$ with $\mathcal{D}_\mu T^{\mu\nu}$ defined as above. This shows that the contributions due to Weyl anomaly can be taken into account with slight modifications. In what follows, we will ignore such subtleties due to Weyl anomaly - we will just assume that the energy-momentum tensor is traceless with the presumption that the statements we make can always be suitably modified once trace anomaly is taken into account.

Similarly, the first law of thermodynamics $\mathcal{T} u^\lambda \nabla_\lambda s = (d-1) u^\lambda \nabla_\lambda p - \mu_i u^\lambda \nabla_\lambda \rho_i$ can be written in a conformal form

$$\mathcal{T} u^\lambda \mathcal{D}_\lambda s = (d-1) u^\lambda \mathcal{D}_\lambda p - \mu_i u^\lambda \mathcal{D}_\lambda \rho_i \tag{3.20}$$

where $(d-1)p$ is the energy density of the conformal fluid. [22]

The fluid mechanics is completely specified once the expressions of the energy momentum tensor, the charged currents and the entropy current in terms of the velocity, temperature and the chemical potentials. The conventional discussion on relativistic hydrodynamics(say as given by Landau and Lifshitz[21]) can be adopted to the case of conformal fluids with the additional condition that the energy momentum tensor of a conformal fluid is traceless. The energy-momentum tensor, the charged currents and the entropy current of the fluid are usually divided into a non-dissipative part and a dissipative part.

$$\begin{aligned} T^{\mu\nu} &= \; p\,(g^{\mu\nu} + d\; u^\mu u^\nu) + \pi^{\mu\nu} \\ J_i^\mu &= \rho_i u^\mu + \nu_i^\mu \\ J_S^\mu &= s u^\mu + J_{S,\text{diss}}^\mu \end{aligned} \tag{3.21}$$

where we take the visco-elastic stress $\pi^{\mu\nu}$ to be transverse ($u_\mu \pi^{\mu\nu} = 0$) and traceless ($\pi^\mu{}_\mu = 0$) and the diffusion current ν_i^μ to be transverse ($u_\lambda \nu_i^\lambda = 0$). This in turn implies the following equations

$$\begin{aligned} 0 &= -u_\nu \mathcal{D}_\mu T^{\mu\nu} = (d-1) u^\lambda \mathcal{D}_\lambda p + \pi^{\mu\nu}\sigma_{\mu\nu} \\ 0 &= \mathcal{D}_\lambda J_i^\lambda = u^\lambda \mathcal{D}_\lambda \rho_i + \mathcal{D}_\lambda \nu_i^\lambda \end{aligned} \tag{3.22}$$

We can now use the first law of thermodynamics (3.20) to conclude

$$\mathcal{T}\mathcal{D}_\mu J_S^\mu = -\pi^{\mu\nu}\sigma_{\mu\nu} + \mu_i \mathcal{D}_\lambda \nu_i^\lambda + \mathcal{T}\mathcal{D}_\mu J_{S,\text{diss}}^\mu \geq 0 \tag{3.23}$$

Now we can write down the most general form of the dissipative currents confining ourselves to no more than second derivatives in velocity.[23] For simplicity, we will consider here the case when no charges are present - the generalization to the case when there are conserved charges is straightforward. Hence, a general derivative expansion for the energy-momentum tensor $T^{\mu\nu}$ is given by

$$\begin{aligned} T^{\mu\nu} &= \eta_0 \mathcal{T}^d (g^{\mu\nu} + d u^\mu u^\nu) \\ &\quad + \eta_1 \mathcal{T}^{d-1} \sigma^{\mu\nu} \\ &\quad + \eta_2 \mathcal{T}^{d-2}\; u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \eta_3\; \mathcal{T}^{d-2} [\omega^\mu{}_\lambda \sigma^{\lambda\nu} + \omega^\nu{}_\lambda \sigma^{\lambda\mu}] \\ &\quad + \eta_4\; \mathcal{T}^{d-2} [\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1} \sigma^{\alpha\beta}\sigma_{\alpha\beta}] + \eta_5\; \mathcal{T}^{d-2} [\omega^\mu{}_\lambda \omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1} \omega^{\alpha\beta}\omega_{\alpha\beta}] \\ &\quad - \eta_6\; \mathcal{T}^{d-2} C^\mu{}_\alpha{}^\nu{}_\beta u^\alpha u^\beta \end{aligned} \tag{3.24}$$

---

[22] Note that the additional terms that appear when one converts ∇ to $\mathcal{D}$ in (3.20) cancel out because of Gibbs-Duhem Relation $\mathcal{T}s = (d-1)p + p - \mu_i \rho_i$ where $(d-1)p$ is the energy density of the conformal fluid.

[23] Given the fact that for a conformal fluid $p \sim \mathcal{T}^d$ and the equation of motion $u^\lambda \mathcal{D}_\lambda p \sim \pi^{\mu\nu}\sigma_{\mu\nu}$ we conclude that wherever a single derivative of $\mathcal{T}$ occurs, it can be replaced by a term involving two or more derivatives of the fluid velocity. Hence, for the sake of counting, one derivative of $\mathcal{T}$ should be counted as equivalent to two derivatives of u^μ.

where the first line denotes the non-dissipative part(with the conformal equation of state $p = \eta_0 \mathcal{T}^d$) and the rest denote the visco-elastic stress $\pi^{\mu\nu}$. We show in the appendix (3.6.2) that no more terms appear at this order in the derivative expansion. This derivative expansion in terms of conformally covariant terms was first analyzed in [8] and our discussion here closely parallels theirs.[24].

3.3 Entropy current in Conformal hydrodynamics

Now we can write down the expression for the second law by restricting (3.23) to the case where there are no charges, and then substituting for $\pi^{\mu\nu}$ from (3.24)

$$\begin{aligned}\mathcal{T}\mathcal{D}_\mu J^\mu_S = \mathcal{T}\mathcal{D}_\mu J^\mu_{S,\text{diss}} &- \eta_1 \mathcal{T}^{d-1}\sigma^{\mu\nu}\sigma_{\mu\nu} - \eta_2 \mathcal{T}^{d-2}\sigma_{\mu\nu}\; u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\ &- \eta_4\; \mathcal{T}^{d-2}\sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu} - \eta_5\; \mathcal{T}^{d-2}\sigma_{\mu\nu}\omega^\mu{}_\lambda\omega^{\lambda\nu} \\ &+ \eta_6\; \mathcal{T}^{d-2}\sigma^{\mu\nu}C_{\mu\alpha\nu\beta}u^\alpha u^\beta\end{aligned} \tag{3.25}$$

Now we invoke two identities(see appendix 3.6.1 for the proofs)

$$\begin{aligned}\sigma^{\mu\nu}\omega_\mu{}^\alpha\omega_{\alpha\nu} &= \mathcal{D}_\lambda\left[\frac{\omega^{\mu\nu}\omega_{\mu\nu}}{4}u^\lambda + \frac{\mathcal{D}_\nu\omega^{\lambda\nu}}{2(d-3)}\right] \\ -\sigma^{\mu\nu}C_{\mu\alpha\nu\beta}u^\alpha u^\beta &= \sigma^{\mu\nu}\sigma_\mu{}^\alpha\sigma_{\alpha\nu} + \mathcal{D}_\lambda\left[\frac{2\sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu}}{4}u^\lambda + \frac{u_\mu(-\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2} + \frac{3\mathcal{D}_\nu\omega^{\lambda\nu}}{2(d-3)}\right]\end{aligned} \tag{3.26}$$

to write

$$\begin{aligned}\mathcal{T}\mathcal{D}_\mu J^\mu_S = &-\eta_1\mathcal{T}^{d-1}\sigma^{\mu\nu}\sigma_{\mu\nu} - (\eta_4+\eta_6)\;\mathcal{T}^{d-2}\sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu} + \mathcal{T}\mathcal{D}_\mu J^\mu_{S,\text{diss}} \\ &- \mathcal{T}^{d-2}\mathcal{D}_\lambda\left[\left(\frac{2(\eta_2+\eta_6)\;\sigma^{\mu\nu}\sigma_{\mu\nu} + (\eta_5+\eta_6)\;\omega^{\mu\nu}\omega_{\mu\nu}}{4}\right)u^\lambda\right. \\ &\left.+\frac{\eta_6\;u_\mu(-\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2} + \frac{(\eta_5+3\eta_6)}{2(d-3)}\mathcal{D}_\nu\omega^{\lambda\nu}\right]\end{aligned} \tag{3.27}$$

We now want to propose an expression for the dissipative entropy flux such that the total entropy obeys the second law of thermodynamics. In this chapter, we give a specific proposal for this entropy current which is consistent with the second law.[25] Taking the dissipative entropy flux as

$$\begin{aligned}J^\lambda_{S,\text{diss}} = &\left(\frac{2(\eta_2+\eta_6)\mathcal{T}^{d-3}\;\sigma^{\mu\nu}\sigma_{\mu\nu} + (\eta_5+\eta_6)\mathcal{T}^{d-3}\;\omega^{\mu\nu}\omega_{\mu\nu}}{4}\right)u^\lambda \\ &+ \frac{\eta_6\mathcal{T}^{d-3}\;u_\mu(-\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2} + \frac{(\eta_5+3\eta_6)\mathcal{T}^{d-3}}{2(d-3)}\mathcal{D}_\nu\omega^{\lambda\nu}\end{aligned} \tag{3.28}$$

[24]Refer §3.4 to see how our notation is related to that of [6] and [8].

[25]Note that, the second law alone does not determine the entropy flux uniquely - for example, an additional term with positive divergence can always be added to the dissipative entropy flux without violating the second law. Given this fact, it is important to emphasize that what is being proposed here is just one possible definition of the entropy current.

and keeping only terms with three derivatives or less of velocity[26]

$$\begin{aligned}\mathcal{T}\mathcal{D}_\mu J^\mu_S &= -\eta_1 \mathcal{T}^{d-1}\sigma^{\mu\nu}\sigma_{\mu\nu} - (\eta_4+\eta_6)\,\mathcal{T}^{d-2}\sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu} \\ &= -\eta_1\mathcal{T}^{d-1}\left[\sigma^{\mu\nu} + \frac{\eta_4+\eta_6}{2\eta_1\mathcal{T}}\sigma^\mu{}_\lambda\sigma^{\lambda\nu}\right]\left[\sigma_{\mu\nu} + \frac{\eta_4+\eta_6}{2\eta_1\mathcal{T}}\sigma_\mu{}^\alpha\sigma_{\alpha\nu}\right]\end{aligned} \tag{3.29}$$

from which we conclude that

$$\eta_1 \le 0 \tag{3.30}$$

along with a dissipative current of the form given in equation(3.28) is sufficient to ensure that the conformal fluid obeys the second law[27]

$$\mathcal{T}\mathcal{D}_\mu J^\mu_S = -\eta_1\mathcal{T}^{d-1}\left[\sigma^{\mu\nu} + \frac{\eta_4+\eta_6}{2\eta_1\mathcal{T}}\sigma^\mu{}_\lambda\sigma^{\lambda\nu}\right]\left[\sigma_{\mu\nu} + \frac{\eta_4+\eta_6}{2\eta_1\mathcal{T}}\sigma_\mu{}^\alpha\sigma_{\alpha\nu}\right] \ge 0 \tag{3.31}$$

Hence for a general energy-momentum tensor of the form

$$\begin{aligned}T^{\mu\nu} &= p(g^{\mu\nu} + du^\mu u^\nu) \\ &\quad - 2\eta\left[\sigma^{\mu\nu} - \tau_\pi\; u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} + \tau_\omega(\omega^\mu{}_\lambda\sigma^{\lambda\nu} + \omega^\nu{}_\lambda\sigma^{\lambda\mu})\right] \\ &\quad + \xi_\sigma[\sigma^\mu{}_\lambda\sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}] + \xi_C\; C_{\mu\alpha\nu\beta}u^\alpha u^\beta \\ &\quad + \xi_\omega[\omega^\mu{}_\lambda\omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\omega_{\alpha\beta}]\end{aligned} \tag{3.32}$$

where we have defined

$$\begin{aligned}p = \eta_0\mathcal{T}^d, \quad -2\eta = \eta_1\mathcal{T}^{d-1}, \quad 2\eta\tau_\pi = \eta_2\mathcal{T}^{d-2} \\ -2\eta\tau_\omega = \eta_3\mathcal{T}^{d-2}, \quad \xi_\sigma = \eta_4\mathcal{T}^{d-2}, \quad \xi_C = -\eta_6\mathcal{T}^{d-2}, \quad \xi_\omega = \eta_5\mathcal{T}^{d-2}\end{aligned} \tag{3.33}$$

the proposed expression for the entropy current is

$$\begin{aligned}J^\lambda_s &= su^\lambda + J^\lambda_{S,\mathrm{diss}} \\ &= \left(s - \frac{2(\xi_C - 2\eta\tau_\pi)\;\sigma^{\mu\nu}\sigma_{\mu\nu} + (\xi_C-\xi_\omega)\;\omega^{\mu\nu}\omega_{\mu\nu}}{4\mathcal{T}}\right)u^\lambda \\ &\quad - \frac{\xi_C u_\mu(-\mathcal{G}^{\mu\lambda} + \mathcal{F}^{\mu\lambda})}{(d-2)\mathcal{T}} - \frac{(3\xi_C - \xi_\omega)}{2(d-3)\mathcal{T}}\mathcal{D}_\nu\omega^{\lambda\nu}\end{aligned} \tag{3.34}$$

$$\text{with}\qquad \mathcal{T}\mathcal{D}_\mu J^\mu_S = 2\eta\left[\sigma^{\mu\nu} + \frac{\xi_C-\xi_\sigma}{4\eta}\sigma^\mu{}_\lambda\sigma^{\lambda\nu}\right]\left[\sigma_{\mu\nu} + \frac{\xi_C-\xi_\sigma}{4\eta}\sigma_\mu{}^\alpha\sigma_{\alpha\nu}\right] \ge 0$$

[26] Since we are working with the divergence of quantities accurate up to second derivatives of velocity, consistency demands that we keep terms involving three derivatives or less. Further, as before, we use the equations of motion to replace a derivative of $\mathcal{T}$ by a term involving two or more derivatives of the fluid velocity.

[27] This section has greatly benefited from my discussions with Shiraz Minwalla regarding the validity of second law for the entropy flux proposed above. I would also like to thank Veronica Hubeny, Giuseppe Policastro, Mukund Rangamani, Dam Thanh Son and Misha Stephanov for commenting on an earlier version of this section.

These expressions completely determine the dynamics of a conformal fluid up to second derivatives in the derivative expansion. We now proceed to apply the above formalism to the constitutive relations of $\mathcal{N}=4$ SYM fluid derived recently using AdS/CFT correspondence.

3.4 $\mathcal{N}=4$ SYM fluid : Energy-momentum and Entropy current

A prominent example of a conformal fluid in four dimensions is the fluid made out of the matter content in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. The flat spacetime stress tensor for the four dimensional conformal fluids with AdS duals (which in particular includes $\mathcal{N}=4$ SYM fluid in the four dimensional Minkowski spacetime) has been calculated recently via AdS/CFT upto second derivative terms [6]. Independently, in [8], its authors wrote down the general derivative expansion for a conformal fluid and determined some of the coefficients occurring in that expansion. In this section, we relate the work done in above references to the formalism developed here.

The expression for the energy-momentum tensor derived in [6] is

$$\begin{aligned} T^{\mu\nu} = &\ p\,(g^{\mu\nu}+4u^\mu u^\nu) \\ &- 2\ \eta\ \sigma^{\mu\nu} + 2\ \eta\ \frac{(\ln 2)\,T^{\mu\nu}_{2a} + 2\ T^{\mu\nu}_{2b} + (2-\ln 2)\left[\frac{1}{3}T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e}\right]}{2\pi\mathcal{T}} \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} p &= \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^4; \qquad \eta = \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^3 \\ \vartheta &= \nabla_\lambda u^\lambda\ ; \qquad a^\mu = u^\lambda\nabla_\lambda u^\mu; \quad l_\mu = \epsilon_{\alpha\beta\gamma\mu}u^\alpha\omega^{\beta\gamma}; \\ \sigma^{\mu\nu} &= P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right) - P^{\mu\nu}\frac{\nabla_\alpha u^\alpha}{3}; \\ T^{\mu\nu}_{2a} &= \frac{\epsilon^{\alpha\beta\gamma\mu}u_\alpha l_\beta \sigma_\gamma{}^\nu + \epsilon^{\alpha\beta\gamma\nu}u_\alpha l_\beta \sigma_\gamma{}^\mu}{2}; \\ T^{\mu\nu}_{2b} &= \sigma^{\mu\alpha}\sigma^\nu_\alpha - \frac{P^{\mu\nu}}{3}\sigma^{\beta\alpha}\sigma_{\alpha\beta}; \\ T^{\mu\nu}_{2c} &= \vartheta\sigma^{\mu\nu}; \quad T^{\mu\nu}_{2d} = a^\mu a^\nu - a_\lambda a^\lambda\frac{P^{\mu\nu}}{3}; \\ T^{\mu\nu}_{2e} &= P^{\mu\alpha}P^{\nu\beta}u^\lambda\nabla_\lambda\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right) - \frac{P^{\mu\nu}}{3}P^{\beta\gamma}u^\lambda\nabla_\lambda\left(\nabla_\beta u_\gamma\right); \end{aligned} \tag{3.36}$$

where $\epsilon_{0123} = -\epsilon^{0123} = 1$ and we are working in flat co-ordinates of the Minkowski spacetime. The above expression can be rewritten in terms of manifestly conformal observables as follows.

$$\begin{aligned} T^{\mu\nu}_{2a} &= -\omega^\mu{}_\lambda\sigma^{\lambda\nu} - \omega^\nu{}_\lambda\sigma^{\lambda\mu} \qquad , \qquad T^{\mu\nu}_{2b} = \sigma^{\mu\alpha}\sigma_\alpha{}^\nu - \frac{P^{\mu\nu}}{3}\sigma^{\beta\alpha}\sigma_{\alpha\beta} \\ \frac{1}{3}T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e} &= P^{\mu\alpha}P^{\nu\beta}u^\lambda\nabla_\lambda\sigma_{\alpha\beta} + \frac{\vartheta}{d-1}\sigma_{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}u^\lambda\mathcal{D}_\lambda\sigma_{\alpha\beta} = u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} \end{aligned} \tag{3.37}$$

The stress tensor becomes

$$
\begin{aligned}
T^{\mu\nu} = &\ p\,(g^{\mu\nu} + 4u^\mu u^\nu) \\
&- 2\ \eta\ \left[\sigma^{\mu\nu} - \frac{(2-\ln 2)}{2\pi\mathcal{T}} u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \frac{(\ln 2)}{2\pi\mathcal{T}}(\omega^\mu{}_\lambda \sigma^{\lambda\nu} + \omega^\nu{}_\lambda \sigma^{\lambda\mu})\right] \\
&+ \frac{4\ \eta}{2\pi\mathcal{T}}[\sigma^{\mu\lambda}\sigma_\lambda{}^\nu - \frac{P^{\mu\nu}}{3}\sigma^{\alpha\beta}\sigma_{\alpha\beta}]
\end{aligned}
\tag{3.38}
$$

This expression matches[28] with the expression in (3.32) provided we take

$$
\begin{gathered}
p = \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^4\ ; \qquad \eta = \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^3\ ; \\
\tau_\pi = \frac{2-\ln 2}{2\pi\mathcal{T}}\ ; \qquad \tau_\omega = \frac{\ln 2}{2\pi\mathcal{T}}\ ; \qquad \xi_\sigma = \xi_C = \frac{4\ \eta}{2\pi\mathcal{T}}\ ; \qquad \xi_\omega = 0\ .
\end{gathered}
\tag{3.39}
$$

where we have also included the value of the curvature coupling ξ_C which was calculated by the authors of [8].

Now, we proceed to compare the results of [8] to the results derived here. Translated into notations of this chapter[29] their expression (See Eqn.(3.11) of [8]) reads

$$
\begin{aligned}
\pi^{\mu\nu} = &- 2\eta\sigma^{\mu\nu} + 2\eta\tau_\pi\ u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} - \kappa[P^{\mu\lambda}P^{\nu\sigma}R_{\lambda\sigma} + (d-2)P^{\mu\lambda}P^{\nu\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta \\
&- \frac{P^{\mu\nu}}{d-1}(P^{\lambda\sigma}R_{\lambda\sigma} + (d-2)P^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta)] \\
&+ 4\lambda_1(\sigma^\mu{}_\lambda\sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}) + 4\lambda_2(\omega^\mu{}_\lambda\sigma^{\lambda\nu} + \omega^\nu{}_\lambda\sigma^{\lambda\mu}) \\
&+ \lambda_3(\omega^\mu{}_\lambda\omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\omega_{\alpha\beta})
\end{aligned}
\tag{3.40}
$$

with

$$
p = \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^4\ ; \qquad \eta = \frac{N_c^2}{8\pi^2}(\pi\mathcal{T})^3\ ; \qquad \tau_\pi = \frac{2-\ln 2}{2\pi\mathcal{T}}\ ; \qquad \lambda_1 = \frac{\eta}{2\pi\mathcal{T}}\ ; \qquad \kappa = \frac{\eta}{\pi\mathcal{T}}\ ;
$$

and the parameters $\lambda_{2,3}$ were left undetermined in [8]. By inspection, we conclude that the above expression satisfies[30] the conditions we laid down in (3.30).The above expression is completely consistent with the coefficients we derived above in (3.39). Hence, the second-order hydrodynamics of $\mathcal{N} = 4$ SYM fluid is completely summarized by (3.39).

[28]Note that the calculation in [6] was done for flat spacetime and hence the curvature term does not appear in their derivation.

[29]Note that the $\sigma_{\mu\nu}$ of [8] is twice that of ours and their curvature tensors are negative of the curvature tensors defined in this chapter.

[30] We have invoked the identity (which follows by applying projection operators to the the definition of Weyl tensor in (3.11))

$$
\begin{aligned}
&P^{\mu\lambda}P^{\nu\sigma}R_{\lambda\sigma} + (d-2)P^{\mu\lambda}P^{\nu\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta - \frac{P^{\mu\nu}}{d-1}(P^{\lambda\sigma}R_{\lambda\sigma} + (d-2)P^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta) \\
&\quad = (d-2)C_{\mu\alpha\nu\beta}u^\alpha u^\beta
\end{aligned}
$$

Now, we can use the discussion in our previous section to calculate the entropy current for $\mathcal{N}=4$ SYM fluid. Using the equation of state $\mathcal{T}s = p\ d = 4p = 4\pi\eta\mathcal{T}$ for a conformal fluid and (3.34) we get

$$J_s^\lambda = 4\pi\eta\left[u^\lambda - \frac{[(\ln 2)\sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu}]\,u^\lambda + 2u_\mu(\mathcal{G}^{\mu\lambda} + \mathcal{F}^{\mu\lambda}) + 6\mathcal{D}_\nu\omega^{\lambda\nu}}{8(\pi\mathcal{T})^2}\right].$$
$$\text{with}\qquad \mathcal{T}\mathcal{D}_\mu J_S^\mu = 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} \geq 0 \tag{3.41}$$

This expression gives the the next to leading order corrections to the holographic result $J_s^\lambda = 4\pi\eta u^\lambda$ of Kovtun, Son and Starinets[22].

Note that our proposal for the entropy current was motivated in an indirect way - by first finding the holographic energy-momentum tensor and then guessing the entropy current from it by demanding second law. It would be interesting to do a direct gravity computation of the entropy current that checks this proposal. Further, the rate of entropy production takes a very simple form in the case of $\mathcal{N}=4$ SYM fluid - the total entropy production is completely given by a term quadratic in shear strain rate $\sigma_{\mu\nu}$ and there is no contribution at the next order. This fact can be traced to an interesting fact that $\xi_\sigma = \xi_C$ for $\mathcal{N}=4$ SYM.

We would now like to give a heuristic reason for why we might expect the entropy production to take such a simpler form. Notice that the additional contribution to the entropy production(over and above the standard shear viscosity part) comes from a viscoelastic stress of the form $\pi^{\mu\nu} \sim \sigma^\mu{}_\lambda\sigma^{\lambda\nu}$. The rate of energy transfer by such a stress is $\sigma_{\mu\nu}\pi^{\mu\nu} \sim \sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu}$. If this energy transfer was irreversible, this would contribute to an entropy production $-\mathcal{T}^{-1}\sigma_{\mu\nu}\pi^{\mu\nu}$ which is precisely the term which we arrived at in the last section.

However, the energy transfer by a stress of the form $\pi \sim \sigma\sigma$ is reversible - in particular, for such a stress, the rate of work done $\pi\sigma$ reverses sign if we reverse the fluid flow. If we assume that such a reversible energy transfer cannot contribute to entropy production, then either such a term can be absorbed into a redefinition of the $J^\mu_{\text{S,diss}}$ or the coefficient of such a contribution should vanish. The second possibility immediately yields the condition $\xi_\sigma = \xi_C$. This, however is a very heuristic line of reasoning and it would be interesting to know how far it is valid. In principle, it should be possible to extend the holographic calculation of ξ_C and ξ_σ to arbitrary dimensional AdS gravity and check whether the relation $\xi_c = \xi_\sigma$ continues to hold.

In the next section, we compare and contrast the formalism used in this chapter with the conventional theories of relativistic hydrodynamics. In particular, we would be interested in comparison with the conventional Israel-Stewart formalism.

3.5 Israel-Stewart formalism

In this section, we give an extremely brief and non-exhaustive review of the conventional theories of relativistic hydrodynamics [23, 24] and discuss how the work presented in this chapter fits into that framework.

The first theories of relativistic viscous hydrodynamics are due to Eckart[25], Landau and Lifshitz [21]. These classical theories which are simple generalizations of their non-relativistic counterparts, assume a linear constitutive relation between the viscous stress $\pi^{\mu\nu}$ and the strain rate $\sigma^{\mu\nu}$. This linear approximation (called the Newtonian approximation) is the most familiar model in dissipative hydrodynamics and the fluids which obey such a relation are called Newtonian fluids.

Such a linear theory, however, leads to parabolic equations for the dissipative fluxes and predict very large speeds of propagation in situations with steep gradients, in contradiction with relativity and causality. It was noticed by many authors including Grad, Muller, Israel[26] and Stewart[27, 28] that one can easily solve this problem by including terms involving higher derivative corrections to the constitutive relations.[31] The most simple extension is to add a non-zero relaxation time in the equation thus converting the problem into a hyperbolic system of equations. [32] The resultant theory is called as causal viscous hydrodynamics or Extended Irreversible Thermodynamics(EIT) or just Israel-Stewart theory.[33]

This approach outlined above differs from the approach adopted here and elsewhere[6, 8] in the holographic studies of $\mathcal{N} = 4$ SYM. In particular, some of the terms appearing in the general derivative expansion of conformal fluids are absent in the conventional Israel-Stewart formalism[34].

One way of formulating Israel-Stewart theory is to consider dissipative fluxes like viscous stress and heat flow as new thermodynamic variables and treat entropy as a function of these new variables. In particular, one formulates the dynamics of such fluxes in a way that is consistent with the second law of thermodynamics. For a conformal fluid with no conserved charges, the viscoelastic stress in Israel-Stewart theory obeys an equation of the form[35]

$$\pi^{\mu\nu} + \tau_\pi u^\lambda \mathcal{D}_\lambda \pi^{\mu\nu} = -2\ \eta\ \sigma^{\mu\nu} + \tau_\omega(\omega^\mu{}_\lambda \pi^{\lambda\nu} + \omega^\nu{}_\lambda \pi^{\lambda\mu}) \tag{3.42}$$

[31] Many authors including Geroch[29] have argued that the large speeds of propagation might not be a problem if the gradients required to produce them are so steep that they are beyond the domain of validity of hydrodynamics (We remind the reader that the hydrodynamics ceases to be valid if the ratio of mean free path to the length scale at consideration (i.e., the Knudsen number) is larger than one). But, this argument might not apply to all fluids - see [30, 31] for further discussion of this issue.

[32] If one is interested in rotational flows, one can further add other terms involving vorticity $\omega_{\mu\nu}$ and cross terms involving other hydrodynamic variables.

[33] Note that, there are other alternative solutions to the problem of causality in Newtonian hydrodynamics. One such class of models termed *divergence type theories* were discussed by Geroch and Lindblom[32] and under quite general conditions, these class of theories exhibit finite speeds of propagation[33].

[34] Further, the authors of the reference [8] argue that some of these terms would be absent even in a systematic derivation of Israel-Stewart formalism from Relativistic Kinetic theory via moment closures.

[35] Note that often in the literature, τ_ω is taken to be equal to τ_π. We refrain from making such an identification here in order to facilitate easy comparison.

so that one can prove a version of the second law

$$\begin{aligned}
J_s^\lambda &= \left(s - \frac{\tau_\pi}{4\eta\mathcal{T}}\pi^{\mu\nu}\pi_{\mu\nu}\right)u^\lambda \\
\mathcal{D}_\lambda J_s^\lambda &= \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta\mathcal{T}} \geq 0
\end{aligned} \tag{3.43}$$

There is now a wide literature devoted to the analysis of the equations above and this formalism has been recently applied to the phenomenology of heavy-ion collisions.[36]

We can take the above equations and eliminate $\pi^{\mu\nu}$ in favor of $\sigma^{\mu\nu}$. We get the following expression which is exact up to second derivatives

$$\pi^{\mu\nu} = -2\ \eta \left[\sigma^{\mu\nu} - \tau_\pi u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \tau_\omega(\omega^\mu{}_\lambda\sigma^{\lambda\nu} + \omega^\nu{}_\lambda\sigma^{\lambda\mu})\right] \tag{3.44}$$

Comparing the equations so obtained with the equation(3.32) , it is clear that an Israel-Stewart conformal fluid is a fluid with ξ_σ, ξ_C and ξ_ω set to zero. Using the above expression, following the method employed in §3.3, we can define an entropy current associated with this fluid obeying the second law.[37]

However, as the previous sections make it clear, the Israel-Stewart conformal fluids form only a subset of conformal fluids. And more importantly, $\mathcal{N} = 4$ SYM fluid lies outside the subset since it has $\xi_\sigma = \xi_C \neq 0$. $\mathcal{N} = 4$ SYM fluid has a shear-shear coupling (and a coupling to the Weyl curvature) which is absent in the conventional Israel Stewart formalism. Hence, the approach developed in the study of $\mathcal{N} = 4$ SYM fluid should be looked upon as a generalization of the Israel Stewart formalism and the entropy current in the equation(3.34) should be treated as a generalization of the Israel-Stewart expression in the equation(3.43).

The main difference between the two formalisms lies in the way the viscoelastic stress is treated. As far as the contribution of the viscoelastic stress to the entropy current is concerned, Israel-Stewart formalism takes an extended thermodynamic point of view by assuming that all sources of viscoelastic stress contribute equally to the entropy current, whereas the entropy current proposed in this chapter treats different sources of visco-elastic stress differently. Rather than assuming that the entropy density is solely a function of $\pi^{\mu\nu}$, the entropy current is allowed to be a general function of the fluid velocity and its derivatives. Note that, despite going out of Israel Stewart formalism, we have succeeded in defining an entropy current which is consistent with the second law. [38]

3.6 Appendices

3.6.1 Some useful identities

In this appendix, we prove some identities that were used in the main body of this chapter. In particular, we want to sketch the proof of the equations quoted in equation(3.26).

---

[36] A non-exhaustive list of references include [34–40]

[37] Note however that the $J^\lambda_{s,diss}$ so obtained is the negative of what would be naively expected from equation(3.43). This apparent discrepancy can be traced to the ambiguity in the definition of $J^\lambda_{\mathrm{S,diss}}$.

[38] The author thanks Shiraz Minwalla for pointing out this distinction and for discussions about related issues.

First, we use the definition of $\mathcal{R}_{\mu\alpha\nu}{}^{\lambda}$ in terms of the commutator to write

$$\begin{aligned} u^{\alpha}(\mathcal{R}_{\mu\alpha\nu}{}^{\lambda}u_{\lambda}+\mathcal{F}_{\mu\alpha}u_{\nu}) &= -u^{\alpha}[\mathcal{D}_{\mu},\mathcal{D}_{\alpha}]u_{\nu} \\ &= -\mathcal{D}_{\mu}(u^{\alpha}\mathcal{D}_{\alpha}u_{\nu})+(\mathcal{D}_{\mu}u^{\alpha})(\mathcal{D}_{\alpha}u_{\nu})+u^{\alpha}\mathcal{D}_{\alpha}(\mathcal{D}_{\mu}u_{\nu}) \\ &= \sigma_{\mu}{}^{\alpha}\sigma_{\alpha\nu}+\sigma_{\mu}{}^{\alpha}\omega_{\alpha\nu}-\sigma_{\nu}{}^{\alpha}\omega_{\alpha\mu}+\omega_{\mu}{}^{\alpha}\omega_{\alpha\nu}+u^{\alpha}\mathcal{D}_{\alpha}(\sigma_{\mu\nu}+\omega_{\mu\nu}) \end{aligned} \tag{3.45}$$

Next, we multiply the expression above with $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ respectively, and simplify the resulting expressions using the curvature identities in §3.1 to get

$$\begin{aligned} \sigma^{\mu\nu}C_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta}-\sigma^{\mu\nu}\mathcal{S}_{\mu\nu} &= \sigma^{\mu\nu}\sigma_{\mu}{}^{\alpha}\sigma_{\alpha\nu}+\sigma^{\mu\nu}\omega_{\mu}{}^{\alpha}\omega_{\alpha\nu}+\sigma^{\mu\nu}u^{\alpha}\mathcal{D}_{\alpha}\sigma_{\mu\nu} \\ -\frac{1}{2}\omega^{\mu\nu}\mathcal{F}_{\mu\nu} &= -2\sigma_{\mu}{}^{\alpha}\omega_{\alpha\nu}\omega^{\nu\mu}+\omega^{\mu\nu}u^{\alpha}\mathcal{D}_{\alpha}\omega_{\mu\nu} \end{aligned} \tag{3.46}$$

The next step is to derive another identity which directly follows from the reduced Bianchi identity (See (3.9))

$$\begin{aligned} \mathcal{D}_{\lambda}\left[\frac{u_{\mu}(\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2}\right] &= \frac{(\mathcal{D}_{\lambda}u_{\mu})(\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2} \\ &= \frac{(\mathcal{D}_{\lambda}u_{\mu})(\mathcal{G}^{\mu\lambda}+\frac{d}{2}\mathcal{F}^{\mu\lambda}-\frac{d-2}{2}\mathcal{F}^{\mu\lambda})}{d-2} \\ &= \frac{\sigma_{\lambda\mu}(\mathcal{G}^{\mu\lambda}+\frac{d}{2}\mathcal{F}^{\mu\lambda})}{d-2}-\frac{1}{2}\omega_{\lambda\mu}\mathcal{F}^{\mu\lambda} \\ &= \sigma^{\mu\nu}\mathcal{S}_{\mu\nu}+\frac{1}{2}\omega_{\mu\nu}\mathcal{F}^{\mu\nu} \end{aligned} \tag{3.47}$$

where we have used the fact that $\mathcal{G}^{\mu\lambda}+\frac{d}{2}\mathcal{F}^{\mu\lambda}$ is a symmetric tensor.

We will need one more identity to finish the proof.

$$\begin{aligned} \mathcal{D}_{\mu}\left[\frac{\mathcal{D}_{\nu}\omega^{\mu\nu}}{d-3}\right] &= \frac{1}{2(d-3)}[\mathcal{D}_{\mu},\mathcal{D}_{\nu}]\omega^{\mu\nu} \\ &= \frac{3\mathcal{F}_{\mu\nu}\omega^{\mu\nu}+\mathcal{R}_{[\mu\nu]}\omega^{\mu\nu}}{2(d-3)} = -\frac{1}{2}\mathcal{F}_{\mu\nu}\omega^{\mu\nu} \end{aligned} \tag{3.48}$$

Using the above identities, it is now straightforward to get the equations quoted in (3.26).

$$\begin{aligned} \sigma^{\mu\nu}\omega_{\mu}{}^{\alpha}\omega_{\alpha\nu} &= \mathcal{D}_{\lambda}\left[\frac{\omega^{\mu\nu}\omega_{\mu\nu}}{4}u^{\lambda}+\frac{\mathcal{D}_{\nu}\omega^{\lambda\nu}}{2(d-3)}\right] \\ \sigma^{\mu\nu}\mathcal{C}_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta} &= \sigma^{\mu\nu}\sigma_{\mu}{}^{\alpha}\sigma_{\alpha\nu}+\mathcal{D}_{\lambda}\left[\frac{2\sigma^{\mu\nu}\sigma_{\mu\nu}+\omega^{\mu\nu}\omega_{\mu\nu}}{4}u^{\lambda}+\frac{u_{\mu}(\mathcal{G}^{\mu\lambda}+\mathcal{F}^{\mu\lambda})}{d-2}+\frac{3\mathcal{D}_{\nu}\omega^{\lambda\nu}}{2(d-3)}\right] \end{aligned} \tag{3.49}$$

3.6.2 Conformal Energy-Momentum tensor

In this appendix, we list all the terms that can appear in the energy-momentum tensor of a conformal fluid and show that only a few of them are linearly independent.

In order to write down the most general derivative expansion of the viscoelastic stress $\pi^{\mu\nu}$, we list below all the Weyl- covariant second- rank tensors which are symmetric, transverse and traceless.

$$
\begin{aligned}
&\sigma^{\mu\nu}\ , \quad u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu}\ , \quad [\omega^\mu{}_\lambda \sigma^{\lambda\nu} + \omega^\nu{}_\lambda \sigma^{\lambda\mu}],\\
&[\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}]\ , \quad [\omega^\mu{}_\lambda \omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\omega_{\alpha\beta}],\\
&C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad [P^{\mu\lambda}P^{\nu\sigma}(\mathcal{R}_{\lambda\sigma} + \frac{d}{2}\mathcal{F}_{\lambda\sigma}) - \frac{P^{\mu\nu}}{d-1}P^{\lambda\sigma}\mathcal{R}_{\lambda\sigma}],\\
&[P^{\mu\lambda}P^{\nu\sigma}(\mathcal{R}_{\lambda\alpha\sigma\beta}u^\alpha u^\beta - \frac{1}{2}\mathcal{F}_{\lambda\sigma}) - \frac{P^{\mu\nu}}{d-1}P^{\lambda\sigma}\mathcal{R}_{\lambda\alpha\sigma\beta}u^\alpha u^\beta]
\end{aligned}
\tag{3.50}
$$

Note that, the different terms appearing above are not all independent .

To show that we take the relation

$$
-u^\alpha[\mathcal{D}_\mu, \mathcal{D}_\alpha]u_\nu = -u^\alpha \mathcal{D}_\mu \mathcal{D}_\alpha u_\nu + u^\alpha \mathcal{D}_\alpha \mathcal{D}_\mu u_\nu = (\mathcal{D}_\mu u^\alpha)(\mathcal{D}_\alpha u_\nu) + u^\alpha \mathcal{D}_\alpha(\mathcal{D}_\mu u_\nu)
$$

and project out out the symmetric traceless transverse part to get

$$
\begin{aligned}
&[P^{\mu\lambda}P^{\nu\sigma}(\mathcal{R}_{\lambda\alpha\sigma\beta}u^\alpha u^\beta - \frac{1}{2}\mathcal{F}_{\lambda\sigma}) - \frac{P^{\mu\nu}}{d-1}P^{\lambda\sigma}\mathcal{R}_{\lambda\alpha\sigma\beta}u^\alpha u^\beta]\\
&\qquad = [\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}] + [\omega^\mu{}_\lambda \omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\omega_{\alpha\beta}] + u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu}
\end{aligned}
\tag{3.51}
$$

Further, if we denote by the subscript TT the transverse traceless part, then we have using (3.13)

$$
[\mathcal{R}_{\lambda\sigma} + (d-2)\mathcal{R}_{\lambda\alpha\sigma\beta}u^\alpha u^\beta]_{TT} = [R_{\lambda\sigma} + (d-2)R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta]_{TT} = (d-2)C_{\lambda\alpha\sigma\beta}u^\alpha u^\beta
$$

Hence, the independent terms that occur in a derivative expansion are

$$
\begin{aligned}
&\sigma^{\mu\nu}\ , \quad u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu}\ , \quad [\omega^\mu{}_\lambda \sigma^{\lambda\nu} + \omega^\nu{}_\lambda \sigma^{\lambda\mu}],\\
&[\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}]\ , \quad [\omega^\mu{}_\lambda \omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\omega_{\alpha\beta}],\\
&C_{\mu\alpha\nu\beta}u^\alpha u^\beta
\end{aligned}
\tag{3.52}
$$

and so we obtain the derivative expansion in (3.24).

3.7 Classification

We can now use this Weyl-covariant derivative to enumerate all the Weyl-covariant scalars, transverse vectors (i.e, vectors that are everywhere orthogonal to the fluid velocity field u^μ) and the transverse traceless tensors in the charged hydrodynamics that involve no more than second order derivatives. We will do this enumeration 'on-shell', i.e., we will enumerate those quantities which remain linearly independent even after the equations of motion are taken into account.

The basic fields in the charged hydrodynamics are the fluid velocity u^μ with weight unity, the fluid temperature T with with weight unity and the chemical potentials μ_i with weight unity. This implies that an arbitrary function of μ_i/T is Weyl-invariant and hence one could always multiply a Weyl-covariant tensor by such a function to get another Weyl-covariant tensor. Hence, in the following list only linearly independent fields appear. To make contact with the conventional literature on hydrodynamics we will work with the charge densities n_i (with weight $d-1$) rather than the chemical potentials μ_i. For simplicity, we will confine ourselves to the case where there is only one charge.

At one derivative level,

- Weyl invariant scalars : None
- Weyl invariant transverse vector : $n^{-1}P_\mu^\nu \mathcal{D}_\nu n$.
  In d=4 , we also have $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha} u_\mu \nabla_\nu u_\lambda$.
- Weyl-invariant symmetric traceless transverse tensors : $T\sigma_{\mu\nu}$

At the two derivative level,

- Weyl invariant scalars :

$$\begin{aligned} &T^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad T^{-2}\omega_{\mu\nu}\omega^{\mu\nu}, \quad T^{-2}\mathcal{R}, \\ &T^{-2}n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n \quad \text{and} \quad T^{-2}n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\mathcal{D}_\nu n \end{aligned} \tag{3.53}$$

  In d=4 ,

$$T^{-2}n^{-1}l^\mu \mathcal{D}_\mu n$$

- Weyl-invariant transverse vectors :

$$\begin{aligned} &T^{-1}P_\mu^\nu \mathcal{D}_\lambda \sigma_\nu{}^\lambda, \qquad T^{-1}P_\mu^\nu \mathcal{D}_\lambda \omega_\nu{}^\lambda, \\ &T^{-1}n^{-1}\sigma_\mu{}^\lambda \mathcal{D}_\lambda n \qquad \text{and} \qquad T^{-1}n^{-1}\omega_\mu{}^\lambda \mathcal{D}_\lambda n \end{aligned} \tag{3.54}$$

  In d=4 ,

$$T^{-1}\sigma_{\mu\nu}\ l^\nu$$

  .

- Weyl-invariant symmetric traceless transverse tensors : "

$$\begin{gathered} C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu}, \\ \omega_\mu{}^\lambda \sigma_{\lambda\nu} + \omega_\nu{}^\lambda \sigma_{\lambda\mu}, \quad \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{P_{\mu\nu}}{d-1}\ \sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad \omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{P_{\mu\nu}}{d-1}\ \omega_{\alpha\beta}\omega^{\alpha\beta}, \\ n^{-1}\ \Pi_{\mu\nu}^{\alpha\beta}\ \mathcal{D}_\alpha \mathcal{D}_\beta n, \quad n^{-2}\ \Pi_{\mu\nu}^{\alpha\beta}\ \mathcal{D}_\alpha n\ \mathcal{D}_\beta n. \end{gathered} \tag{3.55}$$

  In d=4,

$$\begin{aligned} &\frac{1}{4}\ \epsilon^{\alpha\beta\lambda}{}_\mu\ \epsilon^{\gamma\theta\sigma}{}_\nu C_{\alpha\beta\gamma\theta}\ u_\lambda u_\sigma, \qquad \frac{1}{2}\epsilon_{\alpha\beta\lambda(\mu}C^{\alpha\beta}{}_{\nu)\sigma}u^\lambda u^\sigma, \quad \mathcal{D}_{(\mu}l_{\nu)}, \\ &n^{-1}\Pi_{\mu\nu}^{\alpha\beta}l_\alpha \mathcal{D}_\beta n, \quad n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_\alpha \mathcal{D}_\beta n. \end{aligned} \tag{3.56}$$

where we have introduced the projection tensor $\Pi^{\alpha\beta}_{\mu\nu}$ which projects out the transverse traceless symmetric part of second rank tensors

$$\Pi^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}\left[P^{\alpha}_{\mu}P^{\beta}_{\nu} + P^{\alpha}_{\nu}P^{\beta}_{\mu} - \frac{2}{d-1}P^{\alpha\beta}P_{\mu\nu}\right]$$

These invariants can now be used to write down the most general $T_{\mu\nu}$ and J^{μ}_{i} consistent with Weyl-covariance. The energy-momentum tensor and the charged currents of the fluid are usually divided into a zero-derivative part and a part involving at least one derivative

$$\begin{aligned} T^{\mu\nu}_{conf} &= \ p\,(g^{\mu\nu} + d\ u^{\mu}u^{\nu}) + \pi^{\mu\nu} \\ J^{\mu}_{i} &= \rho_i u^{\mu} + \nu^{\mu}_{i} \end{aligned} \tag{3.57}$$

where we can take the visco-elastic stress $\pi^{\mu\nu}$ to be transverse ($u_{\mu}\pi^{\mu\nu} = 0$) and traceless ($\pi^{\mu}{}_{\mu} = 0$) and the diffusion current ν^{μ}_{i} to be transverse ($u_{\lambda}\nu^{\lambda}_{i} = 0$). Hence, $\pi^{\mu\nu}$ and ν^{μ}_{i} are linear combination of transverse traceless Weyl-covariant tensors and transverse Weyl-covariant vectors of appropriate weight.

4 Hydrodynamics and Large Rotating Blackholes in AdS

We will turn to the study of large rotating black holes in global AdS_D spaces - which seems like a distinct subject from the CFTs and their hydrodynamic description. As will see in this chapter, there are deep relations between hydrodynamics of CFTs and the blackhole solutions in AdS. We will use the AdS/CFT correspondence to argue that large rotating black holes in global AdS_D spaces are dual to stationary solutions of the relativistic Navier-Stokes equations on S^{D-2}. Reading off the equation of state of this fluid from the thermodynamics of non-rotating black holes, we proceed to construct the nonlinear spinning solutions of fluid mechanics that are dual to rotating black holes. In all known examples, the thermodynamics and the local stress tensor of our solutions are in precise agreement with the thermodynamics and boundary stress tensor of the spinning black holes.

Our fluid dynamical description applies to large non-extremal black holes as well as a class of large non-supersymmetric extremal black holes, but is never valid for supersymmetric black holes. Our results will yield predictions for the thermodynamics of all large black holes in all theories of gravity on AdS spaces, for example, string theory on $AdS_5 \times S^5$ and M theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

The material for this chapter is drawn from the paper[15] written by the author in collaboration with Sayantani Bhattacharyya, Subhaneil Lahiri and Shiraz Minwalla.

4.1 Introduction

In this chapter, we predict certain universal features in the thermodynamics and other classical properties of large rotating black holes in global AdS_D spaces for arbitrary D. Our analysis applies to black holes in any consistent theory of gravity that admits an AdS_D background; for example, IIB theory on $AdS_5 \times S^5$ or M theory on $AdS_7 \times S^4$ or $AdS_4 \times S^7$.

All theories of gravity on an AdS_D background are expected to admit a dual description as a quantum field theory on $S^{D-2}\times$ time [4, 41]. Moreover, it is expected to be generally true that quantum field theories at sufficiently high energy density admit an effective description in terms of fluid dynamics. Putting together these facts, we propose that large, rotating black holes in arbitrary global AdS_D spaces admit an accurate dual description as rotating, stationary configurations of a conformal fluid on S^{D-2}.

Assuming our proposal is indeed true, we are able to derive several properties of large rotating AdS black holes as follows: We first read off the thermodynamic equation of state of the dual 'fluid' from the properties of large, static, non-rotating AdS black holes. Inputting these equations of state into the Navier-Stokes equations, we are then able to deduce the thermodynamics of rotating black holes. In the rest of this introduction, we will describe our proposal and its consequences, including the tests it successfully passes, in more detail.

Consider a theory of gravity coupled to a gauge field (based on a gauge group of rank c) on AdS_D. In an appropriate limit, the boundary theory is effectively described by conformal fluid dynamics with c simultaneously conserved, mutually commuting $U(1)$ charges R_i ($i = 1 \ldots c$). Conformal invariance and extensivity force the grand canonical partition function of this fluid to take the form

$$\frac{1}{V}\ln \mathcal{Z}_{\rm gc} = T^{d-1}\; h(\zeta/T)\,, \tag{4.1}$$

where ζ represents the set of the c chemical potentials conjugate to the $U(1)$ charges of the fluid, V and T represent the volume and the overall temperature of the fluid respectively and $d = D - 1$ is the spacetime dimensions of the boundary. As we have explained above, the as yet unknown function $h(\zeta/T)$ may be read off from the thermodynamics of large, charged, static black holes in AdS.

The thermodynamic equation of state described above forms an input into the relativistic Navier-Stokes equations that govern the effec tive dynamics of the boundary conformal fluid. The full equations of fluid dynamics require more data than just the equation of state; f or example we need to input dissipative parameters like viscosity. However, fluid dynamics on S^{d-1} admits a distinguished $c+n+1$ parameter set of stationary solutions (the parameters are their energy E, c commuting charges R_i and $n = \text{rank}(SO(d)) = \left[\frac{d}{2}\right]$ commuting angular momenta[39] on S^{d-1}). These solutions are simply the configurations into which any fluid initial state eventually settles down in equilibrium. They have the feature that their form and properties are independent of the values of dissipative parameters.

Although these solutions are nonlinear (i.e. they cannot be thought of as a small fluctuation about a uniform fluid configuration), it turns out that they are simple enough to be determined explicitly. These solutions turn out to be universal (i.e. they are independent of the detailed form of the function $h(\zeta/T)$). Their thermodynamics is incredibly simple; it is summarised by the partition function

$$\ln \mathcal{Z}_{\rm gc} = \ln \text{Tr} \exp\left[-\frac{(H-\zeta_i R_i - \Omega_a L_a)}{T}\right] = \frac{V_d T^{d-1} h(\zeta/T)}{\prod_{a=1}^{n}(1-\Omega_a^2)}\,, \tag{4.2}$$

where H,L_a and Ω_a represent the energy, angular momenta and the angular velocities of the fluid respectively and $V_d = \text{Vol}(S^{d-1})$ is the volume of the sphere S^{d-1}.

We now turn to the gravitational dual interpretation of the fluid dynamical solutions we have described above. A theory of a rank c gauge field, interacting with gravity on AdS$_D$, possesses a $c+n+1$ parameter set of black hole solutions, labelled by the conserved charges described above. We propose that these black holes (when large) are dual to the solutions of fluid dynamics described in the previous paragraph. Our proposal yields an immediate prediction about the thermodynamics of large rotating black holes: the grand canonical partition function of these black holes must take the form of (4.2).

Notice that while the dependence of the partition function (4.2) on ζ/T is arbitrary, its dependence on Ω_a is completely fixed. Thus, while our approach cannot predict thermodynamic properties of the static black holes, it does allow us to predict the thermodynamics of large rotating black holes in terms of the thermodynamics of their static counterparts.[40] Fur-

[39] Here, we use the notation $[x]$ to denote the integer part of the real number x.

[40] The analogue of our procedure in an asymptotically flat space (which we unfortunately do not have) would be a method to deduce the thermodynamics and other properties of the charged Kerr black hole, given the solution of static charged black holes.

ther, our solution of fluid dynamics yields a detailed prediction for the boundary stress tensor and the local charge distribution of the corresponding black hole solution, which may be compared with the boundary stress tensor and currents calculated from black hole solutions (after subtracting the appropriate counterterms [42–49]).

Our proposal is highly reminiscent of the membrane paradigm in black hole physics (see, for instance, [50, 51]). However, we emphasise that our fluid dynamical description of black holes is not a guess; our proposal follows directly from the AdS/CFT correspondence in a precisely understood regime (see [5] for a review of AdS/CFT correspondence).[41]

We have tested the thermodynamical predictions described above on every class of black hole solutions in AdS_D spaces that we are aware of. These solutions include the most general uncharged rotating black holes in arbitrary AdS_D space [16, 52, 53], various classes of charged rotating black holes in $AdS_5 \times S^5$ [19, 54–56], in $AdS_7 \times S^4$ and in $AdS_4 \times S^7$ [57–59]. In the strict fluid dynamical limit, the thermodynamics of each of these black holes exactly reproduces[42] (4.2). In all the cases we have checked, the boundary stress tensor and the charge densities of these black holes (read off from the black hole solutions using the AdS/CFT dictionary) are also in perfect agreement with our fluid dynamical solutions. The agreement described in this paragraph occurs only when one would expect it to, as we now explain in detail.

Recall that the equations of fluid mechanics describe the evolution of local energy densities, charge densities and fluid velocities as functions of spatial position. These equations are applicable only under certain conditions. First, the fluctuations about mean values (of variables like the local energy density) must be negligible. In the situations under study in this chapter, the neglect of fluctuations is well justified by the 'large N' limit of the field theory, dual to the classical limit of the gravitational bulk.

Second, the equations of fluid mechanics assume that the fluid is in local thermodynamic equilibrium at each point in space, even though the energy and the charge densities of the fluid may vary in space. Fluid mechanics applies only when the length scales of variation of thermodynamic variables - and the length scale of curvatures of the manifold on which the fluid propagates - are large compared to the equilibration length scale of the fluid, a distance we shall refer to as the mean free path l_{mfp}.

The mean free path for any fluid may be estimated as [60] $l_{\mathrm{mfp}} \sim \frac{\eta}{\rho}$ where η is the shear viscosity and ρ is the energy density of the fluid. For fluids described by a gravitational dual, $\eta = \frac{s}{4\pi}$ where s is the entropy density [60]. Consequently, for the fluids under study in this chapter, $l_{\mathrm{mfp}} \sim \frac{s}{4\pi\rho}$. As we will see in §§4.3.7, the most stringent bound on l_{mfp}, for the solutions presented in this chapter, comes from requiring that l_{mfp} be small compared to the radius of the S^{d-1}, which we set to unity. Consequently, fluid dynamics should be an

[41] Alternatively, one could regard the agreement between fluid dynamics and gravity described below as a test of the AdS/CFT correspondence (provided we are ready to assume in addition the applicability of fluid mechanics to quantum field theories at high density).

[42] See, however, §§4.6.10 for a puzzle regarding the first subleading corrections for a class of black holes in AdS_5.

accurate description for our solutions whenever $\frac{s}{4\pi\rho} \ll 1$. In every case we have studied, it turns out that this condition is met whenever the horizon radius, R_H, of the dual black hole is large compared to the AdS radius , R_{AdS}. Black holes that obey this condition include all black holes whose temperature is large compared to unity, but also includes large radius extremal black holes in $AdS_5 \times S^5$, $AdS_7 \times S^4$ and $AdS_4 \times S^7$. It, however, never includes supersymmetric black holes in the same backgrounds, whose horizon radii always turn out to be of the same order as the AdS radius.

It follows that we should expect the Navier-Stokes equations to reproduce the thermodynamics of only large black holes. In all the cases we have studied, this is indeed the case. It is possible to expand the formulae of black hole thermodynamics (and the stress tensor and charge distribution) in a power series in R_{AdS}/R_H. While the leading order term in this expansion matches the results of fluid dynamics, we find deviations from the predictions of Navier-Stokes equations at subleading orders.

Starting with the work of Policastro, Son and Starinets [61], there have been several fascinating studies over the last few years, that have computed fluid dynamical dissipative and transport coefficients from gravity (see the review [60] and the references therein). The work reported in this chapter differs from these analyses in several ways. Firstly, the solutions of fluid mechanics we study are nonlinear; in general they cannot be thought of as small fluctuations about the uniform fluid configuration dual to static black holes. Second, all our solutions are stationary: dissipative parameters play no role in our work.

Indeed our work rather follows the same line of investigation as the one applied to plasmaballs and plasmarings in [62, 63]. These investigations used the boundary fluid dynamics to make detailed predictions about the nature and phase structure of the black holes and black rings in Scherk-Schwarz compactified AdS spaces. The predictions of these papers have not yet been quantitatively verified as the corresponding black hole and black ring solutions have not yet been constructed. The perfect agreement between fluid dynamics and gravity in the simpler and better studied context of this chapter lends significant additional support to those predictions of [62, 63] that were made using fluid mechanics.

While in this chapter we have used fluid dynamics to make predictions for black hole physics, the reverse view point may also prove useful. Existing black hole solutions in AdS spaces provide exact equilibrium solutions to the equations of fluid dynamics to all orders in l_{mfp}. A study of the higher order corrections of these solutions (away from the $l_{\text{mfp}} \to 0$ limit) might yield useful information about the nature of the fluid dynamical approximation of quantum field theories.

The plan of this chapter is as follows - In §4.2, we set up the basic fluid mechanical framework necessary for our work. It is followed by §4.3 in which we consider in detail a specific example of rigidly rotating fluid - a conformal fluid in $S^3 \times \mathbb{R}$. A straightforward generalisation gives us a succinct way of formulating fluid mechanics in spheres of arbitrary dimensions in §4.4.

We proceed then to compare the fluid mechanical predictions with various types of black holes in arbitrary dimensions. First, we consider uncharged rotating black holes in arbitrary dimensions in §4.5. Their thermodynamics, stress tensors and charge distributions are computed and are shown to exactly match the fluid mechanical predictions. In §4.6, we turn to the large class of rotating black hole solutions in $AdS_5 \times S^5$. Many different black holes with different sets of charges and angular momenta are considered in the large horizon radius limit and all of them are shown to fit exactly into our proposal in the strict fluid dynamical limit. However we also find deviations from the predictions of the Navier-Stokes equations at first subleading order in $l_{\rm mfp}$ for black holes with all SO(6) Cartan charges nonzero (these deviations vanish when the angular velocities, or one of the SO(6) charges is set to zero). This finding is at odds with naive expectations from fluid dynamics, which predict the first deviations from the Navier-Stokes equations to occur at $\mathcal{O}(l_{\rm mfp}^2)$ and is an as yet unresolved puzzle.

This is followed by §4.7, dealing with large rotating black holes in $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds which are dual to field theories on M2 and M5 branes respectively. The thermodynamics of the rotating black hole solutions in these spaces are derived from their static counterparts using the duality to fluid mechanics and it is shown how the known rotating black hole solutions agree with the fluid mechanical predictions in the large horizon radius limit. In each of these cases, the formulae of black hole thermodynamics deviate from the predictions of the Navier-Stokes equations only at $\mathcal{O}(l_{\rm mfp}^2)$ in accord with general expectations. In the final section, we conclude our work and discuss further directions.

In appendix 4.9.1, we discuss the constraints imposed by conformal invariance on the equations of fluid mechanics. In appendix 4.9.2, we discuss the thermodynamics of free theories on spheres. In appendix 4.9.3, we present our computations of the boundary stress tensor for two classes of black holes in AdS spaces.

4.2 The equations of fluid mechanics

4.2.1 The equations

The fundamental variables of fluid dynamics are the local proper energy density ρ, local charge densities r_i and fluid velocities $u^\mu = \gamma(1, \vec{v})$. Assuming local thermodynamic equilibrium, the rest frame entropy density s, the pressure $\mathcal{P}$, the local temperature $\mathcal{T}$ and the chemical potentials μ_i of the fluid can be expressed as functions of ρ and r_i using the equation of state and the first law of thermodynamics, as in §§4.2.5, §§4.2.6 below. In what follows, we will often find it convenient to express the above thermodynamic quantities in terms of $\mathcal{T}$ and μ_i rather than as functions of ρ and r_i.

The equations of fluid dynamics are simply a statement of the conservation of the stress tensor $T^{\mu\nu}$ and the charge currents J_i^μ.

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0\,, \\ \nabla_\mu J_i^\mu &= \partial_\mu J_i^\mu + \Gamma^\mu_{\mu\alpha} J_i^\alpha = 0\,. \end{aligned} \tag{4.3}$$

4.2.2 Perfect fluid stress tensor

The dynamics of a fluid is completely specified once the stress tensor and charge currents are given as functions of ρ, r_i and u^μ. As we have explained in the introduction, fluid mechanics is an effective description at long distances (i.e, it is valid only when the fluid variables vary on distance scales that are large compared to the mean free path $l_{\rm mfp}$). As a consequence it is natural to expand the stress tensor and charge current in powers of derivatives. In this subsection we briefly review the leading (i.e. zeroth) order terms in this expansion.

It is convenient to define a projection tensor

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{4.4}$$

$P^{\mu\nu}$ projects vectors onto the 3 dimensional submanifold orthogonal to u^μ. In other words, $P^{\mu\nu}$ may be thought of as a projector onto spatial coordinates in the rest frame of the fluid. In a similar fashion, $-u^\mu u^\nu$ projects vectors onto the time direction in the fluid rest frame.

To zeroth order in the derivative expansion, Lorentz invariance and the correct static limit uniquely determine the stress tensor, charge and the entropy currents in terms of the thermodynamic variables. We have

$$\begin{aligned} T^{\mu\nu}_{\rm perfect} &= \rho u^\mu u^\nu + \mathcal{P} P^{\mu\nu}, \\ (J^\mu_i)_{\rm perfect} &= r_i u^\mu, \\ (J^\mu_S)_{\rm perfect} &= s u^\mu, \end{aligned} \tag{4.5}$$

where $\rho = \rho(\mathcal{T}, \mu_i)$ is the rest frame energy density, $s = s(\mathcal{T}, \mu_i)$ is the rest frame entropy density of the fluid and μ_i are the chemical potentials. It is not difficult to verify that in this zero-derivative (or perfect fluid) approximation, the entropy current is conserved. Entropy production (associated with dissipation) occurs only at the first subleading order in the derivative expansion, as we will see in the next subsection.

4.2.3 Dissipation and diffusion

Now, we proceed to examine the first subleading order in the derivative expansion. In the first subleading order, Lorentz invariance and the physical requirement that entropy be non-decreasing determine the form of the stress tensor and the current to be (see, for example, §§14.1 of [24])

$$\begin{aligned} T^{\mu\nu}_{\rm dissipative} &= -\zeta \vartheta P^{\mu\nu} - 2\eta \sigma^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu, \\ (J^\mu_i)_{\rm dissipative} &= q^\mu_i, \\ (J^\mu_S)_{\rm dissipative} &= \frac{q^\mu - \mu_i q^\mu_i}{\mathcal{T}}\,. \end{aligned} \tag{4.6}$$

where

$$
\begin{aligned}
a^\mu &= u^\nu \nabla_\nu u^\mu, \\
\vartheta &= \nabla_\mu u^\mu, \\
\sigma^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta P^{\mu\nu}, \\
q^\mu &= -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T}), \\
q_i^\mu &= -D_{ij} P^{\mu\nu}\partial_\nu\left(\frac{\mu_j}{\mathcal{T}}\right),
\end{aligned}
\tag{4.7}
$$

are the acceleration, expansion, shear tensor, heat flux and diffusion current respectively.

These equations define a set of new fluid dynamical parameters in addition to those of the previous subsection: ζ is the bulk viscosity, η is the shear viscosity, κ is the thermal conductivity and D_{ij} are the diffusion coefficients. Fourier's law of heat conduction $\vec{q} = -\kappa\vec{\nabla}\mathcal{T}$ has been relativistically modified to

$$
q^\mu = -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T}), \tag{4.8}
$$

with an extra term that is related to the inertia of flowing heat. The diffusive contribution to the charge current is the relativistic generalisation of Fick's law.

At this order in the derivative expansion, the entropy current is no longer conserved; however, it may be checked [24] that

$$
\mathcal{T}\nabla_\mu J_S^\mu = \frac{q^\mu q_\mu}{\kappa\mathcal{T}} + \mathcal{T}(D^{-1})^{ij} q_i^\mu q_{j\mu} + \zeta\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{4.9}
$$

As q^μ, q_i^μ and $\sigma^{\mu\nu}$ are all spacelike vectors and tensors, the RHS of (4.9) is positive provided η, ζ, κ and D are positive parameters, a condition we further assume. This establishes that (even locally) entropy can only be produced but never destroyed. In equilibrium, $\nabla_\mu J_S^\mu$ must vanish. It follows that, q^μ, q_i^μ, θ and $\sigma^{\mu\nu}$ each individually vanish in equilibrium.

From the formulae above, we see that the ratio of $T^{\mu\nu}_{\text{dissipative}}$ to $T^{\mu\nu}_{\text{perfect}}$ is of the order $\eta/(R\rho)$ where η is the shear viscosity, ρ is the rest frame energy density and R is the typical length scale of the flow under consideration. Consequently, the Navier-Stokes equations may be thought of as the first term in a series expansion of the microscopic equations in l_{mfp}/R , where $l_{\text{mfp}} \sim \frac{\eta}{\rho}$. In this sense, l_{mfp} plays a role analogous to the string scale in the derivative expansion of the effective action in string theory. This length scale may plausibly be identified with the thermalisation length scale of the fluid.[43]

When studying fluids on curved manifolds (as we will proceed to do in this chapter), one could add generally covariant terms, built out of curvatures, to the stress tensor. For instance, we could add a term proportional to $R^{\mu\nu}$ to the expression for $T^{\mu\nu}$. We will ignore all such terms in this chapter for a reason we now explain. In all the solutions of fluid mechanics that we will study, the length scale over which fluid quantities vary is the same

[43] This may be demonstrated within the kinetic theory, where l_{mfp} is simply the mean free path of colliding molecules, but is expected to apply to more generally to any fluid with short range interactions.

as the length scale of curvatures of the manifold. Any expression built out of a curvature contains at least two spacetime derivatives of the metric; it follows that any contribution to the stress tensor proportional to a curvature is effectively at least two orders subleading in the derivative expansion, and so is negligible compared to all the other terms we have retained in this chapter.

4.2.4 Conformal fluids

We will now specialise our discussion to a conformal fluid – the fluid of the 'stuff' of a conformal field theory in d dimensions. Conformal invariance imposes restrictions on both the thermodynamics of the fluid and the derivative expansion of the stress tensor discussed in the previous subsection.

To start with, conformal invariance requires that the stress tensor be traceless.[44] This requirement relates the pressure of a conformal fluid to its density as $\mathcal{P} = \frac{\rho}{d-1}$ (this requirement may also be deduced from conformal thermodynamics, as we will see in the next subsection) where d is the dimension of the spacetime in which the fluid lives. Further, the tracelessness of the stress tensor also forces the bulk viscosity, ζ, to be zero.

It is easy to verify that these constraints are sufficient to guarantee the conformal invariance of the fluid dynamical equations listed above. Consider a conformal transformation $g_{\mu\nu} = \mathrm{e}^{2\phi}\tilde{g}_{\mu\nu}$ under which fluid velocity, temperature, rest energy density, pressure, entropy density and the charge densities transform as

$$
\begin{aligned}
u^\mu &= \mathrm{e}^{-\phi}\tilde{u}^\mu, \\
\mathcal{T} &= \mathrm{e}^{-\phi}\widetilde{\mathcal{T}}\,, \\
\rho &= \mathrm{e}^{-d\phi}\tilde{\rho}\,, \quad \mathcal{P} = \mathrm{e}^{-d\phi}\widetilde{\mathcal{P}}\,, \\
s &= \mathrm{e}^{-(d-1)\phi}\tilde{s}\,, \\
r_i &= \mathrm{e}^{-(d-1)\phi}\tilde{r}_i\,.
\end{aligned}
$$

It is easy to verify that these transformations induce the following transformations on the stress tensors and currents listed in the previous subsection[45]

$$
\begin{aligned}
T^{\mu\nu} &= \mathrm{e}^{-(d+2)\phi}\widetilde{T}^{\mu\nu}, \\
J_i^\mu &= \mathrm{e}^{-d\phi}\widetilde{J}_i^\mu, \\
J_S^\mu &= \mathrm{e}^{-d\phi}\widetilde{J}_S^\mu.
\end{aligned}
\tag{4.10}
$$

These are precisely the transformation properties that ensure the conformal invariance of the conservation equations (4.3). See appendix 4.9.1 for more details.

[44]More accurately, conformal invariance relates the nonzero trace of the stress tensor to certain curvature forms; for example, in two dimension $T^\mu_\mu = \frac{c}{12}R$ where R is the scalar curvature. However, as we have described above, curvatures are effectively zero in the one derivative expansion studied in this chapter. All formulae through the rest of this chapter and in the appendices apply only upon neglecting curvatures. We thank R. Gopakumar for a discussion of this point.

[45]Note that under such a scaling, the viscosity, conductivity etc. scale as $\kappa = e^{-(d-2)\phi}\tilde{\kappa}$, $\eta = e^{-(d-1)\phi}\tilde{\eta}$, $\mu_i = e^{-\phi}\tilde{\mu}_i$ and $D_{ij} = e^{-(d-2)\phi}\widetilde{D}_{ij}$.

4.2.5 Conformal thermodynamics

In this subsection, we review the thermodynamics of the conformal fluids we discuss below. The notation set up in this subsection will be used through the rest of this chapter.

Define the thermodynamic potential

$$\Phi = \mathcal{E} - \mathcal{T}\mathcal{S} - \mu_i \mathcal{R}_i \ . \tag{4.11}$$

for which the first law of thermodynamics reads

$$\mathrm{d}\Phi = -\mathcal{S}\mathrm{d}\mathcal{T} - \mathcal{P}\mathrm{d}V - \mathcal{R}_i \mathrm{d}\mu_i \,. \tag{4.12}$$

Let us define $\nu_i = \mu_i/\mathcal{T}$. It follows from conformal invariance and extensivity that

$$\Phi = -V\mathcal{T}^d h(\nu)\,, \tag{4.13}$$

where $h(\nu)$ is defined by this expression. All remaining thermodynamic expressions are easily determined in terms of the function $h(\nu_i)$

$$\begin{aligned} \rho &= (d-1)\mathcal{P} = (d-1)\mathcal{T}^d h(\nu)\,, \\ r_i &= \mathcal{T}^{d-1} h_i(\nu)\,, \\ s &= \mathcal{T}^{d-1}(dh - \nu_i h_i)\,, \end{aligned} \tag{4.14}$$

where

$$h_i = \frac{\partial h}{\partial \nu_i}$$

denotes the derivative of h with respect to its i^{th} argument.

4.2.6 A thermodynamic identity

We will now derive a thermodynamic identity that will be useful in our analysis below. Define

$$\Gamma = \mathcal{E} - \mathcal{T}\mathcal{S} + \mathcal{P}V - \mu_i \mathcal{R}_i \ = \Phi + \mathcal{P}V\,, \tag{4.15}$$

the first law of thermodynamics implies that

$$\mathrm{d}\Gamma = -\mathcal{S}\mathrm{d}\mathcal{T} + V\mathrm{d}\mathcal{P} - \mathcal{R}_i \mathrm{d}\mu_i \,. \tag{4.16}$$

Consider scaling the system by a factor $(1+\epsilon)$. Under such a scaling, extensivity implies that

$$\mathrm{d}\Gamma = \epsilon\Gamma\,, \qquad \mathrm{d}\mathcal{T} = \mathrm{d}\mathcal{P} = \mathrm{d}\mu_i = 0\,,$$

which when substituted into (4.16) tells us that $\Gamma = 0$. Then we can divide (4.15) and (4.16) by V to get

$$\begin{aligned} \rho + \mathcal{P} &= s\mathcal{T} + \mu_i r_i\,, \\ \mathrm{d}\mathcal{P} &= s\mathrm{d}\mathcal{T} + r_i \mathrm{d}\mu_i\,. \end{aligned} \tag{4.17}$$

4.3 Equilibrium configurations of rotating conformal fluids on S^3

In this section and in the next, we will determine the equilibrium solutions of fluid dynamics equations for conformal fluids on spheres of arbitrary dimension. In this section, we work out the fluid dynamics on S^3 plus a time dimension in detail.[46] In the next section, we generalise the results of this section to spheres of arbitrary dimension.

4.3.1 Coordinates and conserved charges

Consider a unit S^3 embedded in $\mathbb{R}^4$ as

$$\begin{aligned} x^1 &= \sin\theta\cos\phi_1 \\ x^2 &= \sin\theta\sin\phi_1 \\ x^3 &= \cos\theta\cos\phi_2 \\ x^4 &= \cos\theta\sin\phi_2 \end{aligned} \tag{4.18}$$

with $\theta \in [0, \frac{\pi}{2}]$, $\phi_a \in [0, 2\pi)$. The metric of the spacetime $S^3 \times \mathbb{R}$ is

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta\, d\phi_1^2 + \cos^2\theta\, d\phi_2^2\,. \tag{4.19}$$

This gives the following non-zero Christoffel symbols:

$$\Gamma^{\theta}_{\phi_1\phi_1} = -\Gamma^{\theta}_{\phi_2\phi_2} = -\cos\theta\sin\theta\,, \quad \Gamma^{\phi_1}_{\theta\phi_1} = \Gamma^{\phi_1}_{\phi_1\theta} = \cot\theta\,, \quad \Gamma^{\phi_2}_{\theta\phi_2} = \Gamma^{\phi_2}_{\phi_2\theta} = -\tan\theta\,. \tag{4.20}$$

For the stationary, axially symmetric configurations under consideration, $\partial_t T^{\mu\nu} = \partial_{\phi_a} T^{\mu\nu} = 0$. Using (4.20), (4.3) becomes

$$0 = \nabla_\mu T^{\mu t} = \partial_\theta T^{\theta t} + (\cot\theta - \tan\theta) T^{\theta t}, \tag{4.21}$$

$$0 = \nabla_\mu T^{\mu\theta} = \partial_\theta T^{\theta\theta} + (\cot\theta - \tan\theta) T^{\theta\theta} + \cos\theta\sin\theta\left(T^{\phi_1\phi_1} - T^{\phi_2\phi_2}\right), \tag{4.22}$$

$$0 = \nabla_\mu T^{\mu\phi_1} = \partial_\theta T^{\theta\phi_1} + (\cot\theta - \tan\theta) T^{\theta\phi_1} + 2\cot\theta\, T^{\theta\phi_1}, \tag{4.23}$$

$$0 = \nabla_\mu T^{\mu\phi_2} = \partial_\theta T^{\theta\phi_2} + (\cot\theta - \tan\theta) T^{\theta\phi_2} - 2\tan\theta\, T^{\theta\phi_2}. \tag{4.24}$$

The Killing vectors of interest are ∂_t (Energy) and ∂_{ϕ_a} (SO(4) Cartan angular momenta). Using the formula for the related conserved charge, $\int d^3x\sqrt{-g}\, T^{0\mu} g_{\mu\nu} k^\nu$, we get:

$$\begin{aligned} E &= \int d\theta d\phi_1 d\phi_2\; \cos\theta\sin\theta\, T^{tt}, \\ L_1 &= \int d\theta d\phi_1 d\phi_2\; \cos\theta\sin^3\theta\, T^{t\phi_1}, \\ L_2 &= \int d\theta d\phi_1 d\phi_2\; \cos^3\theta\sin\theta\, T^{t\phi_2}. \end{aligned} \tag{4.25}$$

[46] In this case, the dimensions of the spacetime in which the fluid lives is $d = 3 + 1 = 4$. The number of mutually commuting angular momenta is $n = 2$. The black hole dual lives in AdS space of dimensions $D = d + 1 = 5$.

Assuming $q^\mu = q_i^\mu = 0$ (as will be true for stationary solutions we study in this chapter), the entropy and the R-charges corresponding to the currents in (4.5) are given as

$$\begin{aligned} S &= \int \mathrm{d}\theta \mathrm{d}\phi_1 \mathrm{d}\phi_2 \, \cos\theta \sin\theta \, \gamma s \,, \\ R_i &= \int \mathrm{d}\theta \mathrm{d}\phi_1 \mathrm{d}\phi_2 \, \cos\theta \sin^3\theta \, \gamma r_i \,. \end{aligned} \tag{4.26}$$

4.3.2 Equilibrium solutions

As we have explained in the §§4.2.3, each of the three quantities $\sigma^{\mu\nu}, q^\mu, q_i^\mu$ must vanish on any stationary solution of fluid dynamics. The requirement that $\sigma^{\mu\nu} = 0$ has a unique solution - the fluid motion should be just a rigid rotation. By an $SO(4)$ rotation we can choose the two orthogonal two planes of this rotation as the (1-2) and (3-4) planes (see (4.18)). This implies that $u^\mu = \gamma(1, 0, \omega_1, \omega_2)$ (where we have listed the $(t, \theta, \phi_1, \phi_2)$ components of the velocity) with $\gamma = \left(1 - v^2\right)^{-1/2}$ and $v^2 = \omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta$, for some constants ω_1 and ω_2.

Our equilibrium fluid flow enjoys a symmetry under translations of t, ϕ_1 and ϕ_2; consequently all thermodynamic quantities are functions only of the coordinate θ.

Evaluating the tensors in (4.7), we find

$$\begin{aligned} a^\mu &= (0, -\partial_\theta \ln\gamma, 0, 0), \\ \vartheta &= 0, \\ \sigma^{\mu\nu} &= 0, \\ q^\mu &= -\kappa\gamma \left(0, \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathcal{T}}{\gamma}\right], 0, 0\right), \\ q_i^\mu &= -D_{ij} \left(0, \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mu_j}{\mathcal{T}}\right], 0, 0\right). \end{aligned} \tag{4.27}$$

The requirement that q^μ and q_i^μ vanish forces us to set

$$\mathcal{T} = \tau\gamma \,, \quad \mu_i = \mathcal{T}\nu_i \,, \tag{4.28}$$

for constant τ and ν_i. These conditions completely determine all the thermodynamic quantities as a function of the coordinates on the sphere. We will now demonstrate that this configuration solves the Navier-Stokes equations.

First note that for an arbitrary rigid rotation, the dissipative part of the stress tensor evaluates to

$$T^{\mu\nu}_{\text{dissipative}} = -\kappa\gamma^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \omega_1 & \omega_2 \\ 0 & \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathcal{T}}{\gamma}\right], \tag{4.29}$$

an expression which simply vanishes once we impose (4.28). Consequently, all nonzero contributions to the stress tensor come from the 'perfect fluid piece' and are given by

$$T^{\mu\nu}_{\rm perfect} = \gamma^2 \begin{pmatrix} (\rho + v^2\mathcal{P}) & 0 & \omega_1(\rho+\mathcal{P}) & \omega_2(\rho+\mathcal{P}) \\ 0 & \gamma^{-2}\mathcal{P} & 0 & 0 \\ \omega_1(\rho+\mathcal{P}) & 0 & \omega_1^2\rho + (\csc^2\theta - \omega_2^2\cot^2\theta)\mathcal{P} & \omega_1\omega_2(\rho+\mathcal{P}) \\ \omega_2(\rho+\mathcal{P}) & 0 & \omega_1\omega_2(\rho+\mathcal{P}) & \omega_2^2\rho + (\sec^2\theta - \omega_1^2\tan^2\theta)\mathcal{P} \end{pmatrix}. \tag{4.30}$$

The only non-trivial equation of motion, (4.22), can be written as

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\theta} - \frac{\rho+\mathcal{P}}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = 0\,. \tag{4.31}$$

Now using the thermodynamic identity (4.17) we may recast (4.31) as

$$\gamma s \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathcal{T}}{\gamma}\right] + \gamma r_i \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mu_i}{\gamma}\right] = 0\,, \tag{4.32}$$

an equation which is automatically true from (4.28). Consequently, rigidly rotating configurations that obey (4.28) automatically obey the Navier-Stokes equations.

In a similar fashion, it is easy to verify that all nonzero contributions to the charge currents come from the perfect fluid piece of that current, and that the conservation of these currents holds for our solutions.

In summary the $3+c$ parameter set of stationary solutions to fluid mechanics listed in this subsection (the parameters are τ, ω_a and ν_i where $i = 1\ldots c$) constitute the most general stationary solutions of fluid mechanics.

4.3.3 Stress tensor and currents

Using the equations of state (4.14), we find that

$$\begin{aligned} \rho = 3\mathcal{P} &= 3\tau^4\gamma^4 h(\nu), \\ s &= \tau^3\gamma^3[4h(\nu) - \nu_i h_i(\nu)], \\ r_i &= \tau^3\gamma^3 h_i(\nu). \end{aligned} \tag{4.33}$$

The stress tensor is

$$T^{\mu\nu} = \tau^4 A \gamma^6 \begin{pmatrix} 3+v^2 & 0 & 4\omega_1 & 4\omega_2 \\ 0 & 1-v^2 & 0 & 0 \\ 4\omega_1 & 0 & 3\omega_1^2 + \csc^2\theta - \omega_2^2\cot^2\theta & 4\omega_1\omega_2 \\ 4\omega_2 & 0 & 4\omega_1\omega_2 & 3\omega_2^2 + \sec^2\theta - \omega_1^2\tan^2\theta \end{pmatrix}. \tag{4.34}$$

Charge and entropy currents are given by

$$\begin{aligned} J^\mu_i &= \tau^3\gamma^4 C_i(1,0,\omega_1,\omega_2)\,, \\ J^\mu_S &= \tau^3\gamma^4 B(1,0,\omega_1,\omega_2)\,, \end{aligned} \tag{4.35}$$

where we have defined

$$
\begin{aligned}
A &= h(\nu)\,, \\
B &= 4h(\nu) - \nu_i h_i(\nu)\,, \\
C_i &= h_i(\nu) = \frac{\partial h}{\partial \nu_i}\,.
\end{aligned}
\tag{4.36}
$$

4.3.4 Charges

The energy, angular momentum, entropy and R-charges may now easily be evaluated by integration: we find

$$
\begin{aligned}
E &= \frac{V_4 \tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left[\frac{2\omega_1^2}{1-\omega_1^2} + \frac{2\omega_2^2}{1-\omega_2^2} + 3\right], \\
L_1 &= \frac{V_4 \tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left[\frac{2\omega_1}{1-\omega_1^2}\right], \\
L_2 &= \frac{V_4 \tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left[\frac{2\omega_2}{1-\omega_2^2}\right], \\
S &= \frac{V_4 \tau^3 B}{(1-\omega_1^2)(1-\omega_2^2)}\,, \\
R_i &= \frac{V_4 \tau^3 C_i}{(1-\omega_1^2)(1-\omega_2^2)}\,,
\end{aligned}
\tag{4.37}
$$

where $V_4 = \mathrm{Vol}(S^3) = 2\pi^2$ is the volume of S^3. These formulae constitute a complete specification of the thermodynamics of stationary rotating conformal fluids on S^3.

4.3.5 Potentials

In the previous subsection we have evaluated all the thermodynamic charges of our rotating fluid solutions. It is also useful to evaluate the chemical potentials corresponding to these solutions. To be specific we define these chemical potentials via the grand canonical partition function defined in the introduction

$$
\mathcal{Z}_{\mathrm{gc}} = \mathrm{Tr}\exp\left(\frac{1}{T}(-H + \Omega_a L_a + \zeta_i R_i)\right) = \exp\left(-\frac{E - TS - \Omega_a L_a - \zeta_i R_i}{T}\right), \tag{4.38}
$$

where the last expression applies in the thermodynamic limit. In other words

$$
T = \left(\frac{\partial E}{\partial S}\right)_{L_b, R_j}, \quad \Omega_a = \left(\frac{\partial E}{\partial L_a}\right)_{S, L_b, R_j}, \quad \zeta_i = \left(\frac{\partial E}{\partial R_i}\right)_{S, L_b, R_j}. \tag{4.39}
$$

It is easy to verify that[47]

$$
T = \tau\,, \qquad \Omega_a = \omega_a\,, \qquad \zeta_i = \tau\nu_i\,. \tag{4.40}
$$

[47] We can express $\mathrm{d}E - \tau\mathrm{d}S - \omega_a \mathrm{d}L_a - \tau\nu_i \mathrm{d}R_i$ in terms of $\mathrm{d}\tau, \mathrm{d}\omega_a, \mathrm{d}\nu_i$ and check that it vanishes, or we can check the Legendre transformed statement $\mathrm{d}(E - \tau S - \omega_a L_a - \tau\nu_i R_i) = -S\mathrm{d}\tau - L_a \mathrm{d}\omega_a - R_i \mathrm{d}(\tau\nu_i)$.

Note that T, Ω_a and ζ_i are distinct from $\mathcal{T}$, ω_a and μ_i. While the former quantities are thermodynamic properties of the whole fluid configuration, the latter quantities are local thermodynamic properties of the fluid that vary on the S^3. In a similar fashion, the energy E of the solution is, of course, a distinct concept from the local rest frame energy density ρ which is a function on the sphere. In particular, E receives contributions from the kinetic energy of the fluid as well as its internal energy, $\mathcal{E}$.

4.3.6 Grand canonical partition function

The grand canonical partition function (4.38) is easily computed; we find

$$\ln \mathcal{Z}_{\mathrm{gc}} = \frac{V_4 T^3 h(\zeta/T)}{(1-\Omega_1^2)(1-\Omega_2^2)}\,, \tag{4.41}$$

where $V_4 = V(S^3) = 2\pi^2$ is the volume of S^3.

In other words, the grand canonical partition function of the rotating fluid is obtained merely by multiplying the same object for the non-rotating fluid by a universal angular velocity dependent factor.

4.3.7 Validity of fluid mechanics

A systematic way to estimate the domain of validity of the Navier-Stokes equations would be to list all possible higher order corrections to these equations, and to check under what circumstances the contributions of these correction terms to the stress tensor and currents are small compared to the terms we have retained. Rather than carrying out such a detailed (and worthwhile) exercise, we present in this section a heuristic physical estimate of the domain of validity of fluid dynamics.

Consider a fluid composed of a collection of interacting 'quasiparticles', that move at an average speed v_p and whose collisions are separated (on the average) by the distance l_{mfp} in the fluid rest frame. Consider a particular quasiparticle that undergoes two successive collisions: the first at the coordinate location x_1 and subsequently at x_2. In order for the fluid approximation to hold, it must be that

1. The fractional changes in thermodynamic quantities between the two collision points (e.g. $[T(x_1) - T(x_2)]/T(x_1)$) are small. This condition is necessary in order for us to assume local thermal equilibrium.

2. The distance between the two successive collisions is small compared to the curvature/compactification scales of the manifold on which the fluid propagates. This approximation is necessary, for example, in order to justify the neglect of curvature corrections to the Navier-Stokes equations.

Let us now see when these two conditions are obeyed on our solutions. Recall that the local temperatures in our solutions take the form $\mathcal{T} = T\gamma$ where T is the overall temperature of the solution. If we treat the free path $l_{\rm mfp}$ as a function of temperature and chemical potentials, conformal invariance implies that

$$l_{\rm mfp}(\mathcal{T},\nu_i) = \frac{1}{\gamma} l_{\rm mfp}(T,\nu_i)\,.$$

Hence, the first condition listed above is satisfied when the fractional variation in (say) the temperature is small over the rest frame mean free path $l_{\rm mfp}(\mathcal{T},\nu_i)$, i.e. provided

$$\frac{l_{\rm mfp}(T,\nu_i)}{\gamma} \ll \gamma \left(\frac{\partial\gamma}{\partial\theta}\right)^{-1}, \tag{4.42}$$

which must hold for all points of the sphere.[48] The strictest condition one obtains from this is

$$l_{\rm mfp}(T,\nu_i) \ll \frac{1}{\left|\sqrt{1-\omega_1^2}-\sqrt{1-\omega_2^2}\right|}\,. \tag{4.43}$$

It turns out that the second condition listed above is always more stringent, especially when applied to fluid quasiparticles whose rest frame motion between two collisions is in the same direction as the local fluid velocity. It follows from the formulae of Lorentz transformations that the distance on the sphere between two such collisions is $l_{\rm mfp}(\mathcal{T},\nu_i)\gamma(1+v/v_p) = l_{\rm mfp}(T,\nu_i)(1+v/v_p)$, where v_p is the quasiparticle's velocity in the rest frame of the fluid and v the fluid velocity. As the factor $(1+v/v_p)$ is bounded between 1 and 2, we conclude that the successive collisions happen at distances small compared to the radius of the sphere provided

$$l_{\rm mfp}(T,\nu_i) \ll 1\,. \tag{4.44}$$

Hence, we conclude that the condition (4.44) (which is always more stringent than (4.43)) is the condition for the applicability of the equations of fluid mechanics.

Of course the model (of interacting quasiparticles) that we have used to obtain (4.44) need not apply to the situations of our interest. However the arguments that led to (4.44) were essentially kinematical which leads us to believe that the result will be universal. Nonetheless, it would be useful to verify this result by performing the detailed analysis alluded to at the beginning of this subsection.

4.4 Rotating fluids on spheres of arbitrary dimension

We now generalise the discussion of the previous section to the study of conformal fluids on spheres of arbitrary dimension.

[48] Recall that all variations in the temperature are perpendicular to fluid velocities, so that the typical scale of variation in both the rest frame and the lab frame coincide.

Let us embed S^{2n} in $\mathbb{R}^{2n+1}$ as

$$\begin{aligned} x^{2a-1} &= \left(\prod_{b=1}^{a-1}\cos\theta_b\right)\sin\theta_a\cos\phi_a\,, \\ x^{2a} &= \left(\prod_{b=1}^{a-1}\cos\theta_b\right)\sin\theta_a\sin\phi_a\,, \\ x^{2n+1} &= \left(\prod_{b=1}^{n}\cos\theta_b\right), \end{aligned} \tag{4.45}$$

Where $\theta_n \in [0,\pi]$, all other $\theta_a \in [0,\frac{\pi}{2}]$ and $\phi_a \in [0,2\pi)$. Any products with the upper limit smaller than the lower limit should be set to one. Although we appear to have specialised to even dimensional spheres above, we can obtain all odd dimensional sphere, S^{2n-1}, simply by setting $\theta_n = \pi/2$ in all the formulae of this section.

The metric on $S^{2n}\times$ time is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{a=1}^{n}\left(\prod_{b=1}^{a-1}\cos^2\theta_b\right)\mathrm{d}\theta_a^2 + \sum_{a=1}^{n}\left(\prod_{b=1}^{a-1}\cos^2\theta_b\right)\sin^2\theta_a\mathrm{d}\phi_a^2\,. \tag{4.46}$$

We choose a rigidly rotating velocity

$$\begin{gathered} u^t = \gamma \qquad u^{\theta_a} = 0 \qquad u^{\phi_a} = \gamma\omega_a \\ \gamma = (1-v^2)^{-1/2} \qquad v^2 = \sum_{a=1}^{n}\left(\prod_{b=1}^{a-1}\cos^2\theta_b\right)\sin^2\theta_a\omega_a^2 \end{gathered} \tag{4.47}$$

As in §§4.3.2, the equations of motion are solved, without dissipation, by setting

$$\frac{\mathcal{T}}{\gamma} = \tau = \text{constant}, \qquad \frac{\mu_i}{\mathcal{T}} = \nu_i = \text{constant}, \tag{4.48}$$

which gives the densities

$$\begin{aligned} \rho = (d-1)\mathcal{P} &= (d-1)\tau^d\gamma^d h(\nu_i), \\ s &= \tau^{d-1}\gamma^{d-1}[dh(\nu) - \nu_i h_i(\nu)], \\ r_i &= \tau^{d-1}\gamma^{d-1}h_i(\nu), \end{aligned} \tag{4.49}$$

This gives a stress tensor

$$\begin{gathered} T^{tt} = \tau^d A(d\gamma^{d+2} - \gamma^d) \qquad T^{t\phi_a} = T^{\phi_a t} = \tau^d A d\gamma^{d+2}\omega_a \\ T^{\theta_a\theta_a} = \tau^d A\gamma^d\left(\prod_{b=1}^{a-1}\sec^2\theta_b\right) \\ T^{\phi_a\phi_a} = \tau^d A\left[d\gamma^{d+2}\omega_a^2 + \gamma^d\left(\prod_{b=1}^{a-1}\sec^2\theta_b\right)\csc^2\theta_a\right] \qquad T^{\phi_a\phi_b} = \tau^d A d\gamma^{d+2}\omega_a\omega_b \end{gathered} \tag{4.50}$$

and currents

$$\begin{aligned} J_S^t &= \tau^{d-1} B \gamma^d \quad J_S^{\theta_a} = 0 \quad J_S^{\phi_a} = \tau^{d-1} B \gamma^d \omega_a \,, \\ J_i^t &= \tau^{d-1} C_i \gamma^d \quad J_i^{\theta_a} = 0 \quad J_i^{\phi_a} = \tau^{d-1} C_i \gamma^d \omega_a \,, \end{aligned} \tag{4.51}$$

where

$$\begin{aligned} A &= h(\nu) \,, \\ B &= d h(\nu) - \nu_i h_i(\nu) \,, \\ C_i &= h_i(\nu) \,. \end{aligned} \tag{4.52}$$

Integrating these gives[49]

$$\begin{aligned} E &= \frac{V_d \, \tau^d \, h(\nu)}{\prod_b (1-\omega_b^2)} \left[2 \sum_a \frac{\omega_a^2}{1-\omega_a^2} + d - 1 \right] , \\ S &= \frac{V_d \, \tau^{d-1} [d h(\nu) - \nu_i h_i(\nu)]}{\prod_b (1-\omega_b^2)} \,, \\ L_a &= \frac{V_d \, \tau^d \, h(\nu)}{\prod_b (1-\omega_b^2)} \left[\frac{2\omega_a}{1-\omega_a^2} \right] , \\ R_i &= \frac{V_d \, \tau^{d-1} \, h_i(\nu)}{\prod_b (1-\omega_b^2)} \,, \end{aligned} \tag{4.53}$$

where

$$V_d = \mathrm{Vol}(S^{d-1}) = \frac{2 \cdot \pi^{d/2}}{\Gamma(d/2)} \,.$$

Differentiating these gives

$$T = \tau \qquad \Omega_a = \omega_a \qquad \zeta_i = \tau \nu_i \,. \tag{4.54}$$

and the grand partition function

$$\ln \mathcal{Z}_{\mathrm{gc}} = \frac{V_d \, T^{d-1} \, h(\zeta/T)}{\prod_b (1-\Omega_b^2)} \,. \tag{4.55}$$

As in the previous subsection, the fluid dynamical approximation is expected to be valid provided $l_{\mathrm{mfp}}(T, \nu_i) \ll 1$.

In appendix 4.9.2, we have computed the thermodynamics of a free charged scalar field on a sphere, and compared with the general results of this section.

4.5 Comparison with uncharged black holes in arbitrary dimensions

In the rest of this chapter, we will compare the predictions from fluid dynamics derived above with the thermodynamics, stress tensors and charge distributions of various classes of large rotating black hole solutions in AdS spaces. We start with uncharged rotating black holes on D dimensional AdS spaces (where D is arbitrary), which are dual to rotating configurations of uncharged fluids on spheres of dimension $(D-2)$.

[49] In deriving these formulae we have 'conjectured' that $\int_{S^{d-1}} \gamma^d = \frac{V_d}{\prod_{b=1}^{[d/2]}(1-\omega_b^2)}$. It is easy to derive this formula for odd spheres. We have also analytically checked this formula for S^2 and S^4. We are ashamed, however, to admit that we have not yet found an analytic derivation of this integral for general even spheres.

4.5.1 Thermodynamics and stress tensor from fluid mechanics

In case of uncharged fluids the function $h(\nu)$ in the above section is a constant $h(\nu) = h$. Therefore $h_i(\nu) = \frac{\partial h(\nu)}{\partial \nu_i}$ are all equal to zero. It follows from equations (4.53) and (4.54) that

$$\begin{aligned} E &= \frac{V_d\, T^d\, h}{\prod_b (1-\Omega_b^2)} \left[\sum_a \frac{2\Omega_a^2}{1-\Omega_a^2} + d - 1 \right], \\ S &= \frac{V_d\, T^{d-1} h d}{\prod_b (1-\Omega_b^2)}, \\ L_a &= \frac{V_d\, T^d\, h}{\prod_b (1-\Omega_b^2)} \left[\frac{2\Omega_a}{1-\Omega_a^2} \right], \\ R_i &= 0\,. \end{aligned} \tag{4.56}$$

The partition function is given by

$$\ln \mathcal{Z}_{\rm gc} = \frac{V_d\, T^{d-1}\, h}{\prod_b (1-\Omega_b^2)}\,. \tag{4.57}$$

The stress tensor becomes

$$\begin{gathered} T^{tt} = hT^d (d\gamma^{d+2} - \gamma^d) \qquad T^{t\phi_a} = T^{\phi_a t} = hT^d d\gamma^{d+2}\Omega_a \\ T^{\theta_a\theta_a} = hT^d\gamma^d \left(\prod_{b=1}^{a-1} \sec^2\theta_b \right) \\ T^{\phi_a\phi_a} = hT^d \left[d\gamma^{d+2}\Omega_a^2 + \gamma^d \left(\prod_{b=1}^{a-1} \sec^2\theta_b \right) \csc^2\theta_a \right] \qquad T^{\phi_a\phi_b} = hT^d d\gamma^{d+2}\Omega_a\Omega_b\,. \end{gathered} \tag{4.58}$$

The mean free path in fluid dynamics can be estimated by taking the ratio of shear viscosity to energy density. As mentioned in the introduction, for fluids with gravity duals we can equivalently estimate $l_{\rm mfp}$ by taking the ratio of entropy to 4π times the energy (because of the universal relation $s = 4\pi\eta$).

$$l_{\rm mfp}(T,\nu)|_{\Omega=0} \sim \left[\frac{S}{4\pi E} \right]_{\Omega=0} = \frac{d}{4\pi T(d-1)}. \tag{4.59}$$

Consequently the expansion in $l_{\rm mfp}$ translates simply to an expansion in inverse powers of the temperature of our solutions.

4.5.2 Thermodynamics from black holes

The most general solution for uncharged rotating black holes in AdS_D was obtained in [16, 53]. These solutions are labelled by the $n+1$ parameters[50] a_i and r_+ (these are related to the n

[50] Recall that n denotes the number of commuting angular momenta and is given by the expression $n = \mathrm{rank}\,[SO(D-1)]$ on AdS_D.

angular velocities and the horizon radius (or equivalently the mass parameter) of the black holes). The surface gravity κ and the horizon area A of these black holes are given by[51]

$$\begin{aligned} \kappa &= \begin{cases} r_+(1+r_+^2)\displaystyle\sum_{i=1}^n \frac{1}{r_+^2+a_i^2} - \frac{1}{r_+} & \text{when } D=2n+1\,, \\ r_+(1+r_+^2)\displaystyle\sum_{i=1}^n \frac{1}{r_+^2+a_i^2} - \frac{1-r_+^2}{2r_+} & \text{when } D=2n+2\,, \end{cases} \\ A &= \begin{cases} \dfrac{V_d}{r_+}\displaystyle\prod_{i=1}^n \frac{r_+^2+a_i^2}{1-a_i^2} & \text{when } D=2n+1\,, \\ V_d\displaystyle\prod_{i=1}^n \frac{r_+^2+a_i^2}{1-a_i^2} & \text{when } D=2n+2\,. \end{cases} \end{aligned} \tag{4.60}$$

We will be interested in these formulae in the limit of large r_+. In this limit the parameter m (which appears in the formulae of [16, 53]) and the temperature $T=\kappa/2\pi$ are given as functions of r_+ by

$$\begin{aligned} T &= \left[\frac{(D-1)r_+}{4\pi}\right]\left(1+\mathcal{O}(1/r_+^2)\right), \\ 2m &= r_+^{D-1}\left(1+\mathcal{O}(1/r_+^2)\right). \end{aligned} \tag{4.61}$$

From these equations, it follows that the parameter m is related to the temperature T as

$$2m = T^{D-1}\left[\frac{4\pi}{D-1}\right]^{D-1}\left(1+\mathcal{O}(1/T^2)\right). \tag{4.62}$$

To leading order in r_+, the thermodynamic formulae take the form

$$\begin{aligned} \Omega_i &= a_i\,, \\ E &= \frac{V_{D-1}T^{D-1}}{16\pi G_D \prod_{j=1}^n(1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1}\left[\sum_{i=1}^n \frac{2a_i^2}{1-a_i^2} + D - 2\right], \\ L_i &= \frac{V_{D-1}T^{D-1}}{16\pi G_D \prod_{j=1}^n(1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1}\left[\frac{2a_i}{1-a_i^2}\right], \\ S &= \frac{V_{D-1}T^{D-2}(D-1)}{16\pi G_D \prod_{j=1}^n(1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1}, \\ R_i &= 0\,, \end{aligned} \tag{4.63}$$

where V_{D-1} is the volume of S^{D-2} and G_D is Newton's constant in D dimensions. The corrections to each of these expressions are suppressed by factors of $\mathcal{O}(1/r_+^2) = \mathcal{O}(1/T^2)$

[51]In the expression of κ for even dimension, the sign inside the second term in equation (4.7) of [53] is different form the sign given in equation (4.18) of [16]; we believe the latter sign is the correct one.

relative to the leading order results presented above (i.e. there are no next to leading order corrections).

These thermodynamic formulae listed in (4.63) are in perfect agreement with the fluid mechanics expressions in (4.56) upon making the following identifications: the spacetime dimensions of the boundary theory $d = D-1$, the black hole angular velocities a_i are identified with Ω_a and the constant h is identified as

$$h = \frac{1}{16\pi G_D}\left[\frac{4\pi}{D-1}\right]^{D-1} . \tag{4.64}$$

In the next subsection, we will see that this agreement goes beyond the global thermodynamic quantities. Local conserved currents are also in perfect agreement with the black hole physics.

4.5.3 Stress tensor from rotating black holes in AdS$_D$

The uncharged rotating black holes both in odd dimensions ($D = 2n+1$) and even dimensions ($D = 2n+2$) are presented in detail in [16], equation (E-3) and [53], equation (4.2) . After performing some coordinate transformations (see appendix 4.9.3) that take the metric of that paper to the standard form of AdS$_D$ at the boundary, we have computed the stress tensor of this solution.

Our calculation, presented in appendix 4.9.3 uses the standard AdS/CFT dictionary. In more detail, we foliate the solution in boundary spheres, compute the extrinsic curvature Θ^μ_ν of these foliations near the boundary, subtract off the appropriate counter terms contributions [42–49], and finally multiply the answer by the r^{D-1} to obtain the stress tensor on a unit sphere.

We find that the stress tensor so calculated takes the form (see appendix 4.9.3)

$$\begin{aligned}
\Pi^{tt} &= \frac{2m}{16\pi G_D}[(D-1)\gamma^{D+1} - \gamma^{D-1}] \\
\Pi^{\phi_a\phi_a} &= \frac{2m}{16\pi G_D}[(D-1)\gamma^{D+1}\omega_a^2 + \gamma^{D-1}\mu_a^{-2}] \\
\Pi^{t\phi_a} = \Pi^{\phi_a t} &= \frac{2m}{16\pi G_D}(D-1)\gamma^{D+1}\omega_a \\
\Pi^{\phi_a\phi_b} = \Pi^{\phi_b\phi_a} &= \frac{2m}{16\pi G_D}\gamma^{D+1}\omega_a\omega_b \\
\Pi^{\theta_a\theta_a} &= \frac{2m}{16\pi G_D}\gamma^{D-1}\left(\prod_{b=1}^{a-1}\sec^2\theta_b\right) .
\end{aligned} \tag{4.65}$$

Here $\gamma^{-2} = 1 - \sum_{a=1}^{n}\omega_a^2\mu_a^2$ where $\mu_a = \left(\prod_{b=1}^{a-1}\cos\theta_b\right)\sin\theta_a$.

Note that the functional form of these expressions (i.e. dependence of various components of the stress tensor on the coordinates of the sphere) agrees exactly with the predictions of fluid dynamics even at finite values of r_+. In the large r_+ limit (using (4.62) and (4.64)) , we further have

$$\Omega_a = \omega_a \,,$$
$$D-1 = d \,,$$
$$\frac{2m}{16\pi G_D} = T^d h \,.$$

With these identifications, (4.65) coming from gravity agrees precisely with (4.58) from fluid mechanics.

We proceed now to estimate the limits of validity of our analysis above. From the black hole side, since we have expanded the formulae of black hole thermodynamics in $1/r_+$ to match them with fluid mechanics, this analysis is valid if r_+ is large. From the fluid mechanics side, we expect corrections of the order of $l_{\rm mfp}$. To estimate $l_{\rm mfp}$ in this case, we substitute (4.61) into (4.59) to get

$$l_{\rm mfp} \sim \frac{1}{r_+(d-1)} \ll 1 \,.$$

Hence, we see that the condition from fluid mechanics is exactly the same as taking large horizon radius limit: the expansion of black hole thermodynamics in a power series in $\frac{1}{r_+}$ appears to be exactly dual to the fluid mechanical expansion as a power series in $l_{\rm mfp}$.

4.6 Comparison with black holes in $AdS_5 \times S^5$

Large N, $\mathcal{N}=4$ Yang-Mills, at strong 't Hooft coupling on $S^3 \times \mathbb{R}$, is dual to classical gravity on $AdS_5 \times S^5$. Hence, we can specialise the general fluid dynamical analysis presented above to the study of equilibrium configurations of the rotating $\mathcal{N}=4$ plasma on S^3 and then compare the results with the physics of classical black holes in $AdS_5 \times S^5$.

Large black holes in $AdS_5 \times S^5$ are expected to appear in a six parameter family, labelled by three SO(6) Cartan charges ($c=3$), two $SO(4)$ rotations ($n=2$) and the mass. While the most general black hole in $AdS_5 \times S^5$ has not yet been constructed, several sub-families of these black holes have been determined.

In this section, we will compare the thermodynamic predictions of fluid mechanics with all black hole solutions that we are aware of and demonstrate that the two descriptions agree in the large horizon radius limit. For one class of black holes we will also compare black hole stress tensor and charge distributions with that of the fluid mechanics and once again find perfect agreement (in the appropriate limit).

We begin this section with a review of the predictions of fluid mechanics for strongly coupled $\mathcal{N}=4$ Yang-Mills on S^3. Note that this is a special case of the conformal fluid dynamics of previous sections with $d = D-1 = 4$.

4.6.1 The strongly coupled $\mathcal{N}=4$ Yang-Mills Plasma

The gravity solution for SO(6) charged black branes (or, equivalently, large SO(6) charged but non-rotating black holes in $AdS_5 \times S^5$) has been used to extract the equation of state of $\mathcal{N} = 4$ Yang-Mills (see [64, §2] for the thermodynamic expressions in the infinite radius limit).

Rather than listing all the thermodynamic variables, we use the earlier parametrisation of (4.14) to state our results. The thermodynamics of the $\mathcal{N} = 4$ Yang-Mills is described by the following equations[52]

$$\begin{aligned}
h(\nu) = \frac{\mathcal{P}}{\mathcal{T}^4} &= 2\pi^2 N^2 \frac{\prod_j (1+\kappa_j)^3}{(2 + \sum_j \kappa_j - \prod_j \kappa_j)^4}\,, \\
\nu_i = \frac{\mu_i}{\mathcal{T}} &= \frac{2\pi \prod_j (1+\kappa_j)}{\left(2 + \sum_j \kappa_j - \prod_j \kappa_j\right)} \frac{\sqrt{\kappa_i}}{1+\kappa_i}\,, \\
h_i(\nu) = \frac{r_i}{\mathcal{T}^3} &= \frac{2\pi N^2 \prod_j (1+\kappa_j)^2}{\left(2 + \sum_j \kappa_j - \prod_j \kappa_j\right)^3} \sqrt{\kappa_i}\,.
\end{aligned} \tag{4.66}$$

where the auxiliary parameters κ_i have a direct physical interpretation in terms of entropy and charge densities (see §2 of [64]) -

$$\kappa_i = \frac{4\pi^2 R_i^2}{S^2}\,. \tag{4.67}$$

κ_i are constrained by $\kappa_i \geq 0$ and by the condition[53]

$$\frac{2 + \sum_j \kappa_j - \prod_j \kappa_j}{\prod_j (1+\kappa_j)} = \left[\sum_j \frac{1}{1+\kappa_j} - 1\right] \geq 0.$$

It follows from (4.67) that κ_i is finite for configurations with finite charge and non-zero entropy. The configurations with $\kappa_i \to \infty$ (for any i) are thermodynamically singular, since in this limit, the i^{th} charge density is much larger than the entropy density. Hence, in the following, we shall demand that κ_i be finite.

The general analysis presented before now allows us to construct the most general stationary solution of the $\mathcal{N} = 4$ fluid rotating on a 3-sphere. The thermodynamic formulae and currents of these solutions follow from (4.35), (4.34) and (4.37) upon setting

$$\begin{aligned}
A = h(\nu) &= 2\pi^2 N^2 \frac{\prod_j (1+\kappa_j)^3}{(2 + \sum_j \kappa_j - \prod_j \kappa_j)^4}\,, \\
B = 4h(\nu) - \nu_i h_i(\nu) &= 4\pi^2 N^2 \frac{\prod_j (1+\kappa_j)^2}{(2 + \sum_j \kappa_j - \prod_j \kappa_j)^3}\,, \\
C_i = h_i(\nu) &= 2\pi N^2 \sqrt{\kappa_i} \frac{\prod_j (1+\kappa_j)^2}{(2 + \sum_j \kappa_j - \prod_j \kappa_j)^3}\,,
\end{aligned} \tag{4.68}$$

[52] Note that our convention for the gauge field differs from [64, §2] by a factor of $\sqrt{2}$.

[53] Which is obtained by requiring that the temperature $\mathcal{T} \geq 0$ in the expression for $\mathcal{T}$ in [64, §2].

which leads to

$$\zeta_i = \frac{2\pi T \prod_j (1+\kappa_j)}{\left(2 + \sum_j \kappa_j - \prod_j \kappa_j\right)} \frac{\sqrt{\kappa_i}}{1+\kappa_i}\,, \tag{4.69}$$

and

$$\ln \mathcal{Z}_{\rm gc} = \frac{2\pi^2 N^2 V_4 T^3 \prod_j (1+\kappa_j)^3}{(1-\Omega_1^2)(1-\Omega_2^2)\left(2 + \sum_j \kappa_j - \prod_j \kappa_j\right)^4}\,, \tag{4.70}$$

where we have used the notation $V_4 = \mathrm{Vol}(S^3) = 2\pi^2$ as before.

As before, the mean free path in fluid mechanics can be estimated as

$$\begin{aligned} l_{\rm mfp} \sim \left[\frac{S}{4\pi E}\right]_{\Omega=0} = \frac{B}{(d-1)4\pi T A} &= \frac{\left(2 + \sum_j \kappa_j - \prod_j \kappa_j\right)}{6\pi T \prod_j (1+\kappa_j)} \\ &= \frac{1}{6\pi T}\left[\sum_j \frac{1}{1+\kappa_j} - 1\right]. \end{aligned} \tag{4.71}$$

4.6.2 The extremal limit

The strongly coupled $\mathcal{N}=4$ Yang-Mills plasma has an interesting feature; it has interesting and nontrivial thermodynamics even at zero temperature. In this subsection, we investigate this feature and point out that it implies the existence of interesting zero temperature solutions of fluid dynamics which will turn out to be dual to large, extremal black holes.

Thermodynamics In the above section, we presented the thermodynamics of strongly coupled $\mathcal{N}=4$ Yang-Mills plasma in terms of the parameters κ_i. These parameters are constrained by the conditions $\kappa_i \geq 0$ and $\sum_i \frac{1}{1+\kappa_i} \geq 1$ with κ_i finite. In order to visualise the allowed range over which the variables κ_i's can vary, it is convenient to define a new set of variables

$$\begin{aligned} X_i &= \frac{1}{1+\kappa_i}\,, \quad X_i = X,\ Y,\ Z, \\ \chi &= \frac{T}{X+Y+Z-1}\,. \end{aligned} \tag{4.72}$$

The constraints $\kappa_i \geq 0$ and $\sum_i \frac{1}{1+\kappa_i} \geq 1$ with κ_i finite translate into the constraints $0 < X_i \leq 1$ and $X+Y+Z \geq 1$. Geometrically, this is just the statement that X_i's can lie anywhere inside the cube shown in fig.1, away from the planes $X_i = 0$ and on or above the plane $X+Y+Z=1$.

The energy density, the entropy density and the charge densities of the Yang-Mills plasma may be rewritten as a function of X, Y, Z and χ as

$$\begin{aligned} \rho &= 6\pi^2 N^2 XYZ\chi^4, \qquad s = 4\pi^2 N^2 XYZ\chi^3, \\ r_i &= 2\pi N^2 XYZ\chi^3 \sqrt{\frac{1-X_i}{X_i}}. \end{aligned} \tag{4.73}$$

The condition for the validity of fluid mechanics becomes

$$l_{\rm mfp} \sim \frac{1}{6\pi\chi} \ll 1 \qquad \text{or} \qquad \chi \gg 1\,. \tag{4.74}$$

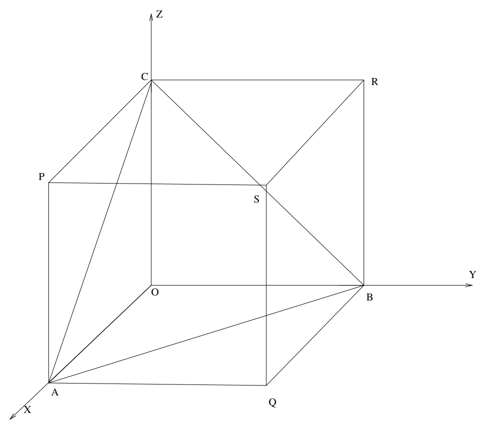

Figure 1. The space of allowed κ_i's. The axes correspond to $X = \frac{1}{1+\kappa_1}$, $Y = \frac{1}{1+\kappa_2}$ and $Z = \frac{1}{1+\kappa_3}$. The X_i's can lie anywhere in the cube outside the "extremal" plane $X + Y + Z = 1$.

Consider now the case in which χ is large, but finite and X, Y, Z take values close to the interior of the triangle ABC in fig.1. From (4.72) and (4.73), it is evident that this is equivalent to taking an extremal limit $T \to 0$ with appropriate chemical potentials. All thermodynamic quantities listed above are smooth in this limit and the fluid mechanics continues to be valid.

The $\mathcal{N} = 4$ Yang-Mills plasma with three nonzero R-charges always has a nonsingular extremal limit. In the case that one of the charges say r_3 is zero, then we are constrained to move on the $X_3 = 1$ plane in the space of X_i's. Hence, we can never approach the 'extremal triangle' $X + Y + Z = 1$.[54] Thus, we have no nonsingular extremal limit if any one of the three R-charges is zero. By a similar argument, no nonsingular extremal limit exists if two of the R-charges were zero.

We note that Gubser and Mitra have previously observed that charged black branes near extremality are sometimes thermodynamically unstable [65]. Although we have not performed a careful analysis of the thermodynamic stability of the charged fluids we study in this chapter (see however [64]), we suspect that these fluids all have Gubser-Mitra type thermodynamic instabilities near extremality. If this is the case, the near extremal fluid solutions we study in this section and the next – and the black holes that these are dual to – are presumably unstable to small fluctuations. Whether stable or not, these configurations are valid solutions of fluid dynamics. We postpone a serious discussion of stability to future work.[55]

[54] Remember that we have already excluded, on physical grounds, the point $X_1 = X_2 = 0,\ X_3 = 1$ which lies in the intersection of $X_3 = 1$ plane and the extremal plane $X + Y + Z = 1$.

[55] We thank Sangmin Lee for discussion of these issues.

Fluid mechanics The thermodynamic expressions for the charges of a rotating Yang-Mills plasma take the form

$$\begin{aligned}
E &= \frac{2\pi^2 N^2 XYZV_4}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{2\omega_1^2}{1-\omega_1^2}+\frac{2\omega_2^2}{1-\omega_2^2}+3\right]\left[\frac{T}{X+Y+Z-1}\right]^4,\\
L_1 &= \frac{2\pi^2 N^2 XYZV_4}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{2\omega_1}{1-\omega_1^2}\right]\left[\frac{T}{X+Y+Z-1}\right]^4,\\
L_2 &= \frac{V_4 \tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{2\omega_2}{1-\omega_2^2}\right]\left[\frac{T}{X+Y+Z-1}\right]^4,\\
S &= \frac{4\pi^2 N^2 XYZV_4}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{T}{X+Y+Z-1}\right]^3,\\
R_i &= \frac{2\pi N^2 XYZV_4}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{T}{X+Y+Z-1}\right]^3\sqrt{\frac{1-X_i}{X_i}},
\end{aligned} \tag{4.75}$$

and the mean free path

$$l_{\mathrm{mfp}} \sim \frac{X+Y+Z-1}{6\pi T} \ll 1\,. \tag{4.76}$$

We see that all thermodynamical charges of our rotating fluid configurations are nonsingular, and that fluid mechanics is a valid approximation for these solutions, in the extremal limit described in the previous subsection, provided only that $\chi \gg 1$.[56]

The solution so obtained describes a rotating fluid whose local temperature vanishes everywhere, but whose rest frame charge density is a function of location on the S^3 (it scales like γ^3). As we will see below these extremal configurations of rotating fluid on S^3 are exactly dual to large, rotating, extremal black holes in AdS$_5$.

4.6.3 Predictions from fluid mechanics in special cases

As mentioned in the beginning of this section, the most general black hole in $AdS_5 \times S^5$ has not yet been constructed, but several subfamilies of these black holes are known. To facilitate the comparison between fluid mechanics on S^3 on one hand and these subfamilies of black holes on the other, in this subsection, we specialise the general predictions of the previous subsection to various specific cases.

All SO(6) charges equal: arbitrary angular velocities Consider first the case of a fluid with equal SO(6) charges (with the rotational parameters arbitrary). That is we set $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$ in the general formulae above. Noting that $(2+3\kappa-\kappa^3) = (\kappa+1)^2(2-\kappa)$

[56]In greater generality, in order for fluid mechanics to be a valid approximation for our solutions it is necessary that either $T \gg 1$ (which is by itself sufficient) or that $X+Y+Z-1 \to 0$ (under which condition the ratio χ of the previous section must be large and (conservatively) none of X, Y or Z be very small).

we find that the stress tensor and currents are given by (4.34) and (4.35) with

$$\begin{aligned} A &= \frac{2\pi^2 N^2 (1+\kappa)}{(2-\kappa)^4}\,, \\ B &= \frac{4\pi^2 N^2}{(2-\kappa)^3}\,, \\ C_i &= \frac{2\pi N^2\sqrt{\kappa}}{(2-\kappa)^3}\,. \end{aligned} \tag{4.77}$$

The thermodynamics can be summarised by

$$\zeta_i = \frac{2\pi T\sqrt{\kappa}}{(2-\kappa)}\,, \qquad \ln \mathcal{Z}_{\rm gc}(T,\Omega,\zeta) = \frac{2\pi^2 N^2 V_4 T^3 (1+\kappa)}{(1-\Omega_1^2)(1-\Omega_2^2)(2-\kappa)^4}\,. \tag{4.78}$$

The formula for mean free path (4.71) reduces to

$$l_{\rm mfp} \sim \frac{1}{6\pi T}\left[\frac{2-\kappa}{1+\kappa}\right]. \tag{4.79}$$

Let us specialise the extremal thermodynamics of $\mathcal{N} = 4$ Yang-Mills fluid presented before to this case. In terms of the variables introduced in §§4.6.2, we have $X = Y = Z$ which is a straight line in the X_i space. The extremal limit is obtained when this line cuts the extremal plane $X + Y + Z = 1$, i.e, at the point $X = Y = Z = 1/3$. This corresponds to the extremal limit $\kappa \to 2$.

More explicitly, in the extremal limit

$$T \to 0\,, \qquad (2-\kappa) = \frac{T}{\chi}\,, \tag{4.80}$$

with χ large but finite. The thermodynamic quantities obtained by differentiating the grand partition function (4.93),

$$\begin{aligned} S &= \frac{2N^2\pi^2 V_4 T^3}{(2-\kappa)^3}\frac{1}{(1-a^2)(1-b^2)} \\ L_1 &= \frac{4N^2\pi^2 V_4 T^4(1+\kappa)}{(2-\kappa)^4}\frac{a}{(1-a^2)^2(1-b^2)} \\ L_2 &= \frac{4N^2\pi^2 V_4 T^4(1+\kappa)}{(2-\kappa)^4}\frac{b}{(1-a^2)(1-b^2)^2} \\ R &= \frac{2\pi N^2 V_4 T^3\sqrt{\kappa}}{(2-\kappa)^3}\frac{1}{(1-a^2)(1-b^2)} \\ E &= \frac{2\pi^2 N^2 V_4 T^4(1+\kappa)}{(2-\kappa)^4}\frac{[4-(1+a^2)(1+b^2)]}{(1-a^2)^2(1-b^2)^2}\,, \end{aligned} \tag{4.81}$$

are all smooth; and they describe a fluid configuration whose energy, angular momentum, charge and entropy scale as $N^2\chi^4$, $N^2\chi^4$, $N^2\chi^3$ and $N^2\chi^3$ respectively.

Independent SO(6) charges: equal rotations Consider the special case $\omega_1 = \omega_2 = \Omega$ (the three SO(6)) chemical potentials are left arbitrary). The stress tensor and currents are given by (4.34) and (4.35) with

$$\begin{aligned} A &= 2\pi^2 N^2 \frac{\prod_j (1+\kappa_j)^3}{(2+\sum_j \kappa_j - \prod_j \kappa_j)^4}\,, \\ B &= 4\pi^2 N^2 \frac{\prod_j (1+\kappa_j)^2}{(2+\sum_j \kappa_j - \prod_j \kappa_j)^3}\,, \\ C_i &= 2\pi N^2 \sqrt{\kappa_i} \frac{\prod_j (1+\kappa_j)^2}{(2+\sum_j \kappa_j - \prod_j \kappa_j)^3}\,, \\ \gamma &= \frac{1}{\sqrt{1-\Omega^2}}\,, \qquad v = \Omega\,. \end{aligned} \tag{4.82}$$

The thermodynamics can be summarised by

$$\zeta_i = \frac{2\pi T \prod_j (1+\kappa_j)}{\left(2+\sum_j \kappa_j - \prod_j \kappa_j\right)} \frac{\sqrt{\kappa_i}}{1+\kappa_i}\,, \qquad \ln \mathcal{Z}_{\text{gc}} = \frac{2\pi^2 N^2 V_4 T^3 \prod_j (1+\kappa_j)^3}{(1-\Omega^2)^2 \left(2+\sum_j \kappa_j - \prod_j \kappa_j\right)^4}\,. \tag{4.83}$$

The expression for mean free path (4.71) reduces to

$$l_{\text{mfp}} \sim \frac{1}{6\pi T} \left[\sum_j \frac{1}{1+\kappa_j} - 1 \right]. \tag{4.84}$$

The extremal limit $\sum_j (1+\kappa_j)^{-1} \to 1$ with all κ_i kept finite, is nonsingular, and yields solutions that are well described by fluid dynamics when l_{mfp} is small.

Two equal nonzero SO(6) charges: arbitrary angular velocities Consider now the case when $\kappa_1 = \kappa_2 = \kappa$, $\kappa_3 = 0$. We find that the stress tensor and currents are given by (4.34) and (4.35) with

$$\begin{aligned} A &= \frac{\pi^2 N^2 (1+\kappa)^2}{8}\,, \\ B &= \frac{\pi^2 N^2 (1+\kappa)}{2}\,, \\ C_1 = C_2 &= \frac{\pi N^2 \sqrt{\kappa}(1+\kappa)}{4}\,, \\ C_3 &= 0\,. \end{aligned} \tag{4.85}$$

The thermodynamics can be summarised by

$$\zeta_1 = \zeta_2 = \pi T \sqrt{\kappa}\,, \qquad \ln \mathcal{Z}_{\text{gc}}(T, \Omega, \zeta) = \frac{\pi^2 N^2 V_4 T^3 (1+\kappa)^2}{8(1-\Omega_1^2)(1-\Omega_2^2)}\,. \tag{4.86}$$

The expression for mean free path, from (4.71), is

$$l_{\text{mfp}} \sim \frac{1}{3\pi T (1+\kappa)}\,. \tag{4.87}$$

It follows from (4.87) that fluid mechanics is a good approximation when T is large. Though this equation would appear to suggest that the fluid dynamical approximation is also valid (for instance) at fixed T and large κ, we have emphasised before, the limit of large κ is thermodynamically suspect. Conservatively, thus, fluid mechanics applies only at large temperatures.

A single nonzero charge: arbitrary angular velocities We now set $\kappa_1 = \kappa$, $\kappa_2 = \kappa_3 = 0$ leaving angular velocities arbitrary. The stress tensor and currents are given by (4.34) and (4.35) with

$$\begin{aligned} A &= \frac{2\pi^2 N^2 (1+\kappa)^3}{(2+\kappa)^4}\,, \\ B &= \frac{4\pi^2 N^2 (1+\kappa)^2}{(2+\kappa)^3}\,, \\ C_1 &= \frac{2\pi N^2 \sqrt{\kappa}(1+\kappa)^2}{(2+\kappa)^3}\,, \\ C_2 = C_3 &= 0\,. \end{aligned} \tag{4.88}$$

The thermodynamics can be summarised by

$$\zeta = \frac{2\pi T \sqrt{\kappa}}{(2+\kappa)}\,, \qquad \ln \mathcal{Z}_{\rm gc}(T,\Omega,\zeta) = \frac{2\pi^2 N^2 V_4 T^3 (1+\kappa)^3}{(1-\Omega_1^2)(1-\Omega_2^2)(2+\kappa)^4}\,. \tag{4.89}$$

The mean free path, from (4.71) is given by

$$l_{\rm mfp} \sim \frac{1}{6\pi T}\left[\frac{2+\kappa}{1+\kappa}\right]. \tag{4.90}$$

As in the previous subsection, this particular case does not admit thermodynamically nonsingular zero temperature (or extremal) configurations.

4.6.4 Black holes with all R-charges equal

Having derived the fluid mechanics predictions for various different black holes, we now proceed to examine the black hole solutions. First, we will focus on the case of black holes with arbitrary angular momenta in AdS_5 but equal SO(6) charges. The relevant solution has been presented in [19].

Thermodynamics The black holes presented in [19] are labelled by two angular velocities a, b, and three more parameters q, m and r_+. These five parameters are not all independent; they are constrained by one equation relating horizon radius to the parameter m ($\Delta_r = 0$ in that paper). We thus have a four parameter set of black holes.[57]

[57]We work in conventions in which the AdS radius and hence the parameter g of [19] is set to unity.

The relatively complicated black hole thermodynamic formulae of [19] simplify if the parameter r_+ (which may be interpreted as the horizon radius) is taken to be large. In particular, consider the limit

$$r_+ \gg 1 \qquad \text{and} \quad y = q/r_+^3 \quad \text{fixed.} \tag{4.91}$$

In this limit, to leading order,we have

$$\begin{aligned} T &= \frac{r_+}{2\pi}(2 - y^2)\,, \\ 2m &= r_+^4(1 + y^2)\,. \end{aligned} \tag{4.92}$$

From the positivity of T and r_+ it follows immediately that $0 \leq y^2 \leq 2$.

Multiplying all thermodynamic integrals in [19] by $\frac{R_{\text{AdS}}^3}{G_5} = \frac{2N^2}{\pi}$ and noting that our charge R is equal to their $Q/\sqrt{3}$, the black hole thermodynamic formulae reduce to (to leading order in r_+)

$$\begin{aligned} \Omega_1 &= a\,, \\ \Omega_2 &= b\,, \\ \zeta_i &= \frac{2\pi y T}{(2 - y^2)}\,, \\ \ln \mathcal{Z}_{\text{gc}} &= \frac{2\pi^2 N^2 (1 + y^2)}{(2 - y^2)^4} \left[\frac{V_4 T^3}{(1 - \Omega_1^2)(1 - \Omega_2^2)} \right]. \end{aligned} \tag{4.93}$$

Once we identify the black hole parameter y^2 with the fluid parameter κ, these formula take precisely the form of fluid mechanics formulae (4.78) with the equation of state coming from (4.77).[58]

We can now compute the fluid mechanical mean free path l_{mfp} as a function of bulk black hole parameters. From equations (4.92) and (4.79), we find (assuming that r_+ is large)

$$l_{\text{mfp}} \sim \frac{1}{3r_+(1 + \kappa)}\,.$$

As $1+\kappa = 1+y^2$ is bounded between 1 and 2, it appears from this equation that the expansion in powers of $1/r_+$ is simply identical to the fluid dynamical expansion in powers of l_{mfp}. This explains why black hole thermodynamics agrees with the predictions of the Navier-Stokes equations when (and only when) r_+ is large.

---

[58] The functions $h(\nu)$ and its derivatives have simple expressions as functions of bulk parameters. Comparing with (4.64) we find

$$\begin{aligned} h(\nu) &= \frac{2N^2}{\pi} \left[\frac{2m}{16\pi T^4} \right] = \frac{m}{8\pi G_5 T^4} \\ h_i(\nu) &= \frac{2N^2}{\pi} \frac{q}{8\pi T^3} = \frac{q}{8\pi G_5 T^3} \end{aligned} \tag{4.94}$$

Stress tensor and charge currents In appendix 4.9.3, we have computed the boundary stress tensor corresponding to this black hole solution (by foliating the space into S^3 's at infinity, computing the extrinsic curvature of these sections, and subtracting the appropriate counterterms). At leading order in $\frac{1}{r_+}$

$$\begin{aligned}
\Pi^{tt} &= \frac{m}{8\pi G_5}\gamma^4(4\gamma^2-1) \\
\Pi^{\phi\phi} &= \frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2a^2+\frac{1}{\sin^2\theta}\right) \\
\Pi^{\psi\psi} &= \frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2a^2+\frac{1}{\cos^2\theta}\right) \\
\Pi^{t\phi} &= \Pi^{\phi t} = \frac{4m}{8\pi G_5}a\gamma^6 \\
\Pi^{t\psi} &= \Pi^{\psi t} = \frac{4m}{8\pi G_5}b\gamma^6 \\
\Pi^{\phi\psi} &= \Pi^{\psi\phi} = \frac{4m}{8\pi G_5}ab\gamma^6 \\
\Pi^{\theta\theta} &= \frac{m}{8\pi G_5}\gamma^4.
\end{aligned} \tag{4.95}$$

In a similar fashion, the charge currents on S^3 may be computed from $J_i^\mu = -r^4 g^{\mu\nu}A_\nu|_{r\to\infty}$ where the indices μ,ν are tangent to the $S^3\times$ time foliations and and the bulk gauge field A_ν is given in the equation (2) of [19]. We find

$$\begin{aligned}
J_1^t &= J_2^t = J_3^t = \frac{q}{8\pi G_5}\gamma^4 \\
J_1^\theta &= J_2^\theta = J_3^\theta = 0 \\
J_1^\phi &= J_2^\phi = J_3^\phi = \frac{q}{8\pi G_5}\gamma^4 a \\
J_1^\psi &= J_2^\psi = J_3^\psi = \frac{q}{8\pi G_5}\gamma^4 b\,.
\end{aligned} \tag{4.96}$$

Using (4.94), it is evident that the expressions in (4.96) are in precise agreement with the predictions (4.51) of fluid dynamics.

4.6.5 Black holes with independent SO(6) charges and two equal rotations

The most general (five parameter) black hole solutions with the two angular velocities set equal can be found in [54]. The thermodynamics of these black holes was computed in [59].

The black hole solutions depend on the parameters $\delta_1, \delta_2, \delta_3, a, m, r_+$ that are related by the equation $Y(r) = 0$. The thermodynamics of these black holes simplify in the limit

$$r_+ \gg 1\,, \qquad \frac{2ms_i^2}{r_+^2} = H_i - 1 \quad \text{fixed.}$$

Then solving the equation $Y = 0$ in this limit, one can express m as

$$2m = \frac{(H_1H_2H_3)r_+^4}{(1-a^2)}\,.$$

The various thermodynamic quantities in this limit[59] (after multiplying integrals by $\frac{R_{\rm AdS}^3}{G_5} = \frac{2N^2}{\pi}$) can be summarised by

$$
\begin{aligned}
\Omega_1 = \Omega_2 = a\,, \qquad T &= \frac{r_+\sqrt{1-a^2}}{2\pi}\left(\sum_j H_j^{-1} - 1\right)\prod_j \sqrt{H_j}\,, \\
\zeta_i = r_+\sqrt{1-a^2}\left(\frac{\sqrt{H_i-1}}{H_i}\right)\prod_j \sqrt{H_j} &= \frac{2\pi T}{\sum_j H_j^{-1}-1}\left(\frac{\sqrt{H_i-1}}{H_i}\right)\,, \\
\ln \mathcal{Z}_{\rm gc} = \frac{\pi N^2 r_+^3}{2\sqrt{1-a^2}}\left(\frac{\prod_j \sqrt{H_j}}{\sum_j H_j^{-1}-1}\right) &= \frac{4\pi^4 N^2 T^3}{(1-\Omega^2)^2\left(\prod_j H_j\right)\left(\sum_j H_j^{-1}-1\right)^4}
\end{aligned}
\tag{4.97}
$$

These expressions match with (4.83) if κ_i is identified with $H_i - 1$, demonstrating perfect agreement between black hole and fluid dynamical thermodynamics.

Translating the estimate for the mean free path into the black hole variables, we find

$$
l_{\rm mfp} \sim \frac{1}{3r_+\prod_j \sqrt{H_j}} \ll 1\,,
$$

(an equation that is valid only in the large r_+ limit). Notice that $l_{\rm mfp}$ is automatically small in the large r_+ limit, explaining why black hole thermodynamics agrees with the predictions of the Navier-Stokes equations in this limit.

Notice that the fluid mechanical expansion parameter $l_{\rm mfp}$ appears to differ from the expansion parameter of black hole thermodynamics used above, $1/r_+$, by a factor of $1/\sqrt{\prod_i H_i}$. When the three charges of the black hole are in any fixed ratio $a : b : c$, with none of a, b or c either zero or infinity, it may easily be verified that this additional factor is bounded between a nonzero number (which depends on a, b, c) and unity. In this case the two expansion parameters - $l_{\rm mfp}$ and $1/r_+$ - are essentially the same.

However when one of the black hole charges (say R_1) vanishes H_2 and/or H_3 can formally take arbitrarily large values. In this extreme limit $l_{\rm mfp}$ appears to differ significantly from the bulk expansion parameter $1/r_+$. However large H_i implies large κ_i, a limit that we have argued above to be thermodynamically singular. Keeping away from the suspicious large κ_i limit, it is always true that $l_{\rm mfp}$ is essentially identical $1/r_+$, the parameter in which we have expanded the formulas of black hole thermodynamics.

Finally we emphasise that the black hole studied in this subsection include a large class of perfectly nonsingular zero temperature or extremal black holes with finite κ_i and large r_+ which perfectly reproduce the predictions of extremal fluid mechanics of §§4.6.2.

[59] We believe that [59] has a typo: (3.10) should read $\Phi_i = \frac{2m}{r^2 H_i}(s_i c_i + \frac{1}{2}a\Omega(c_i s_j s_k - s_i c_j c_k))$. Note that they also use coordinates $\psi = \phi_1 + \phi_2$ and $\varphi = \phi_1 - \phi_2$ so that $\Omega\partial_\psi = \frac{\Omega}{2}\partial_{\phi_1} + \frac{\Omega}{2}\partial_{\phi_2}$ so that $\Omega_a = \frac{\Omega}{2}$.

In more detail, the thermodynamical quantities of a general solution in this subsection is given in terms of X, Y, Z (defined as in (4.72)) as

$$\begin{aligned}
S &= \frac{N^2 \pi r_+^3}{\sqrt{XYZ(1-a^2)}}\,, \\
E &= \frac{N^2 r_+^4 (3+a^2)}{4XYZ(1-a^2)}\,, \\
L &= \frac{N^2 r_+^4 a}{2XYZ(1-a^2)}\,, \\
R_i &= \frac{N^2 r_+^3}{2\sqrt{XYZ(1-a^2)}}\sqrt{\frac{1-X_i}{X_i}}\,, \\
\zeta_i &= r_+ \sqrt{\frac{X_i(1-X_i)(1-a^2)}{XYZ}}\,.
\end{aligned} \tag{4.98}$$

From these expressions, together with the formula for temperature in (4.97) it follows that the limit $X + Y + Z \to 1$ (with none of X, Y, Z zero) is extremal (the temperature goes to zero) and non-singular (all thermodynamic quantities are finite and well defined). Note that r_+ is an arbitrary parameter for these extremal black holes. When r_+ is large the fluid dynamical description is valid. The black holes so obtained are exactly dual to the extremal fluid configurations described in §§4.6.2.

4.6.6 Black holes with two equal large R-charges and third R-charge small

Chong et al. [55] have determined a class of black hole solutions with two SO(6) charges held equal, while the third charge is varied as a function of these two equal charges. In the large radius limit, it turns out that this third charge is negligible compared to the first two, so for our purposes these solutions can be thought of as black holes with two equal SO(6) charges, with arbitrary rotations and the third SO(6) charge set to zero. The parameters of this black hole solution are a, b, m, r_+, s, which are related by the equation $X(r_+) = 0$.

Black hole formulae simplify in the limit

$$r_+ \gg 1 \quad \text{and} \quad k = \frac{2ms^2}{r_+^2} \quad \text{fixed,}$$

in units where the inverse AdS radius $g = 1$, which leads to

$$2m = r_+^4 (1+k)^2\,.$$

Multiplying all thermodynamic integrals in [55] by $\frac{R_{\text{AdS}}^3}{G_5} = \frac{2N^2}{\pi}$, in this limit, the thermodynamics can be summarised by

$$\begin{aligned}
\Omega_1 = a\,, \quad \Omega_2 = b\,, \quad T = \frac{r_+}{\pi}\,, \\
\zeta_1 = \zeta_2 = \pi T \sqrt{k}\,, \qquad \zeta_3 \sim \mathcal{O}\left(\frac{1}{r_+^2}\right)\,, \\
\ln \mathcal{Z}_{\text{gc}} = \frac{\pi^2 N^2 V_4 T^3 (1+k)^2}{8(1-a^2)(1-b^2)}\,.
\end{aligned} \tag{4.99}$$

Note that ζ_3 and R_3 are subleading in r_+. These formulae are in perfect agreement with (4.86) if we identify

$$\kappa = k\,.$$

From the expression for the temperature, it follows that all extremal or zero temperature black holes have $r_+ = 0$. Consequently all extremal black holes (of the class of black holes described in this subsection) are singular, dual to the fact that the fluid mechanics has no thermodynamically nonsingular zero temperature solutions.

Translating the estimate for the fluid dynamical mean free path into the black hole variables we find (assuming $r_+ \gg 0$)

$$l_{\rm mfp} \sim \frac{1}{3r_+(1+\kappa)}\,.$$

It follows that the fluid dynamical expansion parameter is essentially the same as $1/r_+$, provided we stay away from the thermodynamically suspect parameter regime of large κ .

4.6.7 Black holes with two R-charges zero

The solution for the most general black hole with two R-charges set to zero relevant solution has was presented in [56]. The parameters of this black hole are x_0, m, δ, a, b related by $X(x_0) = 0$.

The thermodynamics of these black holes simplifies in the limit

$$x_0 \gg 1\,, \qquad y = \sqrt{x_0}\,\delta \quad \text{fixed,}$$

in units where $g = 1$, which leads to

$$2m = \frac{x_0^2}{(1-y^2)}\,.$$

This gives an upper bound on y: $y \le 1$.

Multiplying all thermodynamic integrals in [56] by $\frac{R_{\rm AdS}^3}{G_5} = \frac{2N^2}{\pi}$, in this limit, the thermodynamic formulae can be summarised by

$$\begin{aligned}
\Omega_1 &= a\,, \qquad\qquad \Omega_2 = b\,,\\
T &= \frac{\sqrt{x_0}(2-y^2)}{2\pi\sqrt{1-y^2}}\,, \qquad \zeta = \sqrt{x_0}\,y = \frac{2\pi T y\sqrt{1-y^2}}{2-y^2}\,,\\
\ln \mathcal{Z}_{\rm gc} &= \frac{x_0^{3/2}\pi N^2}{2\sqrt{1-y^2}(2-y^2)(1-a^2)(1-b^2)} = \frac{4\pi^4 N^2 T^3(1-y^2)}{(1-\Omega_1^2)(1-\Omega_2^2)(2-y^2)^4}
\end{aligned} \tag{4.100}$$

Upon identifying $\kappa = \frac{y^2}{1-y^2}$, we find perfect agreement with (4.89). Under this identification, the expression for temperature becomes

$$T = \frac{\sqrt{x_0}(2+\kappa)}{2\pi\sqrt{1+\kappa}}\,.$$

As in the subsection above, it follows immediately from this equation that the black hole temperature vanishes only for the singular black holes with $x_0 = 0$. This matches with the fact that there are no nonsingular extremal fluid dynamical solutions in this case.

The fluid dynamical mean free path may be evaluated as a function of bulk parameters as

$$l_{\rm mfp} \sim \frac{1}{3\sqrt{x_0(1+\kappa)}}\,.$$

Note that $l_{\rm mfp}$ is small whenever $\sqrt{x_0} = r_+$ is large, an observation that explains the agreement of black hole thermodynamics in the large r_+ limit with the Navier-Stokes equations. In more generality we see that $l_{\rm mfp}$ is essentially the same as $1/r_+$, provided we keep away from the thermodynamically suspicious parameter regime of κ large.

4.6.8 Extremality and the attractor mechanism

As discussed in the previous subsections, there exists a duality between extremal large rotating AdS black holes on one hand and the extremal configurations of the fluid dynamics on the other. This implies that the thermodynamic properties of these large rotating extremal black holes are completely determined by the corresponding properties of large static extremal black holes. As an application of this observation, let us recall the suggestion [66–68] that the attractor mechanism for black holes implies the non-renormalisation of the entropy of all extremal configurations, as a function of the 't Hooft coupling λ. It follows immediately from the fluid mechanical description at large charges, that were any such non-renormalisation theorem be proved for static extremal configurations, it would immediately imply a similar result for rotating extremal configurations.

4.6.9 BPS bound and supersymmetric black holes

All solutions of IIB supergravity on $AdS_5 \times S^5$, and all configurations of $\mathcal{N}=4$ Yang-Mills on S^3 obey the BPS bound

$$E \geq L_1 + L_2 + \sum_i R_i = L_1 + L_2 + 3R\,. \tag{4.101}$$

Within the validity of the fluid dynamical approximation, described in this chapter,

$$E - L_1 - L_2 = 2\pi^2 T^4 A \frac{3+\omega_1+\omega_2-\omega_1\omega_2}{(1+\omega_1)(1+\omega_2)}\,; \tag{4.102}$$

notice that the RHS of this equation is positive definite. The BPS bound is obeyed provided

$$\tau A \frac{3+\omega_1+\omega_2-\omega_1\omega_2}{(1+\omega_1)(1+\omega_2)} \geq C_i\,. \tag{4.103}$$

Plugging in the explicit expressions for A and C_i from (4.77), we find this condition is satisfied provided

$$r_+ = \frac{2\pi T}{2-\kappa} \geq \frac{\sqrt{\kappa}(1+\omega_1)(1+\omega_2)}{(1+\kappa)(3+\omega_1+\omega_2-\omega_1\omega_2)}\,. \tag{4.104}$$

The RHS of (4.104) is of order unity. It follows that (4.104) is saturated only when r_+ of unit order. It follows that when $r_+ \gg 1$ (so that fluid dynamics is a valid approximation) the BPS bound is always obeyed as a strict inequality. Supersymmetric black holes are never reliably described within fluid mechanics.[60] The extremal black holes with large horizon radius, that are well described by fluid mechanics[61] (see the previous subsection) are always far from supersymmetry.

We have noted above that a large class of extremal configurations in strongly interacting Yang-Mills – all those that admit a fluid dynamic description – are not BPS. This is in sharp contrast with the results of computations in free Yang-Mills theory, in which all extremal configurations are supersymmetric [69]. This difference is related to the fact, noted previously, the divergent mean free path prevents a fluid mechanical description from applying to free theories. A practical manifestation of this fact is that the function $h(\nu)$, which appears in the analysis of free Yang-Mills in equation (5.2) of [69], and plays the role of r_+ in our discussion here, is always of order unity for all allowed values of the chemical potential, and so can never become large.

4.6.10 Fluid dynamics versus black hole physics at next to leading order

As we have explained above, the formulae for all thermodynamic charges and potentials of black holes of temperature T and chemical potentials ν_i, in $AdS_5 \times S^5$, may be expanded as a Taylor series in $1/r_+ \sim l_{\rm mfp}(T, \nu_i)$. As we have verified above, for every known family of large AdS black holes, the leading order results in this expansion perfectly match the predictions of the Navier-Stokes equations. Higher order terms in this expansion represent corrections to Navier-Stokes equations. In this subsection we investigate the structure of these corrections.

Let us first investigate the case of black holes with at least one SO(6) charge set equal to zero (the black holes studied in §§4.6.6 and §§4.6.7). It is not difficult to verify that the first deviations from the large radius thermodynamics of these black holes occur at $\mathcal{O}(1/r_+^2) \sim l_{\rm mfp}^2$. This result is in perfect accord with naive expectations from fluid mechanics. As we have explained above, the fluid dynamical configurations presented in this chapter are exact solutions to the equations of fluid mechanics with all one derivative terms, i.e. to the first order in $l_{\rm mfp}$. In general we would expect our solutions (and their thermodynamics) to be modified at $\mathcal{O}(l_{\rm mfp}^2)$, exactly as we find from the black hole formulae.

However when we turn our attention to black holes with all three SO(6) charges nonzero we run into a bit of a surprise. It appears that the thermodynamics (and stress tensor and charge currents) of these black holes receives corrections at order $\mathcal{O}(1/r_+) \sim l_{\rm mfp}$. This result

[60] Although it is possible to make the energy of supersymmetric black holes parametrically larger than their entropy, this is achieved by scaling either ω_1 or ω_2 to unity with r_+ kept at unit order. It is easy to verify that in this limit the local, rest frame mean free path of the fluid is of unit order in regions of the S^3 and so fluid mechanics may not be used to describe these configurations.

[61] Note that the 'physical' radius $(\text{Area})^{1/3}$ of the black hole is distinct from the parameter r_+ which determines the validity of fluid dynamics. The physical radius can be made arbitrarily large, nevertheless fluid mechanics is only valid if r_+ is large.

is a surprise because, for the reason we have explained in the previous paragraph, we would have expected the first corrections to our fluid mechanical configuration to occur at $\mathcal{O}(l_{\rm mfp}^2)$.

We do not have a satisfactory resolution to this puzzle. In this subsection we will simply present the expressions for the first order corrections to black hole thermodynamics in a particular case (the case of black holes with all SO(6) charges equal), and leave the explanation of these formulae to future work.

As we have mentioned above, the thermodynamics of a charged rotating black hole in $AdS_5 \times S^5$ with three equal charges and two different angular momenta can be found in [19]. To calculate next to leading order (NLO) corrections to the thermodynamics of large black holes, we systematically expand the thermodynamic quantities.

We find it convenient to shift to a new parametrisation in which there are no NLO corrections to the intensive quantities. This allows us to cast the NLO corrections entirely in terms of the intensive quantities. The parameters we choose are related to the parameters in [19] in the following way

$$\begin{aligned} a &= \omega_a - \frac{\sqrt{\kappa}(1-\omega_a^2)\omega_b}{\ell}\,, \\ b &= \omega_b - \frac{\sqrt{\kappa}(1-\omega_b^2)\omega_a}{\ell}\,, \\ r_+ &= \ell \ + \sqrt{\kappa}\omega_a\omega_b\,, \\ q &= \sqrt{\kappa}\ell^3 + 3\kappa\ell^2\omega_a\omega_b\,. \end{aligned} \tag{4.105}$$

In terms of these parameters, the intensive quantities can be written as

$$\begin{aligned} \Omega_a &= \omega_a + \mathcal{O}\left[\frac{1}{\ell^2}\right], \\ \Omega_b &= \omega_b + \mathcal{O}\left[\frac{1}{\ell^2}\right], \\ T &= \left[\frac{2-\kappa}{2\pi}\right]\ell + O\left[\frac{1}{\ell}\right], \\ \nu &= \frac{2\pi\sqrt{\kappa}}{2-\kappa} + \mathcal{O}\left[\frac{1}{\ell^2}\right], \end{aligned} \tag{4.106}$$

where we have calculated up to NLO and confirmed that the intensive quantities do not get corrected in this order.

This in turn means that the new parameters can be directly interpreted in terms of the intensive quantities.

$$\omega_a = \Omega_a + \mathcal{O}\left[l_{\rm mfp}^2\right], \qquad \omega_b = \Omega_b + \mathcal{O}\left[l_{\rm mfp}^2\right],$$

where $l_{\rm mfp} \sim \frac{2-\kappa}{T}$.

$$\ell = T\left[\frac{\sqrt{\pi^2+2\nu^2}+\pi}{2}\right] + \mathcal{O}\left[\frac{1}{T^2}\right], \qquad \sqrt{\kappa} = \frac{\sqrt{\pi^2+2\nu^2}-\pi}{\nu} + \mathcal{O}\left[\frac{1}{T^2}\right].$$

Now, we calculate NLO corrections to the extensive quantities in terms of the new parameters.

$$
\begin{aligned}
2m &= (1+\kappa)\ell^4 + 4\sqrt{\kappa}(1+\kappa)\omega_a\omega_b\ell^3 + \mathcal{O}[\ell^2]\,,\\
S &= \frac{T^3}{G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{4\pi^5}{(2-\kappa)^3} + \mathcal{O}\left[\frac{1}{T^2}\right]\right],\\
L_a &= \frac{T^4}{G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{2\omega_a}{1-\omega_a^2}\left[\frac{2\pi^5(1+\kappa)}{(2-\kappa)^4}\right] - \frac{\pi\nu^3\omega_b}{4T}\left[\frac{1+\omega_a^2}{1-\omega_a^2}\right] + \mathcal{O}\left[\frac{1}{T^2}\right]\right],\\
L_b &= \frac{T^4}{G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{2\omega_b}{1-\omega_b^2}\left[\frac{2\pi^5(1+\kappa)}{(2-\kappa)^4}\right] - \frac{\pi\nu^3\omega_a}{4T}\left[\frac{1+\omega_b^2}{1-\omega_b^2}\right] + \mathcal{O}\left[\frac{1}{T^2}\right]\right],\\
R &= \frac{T^3}{G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{2\pi^4\sqrt{\kappa}}{(2-\kappa)^3} - \frac{\pi\nu^2}{4T}\omega_a\omega_b + \mathcal{O}\left[\frac{1}{T^2}\right]\right],\\
E &= \frac{T^4}{G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{2\pi^5(1+\kappa)}{(2-\kappa)^4}\left[\frac{2}{1-\omega_a^2} + \frac{2}{1-\omega_b^2} - 1\right]\right.\\
&\qquad\left. - \frac{\pi\nu^3\omega_a\omega_b}{4T}\left[\frac{2}{1-\omega_a^2} + \frac{2}{1-\omega_b^2}\right] + \mathcal{O}\left[\frac{1}{T^2}\right]\right],
\end{aligned}
\tag{4.107}
$$

where $G_5 = \pi R_{\mathrm{AdS}}^3/(2N^2)$ is the Newton's constant in AdS$_5$.

In particular, the subleading terms can be isolated and written as

$$
\begin{aligned}
\Delta S &= 0\,,\\
\Delta E &= -\frac{\pi\zeta^3\omega_a\omega_b}{4G_5(1-\omega_a^2)(1-\omega_b^2)}\left[\frac{2}{1-\omega_a^2} + \frac{2}{1-\omega_b^2}\right],\\
\Delta L_a &= -\frac{\pi\zeta^3\omega_b(1+\omega_a^2)}{4G_5(1-\omega_a^2)^2(1-\omega_b^2)}\,,\\
\Delta L_b &= -\frac{\pi\zeta^3\omega_a(1+\omega_b^2)}{4G_5(1-\omega_a^2)(1-\omega_b^2)^2}\,,\\
\Delta R &= -\frac{\pi\zeta^2\omega_a\omega_b}{4G_5(1-\omega_a^2)(1-\omega_b^2)}\,,\\
\Delta \ln \mathcal{Z}_{\mathrm{gc}} &= \frac{\pi\zeta^3\omega_a\omega_b}{4G_5T(1-\omega_a^2)(1-\omega_b^2)}\,.
\end{aligned}
\tag{4.108}
$$

4.7 Comparison with black holes in $AdS_4 \times S^7$ and $AdS_7 \times S^4$

In this section we compare solutions of rotating fluids of the M5 or M2 brane conformal field theory on S^2 or S^5 to the classical physics of black holes in M theory on $AdS_4 \times S^7$ and $AdS_7 \times S^5$ respectively. Our results turn out to be qualitatively similar to those of the previous section with one difference: the puzzle regarding the next to leading order agreement between fluid dynamics and black hole physics seems to be absent in this case.

4.7.1 Predictions from fluid mechanics

The equations of state of the strongly coupled M2 and M5 brane fluids were computed from spinning brane solutions in [70]. Our parameters are related to theirs by $\kappa_i = l_i^2/r_H^2$.

M2 branes We define our R-charges to be half of the angular momenta of [70] to agree with gauged supergravity conventions. The equation of state is

$$\begin{aligned}
h(\nu) &= \frac{4\pi^2(2N)^{3/2}\prod_j(1+\kappa_j)^{5/2}}{3(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^3}\,,\\
\nu_i &= \frac{4\pi\prod_j(1+\kappa_j)}{(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)}\left(\frac{\sqrt{\kappa_i}}{1+\kappa_i}\right),\\
h_i(\nu) &= \frac{\pi(2N)^{3/2}\prod_j(1+\kappa_j)^{3/2}}{3(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^2}\sqrt{\kappa_i}\,,
\end{aligned} \tag{4.109}$$

where $i,j,k = 1\dots4$.

The stress tensor and currents are given by (4.50) and (4.51) with

$$\begin{aligned}
A &= \frac{4\pi^2(2N)^{3/2}\prod_j(1+\kappa_j)^{5/2}}{3(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^3}\,,\\
B &= \frac{4\pi^2(2N)^{3/2}\prod_j(1+\kappa_j)^{3/2}}{3(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^2}\,,\\
C_i &= \frac{\pi(2N)^{3/2}\prod_j(1+\kappa_j)^{3/2}}{3(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^2}\sqrt{\kappa_i}\,.
\end{aligned} \tag{4.110}$$

The thermodynamics can be summarised by

$$\begin{aligned}
\zeta_i &= \frac{4\pi T\prod_j(1+\kappa_j)}{(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)}\left(\frac{\sqrt{\kappa_i}}{1+\kappa_i}\right),\\
\ln\mathcal{Z}_{\text{gc}} &= \frac{16\pi^3(2N)^{3/2}T^2\prod_j(1+\kappa_j)^{5/2}}{3(1-\Omega^2)(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j)^3}\,.
\end{aligned} \tag{4.111}$$

The mean free path in fluid dynamics is given by

$$\begin{aligned}
l_{\text{mfp}} \sim \left[\frac{S}{4\pi E}\right]_{\Omega=0} &= \frac{B}{(d-1)4\pi TA} = \frac{\left(3+2\sum_j\kappa_j+\sum_{j<k}\kappa_j\kappa_k-\prod_j\kappa_j\right)}{8\pi T\prod_j(1+\kappa_j)}\\
&= \frac{1}{8\pi T}\left[\sum_j\frac{1}{1+\kappa_j}-1\right].
\end{aligned} \tag{4.112}$$

This simplifies when the charges are pairwise equal, $\kappa_3=\kappa_1$ and $\kappa_4=\kappa_2$. In this case, with $i=1,2$:

$$\begin{aligned}
\zeta_i &= \frac{4\pi T\prod_j(1+\kappa_j)}{(3+\sum_j\kappa_j-\prod_j\kappa_j)}\left(\frac{\sqrt{\kappa_i}}{1+\kappa_i}\right),\\
\ln\mathcal{Z}_{\text{gc}} &= \frac{16\pi^3(2N)^{3/2}T^2\prod_j(1+\kappa_j)^2}{3(1-\Omega^2)(3+\sum_j\kappa_j-\prod_j\kappa_j)^3}\,.
\end{aligned} \tag{4.113}$$

and the mean free path becomes

$$l_{\mathrm{mfp}} \sim \frac{\left(3+\sum_j \kappa_j - \prod_j \kappa_j\right)}{8\pi T \prod_j (1+\kappa_j)} = \frac{1}{8\pi T}\left[\sum_i \frac{2}{1+\kappa_i} - 1\right]. \tag{4.114}$$

It is evident that the thermodynamic equations of state listed above allow a set of extremal fluid configurations very similar to those discussed in §§4.6.2. The analysis of §§4.6.2 can be easily extended to fluids on S^2.

M5 branes We define our R-charges to be twice the angular momenta of [70] to agree with gauged supergravity conventions. The equation of state is

$$\begin{aligned}
h(\nu) &= \frac{64\pi^3 N^3 \prod_j (1+\kappa_j)^4}{3(3+\sum_j \kappa_j - \prod_j \kappa_j)^6}\,, \\
\nu_i &= \frac{2\pi \prod_j (1+\kappa_j)}{(3+\sum_j \kappa_j - \prod_j \kappa_j)}\left(\frac{\sqrt{\kappa_i}}{1+\kappa_i}\right), \\
h_i(\nu) &= \frac{128\pi^2 N^3 \prod_j (1+\kappa_j)^3}{3(3+\sum_j \kappa_j - \prod_j \kappa_j)^5}\sqrt{\kappa_i}\,,
\end{aligned} \tag{4.115}$$

where $i = 1, 2$.

The stress tensor and currents are given by (4.50) and (4.51) with

$$\begin{aligned}
A &= \frac{64\pi^3 N^3 \prod_j (1+\kappa_j)^4}{3(3+\sum_j \kappa_j - \prod_j \kappa_j)^6}\,, \\
B &= \frac{128\pi^3 N^3 \prod_j (1+\kappa_j)^3}{3(3+\sum_j \kappa_j - \prod_j \kappa_j)^5} \\
C_i &= \frac{128\pi^2 N^3 \prod_j (1+\kappa_j)^3}{3(3+\sum_j \kappa_j - \prod_j \kappa_j)^5}\sqrt{\kappa_i}\,.
\end{aligned} \tag{4.116}$$

The thermodynamics can be summarised by

$$\begin{aligned}
\zeta_i &= \frac{4\pi T \prod_j (1+\kappa_j)}{(3+2\sum_j \kappa_j + \sum_{j<k} \kappa_j\kappa_k - \prod_j \kappa_j)}\left(\frac{\sqrt{\kappa_i}}{1+\kappa_i}\right), \\
\ln \mathcal{Z}_{\mathrm{gc}} &= \frac{64\pi^6 N^3 T^5 \prod_j (1+\kappa_j)^4}{3\prod_a (1-\Omega_a^2)(3+\sum_j \kappa_j - \prod_j \kappa_j)^3}\,.
\end{aligned} \tag{4.117}$$

The mean free path in fluid dynamics is given by

$$\begin{aligned}
l_{\mathrm{mfp}} \sim \left[\frac{S}{4\pi E}\right]_{\Omega=0} &= \frac{B}{(d-1)4\pi T A} = \frac{\left(3+\sum_j \kappa_j - \prod_j \kappa_j\right)}{10\pi T \prod_j (1+\kappa_j)} \\
&= \frac{1}{10\pi T}\left[\sum_j \frac{2}{1+\kappa_j} - 1\right].
\end{aligned} \tag{4.118}$$

In the case that the three rotation parameters are equal, $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$, we have $\gamma = (1-\Omega^2)^{-1/2}$ and

$$\begin{aligned}
\zeta_i &= \frac{4\pi T \prod_j (1+\kappa_j)}{(3 + 2\sum_j \kappa_j + \sum_{j<k} \kappa_j \kappa_k - \prod_j \kappa_j)} \left(\frac{\sqrt{\kappa_i}}{1+\kappa_i} \right), \\
\ln \mathcal{Z}_{\text{gc}} &= \frac{64\pi^6 N^3 T^5 \prod_j (1+\kappa_j)^4}{3(1-\Omega^2)^3 (3 + \sum_j \kappa_j - \prod_j \kappa_j)^3} .
\end{aligned} \tag{4.119}$$

It is evident that the thermodynamic equations of state listed above allow a set of extremal fluid configurations very similar to those discussed in §§4.6.2. The analysis of §§4.6.2 can be easily extended to fluids on S^5.

4.7.2 Black holes in AdS$_4$ with pairwise equal charges

The relevant solution was found in [58]. Its thermodynamics have been computed in [59]. We consider the limit of large r_+ with $\frac{2ms_i^2}{r_+} = k_i$ fixed. In this limit m can be written as

$$m = \frac{r_+^3}{2}(1+k_1)^2(1+k_2)^2,$$

and therefore $s_i \sim \frac{1}{r_+}$.

After multiplying integrals by $\frac{R_{\text{AdS}}^2}{G_4} = \frac{(2N)^{3/2}}{3}$, the thermodynamic quantities can be expressed as

$$\begin{aligned}
T &= \frac{r_+(3 + \sum_j k_j - \prod_j k_j)}{4\pi}, \qquad \Omega = a\,, \\
\zeta_1 = \zeta_3 &= 4\pi T \frac{(1+k_2)\sqrt{k_1}}{(3 + \sum_j k_j - \prod_j k_j)}, \quad \zeta_2 = \zeta_4 = 4\pi T \frac{(1+k_1)\sqrt{k_2}}{(3 + \sum_j k_j - \prod_j k_j)}, \\
\ln \mathcal{Z}_{\text{gc}} &= \frac{16\pi^3 (2N)^{3/2} T^2}{3} \left(\frac{\prod_j (1+k_j)^2}{(3 + \sum_j k_j - \prod_j k_j)^3} \right) \frac{1}{1-a^2} .
\end{aligned} \tag{4.120}$$

If one identifies $\kappa_i = k_i$, then these formulae match with (4.113). It is not difficult to verify that the first corrections to the thermodynamical equations above occur at $\mathcal{O}(1/r_+^2)$.

It is clear from (4.120) that the black holes of this subsection admit a zero temperature (extremal) limit with nonsingular thermodynamics at any every value of r_+. These extremal black holes are dual to extremal solutions of fluid dynamics analogous to those described in the previous section in the context of $\mathcal{N} = 4$ Yang-Mills.

The fluid dynamical mean free path may easily be computed as a function of black hole parameters. From (4.114) we find

$$l_{\text{mfp}} \sim \frac{1}{2r_+ \prod_j (1+\kappa_j)} .$$

As in the previous section, the $l_{\text{mfp}} \sim 1/r_+$ away from thermodynamically suspect limits of parameters.

4.7.3 Black holes in AdS$_7$ with equal rotation parameters

The relevant solution was found in [57]. Its thermodynamics have been computed in [59].[62]

We set the parameter g in [59] to be unity and consider the limit

$$\rho_+ \gg 1 \,, \qquad \text{and} \quad H_i = 1 + \frac{2ms_i^2}{\rho_+^6} \quad \text{fixed,}$$

where i=1,2. In this limit, the parameter m is given by

$$2m = \rho_+^6 H_1 H_2 \,.$$

In this limit, after multiplying integrals by $\frac{R_{\rm AdS}^5}{G_7} = \frac{16N^3}{3\pi^2}$, the thermodynamics can be summarised by

$$\begin{aligned}
\Omega &= a\,, & T &= \frac{\rho_+}{2\pi}\left(\frac{2\sum_j H_j - \prod_j H_j}{\prod_j \sqrt{H_j}}\right), \\
\zeta_1 &= 2\pi T \frac{H_2\sqrt{H_1 - 1}}{2\sum_j H_j - \prod_j H_j}\,, & \zeta_2 &= 2\pi T \frac{H_1\sqrt{H_2 - 1}}{2\sum_j H_j - \prod_j H_j}\,, \\
& & \ln \mathcal{Z}_{\rm gc} &= \frac{64\pi^6 N^3 T^5}{3(1-\Omega^2)^3}\left(\frac{\prod_j H_j^4}{(2\sum_j H_j - \prod_j H_j)^6}\right).
\end{aligned} \tag{4.121}$$

These formulae agree with (4.119) upon identifying $\kappa_i = H_i - 1$. The first corrections to these thermodynamical formulae occur at $\mathcal{O}(1/r_+^2)$. Using this identification we can rewrite the expression for the temperature as

$$T = \frac{\rho_+ \prod_j \sqrt{1+\kappa_j}}{2\pi}\left(\sum_j \frac{2}{1+\kappa_j} - 1\right).$$

It follows that the black holes studied in this subsection admit smooth extremal limits at any value of ρ_+. Extremal black holes with large ρ_+ (and with no κ_i arbitrarily large) are dual to extremal solutions of fluid mechanics.

Expressing the fluid mechanical mean free path (4.118) as a function of black hole parameters we find

$$l_{\rm mfp} \sim \frac{1}{5\rho_+ \prod_j \sqrt{1+\kappa_j}}\,.$$

Once again $l_{\rm mfp} \sim 1/r_+$, away from thermodynamically suspect limits.

[62] We believe that [59] has the following typos: equation (4.7) should read

$$S = \frac{\pi^3(r^2+a^2)\sqrt{f_1}}{4\Xi^3} \quad T = \frac{Y'}{4\pi r(r^2+a^2)\sqrt{f_1}} \quad \Phi_i = \frac{2ms_i}{\rho^4 \Xi H_i}\left[\Xi_- \alpha_i + \beta_i(\Omega - g)\right].$$

4.8 Discussion

As we have explained in this chapter, the classical properties of large black holes in AdS spaces enjoy a large degree of universality, summarised by (4.2). However the reasoning that led to (4.2) applies equally to all classical theories of gravity, not just to those theories that are governed by the two derivative effective action. For instance, $\mathcal{N}=4$ Yang-Mills theory at finite λ is dual to IIB theory on $AdS_5 \times S^5$ of finite radius in string units. Even though thermodynamics of black holes in this background will receive contributions from each of the infinite sequence of α' corrections to the Einstein action, we expect (4.2) to be exact in the large horizon radius limit.[63]

We find it particularly interesting that (at least in several particular contexts) our fluid dynamical picture applies not just to non-extremal black holes but also to large radius extremal black holes. This fact might allow us to make connections between our approach and the interesting recent investigations of the properties of extremal black holes. In particular, Astefanesei, Goldstein, Jena, Sen and Trivedi [68] have recently argued that the attractor mechanism applies to rotating extremal black holes, and have derived a differential equation that determines the attractor geometry (and gauge field distribution, etc.) of the near horizon region of such black holes. It would be very i nteresting to investigate the connection, if any, between these rotating attractor equations and our equations of rotating fluid dynamics.

In the next chapter, we will explain purely in bulk terms, why our calculations in this chapter works. Roughly speaking, we will show that the metric of a black hole in global AdS and in the large radius limit, as a superposition of patches of the metric of black branes of various different temperatures and moving at various different velocities, where the temperatures and velocities are given by the solutions to the fluid dynamical equations presented in this chapter.

4.9 Appendices

4.9.1 Conformal fluid mechanics

Consider a conformal fluid in d dimensions. We seek the conformal transformations of various observables of such a fluid. USing the results from the previous chapter, we can derive the transformation of various hydrodynamic quantities

$$\begin{aligned}
\vartheta &= \nabla_\mu u^\mu = \mathrm{e}^{-\phi}\left[\tilde{\vartheta} + (d-1)\tilde{u}^\sigma \partial_\sigma \phi\right], \\
a^\nu &= u^\mu \nabla_\mu u^\nu = \mathrm{e}^{-2\phi}\left[\tilde{a}^\nu + \widetilde{P}^{\nu\sigma}\partial_\sigma\phi\right], \\
\sigma^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta P^{\mu\nu} = \mathrm{e}^{-3\phi}\tilde{\sigma}^{\mu\nu}, \\
\omega^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu - P^{\nu\lambda}\nabla_\lambda u^\mu\right) = \mathrm{e}^{-3\phi}\widetilde{\omega}^{\mu\nu}.
\end{aligned} \tag{4.122}$$

[63] Away from the supergravity limit, the mean free path $l_{\mathrm{mfp}} = \nu/\rho$ is expected to be given by $f(\lambda)s/\rho$ where $f(\lambda)$ is a monotonically decreasing function that interpolates between infinity at $\lambda = 0$ to unity at infinite *lambda*. Thus the condition for the validity of fluid mechanics is modified at finite λ; in the uncharged case, for instance, it is $T \gg f(\lambda)$.

Further, the transformation of the temperature and the chemical potential can be written as $\mathcal{T} = e^{-\phi}\widetilde{\mathcal{T}}$ and $\mu = e^{-\phi}\tilde{\mu}$. The transformation of spatial gradient of temperature (appearing in the Fourier law of heat conduction) is

$$P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T}) = e^{-3\phi}\widetilde{P}^{\mu\nu}(\partial_\nu \widetilde{\mathcal{T}} + \tilde{a}_\nu \widetilde{\mathcal{T}})\,.$$

The viscosity, conductivity etc. scale as $\kappa = e^{-(d-2)\phi}\tilde{\kappa}$, $\eta = e^{-(d-1)\phi}\tilde{\eta}$, $\mu_i = e^{-\phi}\tilde{\mu}_i$ and $D_{ij} = e^{-(d-2)\phi}\widetilde{D}_{ij}$.

For a fluid with c charges, there are $2c+2$ vector quantities involving no more than a single derivative which transform homogeneously[64]. They are

$$u^\mu, \quad \partial_\mu \nu_i, \quad \partial_\mu \mathcal{T} + \left(a_\mu - \frac{\vartheta}{d-1}u_\mu\right)\mathcal{T}, \quad u^\mu u^\sigma \partial_\sigma \nu_i \quad \text{and} \quad \left(u^\sigma \partial_\sigma \mathcal{T} + \frac{\vartheta}{d-1}\mathcal{T}\right)u^\mu.$$

In the kind of solutions we consider in this chapter, all of them vanish except u^μ.

The transformation of the stress tensor is $T^{\mu\nu} = e^{-(d+2)\phi}\widetilde{T}^{\mu\nu}$, from which it follows that

$$\nabla_\mu T^{\mu\nu} = e^{-(d+2)\phi}(\widetilde{\nabla}_\mu \widetilde{T}^{\mu\nu} - \tilde{g}_{\lambda\sigma}\widetilde{T}^{\lambda\sigma}\tilde{g}^{\nu\sigma}\partial_\sigma\phi)\,.$$

So, for the stress tensor to be conserved in both the metrics, it is necessary that $T^{\mu\nu}$ is traceless.

To consider the possible terms that can appear in the stress tensor, we should look at the traceless symmetric second rank tensors which transform homogeneously. The tensors formed out of single derivatives which satisfy the above criterion are easily enumerated. For a fluid with c charges, there are $2c+4$ such tensors and they are

$$\begin{aligned}
&u^\mu u^\nu + \frac{1}{d}g^{\mu\nu}, \quad \sigma^{\mu\nu}, \quad q^\mu u^\nu + q^\nu u^\mu, \quad \left(u^\sigma \partial_\sigma \mathcal{T} + \frac{\vartheta}{d-1}\mathcal{T}\right)\left(u^\mu u^\nu + \frac{1}{d}g^{\mu\nu}\right), \\
&\frac{1}{2}\left(u^\mu \partial^\lambda \nu_i + u^\lambda \partial^\mu \nu_i\right) - \frac{g^{\mu\nu}}{d}u^\sigma \partial_\sigma \nu_i \quad \text{and} \quad u^\sigma \partial_\sigma \nu_i \left(u^\mu u^\nu + \frac{1}{d}g^{\mu\nu}\right).
\end{aligned} \tag{4.123}$$

Among these possibilities, the stress tensor we employ just contains the tensors in the first line. It can be shown that the other tensors which appear in the above list can be removed by a redefinition of the temperature etc. Even if they were to appear in the stress tensor, for the purposes of this chapter, it suffices to notice that all such tensors except $u^\mu u^\nu + \frac{1}{d}g^{\mu\nu}$ vanish on our solutions. Hence, they would not contribute to any of the thermodynamic integrals evaluated on our solutions.

4.9.2 Free thermodynamics on spheres

In (4.55) above, we have presented a general expression for the grand canonical partition function for any conformal fluid on a sphere. In this appendix , we compare this expression with the conformal thermodynamics of a free complex scalar field on a sphere.

[64]In the following analysis, we will neglect pseudo-tensors which can be formed out of $\epsilon_{\mu\nu\ldots}$. Additional tensors appear if such pseudo-tensors are included in the analysis.

Strictly speaking, the fluid dynamical description never applies to free theories on a compact manifold, as the constituents of a free gas have a divergent mean free path (they never collide). Nonetheless, as we demonstrate in this subsection, free thermodynamics already displays some of the features of (4.55) - in its dependence on angular velocities, for example - together with certain pathologies unique to free theories.

Consider a free complex scalar field on $S^{d-1}\times$ time. This system has a $U(1)$ symmetry, under which ϕ has unit charge and ϕ^* has charge minus one. We define the 'letter partition function' [71] Z_{let} as $\mathrm{Tr}\exp\left[-\beta H+\nu R+\beta\Omega_a L_a\right]$ evaluated over all spherical harmonic modes of the scalar field

$$Z_{\text{let}} = (\mathrm{e}^{\nu}+\mathrm{e}^{-\nu})\mathrm{e}^{-\beta\frac{d-2}{2}}\left(\frac{1-\mathrm{e}^{-2\beta}}{\prod_{a=1}^{n}(1-\mathrm{e}^{-\beta-\beta\Omega_a})(1-\mathrm{e}^{-\beta+\beta\Omega_a})}\right) \tag{4.124}$$

(this formula, and some of the others in this section, are valid only for even d; the generalisation to odd d is simple). We will now examine the high temperature limit of the grand-canonical partition function separately for $\nu=0$ and $\nu\neq 0$.

Zero chemical potential: ($\nu=0$) case The second quantised partition function, $\mathcal{Z}_{\text{gc}}$ for the scalar field on the sphere is given by

$$\mathcal{Z}_{\text{gc}} = \exp\left(\sum_N \frac{Z_{\text{let}}(N\beta, N\nu, \Omega_a)}{N}\right). \tag{4.125}$$

For small β, we have

$$Z_{\text{let}} \approx \frac{4}{\beta^{d-1}\prod_a(1-\Omega_a^2)}\,.$$

It follows that[65]

$$\ln \mathcal{Z}_{\text{gc}} = \frac{4\zeta(d)}{\beta^{d-1}\prod_a(1-\Omega_a^2)}\,. \tag{4.126}$$

Upon identifying $V_d h|_{\nu=0} = 4\zeta(d)$, we find that (4.125) is in perfect agreement with (4.55).

Nonzero chemical potential: ($\nu\neq 0$) case The high temperature limit of the thermodynamics of a free, charged, massless field is complicated by the occurrence of Bose condensation. This phenomenon occurs already when $\omega_a=0$; this is the case we first focus on.

It is useful to rewrite the letter partition function as

$$Z_{\text{let}} = (2\cosh\nu)\ \ \mathrm{e}^{-\beta\frac{d-2}{2}}\sum_N m(N)\mathrm{e}^{-\beta N}, \tag{4.127}$$

where $m(N)\approx 2N^{d-2}/(d-2)!$ for $N\gg 1$. The logarithm of the grand canonical partition function may then be written as a sum over Bose factors (one per 'letter')

$$\ln \mathcal{Z}_{\text{gc}} = -\sum_N m(N)\left[\ln(1-\mathrm{e}^{-\beta(N+(d-2)/2)+\nu}) + \ln(1-\mathrm{e}^{-\beta(N+(d-2)/2)-\nu})\right]. \tag{4.128}$$

[65] This formula has been derived before in many contexts, for example [72] have derived this in $d=4$ and compared it with the thermodynamics of black holes in AdS_5.

The total charge in this ensemble is given by

$$R = \frac{\partial}{\partial \nu} \ln \mathcal{Z}_{\rm gc} = \sum_N m(N) \left(\frac{1}{{\rm e}^{\beta(N+(d-2)/2)-\nu} - 1} - \frac{1}{{\rm e}^{\beta(N+(d-2)/2)+\nu} - 1} \right). \tag{4.129}$$

In order to compare with fluid dynamics, we should take β to zero while simultaneously scaling to large R as $R = \frac{q}{\beta^{d-1}}$ with q held fixed. As we will see below, in order to make the total charge R large, we will have to choose the chemical potential to be large. However it is clear from (4.128) that $|\nu| < \beta(d-2)/2$. Consequently, the best we can do is to set $\nu = \beta((d-2)/2) - \epsilon$ where ϵ will be taken to be small. We are interested in the limit when β is also small. We may approximate (4.129) by

$$\frac{q}{\beta^{d-1}} = \frac{1}{\epsilon} - \frac{1}{{\rm e}^{\beta(d-2)-\epsilon} - 1} + \sum_{N=1}^{\infty} \left(\frac{1}{{\rm e}^{\beta N+\epsilon} - 1} - \frac{1}{{\rm e}^{\beta(N+(d-2))-\epsilon} - 1} \right). \tag{4.130}$$

The only solution to (4.130) is

$$\epsilon = \frac{\beta^{d-1}}{q} \left(1 + \mathcal{O}(\beta)\right).$$

Substituting this solution into the partition function, we find

$$\ln Z_q = \frac{4\zeta(d)}{\beta^{d-1}} \left(1 + \mathcal{O}(\beta)\right). \tag{4.131}$$

Consequently, to leading order the partition function is independent of the charge q ! What is going on here is that almost all of the charge of the system resides in a Bose condensate of the zero mode of the field ϕ. This zero mode contributes very little entropy or energy to the system at leading order in β.[66] At high temperatures, the zero mode is simply a sink that absorbs the system charge, leaving the other thermodynamic parameters unaffected.

Upon generalising our analysis to include angular velocities, we once again find that the leading order partition function (in the limit of high temperatures and a charge $R = q/\beta^{d-1}$) is independent of q and in fact is given by (4.126). Consequently, there is a slightly trivial (or pathological) sense in which the thermodynamics of a free charged scalar field agrees with the predictions of fluid mechanics - we find agreement upon setting $h(\nu)$ to a constant.

4.9.3 Stress tensors from black holes

According to the usual AdS/CFT dictionary, the boundary stress tensor on S^{d-1}, corresponding to any finite energy solution about an AdS$_D$ background of gravity may be read off from the metric near the boundary, using the following procedure [42–49]. First we foliate the spacetime near the boundary into a one parameter set of d geometries, each of which is metrically conformal to $S^{d-1} \times \mathbb{R}$, to leading order in deviations from the boundary. We will

[66] In particular, the contribution of the zero mode to the energy is proportional to the charge, which is suppressed by a factor of β relative to the contribution to the energy from nonzero modes.

find it convenient to use coordinates such that the leading order metric in the neighbourhood of the boundary takes the form

$$ds^2 = -r^2 dt^2 + r^2 d\Omega_{d-1}^2 + \frac{dr^2}{r^2} \tag{4.132}$$

(here $r = \infty$ is the boundary; this metric has corrections at subleading orders in $\frac{1}{r^2}$).

In these coordinates, our foliation surfaces are simply given by $r = \text{const}$. We next compute the extrinsic curvature $\Theta_\mu^\nu = -\nabla_\mu n^\nu$ on these surfaces, where n^ν is the unit outward normal to these surfaces. The boundary stress tensor for the dual field theory on a unit sphere is given by [42–49] as[67]

$$\Pi_\nu^\mu = \lim_{r\to\infty} \frac{r^{D-1}}{8\pi G_D} \left(\Theta_\nu^\mu - \delta_\nu^\mu \Theta\right), \tag{4.133}$$

where the coordinates μ, ν go over time and the angles on S^{D-1}.

The stress tensor as defined above will contain some terms which are independent of mass and charge of the black hole. These are the terms that are nonzero even on the vacuum AdS background and they diverge in the limit $r \to \infty$. These terms are all precisely cancelled, up to a zero point Casimir energy, by counter terms presented in §2 of [42]. We will simply ignore all such terms below; consequently, the stress tensors we present in this chapter should be thought of as the field theory stress tensors with the contribution from the Casimir energy subtracted out.[68]

In order to compute the stress tensor in (4.133), we must retain subleading corrections to the metric in (4.132). However, only those corrections that are subleading at $\mathcal{O}(\frac{1}{r^{D-1}})$ (relative to the leading order metric in (4.132)) contribute to (4.133).

In order to compute the stress tensor corresponding to two classes of black hole solutions below, we adopt the following procedure. First, we find a coordinate change that casts the metric at infinity in the form (4.132) at leading order. Next we compute all the subleading corrections to the metric at order $\mathcal{O}(\frac{1}{r^{D-1}})$. Finally, we use these corrections to compute the extrinsic curvature Θ_ν^μ and then Π_ν^μ using (4.133).

Stress tensor from rotating uncharged black holes in AdS$_D$

$D = 2n+1$ The most general rotating black hole in an odd dimensional AdS space is given by equation (E.3) of [16] (we specialise to odd dimensions by setting the parameter ϵ in that equation to zero). The metric presented in [16] uses as coordinates

1. The n Killing azimuthal angles ϕ_i along which the black hole rotates. These may be identified with the coordinates ϕ_i in §4.4 of this chapter.

[67] We rescale the stress tensor of [42] by a factor of $\frac{1}{8\pi G_D}$ in order that the energy of our solutions is given by $\int \sqrt{\gamma}\, \Pi_0^0$ with no extra normalisation factor.

[68] Recall that the full stress tensor of a general d dimensional conformal field theory is not traceless on an arbitrary manifold; however the trace is given by a function of the manifold curvature independent of the field theory configuration. It follows that our stress tensor with Casimir contribution subtracted must be traceless, as indeed it will turn out to be.

2. n other unspecified variables (called 'direction cosines') μ_i subject to the constraint $\sum_i \mu_i^2 = 1$. These may be thought of as the remaining $n-1$ coordinates on S^{2n-1}.

3. The radial variable r and timelike variable t.

In order to cast the metric of [16] into the form (4.132) near the boundary, we perform the following change of coordinates

$$\tilde{r}^2 = \sum_{i=1}^{n} \frac{(r^2+a_i^2)\mu_i^2}{1-a_i^2}\,, \qquad \tilde{r}^2\tilde{\mu}_i^2(1-a_i^2) = (r^2+a_i^2)\mu_i^2. \tag{4.134}$$

Note that

$$\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} \tilde{\mu}_i^2 = 1\,.$$

This equation may be solved by writing $\tilde{\mu}_i$ as functions of the $n-1$ variables θ_j (which may then be identified with the coordinates used in §4.4)

$$\tilde{\mu}_i = \left(\prod_{j=1}^{i-1} \cos^2\theta_j\right) \sin^2\theta_i\,.$$

In these coordinates, the metric in the neighbourhood of $r \to \infty$ becomes

$$\begin{aligned} \mathrm{d}s^2 = &-(1+\tilde{r}^2)\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2\sum_{i=1}^{n}(\mathrm{d}\tilde{\mu}_i^2 + \tilde{\mu}_i^2\mathrm{d}\phi_i^2) \\ &+\frac{2m}{\tilde{r}^{2n-2}}\gamma^{2(n+1)}\mathrm{d}t^2 + \frac{2m}{\tilde{r}^{2n+2}}\gamma^{2n}\mathrm{d}\tilde{r}^2 \\ &-\sum_{i=1}^{n}\frac{4ma_i\tilde{\mu}_i^2}{\tilde{r}^{2n-2}}\gamma^{2(n+1)}\mathrm{d}t\mathrm{d}\phi_i + \sum_{i=1,j=1}^{n}\frac{2ma_ia_j\tilde{\mu}_i^2\tilde{\mu}_j^2}{\tilde{r}^{2n-2}}\gamma^{2(n+1)}\mathrm{d}\phi_i\mathrm{d}\phi_j\,, \end{aligned} \tag{4.135}$$

where we have retained all terms that are subleading up to $\mathcal{O}(\frac{1}{\tilde{r}^{D-1}})$ compared to the metric of pure AdS. Here $\gamma^{-2} = 1 - \sum_{i=1}^{n} a_i^2\tilde{\mu}_i^2$ and $\sum_{i=1}^{n}\tilde{\mu}_i^2\mathrm{d}\tilde{\mu}_i^2 = \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1}\cos^2\theta_j\right)\mathrm{d}\theta_i^2$ as in (4.46).

Note that this metric separates into two parts; the first piece (on the first line of (4.135)) is the metric of pure AdS space while the terms of the remaining lines represent correction proportional to the mass m.

The normal vector is given by $n_{\tilde{r}} = \frac{1}{\sqrt{g^{\tilde{r}\tilde{r}}}}$ (with all other components zero). As our metric contains no terms that mix r with other coordinates at leading order, this is the same as

$$n^{\tilde{r}} = \frac{1}{\sqrt{g_{\tilde{r}\tilde{r}}}} = \tilde{r}\left(1+\frac{1}{\tilde{r}^2}\right)^{\frac{1}{2}}\left[1 - \frac{m}{\tilde{r}^{2n}}\gamma^{2n}\right].$$

Since the normal vector has only the $\tilde{r}$ component and since the component is a function of $\tilde{r}$ only, to compute the extrinsic curvature tensor Θ_λ^ν one needs only those components of Γ

that are of the form $\Gamma^{\nu}_{\lambda\tilde{r}}$. The Christoffel symbols (that are relevant for the calculation of the stress-tensor) as calculated from this metric up to the first subleading term in $\tilde{r}$ are given by

$$\begin{aligned}
\Gamma^t_{t\tilde{r}} &= \frac{\tilde{r}}{\tilde{r}^2+1}\left(1+\frac{2nm}{\tilde{r}^{2n}}\gamma^{2(n+1)}\right) & \Gamma^{\phi_i}_{t\tilde{r}} &= \frac{2nma_i}{\tilde{r}^{2n+1}}\gamma^{2(n+1)} \\
\Gamma^t_{\phi_i\tilde{r}} &= -\frac{2nma_i\tilde{\mu}_i^2}{\tilde{r}^{2n+1}}\gamma^{2(n+1)} & \Gamma^{\phi_i}_{\phi_j\tilde{r}} &= \frac{1}{\tilde{r}}\left(\delta_{ij}-\frac{2nma_ia_j\tilde{\mu}_j}{\tilde{r}^{2n}}\gamma^{2(n+1)}\right) \\
\Gamma^\theta_{\theta\tilde{r}} &= \frac{1}{\tilde{r}}\,.
\end{aligned} \tag{4.136}$$

The extrinsic curvature Θ^ν_λ in this case is given by

$$\Theta^\nu_\lambda = -\Gamma^\nu_{\lambda\tilde{r}}n^{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^{2n+1}}\right).$$

Ignoring all terms in Θ^ν_μ that are independent of mass (for the reasons explained in the introduction to this appendix) we find

$$\begin{aligned}
\Theta^t_t &= -\frac{m\gamma^{D+1}}{\tilde{r}^{D-1}}(D-1-\gamma^{-2}) & \Theta^{\phi_i}_{\phi_i} &= \frac{m\gamma^{D+1}}{\tilde{r}^{D-1}}\left((D-1)a_i^2\tilde{\mu}_i^2+\gamma^{-2}\right) \\
\Theta^t_{\phi_i} &= \frac{(D-1)m\gamma^{D+1}}{\tilde{r}^{D-1}}a_i\tilde{\mu}_i^2 & \Theta^{\phi_i}_{\phi_j} &= \frac{(D-1)m\gamma^{D+1}}{\tilde{r}^{D-1}}a_ia_j\tilde{\mu}_j^2 \quad (i\neq j) \\
\Theta^{\phi_i}_t &= -\frac{(D-1)m\gamma^{D+1}}{\tilde{r}^{D-1}}a_i & \Theta^{\theta_i}_{\theta_i} &= \frac{m\gamma^{(D-1)}}{\tilde{r}^{D-1}}\,.
\end{aligned} \tag{4.137}$$

Here the n has been replaced by $\frac{D-1}{2}$. It may easily be verified that Θ^ν_λ is traceless and therefore the stress tensor is also traceless according to the definition (4.133). Raising one index in Θ by asymptotic AdS metric, normalising it appropriately and then taking the large $\tilde{r}$ limit one can derive the stress tensor as given in (4.65).

$D = 2n+2$ The computation of the boundary stress tensor for the most general uncharged rotating black hole in even dimensional AdS spaces is almost identical to the he analysis presented in the previous subsection. Once again the metric is given in equation (E-3) of [16], where we must set ϵ to 1 to specialise to even dimensions. The coordinates of the black hole solution are similar to those described in the previous subsection, except that there are $n+1$ coordinates μ_i restricted by a single equation $\sum_a \mu_i^2 = 1$. Repeating the computations described in the previous subsection, our final result is once again simply (4.137). In summary (4.137) is correct no matter whether D is odd or even.

Black holes in AdS$_5$ with all R-charges equal In this subappendix, we compute the boundary stress tensor for a class of charged black holes, namely for black holes in AdS$_5$ with all three R-charges equal. Our computation will verify the striking prediction of §4.3 that the functional form of this stress tensor is independent of the black hole charge in the fluid dynamical limit.

The metric for rotating black holes with all R-charges equal is given by (equation (1) of [19])

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{\Delta_{\tilde\theta}[(1+y^2)\rho^2\mathrm{d}t + 2q\nu]\mathrm{d}t}{\Sigma_a\Sigma_b\rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4}\left(\frac{\Delta_\Theta \mathrm{d}t}{\Sigma_a\Sigma_b} - \omega\right)^2 + \frac{\rho^2\mathrm{d}y^2}{\Delta_y}\\ &+\frac{\rho^2\mathrm{d}\tilde\theta^2}{\Delta_{\tilde\theta}} + \frac{y^2+a^2}{\Sigma_a}\sin^2\tilde\theta\mathrm{d}\phi^2 + \frac{y^2+b^2}{\Sigma_b}\cos^2\tilde\theta\mathrm{d}\psi^2,\end{aligned} \tag{4.138}$$

where

$$\begin{aligned}\Delta_y =&\frac{(y^2+a^2)(y^2+b^2)(1+y^2)+q^2+2abq}{y^2} - 2m\,,\\ \rho^2 =&y^2 + a^2\cos^2\tilde\theta + b^2\sin^2\tilde\theta\,,\\ \Delta_{\tilde\theta} =&1 - a^2\cos^2\tilde\theta - b^2\sin^2\tilde\theta\,,\\ \Sigma_a =&1-a^2,\\ \Sigma_b =&1-b^2,\\ f =&2m\rho^2 - q^2 + 2abq\rho^2,\\ \nu =&b\sin^2\tilde\theta\mathrm{d}\phi + a\cos^2\tilde\theta\mathrm{d}\psi\,,\\ \omega =&a\sin^2\tilde\theta\frac{\mathrm{d}\phi}{\Sigma_a} + b\cos^2\tilde\theta\frac{\mathrm{d}\psi}{\Sigma_b}\,.\end{aligned} \tag{4.139}$$

This metric takes the form (4.132) near the boundary, once we perform the change of coordinates

$$\begin{aligned}r^2 =&\frac{y^2(1-a^2\cos^2\tilde\theta - b^2\sin^2\tilde\theta) + a^2\sin^2\tilde\theta + b^2\cos^2\tilde\theta - a^2b^2}{\Sigma_a\Sigma_b}\,,\\ r^2\sin^2\theta =&\frac{(y^2+a^2)\sin^2\tilde\theta}{\Sigma_a}\,,\\ r^2\cos^2\theta =&\frac{(y^2+b^2)\cos^2\tilde\theta}{\Sigma_b}\,.\end{aligned} \tag{4.140}$$

Retaining corrections only to order $\mathcal{O}(1/r^4)$ relative to the leading order metric (4.132), the metric in our new coordinates becomes[69]

$$\begin{aligned}\mathrm{d}s^2 =& -(1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\psi^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{2m}{r^6\Delta_\theta^2}\mathrm{d}r^2 + \frac{2(m+abq)}{r^2\Delta_\theta^3}\mathrm{d}t^2\\ &-\frac{2(2am+bq)\sin^2\theta}{r^2\Delta_\theta^3}\mathrm{d}t\mathrm{d}\phi - \frac{2(2bm+aq)\cos^2\theta}{r^2\Delta_\theta^3}\mathrm{d}t\mathrm{d}\psi + \frac{(2ma^2+2bq)\sin^4\theta}{r^2\Delta_\theta^3}\mathrm{d}\phi^2\\ &+\frac{(2mb^2+2aq)\cos^4\theta}{r^2\Delta_\theta^3}\mathrm{d}\psi^2 + \frac{2(2abm+a^2q+b^2q)\sin^2\theta\cos^2\theta}{r^2\Delta_\theta^3}\mathrm{d}\psi\mathrm{d}\phi\,,\end{aligned} \tag{4.141}$$

[69] The leading behaviour at large r of the pure AdS metric given in terms of $\frac{\mathrm{d}r}{r}$ and $r\mathrm{d}\alpha$ where α represents the coordinates θ, ϕ, ψ, t. In the expression below, we have retained all coefficient terms of these 'forms' that are at most of order $\frac{1}{r^4}$.

where $\Delta_\theta = \gamma^{-2} = 1 - a^2\sin^2\theta - b^2\cos^2\theta$. with γ as in §§4.3.2.

The unit normal vector to constant r slices is given by $n_r = \frac{1}{\sqrt{g^{rr}}}$ (with all other components zero). As our metric contains no terms that mix r with other coordinates at leading order , this is the same as $n^r = \frac{1}{\sqrt{g_{rr}}}$.

The Christoffel symbols (that are relevant for the calculation of the stress-tensor) as calculated from this metric up to the first subleading term (i.e. up to $\mathcal{O}(\frac{1}{r^4})$ terms) in r are given by

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{r}{r^2+1}\left(1 + \frac{4(m+abq)}{r^4(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\right) & \Gamma^\phi_{tr} &= \frac{2(2am+bq)}{r^5(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\\
\Gamma^\psi_{tr} &= \frac{2(2bm+aq)}{r^5(1-a^2\sin^2\theta - b2\cos^2\theta)^3} & \Gamma^t_{\phi r} &= -\frac{2(2am+bq)\sin^2\theta}{r^5(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\\
\Gamma^\phi_{\phi r} &= \frac{1}{r}\left(1 - \frac{4(a^2m+abq)\sin^2\theta}{r^4(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\right) & \Gamma^t_{\psi r} &= -\frac{2(2bm+aq)\cos^2\theta}{r^5(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\\
\Gamma^\psi_{\psi r} &= \frac{1}{r}\left(1 - \frac{4(b^2m+abq)\cos^2\theta}{r^4(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\right) & \Gamma^\theta_{\theta r} &= \frac{1}{r}\\
\Gamma^\phi_{\psi r} &= -\frac{(2abm+a^2q+b^2q)\cos^2\theta}{r^5(1-a^2\sin^2\theta - b^2\cos^2\theta)^3} & \Gamma^\psi_{\phi r} &= -\frac{(2abm+a^2q+b^2q)\sin^2\theta}{r^5(1-a^2\sin^2\theta - b^2\cos^2\theta)^3}\,.
\end{aligned} \tag{4.142}$$

The extrinsic curvature, Θ^ν_μ, is given by

$$\begin{aligned}
\Theta^t_t &= -\left(1+\frac{1}{r^2}\right)^{-\frac{1}{2}}\left(1 + \frac{m\gamma^6}{r^4}(3 + a^2\sin^2\theta + b^2\cos^2\theta)\right) - \frac{4abq\gamma^6}{r^4}\\
\Theta^\phi_\phi &= -\sqrt{1+1/r^2}\left(1 - \frac{m\gamma^6}{r^4}(3a^2\sin^2\theta - b^2\cos^2\theta + 1)\right) + \frac{4abq\gamma^6\sin^2\theta}{r^4}\\
\Theta^\psi_\psi &= -\sqrt{1+1/r^2}\left(1 - \frac{m\gamma^6}{r^4}(3b^2\cos^2\theta - a^2\sin^2\theta + 1)\right) + \frac{4abq\gamma^6\cos^2\theta}{r^4}\\
\Theta^t_\phi &= \frac{2(2am+bq)\gamma^6\sin^2\theta}{r^4} \qquad \Theta^\phi_t = -\frac{2(2am+bq)\gamma^6}{r^4}\\
\Theta^t_\psi &= \frac{2(2bm+aq)\gamma^6\cos^2\theta}{r^4} \qquad \Theta^\psi_t = -\frac{2(2bm+aq)\gamma^6}{r^4}\\
\Theta^\psi_\phi &= \frac{2(2abm+b^2q+a^2q)\gamma^6\sin^2\theta}{r^4} \qquad \Theta^\phi_\psi = \frac{2(2bam+b^2q+a^2q)\gamma^6\cos^2\theta}{r^4}\\
\Theta^\theta_\theta &= \frac{m\gamma^4}{r^4}\,,
\end{aligned} \tag{4.143}$$

where $\gamma^2 = \frac{1}{1-a^2\sin^2\theta-b^2\cos^2\theta}$. Therefore

$$\Theta = \Theta^\alpha_\alpha = 4 + \frac{1}{r^2}\,.$$

It is easily verified that Θ^β_α is traceless when the r dependent divergent terms are cancelled by the counter terms at the limit r going to infinity. After cancelling the divergent terms and then normalising it according to (4.133) the stress tensor is given by

$$
\begin{aligned}
\Pi^{tt} &= \frac{m}{8\pi G_5}\left(\gamma^6(3 + a^2\sin^2\theta + b^2\cos^2\theta) - \frac{4abq}{m}\gamma^6\right) \\
&= \frac{m}{8\pi G_5}\left(\gamma^4(4\gamma^2 - 1) - \frac{4abq}{m}\gamma^6\right) \\
\Pi^{\phi\phi} &= \frac{m}{8\pi G_5}\left(\gamma^6\left(\frac{3a^2\sin^2\theta - b^2\cos^2\theta + 1}{\sin^2\theta}\right) - \frac{4abq}{m}\gamma^6\right) \\
&= \frac{m}{8\pi G_5}\left(\gamma^4\left(4\gamma^2 a^2 + \frac{1}{\sin^2\theta}\right) - \frac{4abq}{m}\gamma^6\right) \\
\Pi^{\psi\psi} &= \frac{m}{8\pi G_5}\left(\gamma^6\left(\frac{3b^2\cos^2\theta - a^2\sin^2\theta + 1}{\cos^2\theta}\right) - \frac{4abq}{m}\gamma^6\right) \\
&= \frac{m}{8\pi G_5}\left(\gamma^4\left(4\gamma^2 a^2 + \frac{1}{\cos^2\theta}\right) - \frac{4abq}{m}\gamma^6\right) \\
\Pi^{t\phi} = \Pi^{\phi t} &= \left(\frac{4m}{8\pi G_5}\right)\left(a - \frac{2bq}{m}\right)\gamma^6 \qquad \Pi^{t\psi} = \Pi^{\psi t} = \left(\frac{4m}{8\pi G_5}\right)\left(b - \frac{2aq}{m}\right)\gamma^6 \\
\Pi^{\phi\psi} = \Pi^{\psi\phi} &= \left(\frac{4m}{8\pi G_5}\right)\left(ab - \frac{2(a^2+b^2)q}{m}\right)\gamma^6 \quad \Pi^{\theta\theta} = \frac{m}{8\pi G_5}\gamma^4.
\end{aligned}
\tag{4.144}
$$

As we have explained in §§4.6.4, $q/m \sim 1/r_+$, and so all terms proportional to q in the equation above are subdominant compared to terms proportional to m in the fluid mechanical limit $r_+ \to \infty$. Dropping all q dependent terms, we find (4.144) matches perfectly with the stress tensor as derived in (4.30) and (4.34) upon identifying $(\phi_1, \phi_2) = (\phi, \psi)$, $(\omega_1, \omega_2) = (a, b)$ and using (4.94).

5 Fluid Gravity Correspondence in Arbitrary Dimensions

In this chapter, we will do an explicit bulk construction of the metric duals to hydrodynamics, thus sharpening the results of the previous chapter. The aim of this chapter is to construct a map from the conformal Navier Stokes equations with holographically determined transport coefficients, in d spacetime dimensions, to the set of asymptotically locally AdS$_{d+1}$ long wavelength solutions of Einstein's equations with a negative cosmological constant, for all $d > 2$.

We will find simple explicit expressions for the stress tensor, the full dual bulk metric and an entropy current of this strongly coupled conformal fluid, to second order in the derivative expansion, for arbitrary $d > 2$. We also rewrite the well known exact solutions for rotating black holes in AdS$_{d+1}$ space in a manifestly fluid dynamical form, generalizing earlier work in $d = 4$. To second order in the derivative expansion, this metric agrees with our general construction of the metric dual to fluid flows.

The material for this chapter is drawn from the paper [7] written by the author in collaboration with Sayantani Bhattacharyya, Ipsita Mandal, Shiraz Minwalla and Ankit Sharma.

5.1 Introduction

The AdS/CFT correspondence establishes a deep connection between quantum field theories and theories of gravity. At generic values of parameters both sides of this equivalence are complicated quantum theories. However, every known example of the AdS/CFT duality admits a large N limit in which the gravitational theory turns classical, and a simultaneous strong 't Hooft coupling limit that suppresses α' corrections to gravitational dynamics. In this limit the AdS/CFT correspondence asserts the equivalence between the effectively classical large N dynamics of the local single trace operators $\rho_n = N^{-1}\mathrm{Tr}\mathbf{O_n}$ of gauge theory and the classical two derivative equations of Einstein gravity interacting with other fields.

The usual rules of the AdS/CFT correspondence establish a one to one map between the bulk fields and the single trace field theory operators; for instance, the bulk Einstein frame graviton maps to the field theory stress tensor. Given a solution of the bulk equations, the evolution of any given trace operator $\rho_n(x^\mu)$ may be read off from the normalizable fall off 'at infinity' of the corresponding bulk field. This dictionary allows us to translate the local and relatively simple bulk equations into unfamiliar and extremely nonlocal equations for the boundary trace operators $\rho_n(x)$. The equations for $\rho_n(x)$ are nonlocal in both space and time; indeed the data for the classical evolution of ρ_n includes an infinite number of time derivatives of ρ_n on an initial slice. Given the complicated and unfamiliar nature of these equations, it is difficult to use our knowledge of bulk dynamics to directly gain intuition for boundary trace dynamics. It would clearly be useful to identify a simplifying limit in order to train intuition.

Some simplification of trace dynamics is achieved by focusing on a universal subsector of gravitational dynamics [6] . We focus on two derivative bulk theories of gravity that admit AdS$_{d+1} \times M_I$ as a solution (here M_I is any internal manifold whose character and properties will be irrelevant for the rest of this chapter). It is easy to convince oneself that every such theory admits a consistent truncation to the Einstein equations with a negative cosmological constant. The only fluctuating field under this truncation is the Einstein frame graviton; all other bulk fieldsare simply set to their background $\mathrm{AdS}_{d+1} \times M_I$ values. This observation implies the existence of a sector of decoupled and universal dynamics of the stress tensor in the corresponding dual field theories. The dynamics is decoupled because all $\rho_n(x)$ other than the stress tensor may consistently be set to zero as the stress tensor undergoes its dynamics, and this dynamics is universal because the evolution of the stress tensor is governed by the same equations of motion in each of these infinite class of strongly coupled CFTs.

While the universal stress tensor dynamics described above is clearly simpler than a general evolution of $\rho_n(x)$ in the dual theory, it is still both complicated and nonlocal. It is useful to take a further limit; to focus on boundary configurations in which the local stress tensor varies on a length scale that is large, at any point, compared to a local equilibration length scale (intuitively, 'mean free path') which is set by the 'rest frame' energy density at the same point (we will make this more precise below). Local field theory intuition suggests that boundary configurations that obey this slow variation condition should be locally thermalized, and consequently well described by the equations of boundary fluid dynamics. Hence, we expect the complicated nonlocal $T_{\mu\nu}$ dynamics to reduce to the familiar boundary Navier Stokes equations of fluid dynamics in this long wavelength limit.

All the expectations spelt out above have been demonstrated to be true for $d = 4, 5$ from a direct analysis of the Einstein equations (See [2, 6, 9, 11, 73, 74]). This analysis has also been generalized to a large extent for general d in a recent paper by Haack and Yarom [12]. In this chapter we continue and complete the analysis of [12] in explicating the connection between the Einstein equations and fluid equations in arbitrary dimensions .

In particular, we implement the programme initiated in [6] to explicitly compute the bulk metric dual to an arbitrary fluid flow (accurate to second order in a boundary derivative expansion). We verify the expressions for the second order stress tensor dual to these flows which was recently derived in [12], study the causal structure of the solutions we derive, determine their event horizons at second order in the derivative expansion, and determine an entropy current for these fluid flows. Further, we compare our results to exact solutions for rotating black holes in global AdS_{d+1} and find perfect match to the expected order. In the rest of this introduction we will more carefully review some of the closely related previous work on this subject in order to place the new results of this chapter in its proper context[70].

The authors of [6] developed a procedure to construct a large class of asymptotically AdS_5 long wavelength solutions to Einstein's equations with a negative cosmological constant. The solutions in [6] were worked out order by order in a boundary derivative expansion, and were parameterized by a four velocity field $u^\mu(x^\mu)$ and a temperature field $T(x^\mu)$. These velocity

[70]Our main aim here is to provide the appropriate background for our work rather than to review the complete expanse of the literature relating hydrodynamics to holography. However, we have included a non-exhaustive list of references pertaining to hydrodynamics in the context of holography in the References section at the end of this chapter.

and temperature fields are further constrained to obey the four dimensional generalized Navier Stokes equations $\nabla_\mu T^{\mu\nu} = 0$ where the stress tensor $T^{\mu\nu}(x^\mu)$ is a local functional of the velocity and the temperature fields. The form of $T^{\mu\nu}(x^\mu)$ was explicitly determined in [6] to second order in a boundary derivative expansion (Some terms in the stress tensor were independently determined by the authors of [8]. Especially notable in this regard are the pioneering work in [75–77].) . Consequently, the construction of [6] may be thought of as an explicit map from the space of solutions of a distinguished set of Navier Stokes equations in $d = 4$ to the space of long wavelength solutions of asymptotically AdS_{d+1} gravity.

The spacetimes derived in [6] were subsequently generalized and studied in more detail. In particular, it was demonstrated in [73] that, subject to mild assumptions, these spacetimes have regular event horizons. In the same paper, the location of this event horizon in the 'radial' direction of AdS_5 was explicitly determined to second order in the derivative expansion and it was found to depend locally on the fluid data at the boundary (via a natural boundary to horizon map generated by ingoing null geodesics). The authors of [73] also constructed a local fluid dynamical 'entropy current' utilizing the pullback of the area form on the horizon onto the boundary. The classic area increase theorem of general relativity was then used to demonstrate the local form of the second law of thermodynamics (i.e., the point wise non negativity of the divergence of this entropy current). On a related note, in [2], a formalism was developed for conformal hydrodynamics which describes the long wavelength limit of a CFT. Using this manifestly Weyl-covariant formalism, many results of [6] and [73] could be cast into a simpler form and Weyl covariance could be used as a powerful tool in classifying the possible forms of the metric, energy momentum tensor and the entropy current.

In [11], the construction of [6] was generalized to spacetimes that are only locally asymptotically AdS_5, i.e. that asymptote to

$$ds_5^2 = \frac{1}{z^2}\left[dz^2 + ds_{3,1}^2\right] \tag{5.1}$$

where $ds_{3,1}^2$ is an arbitrary slowly varying boundary metric, at small z. It is expected that, under the AdS/CFT correspondence, such solutions are a universal subsector of the solution space of the relevant CFTs on the Lorentzian base manifold $M_{3,1}$ with the metric $ds_{3,1}^2$. In agreement with this expectation [11] demonstrated that long wavelength solutions of gravity with asymptotics given by (5.1) were parameterized by a velocity and temperature field on the manifold $M_{3,1}$, subject to a covariant form of the Navier Stokes equations. As an example of this construction, the authors of [11] were able to rewrite the exact asymptotically global AdS_5 Kerr black hole solutions in a very simply manifestly fluid dynamical form, and demonstrate that the expansion of this metric to second order in the derivative expansion is in perfect agreement with the general construction of the metrics dual to fluid dynamics at second order [71].

[71] The authors of [11] also considered the coupling of a slowly varying dilaton to the metric. It would be an interesting exercise to consider generalizing the results of this chapter to a bulk spacetime with dilaton dynamics. However, in this chapter, we will confine ourselves to the case where the dilaton is set to its

All of the results described above were originally worked out for the special case $d = 4$, but some of these results and constructions have since been further generalized. In an early paper Van Raamsdonk [9] generalized the construction of the full second order bulk metric to an arbitrary fluid flow on a flat boundary to $d = 3$ and also computed the holographic fluid dynamical stress tensor to second order in boundary derivatives. Some terms in the second order stress tensor for the uncharged conformal fluid in arbitrary dimensions were calculated using different methods by [10, 78]. Further, $1/\lambda$ and $1/N_c$ corrections to some coefficients have been computed in [74, 79–82].

More recently, Haack and Yarom [12] partially constructed the second order bulk metric to an arbitrary fluid flow in a flat d dimensional boundary (for arbitrary d) and fully computed the dual second order fluid dynamical stress tensor for a flat boundary. In this chapter, we continue the study of Haack and Yarom [12] to generalize all of the work on solutions of pure gravity duals to arbitrary fluid flows in $d = 4$ dimensions (reviewed above) to arbitrary $d > 2$.

This chapter is organized as follows. In section §5.2 below, we begin by briefly explaining the logic of our construction of long wavelength bulk solutions dual to fluid dynamics in the Weyl covariant notation. This is followed by section §5.3 below we present explicit solutions to Einstein equations to second order in the boundary derivative expansion. Our solutions asymptote at small z to

$$ds^2_{d+1} = \frac{1}{z^2}\left[dz^2 + ds^2_{d-1,1}\right] \tag{5.2}$$

where $ds^2_{d-1,1}$ is the arbitrarily specified weakly curved metric on the boundary. Our solutions are parameterized by a boundary d-velocity field $u^\mu(x)$ and a temperature field $T(x)$ where x^μ are the boundary coordinates. These velocity and temperature fields are constrained to obey the d dimensional Navier Stokes equations, $\nabla_\mu T^{\mu\nu} = 0$ where $T^{\mu\nu}$ is a local functional of the velocity and temperature fields. We also present explicit expressions for the boundary stress tensor $T^{\mu\nu}$ dual to our solutions. Our answer can be expressed in an especially simple and manifestly Weyl-covariant form

$$\begin{split} T_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) - 2\eta\sigma_{\mu\nu} \\ &\quad - 2\eta\tau_\omega\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] \\ &\quad + 2\eta b\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] \end{split} \tag{5.3}$$

with

$$b \equiv \frac{d}{4\pi T} \qquad ; \qquad p = \frac{1}{16\pi G_{\text{AdS}} b^d} \qquad ;$$

$$\eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{\text{AdS}} b^{d-1}} \qquad \text{and} \qquad \tau_\omega = b\int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy$$

---

background value. On a related note , we should also mention two recent papers[17, 18] which appeared while this chapter was nearing completion in which the fluid gravity correspondence was extended to a class of charged black holes in AdS_5 with flat boundary.

where p is the pressure, T is the temperature, s is the entropy density and η is the viscosity of the fluid. τ_ω denotes a particular second-order transport coefficient of the fluid, $\sigma_{\mu\nu}$ is the shear strain rate , $\omega_{\mu\nu}$ is the vorticity and $C_{\mu\nu\alpha\beta}$ is the Weyl Tensor of the spacetime in which the fluid lives. Note that our result for the stress tensor agrees with those of Haack and Yarom[12] when restricted to a flat boundary manifold, but also includes an additional term proportional to boundary curvature that vanishes in flat space.

In section §5.4 below , we demonstrate that our solutions all have a regular event horizon, and find an expression for the radial location of that event horizon upto second order in the derivative expansion. We also construct a boundary entropy current J_S that is forced by the area increase theorem of general relativity to obey the equation $\nabla_\mu J^\mu_S \geq 0$. This is followed by section §5.5 where we rewrite the exactly known rotating black hole solutions in global AdS$_{d+1}$ in a manifestly fluid dynamical form. These solutions turn out to be dual to rigid fluid flows on $S^{d-1,1}$ (see [15] for earlier work). These initially complicated looking blackhole metrics admit a rewriting in a rather simple form in the fluid dynamical gauge and variables used in this chapter. In appropriate co-ordinates, the general AdS-Kerr metric [72] assumes the form

$$\begin{aligned} ds^2 = &- 2u_\mu dx^\mu \left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right] dx^\mu dx^\nu \\ &+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu \end{aligned} \tag{5.4}$$

where $\mathcal{A}_\mu$ is the fluid dynamical Weyl-connection and $\mathcal{S}_{\mu\nu}$ is the Weyl-covariantized Schouten tensor introduced in [2]. We demonstrate that the expansion of these solutions to second order in the derivative expansion agrees with our general construction of metrics dual to fluid dynamics. We end this chapter with a discussion of our results and possible generalizations.

5.2 Perturbative Construction of Solutions

In this section, we will briefly review the basic logic that underlies the construction of gravity solutions dual to arbitrary fluid flows. The methodology employed in this chapter is an almost direct generalization of the techniques used in [2, 6, 9, 11, 12, 73, 74]. Consequently in this chapter we will describe the logic of our construction and the details of implementation only briefly, referring the reader to the references above for more details.

5.2.1 Equations of motion and uniform brane solutions

In this chapter we develop a systematic perturbative expansion to solve Einstein's equations with a negative cosmological constant

$$\mathcal{G}_{AB} - \frac{d(d-1)}{2} G_{AB} = 0, \quad M, N = 1 \ldots d+1 \tag{5.5}$$

[72]Note in particular that for $d = 2$, $\sigma_{\mu\nu} = \omega_{\mu\nu} = 0$ and $\mathcal{S}_{\mu\nu}$ term is absent in which case we get the BTZ blackhole in AdS$_3$ as shown in[12].

where $\mathcal{G}_{AB}$ denotes the Einstein tensor of the bulk metric G_{AB}.

One solution of these equations is AdS spacetime of unit radius

$$ds^2 = \frac{dr^2}{r^2} + r^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu\right), \quad \mu, \nu = 1 \ldots d \tag{5.6}$$

Other well known solutions to these equations include boosted black branes which we write here in Schwarzschild like coordinates

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2 f(r)} + r^2 \left(-f(r) u_\mu u_\nu dx^\mu dx^\nu + \mathcal{P}_{\mu\nu} dx^\mu dx^\nu\right) \\ f(r) &= 1 - \frac{1}{(br)^d}, \quad g_{\mu\nu} u^\mu u^\nu = -1, \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad b = \frac{d}{4\pi T} \end{aligned} \tag{5.7}$$

$g_{\mu\nu}$ in (5.7) is an arbitrary constant boundary metric of signature $(d-1, 1)$, while u^μ in the same equation is any constant unit normalized d velocity. Of course any constant metric of signature $(d-1, 1)$ can be set to $\eta_{\mu\nu}$ by an appropriate linear coordinate transformation $x^\mu \to \Lambda^\mu_\nu x^\nu$, and u^μ can subsequently be set to $(1, 0 \ldots 0)$ by a boundary Lorentz transformation. Further b in (5.7) may also be set to unity by a coordinate change; a uniform rescaling of boundary coordinates coupled with a rescaling of r. Thus the $d(d+3)/2$ parameter set of metrics (5.7) are all coordinate equivalent. Nonetheless we will find the general form (5.7) useful below; indeed we will find it useful to write (5.7) in an even more general coordinate redundant form. Consider

$$ds^2 = \frac{\left(d\tilde{r} + \tilde{r}\mathcal{A}_\nu dx^\nu\right)^2}{\tilde{r}^2 f(r)} + \tilde{r}^2 \left(-f(\tilde{b}\tilde{r}) \tilde{u}_\mu \tilde{u}_\nu dx^\mu dx^\nu + \tilde{\mathcal{P}}_{\mu\nu} dx^\mu dx^\nu\right) \tag{5.8}$$

where

$$\tilde{g}_{\mu\nu} = e^{2\phi(x^\mu)} g_{\mu\nu}, \quad \tilde{u}_\mu = e^{\phi(x^\mu)} u_\mu, \quad \tilde{b} = e^{\phi(x^\mu)} b, \tag{5.9}$$

$\phi(x^\mu)$ is an arbitrary function and $g_{\mu\nu}$, u^μ, and b are as defined in the previous equation. This metric is coordinate equivalent to (5.7) under the variable transformation $\tilde{r} \mapsto e^{-\phi} r$. Consequently, the whole function worth of spacetimes (5.8) (taken together with the restrictions (5.9)) are all exact solutions to Einstein's equations and are all coordinate equivalent.

While the metrics (5.8) all describe the same bulk geometry, in this chapter we will give these spacetimes distinct though Weyl equivalent boundary interpretations by regulating them inequivalently near the boundary. We will choose to regulate the spacetimes (5.8) on slices of constant $\tilde{r}$ and consequently regard them as states in a conformal field theory on a space with metric $\tilde{g}_{\mu\nu}(x)$. With this convention, (the non-anomalous part of) the boundary stress tensor dual to (5.8) is given by

$$T_{\mu\nu} = \frac{1}{16\pi G_{\mathrm{AdS}} \tilde{b}^d} \left(\tilde{g}_{\mu\nu} + d\tilde{u}_\mu \tilde{u}_\nu\right) \tag{5.10}$$

which shows that the metric in the Equation(5.8) is dual to a conformal fluid with a pressure $p = 1/(16\pi G_{\mathrm{AdS}} b^d)$ and without any vorticity or shear strain rate. Of course the boundary configurations dual to (5.8) with equal $g_{\mu\nu}, u^\mu, b$ but different values of ϕ are related to each other by boundary Weyl transformations.

Notice that

$$T^\mu_\nu u^\nu = \frac{K}{b^d} u^\mu, \quad K = -\frac{(d-1)}{16\pi G_{\mathrm{AdS}}} \tag{5.11}$$

In other words the velocity field is the unique time like eigenvector of the stress tensor, and the inverse temperature field b is simply related to its eigenvalue. We will use this observation in the next subsection.

5.2.2 Slow variation and bulk tubes and our zero order ansatz

Consider an arbitrary locally asymptotically AdS_{d+1} solution to Einstein's equations (5.5) whose dual boundary stress tensor everywhere has a unique timelike eigenvector. Let this eigenvector (after unit normalization) be denoted by $u^\mu(x)$ and the corresponding eigenvalue by $\frac{K}{b^d}$. We define $u^\mu(x)$ to be the d velocity field dual to our solution, and also define $b(x)$ to be the inverse temperature field dual to our solution.

Let $\delta x(y)$ denote smallest length scale of variation of the stress tensor of the corresponding solution at the point y. We say that the solution is 'slowly varying' if everywhere $\delta x(y) \gg b(y)$. (As will be apparent from our final stress tensor below, $b(y)$ may be interpreted as the effective length scale of equilibration of the field theory at y). Similarly, we say that the boundary metric is weakly curved if $b(y)^2 R(y) \ll 1$ (where $R(y)$ is the curvature scalar, or more generally an estimate of the largest curvature scale in the problem).

In the previous section we described uniform brane solutions of Einstein's equations. In the appropriate Weyl frame the temperature, velocity, boundary metric and hence the stress tensor of those configurations was constant in boundary spacetime. These configurations are exact solutions to Einstein's equations. We will now search for solutions to Einstein equations with slowly varying (rather than constant) boundary stress tensors on a boundary manifold that has a weakly curved (rather than flat) boundary metric. From field theory intuition we expect all such boundary configurations to be locally patchwise equilibriated (but with varying values of the boundary temperature and velocity fields). This suggests that the corresponding bulk solutions should approximately be given by patching together tubes of the uniform black brane solutions. We expect these tubes to start along local patches on the boundary and then extend into the bulk following an ingoing 'radial' curve. However this expectation leaves open an important question: what is the precise shape of the radial curves that our tubes follow?

One guess might be that the tubes follow the lines x^μ =constant in the Schwarzschild coordinates we have employed so far. According to this guess, the bulk metric dual to slowly varying boundary stress tensors and boundary metric is approximately given by

$$ds^2 = \frac{(d\tilde{r} + \tilde{r}\mathcal{A}_\nu dx^\nu)^2}{\tilde{r}^2 f(r)} + \tilde{r}^2 \left(-f(\tilde{b}\tilde{r})\tilde{u}_\mu \tilde{u}_\nu dx^\mu dx^\nu + \tilde{\mathcal{P}}_{\mu\nu} dx^\mu dx^\nu \right) \tag{5.12}$$

where

$$\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \quad \tilde{u}_\mu = e^\phi u_\mu, \quad \tilde{b} = e^\phi b, \tag{5.13}$$

where $g_{\mu\nu}(x)$ is a weakly curved boundary metric, and $u_\mu(x)$ and $b(x)$ are slowly varying boundary functions.

Although this guess seems natural, we believe it is wrong. The technical problem with this guess is that the metric of (5.12) does not in general have a regular future horizon[83] (for particular examples of similar metrics that do not have a regular future horizon see [81, 84, 85].The last two references show a boost-invariant expansion that develops a singular future horizon). In this chapter we will be interested only in regular solutions of Einstein equations; solutions whose (future) singularities are all shielded from the boundary of AdS by regular event horizons. As any perturbation to (5.12) that turns it into a regular space must necessarily be large in the appropriate sense, it follows that (5.12) is not a good starting point for a perturbative expansion of the solutions we wish to find.

There is another more intuitive problem with the proposal that the ansatz (5.12) is dual to boundary fluid dynamics. It is an obvious fact about fluid dynamical evolution that the initial conditions of a fluid may be chosen independent of any 'kick' (forcing) one may choose to apply to the fluid at a later time. It seems reasonable to expect the same property of the bulk solutions dual to fluid dynamics. [73] Now consider kicking a fluid in an arbitrary motion at the point y^μ. The future evolution of the fluid is affected only in the 'fluid causal future' - of y^μ. We call this region $C(y^\mu)$. Note that $C(y^\mu)$ lies within the future boundary light cone of y^μ [74]. Now consider the bulk region $B(y(\mu)$ that consists of the union of all the tubes, referred to above, that originate in the boundary region $C(y^\mu)$. Clearly $B(y^\mu)$ is the part of the bulk spacetime that is affected by our kick at y^μ. Bulk causality implies that $B(y^\mu)$ must lie entirely within the future bulk light cone of y^μ.

This requirement is not met if we generate $B(y^\mu)$ with our tubes that run along lines of constant x^μ in Schwarzschild coordinates. However it is met in a particularly natural way (given the massless nature of the graviton) if our tubes are chosen to run along ingoing null geodesics. [75]

With this discussion in mind, let us consider the ansatz

$$ds^2 = -2u_\mu dx^\mu \left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu + \frac{r^2}{(br)^d} u_\mu u_\nu dx^\mu dx^\nu \tag{5.14}$$

where once again

$$\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}(x), \quad \tilde{u}_\mu = e^{\phi} u_\mu(x), \quad \tilde{b} = e^{\phi} b(x), \tag{5.15}$$

and $g_{\mu\nu}(x)$ is a weakly curved boundary metric, and $u_\mu(x)$ and $b(x)$ are slowly varying boundary functions. When $g_{\mu\nu}$ u_μ and b are all constant, (5.14) is once again simply the uniform brane solution, rewritten in Eddington Finkelstein coordinates; i.e. when $g_{\mu\nu}$ u_μ and b are all constant (5.14) and (5.8) are coordinate equivalent(via large co-ordinate transformations).

[73] In our set up we can kick our fluid at y^μ by varying the boundary metric at y^μ (this induces an effective force on the fluid).

[74] This is strictly true only if we sum all orders in the fluid expansion. Truncation at any finite order could lead to apparent violations of causality over length scales of order $1/T$.

[75] For a related discussion on the desirability of using ingoing null geodesic tubes vis a vis causality violating tubes, see [86, 87].

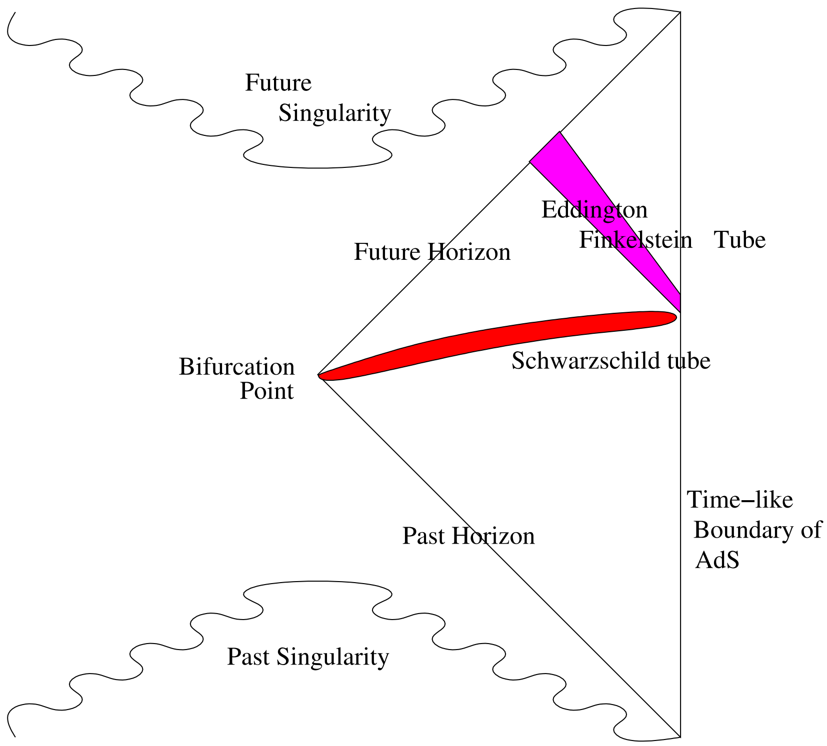

Figure 2. Penrose diagram of the uniform black brane illustrating the causal Eddington-Finkelstein tubes running along ingoing null geodesics . The tubes with $x^{\mu}_{\text{Schwarzschild}}$=constant are also shown. Note that we have suppressed the other regions of the penrose diagram not germane to the discussion in this chapter.

However when $g_{\mu\nu}$ u_{μ} and b are functions of x^{μ} (5.14) and (5.12) are inequivalent and in fact differ qualitatively. As we will demonstrate below, under mild assumptions the metric in (5.14) has a regular event horizon that shields all the boundary from all future singularities in this space. Consequently, this space may (unlike the spacetime in (5.8)) legitimately be used as the first term in the perturbative expansion of a regular solution of Einstein's equations. Moreover the space described in (5.14) approximates the uniform brane solution along tubes of constant x^{μ} in (5.14); such tubes approximately follow null ingoing geodesics in this space.

For all these reasons, in the rest of this chapter we will use (5.14) as the first term in a systematic perturbative expansion of a regular solution to Einstein's equations. The perturbative expansion parameter is $\frac{1}{b\delta x}$ (we assume that the curvature scale in the metric is of the same order as $1/\delta x$). We emphasize that the solutions we find could not be obtained in a legitimate perturbation expansion, starting from (5.8). Several authors have attempted to obtain the bulk metric dual to a 'boost invariant Bjorken fluid flow' starting with the zero order solution described by Janik and Peschanski[75], and correcting it in an expansion in $1/\delta x b$ (that turns into an expansion in $1/\tau^{\frac{2}{3}}$ for those particular solutions). As pointed out in

[86, 87], however, the zeroth order solution of Janik and Peschanski is precisely (5.8) for the particular case of boost invariant flow. Consequently, while the approach of the current work and [6, 9, 11, 12, 74] are similar in spirit to the perturbation procedure initiated by Janik and Peschanski, we differ at a crucial point. While those authors effectively adopt (5.8) as the starting point of their perturbation theory (for the single solution they consider), in our work we adopt the inequivalent and qualitatively different space (5.14) as the starting point of our perturbative expansion.

5.2.3 Perturbation theory at long wavelengths

The logic behind - and the method of implementation of - this perturbative procedure have been described in detail in [6] and also in [9, 11, 12, 74]. It has also been described in those papers how this perturbative procedure establishes a map between solutions of fluid dynamics and regular long wavelength solutions of Einstein gravity with a negative cosmological constant. The discussion in the cited references applies almost without modification to the current work, so we describe it only very briefly.

We start with the ansatz $g_{MN} = g^{(0)}_{MN} + \epsilon g^{(1)}_{MN} + \epsilon^2 g^{(2)}_{MN} + \ldots$. Here $g^{(0)}_{MN}$ is given by (5.14), ϵ is the small parameter of the derivative expansion, and $g^{(k)}_{MN}$ are the corrections to the bulk metric that we will determine with the aid of the bulk Einstein equation.

In implementing our perturbative procedure we adopt a choice of gauge. As in all the metrics described above, we use the coordinates r, x^μ for our bulk spaces. We use x^μ as coordinates that parameterize the boundary and r is a radial coordinate. In order to give precise meaning to our coordinates we need to adopt a choice of gauge. In this chapter we choose the gauge $g_{rr} = 0$ together with $g_{r\mu} = -u_\mu$. The geometrical implication of this gauge choice was discussed in [11], where it was explained that with this choice lines of constant x^μ are ingoing null geodesics along each of which r is an affine parameter. Note that the gauge choice described in this chapter is different in detail from that employed in [6] and also in [9, 11, 12, 74].

The Bulk Einstein equations decompose into 'constraints' on the boundary hydrodynamic data and 'dynamical equations' for the bulk metric along the tubes which are solved order by order in the derivative expansion. The dynamical equations determine the corrections that should be added to our initial metric to make it a solution of the Einstein equations. At each order, we get inhomogeneous linear equations -but, with the same homogeneous parts. These inhomogeneous linear equations obtained from Einstein equations can be solved order by order by imposing regularity at the zeroth order future horizon and appropriate asymptotic fall off at the boundary.These boundary conditions - together with a clear definition of velocity, which fixes the ambiguity of adding zero modes - give a unique solution for the metric, as a function of the original boundary velocity and temperature profile inputted into the metric $g^{(0)}_{MN}$ - order by order in the boundary derivative expansion.

Now, we turn to the 'constraints'. The 'constraints' on the boundary data can be shown to be equivalent to the requirement of the conservation of the boundary stress tensor. Recall that we have already used the dynamical Einstein equations to determine the full bulk metric

- and hence the boundary stress tensor - as a function of the input velocity and temperature fields. It follows that the constraint Einstein equations reduce simply to the equations of fluid dynamics, i.e. the requirement of a conserved stress tensor which, in turn, is a given function of temperature and velocity fields.

It may be worthwhile to reiterate that, as expected from fluid-gravity correspondence, metric duals which solve Einstein equations can be constructed only for those fluid configurations which solve the hydrodynamic equations. In the next section, we will present the metric which is obtained by adopting this procedure.

5.2.4 Weyl Covariance

In this subsection we explain that the bulk metrics dual to fluid dynamics must transform covariantly under boundary 'Weyl' transformations. See [11] for a more detailed explanation of this fact.

To start with we note that our bulk gauge choice (described in the previous subsection) is Weyl covariant. Any metric that obeys that gauge choice can be put in the form

$$ds^2 = -2u_\mu(x)dx^\mu(dr + \mathcal{V}_\nu(r,x)dx^\nu) + \mathfrak{G}_{\mu\nu}(r,x)dx^\mu dx^\nu \tag{5.16}$$

where $\mathfrak{G}_{\mu\nu}$ is transverse, i.e., $u^\mu\mathfrak{G}_{\mu\nu} = 0$. [76]

For later purposes, we note that the inverse of this bulk metric takes the form

$$\begin{aligned} &u^\mu\left[(\partial_\mu - \mathcal{V}_\mu\partial_r)\otimes\partial_r + \partial_r\otimes(\partial_\mu - \mathcal{V}_\mu\partial_r)\right] \\ &\quad + (\mathfrak{G}^{-1})^{\mu\nu}(\partial_\mu - \mathcal{V}_\mu\partial_r)\otimes(\partial_\nu - \mathcal{V}_\nu\partial_r) \end{aligned} \tag{5.18}$$

where the symmetric matrix $(\mathfrak{G}^{-1})^{\mu\nu}$ is uniquely defined by the relations $u_\mu(\mathfrak{G}^{-1})^{\mu\nu} = 0$ and $(\mathfrak{G}^{-1})^{\mu\lambda}\mathfrak{G}_{\lambda\nu} = \delta^\mu_\nu + u^\mu u_\nu \equiv P^\mu_\nu$.

Consider now a bulk-diffeomorphism of the form $r = e^{-\phi}\tilde{r}$ along with a scaling in the temperature of the form $b = e^\phi\tilde{b}$ where we assume that $\phi = \phi(x)$ is a function only of the boundary co-ordinates. The metric and the inverse metric components transform as

$$\mathcal{V}_\mu = e^{-\phi}\left[\tilde{\mathcal{V}}_\mu + \tilde{r}\ \partial_\mu\phi\right], \quad u_\mu = e^\phi\tilde{u}_\mu, \quad \mathfrak{G}_{\mu\nu} = \tilde{\mathfrak{G}}_{\mu\nu} \quad \text{and} \quad (\mathfrak{G}^{-1})^{\mu\nu} = (\tilde{\mathfrak{G}}^{-1})^{\mu\nu} \tag{5.19}$$

Recall however that, within our procedure, the quantities $\mathfrak{G}_{\mu\nu}$ and $\mathcal{V}_\mu$ are each functions of u^μ and b. Now u^μ and b each pick up a factor of e^ϕ under the same diffeomorphism (the transformation of b is determined by examining the action of the diffeomorphism on (5.14)). We conclude that consistency demands that $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ are functions of b and u^μ that respectively transform like a connection/remain invariant under boundary Weyl transformation. It follows immediately that, for instance $\mathfrak{G}_{\mu\nu}$ is a linear sum of the Weyl invariant forms listed

[76] All the Greek indices are raised and lowered using the boundary metric $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = \lim_{r\to\infty} r^{-2}\left[\mathfrak{G}_{\mu\nu} - u_{(\mu}\mathcal{V}_{\nu)}\right] \tag{5.17}$$

and u_μ is the unit time-like velocity field in the boundary, i.e., $g^{\mu\nu}u_\mu u_\nu = -1$.

in section 2, with coefficients that are arbitrary functions of br. Similarly, $\mathcal{V}_\mu - r\mathcal{A}_\mu$ is a linear sum of Weyl-covariant vectors(both transverse and non-transverse) with weight unity.

Symmetry requirements do not constrain the form of these coefficients, which have to be determined via direct calculation. In the next section we simply present the results of such a calculation.

5.3 The bulk metric and boundary stress tensor to second order

5.3.1 The metric dual to hydrodynamics

Using a Weyl-covariant form of the procedure outlined in [6], we find that the final metric can be written in the form

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu \left(dr + r\ A_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right] dx^\mu dx^\nu \\ &+ \frac{1}{(br)^d}(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})u_\mu u_\nu dx^\mu dx^\nu + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right] dx^\mu dx^\nu \\ &- 2(br)^2\left[K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + K_2(br)\frac{u_\mu u_\nu}{(br)^d}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{2(d-1)} - \frac{L(br)}{(br)^d}u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\alpha\sigma^\alpha{}_\lambda\right] dx^\mu dx^\nu \\ &- 2(br)^2 H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] dx^\mu dx^\nu \\ &+ 2(br)^2 H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] dx^\mu dx^\nu \end{aligned} \tag{5.20}$$

We have checked using Mathematica that the above metric solves Einstein equations upto $d = 10$.

The various functions appearing in the metric are defined by the integrals

$$\begin{aligned} F(br) &\equiv \int_{br}^\infty \frac{y^{d-1}-1}{y(y^d-1)}dy \\ H_1(br) &\equiv \int_{br}^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy \\ H_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi y^{d-3}dy\left[1 + (d-1)yF(y) + 2y^2F'(y)\right] \\ &= \frac{1}{2}F(br)^2 - \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi \frac{y^{d-2}-1}{y(y^d-1)}dy \\ K_1(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\int_\xi^\infty dy\ y^2F'(y)^2 \\ K_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\left[1 - \xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1}\right. \\ &\qquad \left. + \Big(2(d-1)\xi^d - (d-2)\Big)\int_\xi^\infty dy\ y^2F'(y)^2\right] \\ L(br) &\equiv \int_{br}^\infty \xi^{d-1}d\xi\int_\xi^\infty dy\ \frac{y-1}{y^3(y^d-1)} \end{aligned} \tag{5.21}$$

Later in this chapter, we will find it convenient to work with other equivalent forms of the above metric. Using

$$\mathcal{S}_{\mu\lambda}u^\lambda = -\frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_\mu + \frac{1}{d-2}\mathcal{D}_\lambda \sigma^\lambda{}_\mu - \frac{\mathcal{R}}{2(d-1)(d-2)}u_\mu + \dots \tag{5.22}$$

we can write

$$\begin{aligned}
ds^2 &= -2u_\mu dx^\mu\left(dr + r\ A_\nu dx^\nu\right) + r^2 g_{\mu\nu}dx^\mu dx^\nu \\
&\quad - \left[\omega_\mu{}^\lambda\omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda\omega^\lambda{}_{(\mu}u_{\nu)} - \frac{1}{d-2}\mathcal{D}_\lambda\sigma^\lambda{}_{(\mu}u_{\nu)} + \frac{\mathcal{R}}{(d-1)(d-2)}u_\mu u_\nu\right]dx^\mu dx^\nu \\
&\quad + \frac{1}{(br)^d}(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})u_\mu u_\nu dx^\mu dx^\nu + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right]dx^\mu dx^\nu \\
&\quad - 2(br)^2\left[K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + K_2(br)\frac{u_\mu u_\nu}{(br)^d}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{2(d-1)} - \frac{L(br)}{(br)^d}u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\alpha\sigma^\alpha{}_\lambda\right]dx^\mu dx^\nu \\
&\quad - 2(br)^2 H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right]dx^\mu dx^\nu \\
&\quad + 2(br)^2 H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} - \sigma_\mu{}^\lambda\omega_{\lambda\nu}\right]dx^\mu dx^\nu
\end{aligned} \tag{5.23}$$

or alternatively the metric can be written in the form (5.16)

$$ds^2 = -2u_\mu dx^\mu(dr + \mathcal{V}_\nu dx^\nu) + \mathfrak{G}_{\mu\nu}dx^\mu dx^\nu \qquad \text{with}$$

$$\begin{aligned}
\mathcal{V}_\mu &= r\mathcal{A}_\mu - \mathcal{S}_{\mu\lambda}u^\lambda - \frac{2L(br)}{(br)^{d-2}}P^\nu_\mu\mathcal{D}_\lambda\sigma^\lambda{}_\nu \\
&\quad - \frac{u_\mu}{2(br)^d}\left[r^2(1-(br)^d) - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} - (br)^2 K_2(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right] + \dots \\
&= r\mathcal{A}_\mu + \frac{1}{d-2}\left[\mathcal{D}_\lambda\omega^\lambda{}_\mu - \mathcal{D}_\lambda\sigma^\lambda{}_\mu + \frac{\mathcal{R}}{2(d-1)}u_\mu\right] - \frac{2L(br)}{(br)^{d-2}}P^\nu_\mu\mathcal{D}_\lambda\sigma^\lambda{}_\nu \\
&\quad - \frac{u_\mu}{2(br)^d}\left[r^2(1-(br)^d) - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} - (br)^2 K_2(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right] + \dots \\
\mathfrak{G}_{\mu\nu} &= r^2 P_{\mu\nu} - \omega_\mu{}^\lambda\omega_{\lambda\nu} \\
&\quad + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right] - 2(br)^2 K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} \\
&\quad - 2(br)^2 H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] \\
&\quad + 2(br)^2 H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] + \dots
\end{aligned} \tag{5.24}$$

Using (5.18), the inverse metric can be calculated . The tensor $(\mathfrak{G}^{-1})^{\mu\nu}$ occurring in the inverse metric can be calculated as

$$
\begin{aligned}
(\mathfrak{G}^{-1})^{\mu\nu} = &\frac{1}{r^2}P^{\mu\nu} + \frac{1}{r^4}\omega^{\mu\lambda}\omega_{\lambda}{}^{\nu} \\
&- \frac{2b^2}{r^2}F(br)\left[\frac{1}{b}\sigma^{\mu\nu} - F(br)\sigma^{\mu}{}_{\lambda}\sigma^{\lambda\nu}\right] + \frac{2b^2}{r^2}K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P^{\mu\nu} \\
&+ \frac{2b^2}{r^2}H_1(br)\left[u^{\lambda}\mathcal{D}_{\lambda}\sigma^{\mu\nu} + \sigma^{\mu}{}_{\lambda}\sigma^{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P^{\mu\nu} + C^{\mu\alpha\nu\beta}u_{\alpha}u_{\beta}\right] \\
&- \frac{2b^2}{r^2}H_2(br)\left[u^{\lambda}\mathcal{D}_{\lambda}\sigma^{\mu\nu} + \omega^{\mu}{}_{\lambda}\sigma^{\lambda\nu} + \omega^{\nu}{}_{\lambda}\sigma^{\mu\lambda}\right] + \ldots
\end{aligned}
\tag{5.25}
$$

We have checked that results of this subsection agree with the hydrodynamic metric duals for $d=4$ derived by the authors of [6] and the $d=3$ metric derived in [9] (in order to match our results with older work that was performed in different gauges we implemented the necessary gauge transformations). In the next subsection, we proceed to derive the stress tensor dual to this metric and compare it against the results available in the literature.

5.3.2 Energy momentum tensor of fluids with metric duals

The dual stress tensor corresponding to the metric in the previous subsection is given by

$$
\begin{aligned}
T_{\mu\nu} = &\, p\left(g_{\mu\nu} + d u_{\mu}u_{\nu}\right) - 2\eta\sigma_{\mu\nu} \\
&- 2\eta\tau_{\omega}\left[u^{\lambda}\mathcal{D}_{\lambda}\sigma_{\mu\nu} + \omega_{\mu}{}^{\lambda}\sigma_{\lambda\nu} + \omega_{\nu}{}^{\lambda}\sigma_{\mu\lambda}\right] \\
&+ 2\eta b\left[u^{\lambda}\mathcal{D}_{\lambda}\sigma_{\mu\nu} + \sigma_{\mu}{}^{\lambda}\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta}\right]
\end{aligned}
\tag{5.26}
$$

with

$$
\begin{aligned}
b = \frac{d}{4\pi T} \qquad &; \qquad p = \frac{1}{16\pi G_{\text{AdS}} b^d} \\
\eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{\text{AdS}} b^{d-1}} \quad &\text{and} \quad \tau_{\omega} = b\int_1^{\infty}\frac{y^{d-2}-1}{y(y^d-1)}dy
\end{aligned}
\tag{5.27}
$$

This result is a generalization to the fluid dynamical stress tensor on an arbitrary curved manifold in general dimension d reported in [2, 6, 8–11] for special values of d and most recently by [12] for flat space in arbitrary dimensions. The values of τ_{ω} for some of the lower dimensions is shown[77] in the table 5.3.2.

[77] More generally, the integral appearing in the expression for τ_{ω} can be evaluated in terms of the derivative of the Gamma function or more directly in terms of 'the harmonic number function' with the fractional argument(as was noted in [13])

$$
\tau_{\omega} = -\frac{b}{d}\left[\gamma_E + \frac{d}{dz}\text{Log }\Gamma(z)\right]_{z=2/d} = -\frac{b}{d}\text{Harmonic}[2/d-1]
$$

For large d, τ_{ω} has an expansion of the form $\tau_{\omega}/b = 1/2 - \pi^2/(3d^2) + \ldots$.

The energy momentum tensor of a general conformal fluid configuration to second order in derivative expansion should assume the form [2, 8]

$$\begin{aligned} T^{\mu\nu} &= p(g^{\mu\nu} + d u^\mu u^\nu) \\ &\quad - 2\eta \left[\sigma^{\mu\nu} - \tau_1\ u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \tau_2 (\omega^\mu{}_\lambda \sigma^{\lambda\nu} + \omega^\nu{}_\lambda \sigma^{\lambda\mu})\right] \\ &\quad + \xi_\sigma [\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1} \sigma^{\alpha\beta}\sigma_{\alpha\beta}] + \xi_C\ C_{\mu\alpha\nu\beta} u^\alpha u^\beta \\ &\quad + \xi_\omega [\omega^\mu{}_\lambda \omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1} \omega^{\alpha\beta}\omega_{\alpha\beta}] + \ldots \end{aligned} \tag{5.28}$$

where Weyl-covariance demands that

$$p \propto b^{-d}\ , \quad \eta \propto b^{1-d}\ , \quad \tau_{1,2} \propto b\ , \quad \xi_{\sigma,C,\omega} \propto b^{2-d} \tag{5.29}$$

Thus, we get

$$\xi_\sigma = \xi_C = 2\eta(\tau_1 + \tau_2) = 2\eta b \quad , \quad \tau_2 = \tau_\omega \quad \text{and} \quad \xi_\omega = 0 \tag{5.30}$$

Note that these relations between $\xi_\sigma, \xi_C, \tau_1$ and τ_2 quoted above are universal in the sense that they hold true for uncharged fluids in arbitrary dimensions with the gravity duals. It would be interesting to check whether these relations between the transport coefficients continue to hold against various possible generalizations including the generalization to fluids with one or more global conserved charge.[78]

Now, we proceed to compare our results against the results already available in the literature. Until now, we have found it convenient to closely follow the parametrisation of the stress tensor in [2]. An alternative parametrisation of the energy-momentum tensor was presented in the section 3.1 of [8] - the parameters $\tau_\Pi, \lambda_{1,2,3}$ and κ defined there can be related to our parameters via the relations

$$\tau_1 = \tau_\Pi \quad , \quad \tau_2 = -\frac{\lambda_2}{2\eta} \quad , \quad \xi_\sigma = 4\lambda_1 \quad , \quad \xi_C = \kappa(d-2) \quad \text{and} \quad \xi_\omega = \lambda_3 \tag{5.31}$$

which in turn gives the value of the transport coefficients as

$$\tau_\Pi = b - \tau_\omega \quad , \quad \lambda_1 = \frac{\eta b}{2} \quad , \quad \lambda_2 = -2\eta\tau_\omega \quad , \quad \lambda_3 = 0 \quad \text{and} \quad \kappa = \frac{2\eta b}{d-2} \tag{5.32}$$

which agrees with all the previous results in the literature [9, 10, 12].

[78] In this context, we would like to note that in the presence of a charge, there are more than one natural convention for the definition of the velocity - velocity can be defined as the unit time-like eigenvector of the energy-momentum tensor (as we have done in the chargeless case) or can be defined alternatively to be the unit time-like vector along the charge current. The former is called the Landau frame velocity and the latter is termed the velocity in the Eckart frame. The transport coefficients defined above can depend crucially on which of these definitions are used.

While this work was nearing completion, the authors of [18] and [17] reported independently the transport coefficients for a particular class of charged black brane configurations with flat boundaries. Interestingly, their coefficients continue to obey $\xi_\sigma = 2\eta(\tau_1 + \tau_2)$ (or equivalently $4\lambda_1 + \lambda_2 = 2\eta\tau_\Pi$) in the Landau frame. As far as we know, the charge dependence of ξ_C is not known yet. Authors of [18] and [17] report $\xi_\omega \neq 0$ for the charged case in the Landau frame.

Value of τ_ω/b for various dimensions		
d	Value of $\tau_\omega/b = \int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy$	τ_ω/b (Numerical)
3	$\frac{1}{2}\left(\text{Log }3 - \frac{\pi}{3\sqrt{3}}\right)$	0.247006...
4	$\frac{1}{2}\text{Log }2$	0.346574...
5	$\frac{1}{4}\left(\text{Log }5 + \frac{2\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}} - \frac{2}{\sqrt{5}}\text{ArcCoth }\sqrt{5}\right)$	0.396834...
6	$\frac{1}{4}\left(\text{Log }3 + \frac{\pi}{3\sqrt{3}}\right)$	0.425803...

5.4 Causal structure and the local entropy current

5.4.1 The event horizon of our solutions

Although our assumptions can almost certainly be greatly relaxed, for the purposes of this section we specialize to boundary metrics that settle down, at late times to either the flat metric on $R^{d-1,1}$ or the flat metric on $S^{d-1}\times$ time and to fluid flows that settle down at late times to uniform brane configurations on $R^{d-1,1}$ or stationary rotating black holes (studied in greater detail ahead) on $S^{d-1}\times$ time. See [73] for a discussion on how the dissipative nature of fluid dynamics makes the last assumption less restrictive than it naively seems.

Now the event horizon of our spacetimes is simply the unique null hypersurface that tends, at late times, to the known event horizons of the late time limit of our solutions. In this subsection we will explain how this clear characterization may be translated into an explicit and local mathematical formula for the event horizon within the derivative expansion.

Recall that our bulk metric is written in the gauge $g_{rr} = 0$, $g_{r\mu} = -u_\mu$, and consequently takes the form

$$ds^2 = -2u_\mu dx^\mu (dr + \mathcal{V}_\nu dx^\nu) + \mathfrak{G}_{\mu\nu} dx^\mu dx^\nu \tag{5.33}$$

where we remind the reader that $\mathfrak{G}_{\mu\nu}$ is transverse and all the Greek indices are raised using the boundary metric $g_{\mu\nu}$. As we have explained before $\mathcal{V}_\mu$ transforms like a connection and $\mathfrak{G}_{\mu\nu}$ is invariant under boundary Weyl transformations.

Let us suppose that the event horizon is given by the equation $\mathcal{S} \equiv r - r_H(x) = 0$. The normal vector ξ_A to this hypersurface is simply the one-form $dS = \xi_A dy^A = dr - \partial_\mu r_H dx^\mu$. This one-form - and its dual normal vector - can be written in a manifestly Weyl covariant (if slightly complicated) form as follows

$$\begin{aligned} \xi_A dy^A &= dS = (dr + \mathcal{V}_\lambda dx^\lambda) - \kappa_\mu dx^\mu \\ \xi^A \partial_A &= G^{AB}\partial_A \mathcal{S} \partial_B = n^\mu(\partial_\mu - \mathcal{V}_\mu \partial_r) - u^\mu \kappa_\mu \partial_r \\ &= n^\mu \left[\partial_\mu + \partial_\mu r_H \partial_r\right] = n^\mu \left[\partial_\mu\right]_{r=r_H} \end{aligned} \tag{5.34}$$

where we have introduced two new Weyl-covariant vectors $\kappa^\mu = e^{-\phi}\tilde{\kappa}^\mu$ and $n^\mu = e^{-\phi}\tilde{n}^\mu$ defined via

$$\begin{aligned} \kappa_\mu &\equiv \partial_\mu r_H + \mathcal{V}_{\mu H} \qquad \text{and} \\ n^\mu &\equiv u^\mu - (\mathfrak{G}_H^{-1})^{\mu\nu}\kappa_\nu \end{aligned} \tag{5.35}$$

We use the subscript H to denote that the functions are to be evaluated at the event-horizon.

If we adopt the boundary co-ordinates x^μ as the co-ordinates on the event horizon, the induced metric on the horizon can be written as

$$ds_H^2 = \left[G_{AB}(y)dy^A dy^B\right]_{r=r_H(x)} \equiv \mathcal{H}_{\mu\nu}(x)dx^\mu dx^\nu \tag{5.36}$$

with

$$\mathcal{H}_{\mu\nu} = \mathfrak{G}_{\mu\nu} - u_{(\mu}\kappa_{\nu)} \tag{5.37}$$

and the null-condition on the horizon, $[G_{AB}]_H \xi^A \xi^B = \mathcal{H}_{\mu\nu}n^\mu n^\nu = 0$ translates to

$$(\mathfrak{G}^{-1})^{\mu\nu}\kappa_\mu\kappa_\nu = 2u^\mu\kappa_\mu \tag{5.38}$$

We now follow [11] to compute the event horizon of our solutions in the derivative expansion. We start from a Weyl-covariant derivative expansion for r_H given by

$$\begin{aligned} r_H &= \frac{1}{b} + b\left(h_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + h_2\omega_{\alpha\beta}\omega^{\alpha\beta} + h_3\mathcal{R}\right) + \dots \\ &= r_H^{(0)} + r_H^{(2)} + \dots \end{aligned} \tag{5.39}$$

Note that, since there is no first order Weyl-covariant scalar,[79] there are no corrections to r_H at the first order in the derivative expansion.

We first compute κ_μ

$$\begin{aligned} \kappa_\mu &= \mathcal{D}_\mu b^{-1} - \mathcal{S}_{\mu\lambda}u^\lambda - 2L_H P_\mu^\nu \mathcal{D}_\lambda \sigma^\lambda{}_\nu \\ &\quad + u_\mu\left[\frac{1}{4}\omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{K_{2H}}{2(d-1)}\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \frac{d}{2b}r_H^{(2)}\right] + \dots \end{aligned} \tag{5.40}$$

Substituting the above into (5.38), we get

$$r_H^{(2)} = \frac{2b}{d}\left[u^\mu(\mathcal{D}_\mu b^{-1} - \mathcal{S}_{\mu\nu}u^\nu) - \frac{1}{4}\omega_{\alpha\beta}\omega^{\alpha\beta} - \frac{K_{2H}}{2(d-1)}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right] \tag{5.41}$$

To bring this to the form (5.39), we use (5.75) and (5.22) to write

$$\begin{aligned} \mathcal{D}_\mu b^{-1} - \mathcal{S}_{\mu\nu}u^\nu &= \left(\frac{2}{d} - \frac{1}{d-2}\right)\mathcal{D}_\lambda\sigma^\lambda{}_\mu + \frac{\mathcal{D}_\lambda\omega^\lambda{}_\mu}{d-2} \\ &\quad - \frac{2}{d-1}\sigma_{\alpha\beta}\sigma^{\alpha\beta}u_\mu + \frac{\mathcal{R}u_\mu}{2(d-1)(d-2)} + \dots \end{aligned}$$

and

$$L_H = \int_1^\infty \xi^{d-1}d\xi \int_\xi^\infty dy\ \frac{y-1}{y^3(y^d-1)} = \frac{1}{2d} \tag{5.42}$$

[79] See [11] for a classification of the possible Weyl-covariant tensors.

which gives us the position of the event horizon as

$$r_{_H} = \frac{1}{b} + b\left(h_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + h_2\omega_{\alpha\beta}\omega^{\alpha\beta} + h_3\mathcal{R}\right) + \dots \tag{5.43}$$

where

$$\begin{split}
h_1 &= \frac{2(d^2+d-4)}{d^2(d-1)(d-2)} - \frac{K_{2H}}{d(d-1)}\\
h_2 &= -\frac{d+2}{2d(d-2)} \qquad \text{and} \quad h_3 = -\frac{1}{d(d-1)(d-2)}\\
\text{with} \qquad K_{2H} &= \int_1^\infty \frac{d\xi}{\xi^2}\Big[1-\xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1}\\
&+2\left((d-1)\xi^d-(d-2)\right)\int_\xi^\infty dy\ y^2F'(y)^2\Big]
\end{split} \tag{5.44}$$

5.4.2 Entropy current as the pullback of Area form

Once the event-horizon is obtained, one can compute an area form on the horizon which when pulled-back to boundary along the ingoing null geodesics gives the entropy current.This general prescription by [73] translates into the following expression for the boundary entropy current[80]

$$\begin{split}
J^\mu_S &= \frac{\sqrt{\det^{(n)}_{d-1}\mathcal{H}}}{4G_{AdS}}n^\mu\\
&= \frac{\sqrt{\det^{(n)}_{d-1}\mathcal{H}}}{4G_{AdS}}\left[u^\mu - (\mathfrak{G}^{-1}_H)^{\mu\nu}\kappa_\nu\right]
\end{split} \tag{5.45}$$

where we will define $\det^{(n)}_{d-1}\mathcal{H}$ in the following.

To define $\det^{(n)}_{d-1}\mathcal{H}$ we will split the boundary co-ordinates x^μ to (v, x^i) and we continue to use the same co-ordinates also on the event horizon. Under this split, the components of the n^μ also spilt into (n^v, n^i). We will denote the $d-1$ dimensional induced metric on the constant v submanifolds of the event horizon by $\mathfrak{h}_{ij}$.Then, we define

$$\sqrt{\det^{(n)}_{d-1}\mathcal{H}} = \frac{\sqrt{\det_{d-1}\mathfrak{h}}}{n^v\sqrt{-\det g}} \tag{5.46}$$

where $g_{\mu\nu}$ is the boundary metric and the expression on the right hand side has been assumed to be pulled back from the horizon to the boundary via the ingoing null-geodesics. Though we have used a particular split to define $\det^{(n)}_{d-1}\mathcal{H}$, it can be shown that the answer that we get in the end is independent of the split(See section 3.3 of [73]). Hence,the expression in (5.45) constitutes a specific proposal for what the entropy current of the boundary fluid

[80]A more detailed justification of this formula can be found in [73]

should be. This construction has the advantage that the second law in the boundary theory is automatically guaranteed by the area increase theorem in the bulk.[81]

By following the procedure just outlined, the dual entropy current of the conformal fluid can be calculated. We get

$$\begin{aligned} 4G_{AdS} b^{d-1} J_S^\mu = u^\mu + b^2 u^\mu \Big[A_1 \ \sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2 \ \omega_{\alpha\beta}\omega^{\alpha\beta} + A_3 \ \mathcal{R} \Big] \\ + b^2 \Big[B_1 \ \mathcal{D}_\lambda \sigma^{\mu\lambda} + B_2 \ \mathcal{D}_\lambda \omega^{\mu\lambda} \Big] + \dots \end{aligned} \tag{5.47}$$

with

$$\begin{aligned} A_1 &= \frac{2}{d^2}(d+2) - \frac{K_{1H} d + K_{2H}}{d} \ , \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2} \\ B_1 &= -2A_3 = \frac{2}{d(d-2)} \end{aligned} \tag{5.48}$$

where $K_{1H}d + K_{2H}$ is given by the integral

$$\begin{aligned} K_{1H} d + K_{2H} = \int_1^\infty \frac{d\xi}{\xi^2} \Big[1 - \xi(\xi-1) F'(\xi) - 2(d-1)\xi^{d-1} \\ +2 \Big((d-1)\xi^d + 1 \Big) \int_\xi^\infty dy \ y^2 F'(y)^2 \Big] \end{aligned} \tag{5.49}$$

5.4.3 Second law and the Rate of entropy production

In the absence of a clear field theoretic microscopic definition, it may be pragmatic to regard the entropy current of fluid dynamics as any local functional of the fluid dynamical variables whose divergence is non negative on every solution to the equations of motion of fluid dynamics, and which integrates to the thermodynamic notion of entropy in equilibrium. According to this characterization the entropy current is *any* local Boltzmann H function, whose monotonic increase characterizes the dissipative irreversibility of fluid flows.

In the previous subsection we have used the dual bulk description to give a 'natural' bulk definition of the entropy current that satisfies all these properties. However, at least at the two derivative level, the construction of the previous subsection is not the unique construction that satisfies the requirements spelt out in the paragraph above.

In this subsection we will take a purely algebraic approach to determine the most general Weyl covariant two derivative entropy current that has a non negative divergence, given the equations of motion derived above. The entropy current of the previous section will turn out to be one of a 4-parameter class of solutions to this constraint.

[81] The Area increase theorem states that under appropriate assumptions the area of a blackhole can never decrease. This statement was proved by Hawking for the case of asymptotically flat spacetimes and is by now standard text book material (see e.g. [88, 89]). This theorem has since been extended to black holes in more general spacetimes (see e.g. [90–92]), including asymptotically AdS spaces (see [93] and references therein for a clear statement to this effect).

The most general entropy current consistent with Weyl covariance[82] can be written as

$$\begin{aligned}4G_{\rm AdS}b^{d-1}J_S^\mu &= u^\mu + b^2 u^\mu\left[A_1\sigma_{\alpha\beta}\sigma^{\alpha\beta}+A_2\omega_{\alpha\beta}\omega^{\alpha\beta}+A_3\mathcal{R}\right]\\ &\quad + b^2\left[B_1\mathcal{D}_\lambda\sigma^{\mu\lambda}+B_2\mathcal{D}_\lambda\omega^{\mu\lambda}\right]+\ldots\end{aligned}\tag{5.50}$$

Since we want to constrain the entropy current upto second order we will need to calculate the divergence of this current. In order to perform the calculation in a Weyl covariant fashion we note that the ordinary divergence $\nabla_\mu J_S^\mu$ can be replaced by the Weyl-covariant divergence $\mathcal{D}_\mu J_S^\mu$ with

$$\mathcal{D}_\mu J_S^\mu \equiv \nabla_\mu J_S^\mu + (w-d)\mathcal{A}_\mu J_S^\mu \tag{5.51}$$

as the conformal weight of any entropy current must be d.

Let us now take the divergence of (5.50). We find

$$\begin{aligned}4G_{\rm AdS}b^{d-1}\mathcal{D}_\mu J_S^\mu &= (d-1)\ b\ u^\mu\mathcal{D}_\mu b^{-1} + b^2u^\mu\mathcal{D}_\mu\left[A_1\sigma_{\alpha\beta}\sigma^{\alpha\beta}+A_2\omega_{\alpha\beta}\omega^{\alpha\beta}+A_3\mathcal{R}\right]\\ &\quad + b^2\mathcal{D}_\mu\left[B_1\mathcal{D}_\lambda\sigma^{\mu\lambda}+B_2\mathcal{D}_\lambda\omega^{\mu\lambda}\right]+\ldots\end{aligned}\tag{5.52}$$

which can in turn be evaluated using the following identities:[83]

$$\begin{aligned}(d-1)\ b\ u^\mu\mathcal{D}_\mu b^{-1} &= -\frac{\sigma_{\mu\nu}T^{\mu\nu}}{pd}\\ u^\mu\mathcal{D}_\mu\left[\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right] &= 2\sigma_{\mu\nu}u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\\ u^\mu\mathcal{D}_\mu\left[\omega_{\alpha\beta}\omega^{\alpha\beta}\right] &= 4\sigma_{\mu\nu}\omega^{\mu\lambda}\omega_\lambda{}^\nu+\omega_{\mu\nu}\mathcal{F}^{\mu\nu}\\ u^\mu\mathcal{D}_\mu\mathcal{R} &= -2\sigma^{\mu\nu}\mathcal{R}_{\mu\nu}+\omega_{\mu\nu}\mathcal{F}^{\mu\nu}+2\mathcal{D}_\mu\mathcal{D}_\nu\sigma^{\mu\nu}-2(d-2)\mathcal{D}_\mu\left[u_\nu\mathcal{F}^{\mu\nu}\right]\\ &= -2(d-2)\sigma^{\mu\nu}\left[\sigma_\mu{}^\lambda\sigma_{\lambda\nu}+\omega_{\mu\lambda}\omega^\lambda{}_\nu+u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}+C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right]\\ &\quad +\omega_{\mu\nu}\mathcal{F}^{\mu\nu}+2\mathcal{D}_\mu\mathcal{D}_\nu\sigma^{\mu\nu}-2(d-2)\mathcal{D}_\mu\left[u_\nu\mathcal{F}^{\mu\nu}\right]\\ \mathcal{D}_\mu\mathcal{D}_\nu\omega^{\mu\nu} &= -\frac{(d-3)}{2}\ \omega_{\mu\nu}\mathcal{F}^{\mu\nu}\end{aligned}\tag{5.53}$$

Substituting for the energy-momentum tensor from (5.26) and keeping only those terms

[82] We assume that there are no pseudo-vector contributions to the entropy current which can possibly appear only in the case of $d=4$. See [11] for an analysis in $d=4$ including pseudovectors.

[83] The first of these identities is just the re-statement of energy conservation $u_\mu\mathcal{D}_\nu T^{\mu\nu}=0$. The rest of them can be obtained by exploiting the properties of various Weyl-covariant quantities which are detailed in [2]. Note however that the curvature tensors used here are the negative of those appearing in [2].

which contain no more than three derivatives, we get[84]

$$
\begin{aligned}
4G_{\text{AdS}}b^{d-1}\mathcal{D}_\mu J^\mu_S = \frac{2b}{d}\sigma^{\mu\nu}\Big[&\sigma_{\mu\nu} - bd(d-2)\left(A_3 - \frac{2A_2}{d-2}\right)\omega_{\mu\lambda}\omega^\lambda{}_\nu \\
&- bd(d-2)\left(A_3 + \frac{1}{d(d-2)}\right)\left(\sigma_\mu{}^\lambda\sigma_{\lambda\nu} + u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right) \\
&+ (A_1 bd + \tau_\omega)\, u^\lambda \mathcal{D}_\lambda\sigma_{\mu\nu}\Big] \\
&+ b^2(B_1 + 2A_3)\mathcal{D}_\mu\mathcal{D}_\nu\sigma^{\mu\nu} + \ldots
\end{aligned}
\tag{5.54}
$$

We rewrite the above expression in a more useful form by isolating the terms that are manifestly non-negative (keeping terms containing no more than three derivatives):

$$
\begin{aligned}
4G_{\text{AdS}}b^{d-1}\mathcal{D}_\mu J^\mu_S = \frac{2b}{d}\Big[&\sigma_{\mu\nu} - \frac{bd(d-2)}{2}\left(A_3 - \frac{2A_2}{d-2}\right)\omega_{\mu\lambda}\omega^\lambda{}_\nu \\
&- \frac{bd(d-2)}{2}\left(A_3 + \frac{1}{d(d-2)}\right)\left(\sigma_\mu{}^\lambda\sigma_{\lambda\nu} + u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right) \\
&+ \frac{1}{2}(A_1 bd + \tau_\omega)\, u^\lambda \mathcal{D}_\lambda\sigma_{\mu\nu}\Big]^2 \\
&+ b^2(B_1 + 2A_3)\mathcal{D}_\mu\mathcal{D}_\nu\sigma^{\mu\nu} + \ldots
\end{aligned}
\tag{5.55}
$$

The second law requires that the right hand side of the above equation be positive semi-definite at every point on the boundary. This gives us the single constraint :

$$B_1 + 2\,A_3 = 0 \tag{5.56}$$

Equation (5.56) is the main result of this subsection. Any Weyl covariant entropy current that obeys the constraint spelt out in (5.56) has a manifestly non negative divergence of the entropy current, keeping only terms to the order of interest.

A particular example of such a current was constructed in [2]. The entropy current proposed in [2] is equivalent to the following proposal for the coefficients

$$\tilde{A}_1 = -\frac{\tau_\omega}{bd}\ , \quad \tilde{A}_2 = -\frac{1}{2d}, \quad \tilde{B}_2 = \frac{1}{d-2}\ , \quad \tilde{B}_1 = -2\tilde{A}_3 = \frac{2}{d(d-2)} \tag{5.57}$$

which yields a simple manifestly non-negative formula for the rate of entropy production $T\mathcal{D}_\mu \tilde{J}^\mu_S = 2\eta\ \sigma_{\mu\nu}\sigma^{\mu\nu} + \ldots$.

Another example of an entropy current whose divergence is non-negative is the entropy current derived from gravity in the previous section using the coefficients appearing in the equation (5.48). Since the values of A_3 and B_1 appearing in (5.48) satisfy the constraint (5.56), we conclude that the entropy current constructed in the previous subsection satisfies the second law. More explicitly, by substituting the values of A and B coefficients from (5.47), we get the rate of entropy production as

$$\mathcal{D}_\mu J^\mu_S = \frac{2\eta}{T}\left[\sigma_{\mu\nu} + \frac{1}{2}(A_1 bd + \tau_\omega)\, u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}\right]^2 + \ldots \tag{5.58}$$

[84]Note, in particular, that $\mathcal{F}_{\mu\nu}$ is zero on-shell upto second order in the derivative expansion(See (5.77)).

where A_1 and τ_ω have been defined in equations (5.47) and (5.27) respectively.

5.5 Black Holes in AdS

5.5.1 AdS Kerr metrics as fluid duals

In the previous sections, we have found the bulk dual to arbitrary fluid dynamical evolutions on the boundary, to second order in the derivative expansion. In this section, we now proceed to test our results against a class of exact solutions of Einstein's equations.

This class of solutions is the set of rotating black holes in the global AdS spaces. The dual boundary stress tensor to these solutions varies on the length scale unity (if we choose our boundary sphere to be of unit radius). On the other hand the temperature of these black holes may be taken to be arbitrarily large. It follows that, in the large temperature limit, these black holes are dual to 'slowly varying' field theory configurations that should be well described by fluid dynamics. All of these remarks, together with nontrivial evidence for this expectation was described in [15]. In this subsection, we will complete the programme initiated in [15] for uncharged blackholes by demonstrating that the full bulk metric of these high temperature rotating black holes agrees in detail with the 2nd order bulk metric determined by our analysis earlier in this chapter. This exercise was already carried out in [11] for the special case $d = 4$.

Consider the AdS-Kerr BHs in arbitrary dimensions - exact solution for the rotating blackholes in general AdS_{d+1} in different coordinates is derived in reference [16]. Following [16], we begin by defining two integers n and ϵ via $d = 2n + \epsilon$ with $\epsilon = d \bmod 2$. We can then parametrise the $d + 1$ dimensional AdS Kerr solution by a radial co-ordinate r, a time co-ordinate $\hat{t}$ along with $d - 1 = 2n + \epsilon - 1$ spheroidal co-ordinates on S^{d-1}. We will choose these spheroidal co-ordinates to be $n+\epsilon$ number of direction cosines $\hat{\mu}_i$ (obeying $\sum_{k=1}^{n+\epsilon} \hat{\mu}_k^2 = 1$) and $n + \epsilon$ azimuthal angles $\hat{\varphi}_i$ with $\hat{\varphi}_{n+1} = 0$ identically. The angular velocities along the different $\hat{\varphi}_i$s are denoted by a_i ($a_{n+1}$is taken to be zero identically).

In this 'altered' Boyer-Lindquist co-ordinates, AdS Kerr metric assumes the form (See equation (E.3) of the [16])

$$
\begin{aligned}
ds^2 = &-W(1+r^2)d\hat{t}^2 + \frac{\mathfrak{F}dr^2}{1-2M/V} + \frac{2M}{V\mathfrak{F}}\left(W d\hat{t} - \sum_{i=1}^{n}\frac{a_i\hat{\mu}_i^2 d\hat{\varphi}_i}{1-a_i^2}\right)^2 \\
&+ \sum_{i=1}^{n+\epsilon}\frac{r^2+a_i^2}{1-a_i^2}\left[d\hat{\mu}_i^2 + \hat{\mu}_i^2 d\hat{\varphi}_i^2\right] - \frac{1}{W(1+r^2)}\left(\sum_{i=1}^{n+\epsilon}\frac{r^2+a_i^2}{1-a_i^2}\hat{\mu}_i d\hat{\mu}_i\right)^2
\end{aligned}
\tag{5.59}
$$

where

$$
W \equiv \sum_{i=1}^{n+\epsilon}\frac{\hat{\mu}_i^2}{1-a_i^2} \quad ; \quad V \equiv r^d\,(1+\frac{1}{r^2})\prod_{i=1}^{n}(1+\frac{a_i^2}{r^2}) \quad \text{and} \quad \mathfrak{F} \equiv \frac{1}{1+r^2}\sum_{i=1}^{n+\epsilon}\frac{r^2\hat{\mu}_i^2}{r^2+a_i^2} \tag{5.60}
$$

We first perform a co-ordinate transformation of the form

$$
d\hat{t} = dt - \frac{dr}{(1+r^2)\,(1-2M/V)} \quad ; \quad d\hat{\varphi}_i = d\varphi_i - \frac{a_i dr}{\left(r^2+a_i^2\right)(1-2M/V)} \tag{5.61}
$$

followed by another transformation of the form

$$\mu_i^2 \equiv \frac{1}{W}\left(\frac{\hat{\mu}_i^2}{1-a_i^2}\right) \quad \text{with} \quad W = \frac{1}{1-\sum_i a_i^2\mu_i^2} \quad , \quad \mathfrak{F} = W\left[\sum_i \frac{\mu_i^2}{1+\frac{a_i^2}{r^2}} - \frac{1}{1+\frac{1}{r^2}}\right] \tag{5.62}$$

to get

$$ds^2 = -2u_\mu dx^\mu(dr + r\mathcal{A}_\nu dx^\nu) + \left[r^2 g_{\mu\nu} + \Sigma_{\mu\nu}\right] dx^\mu dx^\nu + \frac{u_\mu u_\nu}{V\mathfrak{F}b^d} dx^\mu dx^\nu \tag{5.63}$$

where

$$\begin{aligned} u^\mu\partial_\mu &\equiv \partial_t + a_i\partial_{\varphi_i} \quad , \quad \mathcal{A}_\mu = 0 \quad , \quad b \equiv (2M)^{-1/d} \\ g_{\mu\nu} &\equiv W\left[-dt^2 + \sum_i \left(d\mu_i^2 + \mu_i^2 d\varphi_i^2\right)\right] \\ \Sigma_{\mu\nu} &\equiv W\left[-dt^2 + \sum_i a_i^2\left(d\mu_i^2 + \mu_i^2 d\varphi_i^2\right) + \left(\sum_i a_i^2\mu_i d\mu_i\right)^2\right] \end{aligned} \tag{5.64}$$

This expression can be further simplified using the following identities

$$\begin{aligned} \Sigma_{\mu\nu} &= u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu} \\ r^2 V\mathfrak{F} &= \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right] \end{aligned} \tag{5.65}$$

where the determinant of a tensor M^λ_σ is defined by

$$\epsilon_{\mu\nu\ldots}M^\mu_\alpha M^\nu_\beta\ldots = \det\left[M^\lambda_\sigma\right]\epsilon_{\alpha\beta\ldots}$$

Hence, we conclude that the AdS Kerr metric in arbitrary dimensions can be rewritten in the form

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu\left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right] dx^\mu dx^\nu \\ &+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu \end{aligned} \tag{5.66}$$

We have checked this form explicitly using Mathematica till $d = 8$.

This metric can also be written in the form

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu\left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + r^2 g_{\mu\nu}dx^\mu dx^\nu \\ &- \left[\omega_\mu{}^\lambda\omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda\omega^\lambda{}_{(\mu}u_{\nu)} + \frac{1}{(d-1)(d-2)}\mathcal{R}u_\mu u_\nu\right] dx^\mu dx^\nu \\ &+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu \end{aligned} \tag{5.67}$$

or alternatively

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu (dr + \mathcal{V}_\nu dx^\nu) + \mathfrak{G}_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \\
\mathcal{V}_\mu &= r\mathcal{A}_\mu - \mathcal{S}_{\mu\lambda} u^\lambda - \frac{r^2 u_\mu}{2b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]} \\
\mathfrak{G}_{\mu\nu} &= r^2 P_{\mu\nu} - \omega_\mu{}^\lambda \omega_{\lambda\nu}
\end{aligned}
\tag{5.68}
$$

It is easily checked that this metric agrees(upto second order in boundary derivative expansion) with the metric presented in (5.23) in section 4 of this chapter, upon inserting the velocity and temperature fields listed in (5.64).

5.5.2 The Energy momentum tensor and the Entropy Current for the AdS Kerr Black Hole

The exact energy momentum tensor for the AdS Kerr Black Hole described can be computed using the standard counterterm methods. The non-anomalous part of the energy momentum tensor is given by

$$
T_{\mu\nu} = p(g_{\mu\nu} + d u_\mu u_\nu) \quad \text{with} \quad p = \frac{1}{16\pi G_{AdS} b^d} \tag{5.69}
$$

which is consistent with (5.27) if we take into account the fact that $\sigma_{\mu\nu} = 0$ in these configurations.

The equation for the event horizon of the AdS Kerr Black Hole is given by $V = 2M$ or

$$
\begin{aligned}
\frac{1}{b^d} &= r_H^d \,(1 + \frac{1}{r_H^2}) \prod_{i=1}^{n} (1 + \frac{a_i^2}{r_H^2}) \\
&= r_H^d \left[1 + \frac{1 + \sum_i a_i^2}{r_H^2} + \ldots\right]
\end{aligned}
\tag{5.70}
$$

which can be solved for r_H to give

$$
\begin{aligned}
r_H &= \frac{1}{b}\left[1 - \frac{b^2}{d}\left(1 + \sum_i a_i^2\right) + \ldots\right] \\
&= \frac{1}{b} - \frac{d+2}{2d(d-2)} b\ \omega^{\mu\nu}\omega_{\mu\nu} - \frac{b\ \mathcal{R}}{d(d-2)(d-1)} + \ldots
\end{aligned}
\tag{5.71}
$$

which agrees with the expression for the event horizon in (5.43) upon inserting the velocity and temperature field configurations (5.64).

The entropy current for the AdS Kerr blackhole can be directly obtained from (5.45). We have the following exact results :

$$\begin{aligned}
\sqrt{\det{}^{(n)}_{d-1}\mathcal{H}} &= r_{_H}^{d-1}\prod_{i=1}^{n}(1+\frac{a_i^2}{r_{_H}^2}) = \frac{r_{_H}}{b^d(r_{_H}^2+1)} \\
n^\mu\partial_\mu &= \partial_t + \sum_i \frac{r_{_H}^2+1}{r_{_H}^2+a_i^2}a_i\partial_{\varphi_i} = u^\mu\partial_\mu + \sum_i \frac{1-a_i^2}{r_{_H}^2+a_i^2}a_i\partial_{\varphi_i} \\
J_S^\mu\partial_\mu &= \frac{r_{_H}}{4G_{\text{AdS}}b^d(r_{_H}^2+1)}\left[u^\mu\partial_\mu + \sum_i \frac{1-a_i^2}{r_{_H}^2+a_i^2}a_i\partial_{\varphi_i}\right]
\end{aligned} \tag{5.72}$$

These exact results can alternatively be expanded in a derivative expansion. Keeping terms only upto second order in the derivative expansion, we get

$$\begin{aligned}
\sqrt{\det{}^{(n)}_{d-1}\mathcal{H}} &= \frac{1}{b^{d-1}}[1-\frac{d-1}{d}b^2+\frac{b^2}{d}\sum_i a_i^2+\ldots] \\
n^\mu\partial_\mu &= u^\mu\partial_\mu + b^2\sum_i(1-a_i^2)a_i\partial_{\varphi_i}+\ldots
\end{aligned} \tag{5.73}$$

which gives

$$4G_{\text{AdS}}b^{d-1}J_S^\mu = u^\mu\left[1-\frac{b^2}{2d}\omega^{\alpha\beta}\omega_{\alpha\beta}-\frac{b^2\mathcal{R}}{d(d-2)}\right]+\frac{b^2}{d-2}\mathcal{D}_\lambda\omega^{\mu\lambda} \tag{5.74}$$

We have checked this form explicitly using Mathematica till $d=8$.

Comparing the above with (5.47) and remembering that $\sigma^{\alpha\beta}=0$ for the AdS Kerr black hole, we find that our results in the previous sections are consistent with these exact solutions.

5.6 Discussion

In this chapter we have constructed an explicit map from solutions of the (generalized) Navier Stokes equations on a $d-1,1$ dimensional boundary with an arbitrary weakly curved metric $g_{\mu\nu}$ to the space of regular solutions to the Einstein equations with a negative cosmological constant that asymptote, at small z, to $ds^2 = z^{-2}\left[dz^2+g_{\mu\nu}dx^\mu dx^\nu\right]$. We have demonstrated that our solution space is exhaustive locally in solution space. In other words consider a particular bulk solution B that is dual, under the map constructed in this chapter, to a fluid flow F. Then every regular slowly varying bulk solution to Einstein's equations that is infinitesimally separated from B is dual to a fluid flow infinitesimally separated from F.

We have also demonstrated that - subject to certain restrictions on the long time behavior - all the metrics constructed in this chapter have regular event horizons, and have constructed the event horizon manifold of our solutions in this chapter. It would be interesting to relax the restrictions on long time behavior under which this result follows, and simultaneously examine under what conditions these restrictions are dynamically automatic from the equations of fluid dynamics. In particular, as the long time limit of a fluid flow on a static metric is necessarily non dissipative, it would be interesting to fully classify all nondissipative flows on static background geometries. [85]

We have been able to put our construction of the event horizons described above to practical use: by pulling the area form on the event horizon back to the boundary, we have been able to define an entropy current for the dual fluid flow. The divergence of this current is guaranteed to be non negative by the classic area increase theorem of black hole physics. The entropy current we have constructed is a sort of local 'Boltzmann H function' which can, locally, only be created and never destroyed. The local entropy increase theorem establishes the irreversible nature of dual fluid flows. It may be interesting to study the structure of gradient flows generated by this 'entropy function'.

In the next chapter, we will use the insights gathered in this chapter to work out the slightly more complicated case of charged fluids and their gravity duals. For simplicity, we will confine ourselves to the case of CFT_4 : as we will see there are novel transport properties which could be derived by looking at the gravity duals.

5.7 Appendices

5.7.1 d=2

Through the text of this paper we have worked with conformal fluids in $d > 2$ dimensions. In this section we explain that conformal fluid dynamics in $d = 2$ is special and essentially trivial.

To start with note that a traceless stress tensor in d dimensions has $s_d = \frac{d^2+d-2}{2}$ independent components. The assumption of local thermalization in the fluid dynamical limit allows us to work instead with the d variables of fluid dynamics; the velocities and temperature. Now $d < s_d$ for $d > 2$; it is precisely for this reason that fluid dynamics contains physical information beyond the conservation of the stress tensor. However $s_2 = 2$; consequently two dimensional conformal fluid dynamics is simply the assertion of conservation of the two dimensional stress tensor. One may as well work directly with the components of the stress tensor. The general solution to the conservation of the stress tensor in $d = 2$ is of course well known. In a frame in which the boundary metric locally takes the form $ds^2 = e^{2\phi}dx^+dx^-$ (and ignoring anomaly effects in this discussion) the most general conserved and traceless stress tensor is given by $T_{++} = f(x^+)$ and $T_{--} = g(x^-)$ for arbitrary functions f and g. This constitutes the most general solution to 'conformal fluid dynamics' in two dimensions. Note that according to this solution, left and right moving waves do not interact with each other. Consequently two dimensional conformal 'fluid' dynamics is both trivial and a misnomer; conformal fluids in two dimensions do not locally equilibrate.

The triviality of conformal fluid dynamics in two dimensions has a simple gravitational counterpart: every solution of Einstein's equations in two dimensions is locally AdS_3. All

[85] It is natural to guess that this set is exhausted by uniform motion (in the case of the boundary $\mathrm{R}^{d-1,1}$) and rigid rotations (in the case of the boundary $\mathrm{S}^{d-1,1}$), but we are unaware of proofs if any of this intuition. We thank G.Gibbons for discussions on this issue.

generally coordinate inequivalent regular solutions of these equations are the BTZ black holes. (Note that the point mass solutions, studied extensively for instance in [94], have a naked singularity atleast from the purely gravitational point of view). Conformally inequivalent slicings of the same geometry (a la Brown and Hanneaux) generate the left and right moving waves described in the previous subsection. From the bulk point of view these solutions are trivial because they are all (large) diffeomorphism equivalent to static black holes.

There is yet another way to express the triviality of conformal fluid dynamics in two dimensions. It turns out that there are no non-zero Weyl-covariant quantities which can be formed out of velocity/temperature derivatives and hence, as noted by [12, 95], the first order fluid dynamical metric becomes an exact solution of the bulk Einstein equations (see section 4 of [12] for more details). For all the reasons spelt out above, in the rest of our paper we will focus on $d > 2$.

5.7.2 Pointwise solution to dynamics at second order in derivatives

As explained in [6], in order to construct the map from solutions of fluid dynamics to solutions of gravity at second order, we need to 'solve' the equations of fluid dynamics, at a point x^μ to second order in derivatives. While it is of course very difficult to find the general global solutions to fluid dynamics, the corresponding equations are very easily solved at a point. In this Appendix we review the solution of these equations in explicitly Weyl covariant terms. The results of this appendix were utilized in our construction of the bulk metric in section 4.

For solving the bulk constraint equations upto second order, we need ($\mathcal{D}_\mu T^{\mu\nu} = 0$) evaluated upto second order

$$\mathcal{D}_\mu b = 2b^2 \frac{4\pi\eta}{s}\left[\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}u_\mu - \frac{\mathcal{D}_\lambda \sigma^\lambda{}_\mu}{d}\right] + \dots \tag{5.75}$$

where we have introduced the entropy density s of the conformal fluid related to its pressure by $s = pd/T = 4\pi p b$. This can be used to solve for the partial derivatives of b completely in terms of velocity derivatives

$$\begin{aligned}\partial_\mu b &= \mathcal{A}_\mu b + 2b^2 \frac{4\pi\eta}{s}\left[\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}u_\mu - \frac{\mathcal{D}_\lambda \sigma^\lambda{}_\mu}{d}\right] + \dots \\ \partial_\mu\partial_\nu b &= b(\partial_\mu \mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu) + \dots\end{aligned} \tag{5.76}$$

Since the left hand side of the last equation is symmetric in μ and ν, we get an integrability condition

$$\partial_\mu \mathcal{A}_\nu = \partial_\nu \mathcal{A}_\mu + \dots \tag{5.77}$$

Hence, we conclude that to this order we have a valid fluid configuration in a patch around a point P_0 provided we assume

$$\begin{aligned} b &= b_0 + \epsilon b_0 \mathcal{A}_{\nu 0} x^\nu + 2\epsilon^2 b_0^2 \frac{4\pi\eta}{s}\left[\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}u_\mu - \frac{\mathcal{D}_\lambda \sigma^\lambda{}_\mu}{d}\right]_0 \\ &\quad + \epsilon^2 \frac{b_0}{2}\left[\partial_\mu \mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu\right]_0 x^\mu x^\nu + \dots \\ F_{\mu\nu} &\equiv \partial_{[\mu}\mathcal{A}_{\nu]} = 0 + \dots \end{aligned} \tag{5.78}$$

For the metric given in the text to be a solution of the Einstein equations, it is necessary that the velocity/temperature fields obey the above equations of motion with $\eta/s = 1/(4\pi)$.

6 Charged Hydrodynamics in $\mathrm{AdS}_5/\mathrm{CFT}_4$

In this chapter, we will study the metric duals of charged fluids in $d = 4$ spacetime dimensions. We extend the methods decribed in the previous chapter to charged black-branes by determining the metric duals to arbitrary charged fluid configuration up to second order in the boundary derivative expansion. We also derive the energy-momentum tensor and the charge current for these configurations up to second order in the boundary derivative expansion.

We find a new term in the charge current when there is a bulk Chern-Simons interaction thus resolving an earlier discrepancy between thermodynamics of charged rotating black holes and boundary hydrodynamics. We have also confirmed that all our expressions are covariant under boundary Weyl-transformations as expected.

The material for this chapter is drawn from the paper[17] written by the author in collaboration with Nabamita Banerjee, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Suvankar Dutta and P. Surówka. Many of the results in that paper were simultaneously and independently derived by Erdmenger et.al.[18].

6.1 Introduction

The connection between the equations of gravity and fluid dynamics, described in the previous chapter, was demonstrated essentially by use of the method of collective coordinates. The authors of [6, 9, 11, 12, 74] noted that there exists a d parameter set of exact, asymptotically AdS_{d+1} black brane solutions of the gravity equations parameterized by temperature and velocity. They then used the 'Goldstone' philosophy to promote temperatures and velocities to fields. The Navier Stokes equations turn out to be the effective 'chiral Lagrangian equations' of the temperature and velocity collective fields.[86].

Now consider a conformal field theory that has a conserved charge Q in addition to energy and momentum. This is especially an interesting extension of the hydrodynamics of the uncharged fluids since the hydrodynamics of many real fluids has a global conserved charge which is often just the number of particles that make up the fluid. The long distance dynamics of such a system is expected to be determined by the augmented Navier Stokes equations; $\nabla_\mu T^{\mu\nu} = 0$ together with $\nabla_\mu J^\mu_Q = 0$, where the stress tensor and charge current are now given as functions of the temperature, velocity and charge density, expanded to a given order in the derivative expansion. The bulk dual description of a field theory with a conserved charge always includes a propagating Maxwell field. Consequently the AdS/CFT correspondence suggests asymptotically AdS long wavelength solutions of appropriate modifications of the the Einstein Maxwell equation are in one to one correspondence with solutions of the augmented Navier Stokes equations described above.

This expectation of the previous paragraph also fits well with the collective coordinate intuition described above. Recall that the Einstein Maxwell equations have a well known $d+1$ dimensional set of charged black brane solutions, parameterized by the brane temperature,

[86]There exists a large literature in deriving linearise hydrodynamics from AdS/CFT. See([61] - [8]). There have been some recent work on hydrodynamics with higher derivative corrections [80, 81].

charge density and velocity. It seems plausible that the effective Goldstone equations, that arise from the promotion of these $d+1$ dimensional parameters to fields, are simply the augmented Navier Stokes equations. In this chapter we verify the expectations via a direct analysis of the relevant bulk equations. More concretely, we generalize the work out in [6] to set up a perturbative scheme to generate long wavelength solutions of the Einstein Maxwell equations plus a Chern Simons term (see below for more details) order by order in the derivative expansion. We also implement this expansion to second order, and thereby find explicit expressions for the stress tensor and charge current of our dual fluid to second order in the derivative expansion.

In this chapter we work with the Einstein Maxwell equations augmented by a Chern Simon's term. This is because the equations of IIB SUGRA on $\mathrm{AdS}_5\times\mathrm{S}^5$ (which is conjectured to be dual to $\mathcal{N}=4$ Yang Mills) with the restriction of equal charges for the three natural Cartans, admit a consistent truncation to this system. Under this truncation, we get the following action

$$S=\frac{1}{16\pi G_5}\int\sqrt{-g_5}\left[R+12-F_{AB}F^{AB}-\frac{4\kappa}{3}\epsilon^{LABCD}A_LF_{AB}F_{CD}\right] \tag{6.1}$$

In the above action the size of the S_5 has been set to 1. The value of the parameter κ for $\mathcal{N}=4$ Yang Mills is given by $\kappa=1/(2\sqrt{3})$ - however, with a view to other potential applications we leave κ as a free parameter in all the calculations below. Note in particular that our bulk Lagrangian reduces to the true Einstein Maxwell system at $\kappa=0$.

Our expressions for the charge current and the stress tensor of the fluid are complicated, and are listed in detail in subsequent sections. We would however like to point out an important qualitative feature of our result. Already at first order, and at nonzero κ, the charge current includes a term proportional to $l^\alpha\equiv\epsilon^{\mu\nu\lambda\alpha}u_\mu\nabla_\nu u_\lambda$. The presence of this term in the current resolves an apparent mismatch between the predictions of fluid dynamics and the explicit form of charged rotating black holes in IIB supergravity reported in [15]. Note that due to the presence of the ϵ symbol, this term is parity odd. However, when accompanied by a flip in the R-charge of the brane, its sign remains unchanged. Consequently, this term is CP symmetric in agreement with the expectations of CP symmetry of $\mathcal{N}=4$ Yang Mills theory.

As we have explained above, the reduction of boundary field theory dynamics is expected to reduce to field theory dynamics only at long wavelength compared to an effective mean free path or equilibration length scale. All the gravitational constructions of this chapter also work only in the same limit. It is consequently of interest to know the functional form of the equilibration length scale of our conformal fluid as a function of intensive fluid parameters.

In the case of $\mathcal{N}=4$ Yang Mills, it follows from 't Hooft scaling and dimensional analysis that, at large λ, the effective equilibration length scale is given by $l_{mfp}=f(\nu)/T$ where ν is the dimensionless chemical potential conjugate to the conserved charge of the theory and T being the associated temperature. Explicit computation within gravity demonstrates that $f(\nu)$ is of unit order for generic values of ν away from extremality. Consequently, at generic values of ν, all the considerations of this chapter apply only when all fields vary at distances and times that are large compared to the local effective temperature.

6.2 Zeroeth Order Black Brane solution

In this section. we will establish the basic conventions and notations that we will use in the rest of the paper. We start with the five-dimensional action[87]

$$S = \frac{1}{16\pi G_5}\int \sqrt{-g_5}\left[R + 12 - F_{AB}F^{AB} - \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\right] \tag{6.2}$$

which is a consistent truncation of IIB SUGRA Lagrangian on $\text{AdS}_5\times\text{S}^5$ background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = 1/(2\sqrt{3})$ (See for example, [96–107]). However, for the sake of generality (and to keep track of the effects of the Chern-Simons term), we will work with an arbitrary value of κ in the following. In particular, $\kappa = 0$ corresponds to a pure Maxwell theory with no Chern-Simons type interactions.

The field equations corresponding to the above action are

$$\begin{aligned} G_{AB} - 6g_{AB} + 2\left[F_{AC}F^{C}{}_{B} + \frac{1}{4}g_{AB}F_{CD}F^{CD}\right] &= 0 \\ \nabla_B F^{AB} + \kappa\epsilon^{ABCDE}F_{BC}F_{DE} &= 0 \end{aligned} \tag{6.3}$$

where g_{AB} is the five-dimensional metric, G_{AB} is the five dimensional Einstein tensor. These equations admit an AdS-Reisner-Nordström black-brane solution

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 V(r,m,q)\; u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu \\ A &= \frac{\sqrt{3}q}{2r^2}u_\mu dx^\mu, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} u_\mu dx^\mu = -dv; \qquad V(r,m,q) &\equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6}; \\ P_{\mu\nu} &\equiv \eta_{\mu\nu} + u_\mu u_\nu, \end{aligned} \tag{6.5}$$

with $\eta_{\mu\nu} = \text{diag}(- + ++)$ being the Minkowski-metric. Following the procedure elucidated in [6], we shall take this flat black-brane metric as our zeroth order metric/gauge field ansatz and promote the parameters u_μ, m and q to slowly varying fields [88].

[87] We use Latin letters $A, B \in \{r, v, x, y, z\}$ to denote the bulk indices and $\mu, \nu \in \{v, x, y, z\}$ to denote the boundary indices.

[88] Note that the charge we consider here refers to the Maxwell charge $\int_{\partial S} F_{AB}r^A t^B$ in the bulk (where r^A and t^A are respectively the unit radial normal and future pointing time-like normal to the spatial bounday ∂S). In the presence of a Chern-Simons term in the bulk lagrangian (or alternatively, when the boundary global charge is anomalous), there are other notions of charge (like Page charge - see, for example [108]) which are employed in the literature. The Page charge in the bulk would be $\int_{\partial S}\left(F_{AB} + 2\kappa\;\; \epsilon_{AB}{}^{CDE}A_C F_{DE}\right) r^A t^B$ in our notation . These other notions of charge in the bulk mirrors the various possible notions of a global charge

In the course of our calculations, we will often find it convenient to use the following 'rescaled' variables

$$\rho \equiv \frac{r}{R}; \qquad M \equiv \frac{m}{R^4}; \qquad Q \equiv \frac{q}{R^3}; \qquad Q^2 = M - 1 \tag{6.6}$$

where R is the radius of the outer horizon,i.e., the largest positive root of the equation $V = 0$. The Hawking temperature, chemical potential and the charge density of this black-brane are given by[89]

$$T \equiv \frac{R}{2\pi}(2 - Q^2)\ , \qquad \mu \equiv \frac{2\sqrt{3}q}{R^2} = 2\sqrt{3}QR \qquad \text{and} \qquad n \equiv \frac{\sqrt{3}q}{16\pi G_5}. \tag{6.7}$$

In terms of the rescaled variables, the outer and the inner horizon are given by

$$\rho_+ \equiv 1 \qquad \text{and} \qquad \rho_- \equiv \left[\left(Q^2 + 1/4\right)^{1/2} - 1/2\right]^{1/2}$$

and the extremality condition $\rho_+ = \rho_-$ corresponds to $(Q^2 = 2, M = 3)$. We shall assume the black-branes and the corresponding fluids to be non-extremal unless otherwise specified - this corresponds to the regime $0 < Q^2 < 2$ or $0 < M < 3$ which we will assume henceforth.

Using the flat black-brane solutions with slowly varying velocity, temperature and charge fields, our intention is to systematically determine the corrections to the metric and the gauge field in a derivative expansion. More precisely, we expand the metric and the gauge field in terms of derivatives of velocity, temperature and charge fields of the fluid as

$$\begin{aligned} g_{AB} &= g^{(0)}_{AB} + g^{(1)}_{AB} + g^{(2)}_{AB} + \dots \\ A_M &= A^{(0)}_M + A^{(1)}_M + A^{(2)}_M + \dots \end{aligned} \tag{6.8}$$

where $g^{(k)}_{AB}$ and $A^{(k)}_M$ contain the k-th derivatives of the velocity, temperature and the charge fields with

$$\begin{aligned} g^{(0)}_{AB}dx^A dx^B &= -2u_\mu(x)dx^\mu dr - r^2 V(r, m(x), q(x))\ u_\mu(x)u_\nu(x)dx^\mu dx^\nu + r^2 P_{\mu\nu}(x)dx^\mu dx^\nu \\ A^{(0)}_M dx^M &= \frac{\sqrt{3}q(x)}{2r^2} u_\mu(x)dx^\mu. \end{aligned} \tag{6.9}$$

when it is anomalous in the boundary theory. However, in the rest of the paper, we shall not concern ourselves with these subtleties for the following reason - for the solutions in this chapter, F and hence $A \wedge F$ vanishes when restricted to boundary of AdS. In such a case, the boundary anomaly is turned off and the definition of conserved charge in the boundary is unambiguous (Maxwell charge and Page charge become equal for this subset of solutions). In fact, for a specific value of κ , this conserved charge refers to the unique R-charge of the boundary super conformal field theory.

[89] In much of the literature the chemical potential μ is taken to be the potential difference between the boundary and the horizon. However we have chosen a different normalization for μ (and hence the charge density n). we shall elaborate on this point in subsection 6.3.5.

In order to solve the Einstein-Maxwell-Chern-Simons system of equations, it is necessary to work in a particular gauge for the metric and the gauge fields. Following [6], we choose our gauge to be

$$g_{rr} = 0; \quad g_{r\mu} \propto u_\mu \ ; \quad A_r = 0; \quad Tr[(g^{(0)})^{-1} g^{(k)}] = 0. \tag{6.10}$$

Further, in order to relate the bulk dynamics to boundary hydrodynamics, it is useful to parameterise the fluid dynamics in the boundary in terms of a 'fluid velocity' u_μ. In case of relativistic fluids with conserved charges, there are two widely used conventions of how the fluid velocity should be defined. In this paper, we will work with the Landau frame velocity where the fluid velocity is defined with reference to the energy transport. In a more practical sense working in the Landau frame amounts to taking the unit time-like eigenvector of the energy-momentum tensor at a point to be the fluid velocity at that point.

Alternatively, one could work in the 'Eckart frame' where the fluid velocity is defined with reference to the charge transport where the unit time-like vector along the charged current to be the definition of fluid velocity. Though the later is often the more natural convention in the context of charged fluids, we choose to use the Landau's convention for the ease of comparison with the other literature. We will leave the conversion to the more natural Eckart frame to future work.

In the next two sections, we will report in some detail the calculations leading to the determination of the metric and the gauge field up to second order in the derivative expansion. This will enable us to determine the boundary stress tensor and charge current up to the second order.

6.3 First Order Hydrodynamics

In this section, we present the computation of the metric and the gauge field up to first order in derivative expansion, the derivative being taken with respect to the boundary coordinates. We choose the boundary coordinates such that $u^\mu = (1, 0, 0, 0)$ at x^μ. Since our procedure is ultra local therefore we intend to solve for the metric and the gauge field at first order about this special point x^μ. We shall then write the result thus obtained in a covariant form which will be valid for arbitrary choice of boundary coordinates.

In order to implement this procedure we require the zeroth order metric and gauge field expanded up to first order. For this we recall that the parameters m, q and the velocities (β_i) are functions of the boundary coordinates and therefore admit an expansion in terms of the boundary derivatives. These parameters expanded up to first order is given by

$$\begin{aligned} m =& m_0 + x^\mu \partial_\mu m^{(0)} + \dots \\ q =& q_0 + x^\mu \partial_\mu q^{(0)} + \dots \\ \beta_i =& x^\mu \partial_\mu \beta_i^{(0)} + \dots \end{aligned} \tag{6.11}$$

Here $m^{(i)}$, $q^{(i)}$, $\beta^{(i)}$ refers to the i-th order correction to mass, charge and velocities respectively.

The zeroth order metric expanded about x^μ up to first order in boundary coordinates is given by

$$\begin{aligned} {ds^{(0)}}^2 &= 2\,dv\,dr - r^2 V^{(0)}(r)\,dv^2 + r^2\,dx_i\,dx^i \\ &\quad - 2\,x^\mu\,\partial_\mu \beta_i^{(0)}\,dx^i\,dr - 2\,x^\mu \partial_\mu \beta_i^{(0)}\,r^2(1 - V^{(0)}(r))\,dx^i\,dv \\ &\quad - \left(\frac{-x^\mu \partial_\mu\, m^{(0)}}{r^2} + \frac{2q_0 x^\mu \partial_\mu\, q^{(0)}}{r^4} \right) dv^2, \end{aligned} \tag{6.12}$$

where m_0 and q_0 are related to the mass and charge of the background blackbrane respectively and

$$V^{(0)} = 1 - \frac{m_0}{r^4} + \frac{q_0^2}{r^6}.$$

Similarly the zeroth order gauge fields expanded about x^μ up to first order is given by

$$A = -\frac{\sqrt{3}}{2} \left[\left(\frac{q_0 + x^\mu \partial_\mu\, q^{(0)}}{r^2} \right) dv - \frac{q_0}{r^2} x^\mu \partial_\mu\, \beta_i^{(0)}\, dx^i \right] \tag{6.13}$$

Since the background black brane metric preserves an $SO(3)$ symmetry [90], the Einstein-Maxwell equations separate into equations in scalar, transverse vector and the symmetric traceless transverse tensor sectors. This in turn allows us to solve separately for $SO(3)$ scalar, vector and symmetric traceless tensor components of the metric and the gauge field.

6.3.1 Scalars Of $SO(3)$ at first order

The scalar components of first order metric and gauge field perturbations ($g^{(1)}$ and $A^{(1)}$ respectively) are parameterized by the functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ as follows [91]

$$\begin{aligned} \sum_i g_{ii}^{(1)}(r) &= 3r^2 h_1(r), \\ g_{vv}^{(1)}(r) &= \frac{k_1(r)}{r^2} \\ g_{vr}^{(1)}(r) &= -\frac{3}{2} h_1(r) \\ A_v^{(1)}(r) &= -\frac{\sqrt{3} w_1(r)}{2r^2} \end{aligned} \tag{6.14}$$

Note that $g_{ii}^{(1)}(r)$ and $g_{vr}^{(1)}(r)$ are related to each other by the gauge choice $Tr[(g^{(0)})^{-1} g^{(1)}] = 0$.

Constraint equations

We begin by finding the constraint equations that constrain various derivatives velocity,temperature and charge that appear in the first order scalar sector.The constraint equations are obtained

[90] Here we are referring to the $SO(3)$ rotational symmetry in the boundary spatial coordinates.

[91] here i runs over the boundary spatial coordinates, v is the boundary time coordinate and r is the radial coordinate in the bulk

by taking a dot of the Einstein and Maxwell equations with the vector dual to the one form dr. If we denote the Einstein and the Maxwell equations by $E_{AB} = 0$ and $M_{AB} = 0$, then there are three constraint relations.

Two of them come from Einstein equations. They are given by

$$g^{rr}E_{vr} + g^{rv}E_{vv} = 0 \ , \tag{6.15}$$

and

$$g^{rr}E_{rr} + g^{rv}E_{vr} = 0 \ , \tag{6.16}$$

and the third constraint relation comes from Maxwell equations and is given by

$$g^{rr}M_r + g^{rv}M_v = 0 \ . \tag{6.17}$$

Equation (6.15) reduces to

$$\partial_v m^{(0)} = -\frac{4}{3}m_0\partial_i\beta_i^{(0)} \ . \tag{6.18}$$

which is same as the conservation of energy in the boundary at the first order in the derivative expansion, i.e., the above equation is identical to the constraint (scalar component of the constraint in this case)

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ . \tag{6.19}$$

on the allowed boundary data.

The second constraint equation (6.16) in scalar sector implies a relation between $h_1(r)$ and $k_1(r)$.

$$2\partial_i\beta_i^{(0)}r^5 + 12r^6h_1(r) + 4q_0w_1(r) - m_0r^3h_1'(r) + 3r^7h_1'(r) - r^3k_1'(r) - 2q_0rw_1'(r) = 0. \tag{6.20}$$

The constraint relation coming from Maxwell equation (See Eq. (6.17)) gives

$$\partial_v \ q^{(0)} = -q_0 \ \partial_i\beta_i^{(0)} \ . \tag{6.21}$$

This equation can be interpreted as the conservation of boundary current density at the first order in the derivative expansion.

$$\partial_\mu J^\mu_{(0)} = 0. \tag{6.22}$$

We now proceed to find the scalar part of the metric dual to a fluid configuration which obeys the above constraints.

Dynamical equations and their solutions

Among the Einstein equations four are $SO(3)$ scalars (namely the vv, rv, rr components and the trace over the boundary spatial part). Further the r and v-components of the Maxwell equations constitute two other equations in this sector. Two specific linear combination of the rr and vv components of the Einstein equations constitute the two constraint equations in (6.18). Further, a linear combination of the r and v-components of the Maxwell equations appear as a constraint equation in (6.21). Now among the six equations in the scalar sector we can use any three to solve for the unknown functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ and we must make sure that the solution satisfies the rest. The simplest two equations among these dynamical equations are

$$5h_1'(r) + rh_1''(r) = 0. \tag{6.23}$$

which comes from the rr-component of the Einstein equation and

$$6q_0h_1'(r) + w_1'(r) - rw_1''(r) = 0. \tag{6.24}$$

which comes from the r-components of the Maxwell equation. We intend to use these dynamical equations (6.23), (6.24) along with one of the constraint equations in (6.18) to solve for the unknown functions $h_1(r)$, $k_1(r)$ and $w_1(r)$.

Solving (6.23) we get

$$h_1(r) = \frac{C^1_{h_1}}{r^4} + C^2_{h_1}, \tag{6.25}$$

where $C^1_{h_1}$ and $C^2_{h_1}$ are constants to be determined. We can set $C^2_{h_1}$ to zero as it will lead to a non-normalizable mode of the metric. We then substitute the solution for $h_1(r)$ from (6.25) into (6.24) and solve the resultant equation for $w_1(r)$. The solution that we obtain is given by

$$w_1(r) = C^1_{w_1}r^2 + C^2_{w_1} - q_0\frac{C^1_{h_1}}{r^4}. \tag{6.26}$$

Here again $C^1_{w_1}$, $C^2_{w_2}$ are constants to be determined. Again $C^1_{w_1}$ corresponds to a non-normalizable mode of the gauge field and therefore can be set to zero.

Finally plugging in these solutions for $h_1(r)$ and $w_1(r)$ into one of the constraint equations in (6.18) and then solving the subsequent equation we obtain

$$k_1(r) = \frac{2}{3}r^3\partial_i\beta_i^{(0)} + C_{k_1} - \frac{2q0}{r^2}C^2_{w_1} + \left(\frac{2{q_0}^2}{r^6} - \frac{m_0}{r^4}\right)C^1_{h_1} \tag{6.27}$$

Now the constants C_{k_1} and $C^2_{w_1}$ may be absorbed into redefinitions of mass (m_0) and charge (q_0) respectively and hence may be set to zero. Further we can gauge away the constant $C^1_{h_1}$ by the following redefinition of the r coordinate

$$r \to r\left(1 + \frac{C}{r^4}\right),$$

C being a suitably chosen constant.

Thus we conclude that all the arbitrary constants in this sector can be set to zero and therefore our solutions may be summarized as

$$h_1(r) = 0, \quad w_1(r) = 0, \quad k_1(r) = \frac{2}{3}r^3\partial_i\beta_i^{(0)}\ . \tag{6.28}$$

In terms of the first order metric and gauge field this result reduces to

$$\begin{aligned} \sum_i g_{ii}^{(1)}(r) &= 0, \\ g_{vv}^{(1)}(r) &= \frac{2}{3} r \partial_i \beta_i^{(0)}, \\ g_{vr}^{(1)}(r) &= 0, \\ A_v^{(1)}(r) &= 0 \ . \end{aligned} \tag{6.29}$$

Now, we proceed to solving the equations in the vector sector.

6.3.2 Vectors Of $SO(3)$ at first order

The vector components of metric and gauge field $g^{(1)}$ and $A^{(1)}$ are parameterized by the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ as follows

$$\begin{aligned} g_{vi}^{(1)}(r) &= \left(\frac{m_0}{r^2} - \frac{q_0^2}{r^4}\right) j_i^{(1)}(r) \\ A_i^{(1)}(r) &= -\left(\frac{\sqrt{3} q_0}{2r^2}\right) j_i^{(1)}(r) + g_i^{(1)}(r) \end{aligned} \tag{6.30}$$

Now we intend to solve for the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$.

Constraint equations

The constraint equations in the vector sector comes only from the Einstein equation. So there is only one constraint equation in this sector. It is given by

$$g^{rr} E_{ri} + g^{rv} E_{vi} = 0 \tag{6.31}$$

which implies

$$\partial_i m^{(0)} = -4 m_0 \partial_v \beta_i^{(0)}. \tag{6.32}$$

These equations also follow from the conservation of boundary stress tensor at first order. We shall use this constraint equation to simplify the dynamical equations in the vector sector.

Dynamical equations and their solutions

In the vector sector we have two equations from Einstein equations (the ri and vi-components) and one from Maxwell equations (the ith-component) [92].

The dynamical equation obtained from the vi-component of the Einstein equations is given by

$$\left(q_0^2 - 3 m_0 r^2\right) \frac{d j_i^{(1)}(r)}{dr} + 4\sqrt{3} q_0 r^2 \frac{d g_i(1)(r)}{dr} + \left(m_0 r^2 - q_0^2\right) r \frac{d^2 j_i^{(1)}(r)}{dr^2} = -3 r^4 \partial_v \beta_i^{(0)}. \tag{6.33}$$

[92] Note that a linear combination of the ri and vi-components of the Einstein equation appear as the constraint equation in (6.32).

Also the dynamical equation from the ith-component of the Maxwell equation is given by

$$
\begin{aligned}
&r\left[2\left(r^6-m_0r^2+q_0^2\right)\frac{d^2g_i^{(1)}}{dr^2}r^2+\left(6r^7+2m_0r^3-6q_0^2r\right)\frac{dg_i^{(1)}(r)}{dr}\right]\\
&-\sqrt{3}q_0r\left(r^6-m_0r^2+q_0^2\right)\frac{d^2j_i^{(1)}(r)}{dr^2}+\sqrt{3}q_0\left(r^6-3m_0r^2+5q_0^2\right)\frac{dj_i^{(1)}(r)}{dr}\\
&\quad=\sqrt{3}(q_0\partial_v\beta_i^{(0)}+\partial_iq^{(0)})r^3-24q_0^2\kappa rl_i^{(0)},
\end{aligned}
\tag{6.34}
$$

where l_i is defined as

$$
l_i\equiv\epsilon_{ijk}\partial_j\beta_k. \tag{6.35}
$$

Now in order to solve this coupled set of differential equations (6.33) and (6.34) we shall substitute $g_i^{(1)}(r)$ obtained from (6.33) into (6.34) and solve the resultant equation for $j_i^{(1)}(r)$. For any function $j_i^{(1)}(r)$, using (6.33) $g_i^{(1)}(r)$ may be expressed as

$$
g_i^{(1)}(r)=(C_g)_i+\frac{1}{4\sqrt{3}q_0}\left(-\partial_v\beta_i^{(0)}r^3+4m_0j_i^{(1)}(r)-\frac{\left(m_0r^2-q_0^2\right)\frac{dj_i^{(1)}(r)}{dr}}{r}\right). \tag{6.36}
$$

Here $(C_g)_i$ is an arbitrary constant. It corresponds to non normalizable mode of the gauge field and hence may be set to zero.

Substituting this expression for $g_i^{(1)}(r)$ into (6.34) we obtain the following differential equation for $j_i^{(1)}(r)$

$$
\begin{aligned}
&\left(35q_0^4+5r^2\left(r^4-6m_0\right)q_0^2+3m_0r^4\left(3r^4+m_0\right)\right)\frac{dj_i^{(1)}(r)}{dr}\\
&r\left(-11q_0^4-\left(5r^6-14m_0r^2\right)q_0^2-m_0r^4\left(r^4+3m_0\right)\right)\frac{d^2j_i^{(1)}(r)}{dr^2}\\
&+r^2\left(q_0^2-m_0r^2\right)\left(r^6-m_0r^2+q_0^2\right)\frac{d^3j_i^{(1)}(r)}{dr^3}\\
&=\frac{1}{\sqrt{3}}\left(6\sqrt{3}q_0\partial_iq^{(0)}r^4+3\sqrt{3}\partial_v\beta_i^{(0)}\left(5r^6-m_0r^2+q_0^2\right)r^4-144\ r\ l_i^{(0)}q_0^3\kappa\right)
\end{aligned}
\tag{6.37}
$$

The solution to this equation is given by,

$$
\begin{aligned}
j_i^{(1)}(r)=(C_j^1)_i+\frac{(C_j^2)_ir^2}{\frac{m_0}{r^2}-\frac{q_0^2}{r^4}}+\frac{r\ \partial_v\beta_i^{(0)}}{\frac{m_0}{r^2}-\frac{q_0^2}{r^4}}\\
+\frac{\sqrt{3}\ l_i^{(0)}\ q_0^3\kappa}{m_0\left(\frac{m_0}{r^2}-\frac{q_0^2}{r^4}\right)r^4}+\frac{6r^2q_0(\partial_iq^{(0)}+3q_0\partial_v\beta_i^{(0)})}{R^7\left(\frac{m_0}{r^2}-\frac{q_0^2}{r^4}\right)}F_1(\frac{r}{R},\frac{m_0}{R^4}),
\end{aligned}
\tag{6.38}
$$

where again $(C_j^1)_i$ and $(C_j^2)_i$ are arbitrary constants. $(C_j^2)_i$ corresponds to a non-normalizable mode of the metric and so is set to zero. $(C_j^1)_i$ can be absorbed into a redefinition of the velocities and hence is also set to zero.

Here the function $F_1(\frac{r}{R}, \frac{m_0}{R^4})$ is given by[93]

$$F_1(\rho, M) \equiv \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right)\int_\rho^\infty dp \frac{1}{\left(1 - \frac{M}{p^4} + \frac{Q^2}{p^6}\right)^2}\left(\frac{1}{p^8} - \frac{3}{4p^7}\left(1 + \frac{1}{M}\right)\right), \tag{6.39}$$

where $Q^2 = M - 1$.

Substituting this result for $j_i^{(1)}(r)$ into (6.36) we obtain the following expression for $g_i^{(1)}(r)$

$$\begin{aligned} g_i^{(1)}(r) = & \frac{\sqrt{3}r^3\sqrt{R^2\left(m_0 - R^4\right)}}{2\left(m_0(r-R)(r+R) + R^6\right)}(\partial_v\beta_i^{(0)}) + \frac{3R^2\kappa(m_0 - R^4)}{2\left(m_0\left(r^2 - R^2\right) + R^6\right)}l_i \\ & - \frac{\sqrt{3}r^4\left(r\left(m_0\left(r^2 - R^2\right) + R^6\right)F_1^{(1,0)}\left(\frac{r}{R}, \frac{m_0}{R^4}\right) + \left(6R^7 - 6m_0R^3\right)F_1\left(\frac{r}{R}, \frac{m_0}{R^4}\right)\right)}{2R^8\left(m_0\left(r^2 - R^2\right) + R^6\right)}(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)}) \end{aligned} \tag{6.40}$$

where we use the notation $f^{(i,j)}(\alpha, \beta)$ to denote the partial derivative $\partial^{i+j}f/\partial\alpha^i\partial\beta^j$ of the function f.

Plugging back $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ back into (6.30) we conclude that the first order metric and gauge field in the vector sector is given by

$$\begin{aligned} g_{vi}^{(1)}(r) &= r\partial_v\beta_i^{(0)} + \frac{\sqrt{3}\, l_i^{(0)} q_0^3\kappa}{m_0 r^4} + \frac{6r^2}{R^7}q_0(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)})F_1(\frac{r}{R}, \frac{m_0}{R^4}) \\ A_i^{(1)}(r) &= -\frac{\sqrt{3}r^5 F_1^{(1,0)}\left(\frac{r}{R}, \frac{m_0}{R^4}\right)}{2R^8}(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)}) + \frac{3R\kappa\sqrt{m_0 - R^4}\sqrt{R^2\left(m_0 - R^4\right)}}{2m_0 r^2}l_i \end{aligned} \tag{6.41}$$

6.3.3 Tensors Of $SO(3)$ at first order

The tensor components of the first order metric is parameterized by the function $\alpha_{ij}^{(1)}(r)$ such that

$$g_{ij}^{(1)} = r^2\alpha_{ij}^{(1)}. \tag{6.42}$$

The gauge field does not have any tensor components therefore in this sector there is only one unknown function to be determined.

There are no constraint equations in this sector and the only dynamical equation is obtained from the ij-component of the Einstein equation. This equation is given by

$$r\left(r^6 - m_0r^2 + q_0^2\right)\frac{d^2\alpha_{ij}(r)}{dr^2} - \left(-5r^6 + m_0r^2 + q_0^2\right)\frac{d\alpha_{ij}(r)}{dr} = -6\sigma_{ij}^{(0)}r^4 \tag{6.43}$$

where σ_{ij} is given by

$$\sigma_{ij}^{(0)} = \frac{1}{2}\left(\partial_i\beta_j^{(0)} + \partial_j\beta_i^{(0)}\right) - \frac{1}{3}\partial_k\beta_k^{(0)}\delta_{ij}. \tag{6.44}$$

[93]Although the expression for $F_1(\frac{r}{R}, \frac{m_0}{R^4})$ is very complicated but it satisfies some identities. One can use those identities to perform practical calculations with this function.

The solution to equation (6.43) obtained by demanding regularity at the future event horizon and appropriate normalizability at infinity. The solution is given by

$$\alpha^{(1)}_{ij} = \frac{2}{R}\sigma_{ij}F_2(\frac{r}{R}, \frac{m_0}{R^4}), \tag{6.45}$$

where the function $F_2(\rho, M)$ is given by

$$F_2(\rho, M) \equiv \int_\rho^\infty \frac{p\left(p^2+p+1\right)}{(p+1)\left(p^4+p^2-M+1\right)}dp \tag{6.46}$$

with $M \equiv m/R^4$ as before.

Thus the tensor part of the first order metric is determined to be

$$g^{(1)}_{ij} = \frac{2r^2}{R}\sigma_{ij}F_2(\frac{r}{R}, \frac{m_0}{R^4}). \tag{6.47}$$

6.3.4 The global metric and the gauge field at first order

In this subsection, we gather the results of our previous sections to write down the entire metric and the gauge field accurate up to first order in the derivative expansion.

We obtain the metric as

$$\begin{aligned}
ds^2 &= g_{AB}dx^A dx^B \\
&= -2u_\mu dx^\mu dr - r^2 \ V \ u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu \ r\left[u^\lambda\partial_\lambda u_\nu - \frac{\partial_\lambda u^\lambda}{3}u_\nu\right]dx^\nu + \frac{2r^2}{R}F_2(\rho, M)\sigma_{\mu\nu}dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu\left[\frac{\sqrt{3}\kappa q^3}{mr^4}l_\nu + \frac{6qr^2}{R^7}P^\lambda_\nu \mathcal{D}_\lambda q F_1(\rho, M)\right]dx^\nu + \ldots \\
A &= \left[\frac{\sqrt{3}q}{2r^2}u_\mu + \frac{3\kappa q^2}{2mr^2}l_\mu - \frac{\sqrt{3}r^5}{2R^8}P^\lambda_\mu \mathcal{D}_\lambda q F_1^{(1,0)}(\rho, M)\right]dx^\mu + \ldots
\end{aligned} \tag{6.48}$$

where $\mathcal{D}_\lambda$ is the weyl covarient derivative defined in appendix 6.6.1. We also have defined

$$\begin{aligned}
&V \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6}\ ; \qquad l^\mu \equiv \epsilon^{\nu\lambda\sigma\mu}u_\nu\partial_\lambda u_\sigma; \qquad P^\lambda_\mu \mathcal{D}_\lambda q \equiv P^\lambda_\mu\partial_\lambda q + 3(u^\lambda\partial_\lambda u_\mu)q; \qquad \rho \equiv \frac{r}{R} \\
&\sigma^{\mu\nu} \equiv P^{\mu\alpha}P^{\nu\beta}\partial_{(\alpha}u_{\beta)} - \frac{1}{3}P^{\mu\nu}\partial_\alpha u_\alpha; \qquad M \equiv \frac{m}{R^4}; \qquad Q \equiv \frac{q}{R^3}; \qquad Q^2 = M - 1
\end{aligned} \tag{6.49}$$

and

$$\begin{aligned}
F_1(\rho, M) &\equiv \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right)\int_\rho^\infty dp \frac{1}{\left(1 - \frac{M}{p^4} + \frac{Q^2}{p^6}\right)^2}\left(\frac{1}{p^8} - \frac{3}{4p^7}\left(1 + \frac{1}{M}\right)\right) \\
F_2(\rho, M) &\equiv \int_\rho^\infty \frac{p\left(p^2+p+1\right)}{(p+1)\left(p^4+p^2-M+1\right)}dp\ .
\end{aligned} \tag{6.50}$$

6.3.5 The Stress Tensor and Charge Current at first order

In this section, we obtain the stress tensor and the charge current from the metric and the gauge field. The stress tensor can be obtained from the extrinsic curvature after subtraction of the appropriate counterterms [43, 109]. We get the first order stress tensor as

$$T_{\mu\nu} = p(\eta_{\mu\nu} + 4u_\mu u_\nu) - 2\eta\sigma_{\mu\nu} + \dots \tag{6.51}$$

where the fluid pressure p and the viscosity η are given by the expressions

$$p \equiv \frac{MR^4}{16\pi G_5} \qquad ; \qquad \eta \equiv \frac{R^3}{16\pi G_5} = \frac{s}{4\pi} \tag{6.52}$$

where s is the entropy density of the fluid obtained from the Bekenstein formula.

To obtain the charge current, we use

$$J_\mu = \lim_{r\to\infty} \frac{r^2 A_\mu}{8\pi G_5} = n\ u_\mu - \mathfrak{D}\ P^\nu_\mu \mathcal{D}_\nu n + \xi\ l_\mu + \dots \tag{6.53}$$

where the charge density n, the diffusion constant $\mathfrak{D}$ and an additional transport coefficient ξ for the fluid under consideration are given by [94]

$$n \equiv \frac{\sqrt{3}q}{16\pi G_5} \quad ; \quad \mathfrak{D} = \frac{1+M}{4MR} \quad ; \quad \xi \equiv \frac{3\kappa q^2}{16\pi G_5 m} \tag{6.54}$$

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the charge current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid made of $\mathcal{N} = 4$ SYM matter. It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation and we hope to return to this issue in future.

The presence of such an effect was indirectly observed by the authors of [15] where they noted a discrepancy between the thermodynamics of charged rotating AdS black holes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N} = 4$ SYM fluid. In fact, one could go further and compare the first order metric that we have obtained with rotating black hole metrics written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.

[94] Here we have taken the chemical potential $\mu = 2\sqrt{3}QR$ which determines the normalization factor of the charge density n (because thermodynamics tells us $n\mu = 4p - Ts$) which in turn determines the normalization of J_μ. Note that due to the difference in μ with [18], our normalization of J_μ is different from that in [18].

6.4 Second Order Hydrodynamics

In this section we will find out the metric, stress tensor and charge current at second order in derivative expansion. We will follow the same procedure as in [6] but in presence charge parameter q.

The metric and gauge field perturbations at second order that we consider are

$$g^{(2)}_{\alpha\beta}dx^\alpha dx^\beta = -3h_2(r)dvdr + r^2h_2(r)dx^idx_i + \frac{k_2(r)}{r^2}dv^2 + 12r^2j_i^{(2)}dvdx^i + r^2\alpha^{(2)}_{ij}dx^idx^j \quad (6.55)$$

and

$$\begin{aligned} A_v^{(2)} &= -\frac{\sqrt{3}}{2r^2}w_2(r) \\ A_i^{(2)} &= \frac{\sqrt{3}}{2}r^5g_i^{(2)}(r)dx^i \ . \end{aligned} \quad (6.56)$$

Here we have used a little different parameterizations (from first order) for metric and gauge field perturbations in the vector sector. We found that this aids in writting the corresponding dynamical equations for $j_i^{(2)}(r)$ and $g_i^{(2)}(r)$ in a more tractable form (as we will see later).

Like neutral black brane case, here also we will list all the source terms (second order in derivative expansion) which will appear on the right hand side of the constraint dynamical equations in scalar, vector and tensor sectors. These source terms are built out of second derivatives of m, q and β or square of first derivatives of these three fields. We can group these source terms according to their transformation properties under $\mathbf{SO(3)}$ group. A complete list has been provided in table 6.4. In the table the quantities l_i and σ_{ij} are defined to be

$$l_i = \epsilon_{ijk}\partial_j\beta_k \ , \qquad \sigma_{ij} = \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_k) - \frac{1}{3}\delta_{ij}\partial_k\beta_k \ . \quad (6.57)$$

In table 6.4 we have already employed the first order conservation relations i.e. equation 6.19 and 6.20. Using these two relations we have eliminated the first derivatives of m and q. However at second order in derivative expansion we also have the relations

$$\partial_\mu\partial_\nu T^{\mu\nu}_{(0)} = 0 \ , \quad (6.58)$$

and

$$\partial_\lambda\partial_\mu J^\mu_{(0)} = 0 \ . \quad (6.59)$$

The equations (6.58) and (6.59) imply some relations between the second order source

1 of $SO(3)$	**3** of $SO(3)$	**5** of $SO(3)$
$\mathrm{S1} = \partial_v^2 m$	$\mathrm{V1}_i = \partial_i \partial_v m$	$\mathrm{T1}_{ij} = \partial_i \partial_j m - \frac{1}{3}\ \mathrm{s3}\ \delta_{ij}$
$\mathrm{S2} = \partial_v \partial_i \beta_i$	$\mathrm{V2}_i = \partial_v^2 \beta_i$	$\mathrm{T2}_{ij} = \partial_{(i} l_{j)}$
$\mathrm{S3} = \partial^2 m$	$\mathrm{V3}_i = \partial_v l_i$	$\mathrm{T3}_{ij} = \partial_v \sigma_{ij}$
$\mathrm{ST1} = \partial_v \beta_i\, \partial_v \beta_i$	$\mathrm{V4}_i = \frac{9}{5} \partial_j \sigma_{ji} - \partial^2 \beta_i$	$\mathrm{TT1}_{ij} = \partial_v \beta_i\, \partial_v \beta_j - \frac{1}{3}\ \mathrm{ST1}\ \delta_{ij}$
$\mathrm{ST2} = l_i\, \partial_v \beta_i$	$\mathrm{V5}_i = \partial^2 \beta_i$	$\mathrm{TT2}_{ij} = l_{(i}\, \partial_v \beta_{j)} - \frac{1}{3}\ \mathrm{ST2}\ \delta_{ij}$
$\mathrm{ST3} = (\partial_i \beta_i)^2$	$\mathrm{VT1}_i = \frac{1}{3} (\partial_v \beta_i)(\partial_j \beta^j)$	$\mathrm{TT3}_{ij} = 2\, \epsilon_{kl(i}\, \partial_v \beta^k\, \partial_{j)} \beta^l + \frac{2}{3}\ \mathrm{ST2}\ \delta_{ij}$
$\mathrm{ST4} = l_i\, l^i$	$\mathrm{VT2}_i = -\epsilon_{ijk}\, l^j\, \partial_v \beta^k$	$\mathrm{TT4}_{ij} = \partial_k \beta^k\, \sigma_{ij}$
$\mathrm{ST5} = \sigma_{ij}\, \sigma^{ij}$	$\mathrm{VT3}_i = \sigma_{ij}\, \partial_v \beta^j$	$\mathrm{TT5}_{ij} = l_i\, l_j - \frac{1}{3}\ \mathrm{ST4}\ \delta_{ij}$
$\mathrm{QS1} = \partial_v^2 q$	$\mathrm{VT4}_i = l_i\, \partial_j \beta^j$	$\mathrm{TT6}_{ij} = \sigma_{ik}\, \sigma^k_j - \frac{1}{3}\ \mathrm{ST5}\ \delta_{ij}$
$\mathrm{QS2} = \partial_i \partial_i q$	$\mathrm{VT5}_i = \sigma_{ij}\, l^j$	$\mathrm{TT7}_{ij} = 2\, \epsilon_{mn(i}\, l^m\, \sigma^n_{j)}$
$\mathrm{QS3} = l_i \partial_i q$	$\mathrm{QV1}_i = \partial_i \partial_v q$	$\mathrm{QT1}_{ij} = \partial_i \partial_j q - \frac{1}{3}\ \mathrm{QS2}\ \delta_{ij}$
$\mathrm{QS4} = (\partial_i q)^2$	$\mathrm{QV2}_i = \partial_i q \partial_k \beta^k$	$\mathrm{QT2}_{ij} = \partial_{(i} q l_{j)} - \frac{1}{3}\ \mathrm{QS3}\ \delta_{ij}$
$\mathrm{QS5} = (\partial_i q)(\partial_v \beta_i)$	$\mathrm{QV3}_i = \epsilon_{ijk} \partial_j l_k$	$\mathrm{QT3}_{ij} = \partial_{(i} q \partial_{j)} q - \frac{1}{3}\ \mathrm{QS4}\ \delta_{ij}$
	$\mathrm{QV4}_i = \sigma_{ij} \partial_j q$	$\mathrm{QT4}_{ij} = \partial_{(i} q \partial_v \beta_{j)} - \frac{1}{3}\ \mathrm{QS5}\ \delta_{ij}$
	$\mathrm{QV5}_i = \epsilon_{ijk} \partial_v \beta_j \partial_k q$	$\mathrm{QT5}_{ij} = \epsilon_{(ikm} \partial_k q\ \sigma_{mj)}$

Table 1. An exhaustive list of two derivative terms in made up from the mass, charge and velocity fields. In order to present the results economically, we have dropped the superscript on the velocities β_i charge q and the mass m, leaving it implicit that these expressions are only valid at second order in the derivative expansion.

terms which are listed in table 6.4. These relations are

$$
\begin{aligned}
\mathrm{S1} &= \frac{\mathrm{S3}}{3} - \frac{8m}{3}\mathrm{ST1} + \frac{16m}{9}\mathrm{ST3} - \frac{2m}{3}\mathrm{ST4} + \frac{4m}{3}\mathrm{ST5} \\
\mathrm{S2} &= -\frac{1}{4m}\mathrm{S3} + 4\mathrm{ST1} + \frac{1}{2}\mathrm{ST4} - \mathrm{ST5} \\
\mathrm{QS1} &= q\left(-\mathrm{ST1} - \mathrm{S2} + \mathrm{ST3}\right) - \mathrm{QS5} \\
\mathrm{V1}_i &= m\left(-\frac{40}{9}\mathrm{V4}_i - \frac{4}{9}\mathrm{V5}_i + \frac{56}{3}\mathrm{VT1}_i + \frac{4}{3}\mathrm{VT2}_i + \frac{8}{3}\mathrm{VT3}_i\right) \\
\mathrm{V2}_i &= \frac{10}{9}\mathrm{V4}_i + \frac{1}{9}\mathrm{V5}_i - \frac{2}{3}\mathrm{VT1}_i + \frac{1}{6}\mathrm{VT2}_i - \frac{5}{3}\mathrm{VT3}_i \\
\mathrm{V3}_i &= -\frac{1}{3}\mathrm{VT4}_i + \mathrm{VT5}_i \\
\mathrm{QV1}_i &= -q\left(\frac{10}{3}\mathrm{V4}_i + \frac{1}{2}(\mathrm{VT2}_i + 2\mathrm{VT1}_i + 2\mathrm{VT3}_i) + \frac{1}{3}\mathrm{V5}_i\right) \\
&\quad -\mathrm{QV2}_i - \frac{1}{2}\left(2\mathrm{QV4}_i + \mathrm{QV3}_i + \frac{2}{3}\mathrm{QV2}_i\right) \\
\mathrm{T1}_{ij} &= -4m\left(\mathrm{T3}_{ij} + \frac{1}{4}\mathrm{TT5}_{ij} - 4\mathrm{TT1}_{ij} + \frac{1}{3}\mathrm{TT4}_{ij} + \mathrm{TT6}_{ij}\right)
\end{aligned}
\tag{6.60}
$$

With these relation between the source terms we will now solve the Einstein equations and Maxwell equations to find out the constraint and dynamical equations at second order in derivative expansion. As in the first order calculations we shall perform this seperately in various sectors denoting different representation of the boundary rotation group $SO(3)$.

6.4.1 Scalars of $SO(3)$ at second order

We parametrise the metric and the gauge field as follows

$$
\begin{aligned}
\sum_i g^{(2)}_{ii}(r) &= 3r^2 h_2(r), \\
g^{(2)}_{vv}(r) &= \frac{k_2(r)}{r^2} \\
g^{(2)}_{vr}(r) &= -\frac{3}{2}h_2(r) \\
A^{(2)}_v(r) &= -\frac{\sqrt{3}w_2(r)}{2r^2}.
\end{aligned}
\tag{6.61}
$$

Now we intend to solve for the functions $h_2(r), k_2(r)$ and $w_2(r)$.

Constraint Equations

As we have already explained, there are three constraint equations. First two come from Einstein equations (Eq. 6.15 and 6.15) and the third one comes from Maxwell equations (Eq. 6.17). The first constrain from Einstein equations gives

$$
\partial_v m^{(1)} = \frac{2}{3}R^3\ \mathrm{ST5} \tag{6.62}
$$

Second constraint implies relation between $k_2(r)$ and $h_2(r)$. This constraint equation is given by

$$-m_0 h_2'(r) + 3r^4 h_2'(r) + 12r^3 h_2(r) - k_2'(r) + \frac{4q_0 w_2(r)}{r^3} - \frac{2q_0 w_2'(r)}{r^2} = S_C, \tag{6.63}$$

where the source term S_C is given in appendix 6.6.2.

The constraint relation coming from Maxwell equations is given by

$$\begin{aligned}
\partial_v q^{(1)} &= -\frac{3q_0\left(R^4+m_0\right)}{16{m_0}^2 R}\mathrm{S3} + \frac{\left(R^4+m_0\right)}{4m_0 R}\mathrm{QS2} - \frac{6\sqrt{3}{q_0}^2\kappa}{m_0}\mathrm{ST2} \\
&- \frac{\left(m_0-11R^4\right)}{4m_0 R}\mathrm{QS5} - \frac{2\sqrt{3}q_0\kappa}{m_0}\mathrm{QS3} - \frac{q_0}{4m_0 R^3}\mathrm{QS4} \\
&+ \frac{9q_0\left(3R^4+m_0\right)}{4m_0 R}\mathrm{ST1}
\end{aligned} \tag{6.64}$$

Dynamical Equations and their solutions

The Dynamical Equations in the scalar sector (coming from the Einstein equation $E_{rr} = 0$) is given by

$$r h_2''(r) + 5h_2'(r) = S_h \ . \tag{6.65}$$

The source term S_h is explicitly given in appendix 6.6.2.

The second dynamical scalar equation, which comes form the Maxwell equations ($M(r) = 0$), is given by

$$-6q_0 h_2'(r) + r w_2''(r) - w_2'(r) = S_M(r). \tag{6.66}$$

The explicit form of the source term $S_M(r)$ is again given in appendix 6.6.2.

The source terms have the same large r behavior as uncharged case (see [6]) because the charge dependent terms (leading) are more suppressed than that of charge independent terms. So one can follow the same procedure to obtain the solution for $h_2(r)$ and $k_2(r)$. Here we present the result schematically. Firstly, we solve equation (6.65) for the function $h_2(r)$; we obtain

$$h_2(r) = \int \left(\frac{1}{r^5}\left(\int \left(r^4 S_h(r)\right) dr + C_h^{(1)}\right)\right) dr + C_h^{(2)}, \tag{6.67}$$

where $C_h^{(1)}$ and $C_h^{(2)}$ are the constants of integration. We then plug in this solution for $h_2(r)$ in to (6.66). Solving the resultant equation for the w_2 we obtain,

$$w_2(r) = \int \left(r\left(\int \left(\frac{1}{r^2} S_w(r)\right) dr + C_w^{(1)}\right)\right) dr + C_w^{(2)}, \tag{6.68}$$

where again $C_w^{(1)}$ and $C_w^{(2)}$ are integration constants, and the function $S_w(r)$ is

$$S_w(r) = S_M(r) + 6q_0 h_2'(r).$$

Finally, we substitute the functions $h_2(r)$ and $w_2(r)$ solved above, in to (6.63) to obtain the following equation for $k_2(r)$

$$k_2'(r) = (3r^4 - m_0)h_2'(r) + 12r^3 h_2(r) + \frac{4q_0}{r^3}w_2(r) - \frac{2q_0}{r^2}w_2'(r) - S_C \equiv S_k(r). \tag{6.69}$$

This equation can be easily integrated to obtain

$$k_2(r) = \int S_k(r)dr + C_k, \tag{6.70}$$

C_k being the integration constant. All the integration constants in the above solutions are obtained by imposing regularity at the horizon and normalizability of the functions, just as in the first order computation.

6.4.2 Vectors of $SO(3)$ at second order

As given in (6.55) and (6.56), in this sector we parametrize[95] the metric, and the gauge field respectively in the following way

$$\begin{aligned} g_{vi} &= 6r^2 j_i^{(2)}(r) \\ A_i^{(2)} &= \frac{\sqrt{3}}{2} r^2 g_i^{(2)}(r). \end{aligned} \tag{6.71}$$

Constraint Equations

In this sector, the constraint equation comes only from the Einstein equations (6.31). This constraint relation is give by

$$\begin{aligned} \partial_i m^{(1)} &= \frac{10R^3}{9}\mathrm{V4}_i + \frac{10R^3}{9}\mathrm{V5}_i + \frac{10R^3}{3}\mathrm{VT1}_i - \frac{5R^3}{6}\mathrm{VT2}_i \\ &+ \frac{6q_0 R}{m_0 - 3R^4}\mathrm{QV4}_i - \frac{\left(21R^7 - 43m_0R^3\right)}{3\left(m_0 - 3R^4\right)}\mathrm{VT3}_i \ . \end{aligned} \tag{6.72}$$

Dynamical Equations and their solutions

There are two vector dynamical equations. The first equation comes from Einstein equation and is given by

$$q_0 r {g_i^{(2)}}'(r) + 5q_0 g_i^{(2)}(r) + r {j_i^{(2)}}''(r) + 5{j_i^{(2)}}'(r) = (S_E^{\mathrm{vec}})_i(r), \tag{6.73}$$

where $(S_E^{\mathrm{vec}})_i(r)$ is the source terms given in the appendix 6.6.3. The second dynamical equation comes from Maxwell equation and is given by

$$\begin{aligned} \sqrt{3}\Big(&- m_0 r^4 {g_i^{(2)}}''(r) + q_0^2 r^2 {g_i^{(2)}}''(r) + r^8 {g_i^{(2)}}''(r) + {g_i^{(2)}}'(r)\left(-9m_0r^3 + 7q_0^2 r + 13r^7\right) \\ &+ 5g_i^{(2)}(r)\left(-3m_0r^2 + q_0^2 + 7r^6\right) + 12q_0 {j_i^{(2)}}'(r)\Big) = (S_M^{\mathrm{vec}})_i(r) \end{aligned} \tag{6.74}$$

[95]Note that the parametrization of the gauge field at this order is different from the one used for the scalar sector.

where $(S_M^{\text{vec}})_i(r)$ is the other source term the explicit form of which is also given in the appendix 6.6.3. The sources $(S_M^{\text{vec}})_i(r)$ and $(S_E^{\text{vec}})_i(r)$ are expressed in terms of the weyl invariant quantities $(W_v)_i^m$ which are defined in appendix 6.6.1. We can now solve equation (6.73) for the function $g_i^{(2)}(r)$ to obtain

$$g_i^{(2)}(r) = -\frac{{j_i^{(2)}}'(r)}{q_0} + \frac{(W_v)_i^1 + (W_v)_i^2}{6q_0 r^3} - \left(\frac{1}{q_0 r^5}\right)\int_r^\infty x^4\left((S_E^{\text{vec}})_i(r) - \frac{(W_v)_i^1 + (W_v)_i^2}{3x^3}\right)dx, \tag{6.75}$$

where the integrating constant has been chosen by the normalizability condition. Plugging in this solution in to (6.74) we obtain the following effective equation for $j_i^{(2)}(r)$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^7\left(V^{(0)}(r)\right)^2\frac{d}{dr}\left(\frac{1}{V^{(0)}(r)}j_i^{(2)}(r)\right)\right)\right) + S_i(r) = 0, \tag{6.76}$$

where

$$\begin{aligned} S_i(r) = \left(-\frac{1}{\sqrt{3}r^2}\right)\Bigg(&\sqrt{3}\left(r\left(m_0\left(R^2 - r^2\right) + r^6 - R^6\right)(S_E^{\text{vec}})_i'(r)\right. \\ &+(S_E^{\text{vec}})_i(r)\left(m_0\left(R^2 - 3r^2\right) + 7r^6 - R^6\right)\Big) - \sqrt{R^2\left(m_0 - R^4\right)}(S_M^{\text{vec}})_i(r)\Bigg). \end{aligned} \tag{6.77}$$

Finally, the solution to the equation (6.76) is given by

$$\begin{aligned} j_i^{(2)}(r) = &-V^{(0)}(r)\int_r^\infty \frac{1}{x^7\left(V^{(0)}(x)\right)^2}\left(\int_x^\infty y\int_y^\infty S_i^{reg}(z)dzdy\right)dx \\ &- V^{(0)}(r)\int_r^\infty \frac{1}{x^7\left(V^{(0)}(x)\right)^2}\Bigg[C_i^{(j)} - \frac{1}{3(m_0 3R^2)}3R^7\Bigg(\left((W_v)_i^1 + (W_v)_i^4\right)x \\ &- m_0 R^3\left((W_v)_i^1 + 3(W_v)_i^4\right)x - \frac{1}{2}m_0\left((W_v)_i^1 + (W_v)_i^2\right)x^4 + \frac{3}{2}R^4\left((W_v)_i^1 + (W_v)_i^2\right)x^4\Bigg)\Bigg]dx, \end{aligned} \tag{6.78}$$

where again for convenience we have defined

$$S_i^{reg}(z) = \frac{R^3\left(m_0((W_v)_i^1 + 3(W_v)_i^4) - 3R^4((W_v)_i^1 + (W_v)_i^4)\right)}{3z^2\left(m_0 - 3R^4\right)} - S_i(z) - \frac{4}{3}z((W_v)_i^1 + (W_v)_i^2). \tag{6.79}$$

The constant $C_i^{(j)}$ is determined by the regularity at horizon and is given by

$$\begin{aligned} C_i^{(j)} = -\frac{1}{12m_0\left(m_0 - 3R^4\right)}\Bigg(&R^4\Big(m_0^2(9(W_v)_i^1 + 4(W_v)_i^2 + 15(W_v)_i^4) \\ &- 6m_0R^4(6(W_v)_i^1 + 3(W_v)_i^2 + 4(W_v)_i^4) + 9R^8(3(W_v)_i^1 + 2(W_v)_i^2 + (W_v)_i^4)\Big) \\ &- 9R^2\left(m_0^2 - 4m_0R^4 + 3R^8\right)\left(\int_R^\infty S_i^{reg}(x)\,dx\right) + 6m_0\left(m_0 - 3R^4\right)\int_R^\infty y^2 S_i^{reg}(y)\,dy\Bigg), \end{aligned} \tag{6.80}$$

We now have to plug in the source terms (given in Appendix 6.6.3) and perform the integrals to write the solutions explicitly. Since such explicit solution would be very complicated, we do not provide it here. Nevertheless, from the above solution we extract the boundary charge current as we explicate in the following section.

6.4.3 Boundary Charge Current at second order

The charge current at second order in derivative expansion is given by

$$J_\mu^{(2)} = \lim_{r\to\infty} \frac{r^2 A_\mu^{(2)}}{8\pi G_5} \ . \tag{6.81}$$

The gauge field perturbation at this order is parametrised by the function $g_i^{(2)}(r)$. Thus to obtain the charge current density we have to consider the asymptotic limit (i.e. the $r \to \infty$ limit) of the function $g_i^{(2)}(r)$. This function is given by (6.75). The function $j_i^{(2)}(r)$ in that equation is in turn given by (6.78).

If we carefully extract the coefficient of the $1/r^2$ term in the $r \to \infty$ limit of the gauge field (using the equation referred to in the last paragraph) we find that the charge current is given by

$$J_i^{(2)} = \frac{m_0 (W_v)_i^2 - 6C_i^{(j)}}{4\sqrt{3}\sqrt{R^2\left(m_0 - R^4\right)}}, \tag{6.82}$$

the constant $C_i^{(j)}$ being given by the equation (6.80). Plugging in the sources in to equation (6.80) and performing the integrations we find

$$J_i^{(2)} = \left(\frac{1}{8\pi G_5}\right) \sum_{l=1}^{5} \mathcal{C}_l (W_v)_i^l, \tag{6.83}$$

where the coefficients of the Weyl invariant terms $(W_v)_i^l$ are given by [96]

$$\begin{aligned}
\mathcal{C}_1 &= \frac{3\sqrt{3}R\sqrt{M-1}}{8M}, \\
\mathcal{C}_2 &= \frac{\sqrt{3}R(M-1)^{3/2}}{4M^2}, \\
\mathcal{C}_3 &= -\frac{3R\kappa(M-1)}{2M^2}, \\
\mathcal{C}_4 &= \frac{1}{4}\sqrt{3}R\sqrt{M-1}\log(2) + \mathcal{O}(M-1), \\
\mathcal{C}_5 &= -\frac{\sqrt{3}R\sqrt{M-1}\left(M^2 - 48(M-1)\kappa^2 + 3\right)}{16M^2}.
\end{aligned} \tag{6.84}$$

We have expressed the above results in terms of the parameters M and R with $M = m_0/R^4$.

[96] All these coefficients perfectly match with the corresponding coefficients in version 4 of [18]

6.4.4 Tensors Of $SO(3)$ at second order

We now consider the tensor modes at second order. Following the first order calculations we pametrize the traceless symmetric tensor components of the second order metric by the function $\alpha^{(2)}_{ij}(r)$ such that

$$g^{(2)}_{ij} = r^2 \alpha^{(2)}_{ij}(r). \tag{6.85}$$

In this sector there are no constraint equations. However, there is a dynamical equation which we solve in the following subsection.

Dynamical equations and their solutions

The ij-component of the Einstein equation gives the dynamical equation for $\alpha^{(2)}_{ij}(r)$ which is similar to (6.43). However the source term of the differential equation is modified in the second order. Thus, at second order this equation is given by

$$-\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{r}\left(q_0^2 - m_0 r^2 + r^6\right)\frac{d}{dr}\alpha^{(2)}_{ij}(r)\right) = \mathbf{T}_{ij}(r), \tag{6.86}$$

where we write the source in terms of weyl-covariant quantities as follows

$$\mathbf{T}_{ij}(r) = \sum_{l=1}^{9} \tau_l(r)\ WT^{(l)}_{ij}. \tag{6.87}$$

We define the weyl-covariant terms $WT^{(l)}_{ij}$ in appendix 6.6.1. The coefficients $\tau_l(r)$ of these weyl-covariant terms are given in appendix 6.6.4.

The solution to (6.86) which is regular at the outer horizon and normalizable at infinity is given by

$$\alpha^{(2)}_{ij}(r) = \int_r^\infty \left(\left(\frac{\xi}{q_0^2 - m_0\xi^2 + \xi^6}\right)\int_1^\xi \left(\ 2\ \zeta\ \mathbf{T}_{ij}(\zeta)\right) d\zeta\right) d\xi. \tag{6.88}$$

We need to plug in the source from appendix 6.6.4 in to the above equation and perform the integrals to obtain an explicit answer. However, as in the second order vector sector this turns out to be very complicated in general and therefore we do not produce it here. The transport coefficients, however, of the boundary stress tensor at second order in derivative expansion may be obtained only by knowing the function $\alpha^{(2)}_{ij}(r)$ asymptotically (near the boundary). In the next subsection, we compute this boundary stress tensor.

6.4.5 Boundary Stress Tensor at second order

As mentioned earlier in subsection 6.3.5, the AdS/CFT prescription for obtaining the boundary stress tensor from the bulk metric is given by

$$T^\mu_\nu = -\frac{1}{8\pi G_5}\lim_{r\to\infty}\left(r^4\left(K^\mu_\nu - \delta^\mu_\nu\right)\right), \tag{6.89}$$

where K^μ_ν is the extrinsic curvature normal to the constant r surface. Now, as is apparent from the formula, we need to know the asymptotic expansion of the metric perturbation

$\alpha_{ij}^{(2)}(\rho)$ in order to obtain the stress tensor. The asymptotic expansion of the solution (6.88) for $\alpha_{ij}^{(2)}(\rho)$ is given by

$$\alpha_{ij}^{(2)}(\rho) = \frac{1}{r^2}\left(WT_{ij}^{(3)} - \frac{1}{2}WT_{ij}^{(2)} - \frac{1}{4}WT_{ij}^{(4)}\right) + \frac{1}{4r^4}\sum_{l=1}^{9}\mathcal{N}_l \; WT_{ij}^{(l)} + \mathcal{O}\left(\frac{1}{r^5}\right), \tag{6.90}$$

The leading term of this asymptotic expansion gives divergent contributions to the stress tensor which are canceled by divergence arising from the expansion of $g^{(0)} + g^{(1)}$ up to second order.

On plugging in this asymptotic solution for the metric in to the formula (6.89) we obtain

$$T_{\mu\nu} = \left(\frac{1}{16\pi G_5}\right)\sum_{l=1}^{9}\mathcal{N}_l \; WT_{\mu\nu}^{(l)}. \tag{6.91}$$

with $\mathcal{N}_l$ being the transport coefficients at second order in derivative expansion. These transport coefficients are given by

$$\begin{aligned}
\mathcal{N}_1 &= R^2\left(\frac{M}{\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{3+\sqrt{4M-3}}\right)+2\right)\Bigg), \\
\mathcal{N}_2 &= -\frac{MR^2}{2\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{\sqrt{4M-3}+3}\right), \\
\mathcal{N}_3 &= 2R^2, \\
\mathcal{N}_4 &= \frac{R^2}{M}(M-1)\left(12(M-1)\kappa^2 - M\right), \\
\mathcal{N}_5 &= -\frac{(M-1)R^2}{2M}, \\
\mathcal{N}_6 &= \frac{1}{2}(M-1)R^2\Big(\log(8)-1\Big) + \mathcal{O}\left((M-1)^2\right), \\
\mathcal{N}_7 &= \frac{\sqrt{3}(M-1)^{3/2}R^2\kappa}{M}, \\
\mathcal{N}_8 &= 0 \\
\mathcal{N}_9 &= 0.
\end{aligned} \tag{6.92}$$

6.5 Charged Blackhole Solution

We will now turn to the black hole solutions of the five-dimensional action in (6.2) :

$$S = \frac{1}{16\pi G_5}\int\sqrt{-g_5}\left[R + 12 - F_{AB}F^{AB} - \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\right] \tag{6.93}$$

which is a consistent truncation of IIB SUGRA Lagrangian on $AdS_5 \times S^5$ background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = 1/(2\sqrt{3})$. The general blackhole solutions which solves the equations coming out of this action with this special value of κ was found in [19]. Their solution is given by[97]

$$
\begin{aligned}
ds^2 = &-\frac{\left(r^2+1\right)\Delta_\Theta {dt_1}^2}{\left(1-{\omega_1}^2\right)\left(1-{\omega_2}^2\right)} + \frac{2(m-q\omega_1\omega_2)}{\rho^2} - \frac{q^2}{\rho^4}\\
&+ \frac{(\mathrm{d}\psi_1+dt_1\omega_2)^2\left(r^2+{\omega_2}^2\right)\cos^2\Theta}{1-{\omega_2}^2} + \frac{(\mathrm{d}\phi_1+dt_1\omega_1)^2\left(r^2+{\omega_1}^2\right)\sin^2\Theta}{1-{\omega_1}^2}\\
&+ \frac{\rho^2 \mathrm{d}r^2 r^2}{q^2-2\omega_1\omega_2 q - 2mr^2 + \left(r^2+1\right)\left(r^2+{\omega_1}^2\right)\left(r^2+{\omega_2}^2\right)}\\
&+ \frac{\rho^2\mathrm{d}\Theta^2}{\Delta_\Theta} + \frac{2q}{\rho^2}\left(\omega_1(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta + (\mathrm{d}\phi_1+dt_1\omega_1)\omega_2\sin^2\Theta\right)\\
&\times\left[\frac{\Delta_\Theta dt_1}{\left(1-{\omega_1}^2\right)\left(1-{\omega_2}^2\right)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-{\omega_2}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-{\omega_1}^2}\right]\\
\mathbf{A} = &-\frac{\sqrt{3}q}{\rho^2}\left[\frac{\Delta_\Theta dt_1}{\left(1-{\omega_1}^2\right)\left(1-{\omega_2}^2\right)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-{\omega_2}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-{\omega_1}^2}\right]
\end{aligned}
\tag{6.94}
$$

where we use the definitions

$$
\begin{aligned}
\rho^2 &\equiv r^2 + {\omega_1}^2\cos^2\Theta + {\omega_2}^2\sin^2\Theta\\
\Delta_\Theta &\equiv 1 - {\omega_1}^2\cos^2\Theta - {\omega_2}^2\sin^2\Theta
\end{aligned}
\tag{6.95}
$$

After some manipulations (which closely follow the methods outlined in [7, 11]) , we find that the final metric and the gauge field can be written in a manifestly Weyl-covariant form

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu\left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - {\omega_\mu}^\lambda\omega_{\lambda\nu}\right]dx^\mu dx^\nu\\
&\quad + \left[\left(\frac{2m}{\rho^2} - \frac{q^2}{\rho^4}\right)u_\mu u_\nu + \frac{q}{2\rho^2}u_{(\mu}l_{\nu)}\right]dx^\mu dx^\nu\\
\mathbf{A} &= \frac{\sqrt{3}q}{\rho^2}u_\mu dx^\mu \quad ; \quad \rho^2 \equiv r^2 + \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \quad ; \quad l_\mu \equiv \epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma}
\end{aligned}
\tag{6.96}
$$

with

$$
\begin{aligned}
&T_{\mu\nu} = p(g_{\mu\nu}+4u_\mu u_\nu) + 2\kappa l_{(\mu}J_{\nu)} + \frac{1}{64\pi G_{\text{AdS}}}\left(R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{R^2}{12}g_{\mu\nu}\right)\\
&J_\mu = n u_\mu \qquad \text{where} \quad l_\mu \equiv \epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma} \quad ; \quad p \equiv \frac{m}{8\pi G_{\text{AdS}}} \quad \text{and} \qquad n \equiv \frac{\sqrt{3}q}{8\pi G_{\text{AdS}}}
\end{aligned}
\tag{6.97}
$$

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the energy current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid

[97] Note that the parameter q here is the negative of the one used in [19].

made of $\mathcal{N} = 4$ SYM matter. This new non-dissipative transport can be traced back to the Chern-Simons term in the gravity theory which according to the gauge-gravity duality, encodes the information about the global anomalies in the field theory. This suggests that this transport is closely related to the $U(1)^3$ global anomaly in the field theory.

It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation. However, an indirect explanation was provided by the authors of [20], where they give a clever entropic argument which relates this coefficient to the anomaly. This suggests the possibility that such a transport is universal,i.e., it is present in any field theory which has global anomalies and it would be useful to explicitly check whether this is the case by calculating this transport coefficient in a calculable model - say a spin model.

The presence of such an effect was indirectly observed by the authors of [15] where they noted a discrepancy between the thermodynamics of charged rotating AdS blackholes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N} = 4$ SYM fluid. In fact, one could go further and compare the first order metric obtained in [17, 18] with the rotating blackhole metric written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.

6.6 Appendices

6.6.1 Charged conformal fluids and Weyl covariance

Consider the hydrodynamic limit of a 3+1 dimensional CFT with one global conserved charge. The Weyl covariance of the CFT translates into the Weyl covariance of its hydrodynamics. In turn, this implies that the metric dual to fluid configurations of the CFT under consideration should also be invariant under boundary Weyl-transformations [2, 11, 73].

In this section, we use the manifestly Weyl-covariant formalism introduced in [2] to examine the constraints that Weyl-covariance imposes on the conformal hydrodynamics and its metric dual. We begin by introducing a Weyl-covariant derivative acting on a general tensor field $Q^{\mu\ldots}_{\nu\ldots}$ with weight w (by which we mean that the tensor field transforms as $Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\tilde{Q}^{\mu\ldots}_{\nu\ldots}$ under a Weyl transformation of the boundary metric $g_{\mu\nu} = e^{2\phi}g_{\mu\nu}$)

$$\begin{aligned} \mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} \equiv\, & \nabla_\lambda\, Q^{\mu\ldots}_{\nu\ldots} + w\, \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots} \\ & + \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots \\ & - \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda \mathcal{A}_\nu - \delta^\alpha_\nu \mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots \end{aligned} \tag{6.98}$$

where the Weyl-connection $\mathcal{A}_\mu$ is related to the fluid velocity u^μ via the relation

$$\mathcal{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{3} u_\mu \tag{6.99}$$

We can now use this Weyl-covariant derivative to enumerate all the Weyl-covariant scalars, transverse vectors (i.e, vectors that are everywhere orthogonal to the fluid velocity field u^μ) and the transverse traceless tensors in the charged hydrodynamics that involve no more than second order derivatives. We will do this enumeration 'on-shell', i.e., we will enumerate those quantities which remain linearly independent even after the equations of motion are taken into account. Our discussion here will closely parallel the discussion in section 4.1 of [11] where a similar question was answered in the context of uncharged hydrodynamics coupled to a scalar with weight zero. However, we will use a slightly different basis of Weyl-covariant tensors which is more suited for purposes of this chapter.

The basic fields in the charged hydrodynamics are the fluid velocity u^μ with weight unity, the fluid temperature T with with weight unity and the chemical potential μ with weight unity. This implies that an arbitrary function of μ/T is Weyl-invariant and hence one could always multiply a Weyl-covariant tensor by such a function to get another Weyl-covariant tensor. Hence, in the following list only linearly independent fields appear. To make contact with the conventional literature on hydrodynamics we will work with the charge density n (with weight 3) rather than the chemical potential μ.

At one derivative level, there are no Weyl invariant scalars or pseudo-scalars. The only Weyl invariant transverse vector is $n^{-1}P_\mu^\nu\mathcal{D}_\nu n$. Finally, the only Weyl-invariant transverse pseudo-vector l_μ and only one Weyl-invariant symmetric traceless transverse tensor $T\sigma_{\mu\nu}$.

At the two derivative level, there are five independent Weyl-invariant scalars[98]

$$T^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad T^{-2}\omega_{\mu\nu}\omega^{\mu\nu}, \quad T^{-2}\mathcal{R}, \quad T^{-2}n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n \quad \text{and} \quad T^{-2}n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\mathcal{D}_\nu n \tag{6.101}$$

one Weyl-invariant pseudo-scalar $T^{-2}n^{-1}l^\mu\mathcal{D}_\mu n$ and four independent Weyl-invariant transverse vectors

$$T^{-1}P_\mu^\nu\mathcal{D}_\lambda\sigma_\nu{}^\lambda, \qquad T^{-1}P_\mu^\nu\mathcal{D}_\lambda\omega_\nu{}^\lambda, \qquad T^{-1}n^{-1}\sigma_\mu{}^\lambda\mathcal{D}_\lambda n \qquad \text{and} \qquad T^{-1}n^{-1}\omega_\mu{}^\lambda\mathcal{D}_\lambda n \tag{6.102}$$

and one Weyl-invariant transverse pseudo-vector $T^{-1}\sigma_{\mu\nu}\, l^\nu$.

There are eight Weyl-invariant symmetric traceless transverse tensors -

$$\begin{gathered} u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}, \quad \omega_\mu{}^\lambda\sigma_{\lambda\nu}+\omega_\nu{}^\lambda\sigma_{\lambda\mu}, \quad \sigma_\mu{}^\lambda\sigma_{\lambda\nu}-\frac{P_{\mu\nu}}{3}\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad \omega_\mu{}^\lambda\omega_{\lambda\nu}+\frac{P_{\mu\nu}}{3}\,\omega_{\alpha\beta}\omega^{\alpha\beta}, \\ n^{-1}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha\mathcal{D}_\beta n, \quad n^{-2}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha n\,\mathcal{D}_\beta n, \quad C_{\mu\alpha\nu\beta}u^\alpha u^\beta \quad \text{and} \quad \frac{1}{4}\,\epsilon^{\alpha\beta\lambda}{}_\mu\,\epsilon^{\gamma\theta\sigma}{}_\nu C_{\alpha\beta\gamma\theta}\,u_\lambda u_\sigma. \end{gathered} \tag{6.103}$$

[98]We shall follow the notations of [2] in the rest of this section(except for the curvature tensors which differ by a sign from the curvature tensors in [2]. In particular, we recall the following definitions

$$\begin{aligned} &\mathcal{R}=R+6\nabla_\lambda\mathcal{A}^\lambda-6\mathcal{A}_\lambda\mathcal{A}^\lambda\ ; \qquad &&\mathcal{D}_\mu u_\nu=\sigma_{\mu\nu}+\omega_{\mu\nu} \\ &\mathcal{D}_\lambda\sigma^{\mu\lambda}=\nabla_\lambda\sigma^{\mu\lambda}-3\mathcal{A}_\lambda\sigma^{\mu\lambda}\ ; \qquad &&\mathcal{D}_\lambda\omega^{\mu\lambda}=\nabla_\lambda\omega^{\mu\lambda}-\mathcal{A}_\lambda\omega^{\mu\lambda} \end{aligned} \tag{6.100}$$

Note that in a flat space-time, R is zero but $\mathcal{R}$ is not.

where we have introduced the projection tensor $\Pi^{\alpha\beta}_{\mu\nu}$ which projects out the transverse traceless symmetric part of second rank tensors

$$\Pi^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}\left[P^{\alpha}_{\mu}P^{\beta}_{\nu} + P^{\alpha}_{\nu}P^{\beta}_{\mu} - \frac{2}{3}P^{\alpha\beta}P_{\mu\nu}\right]$$

and $C_{\mu\nu\alpha\beta}$ is the boundary Weyl curvature tensor. Further, there are four Weyl-invariant symmetric traceless transverse pseudo-tensors

$$\mathcal{D}_{(\mu}l_{\nu)}, \quad n^{-1}\Pi^{\alpha\beta}_{\mu\nu}l_\alpha\mathcal{D}_\beta n, \quad n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_\alpha\mathcal{D}_\beta n \quad \text{and} \quad \frac{1}{2}\epsilon_{\alpha\beta\lambda(\mu}C^{\alpha\beta}{}_{\nu)\sigma}u^\lambda u^\sigma. \tag{6.104}$$

We will now restrict ourselves to the case where the boundary metric is flat. In this case the last two tensors appearing in (6.103) and the last tensor appearing in (6.104) are identically zero whereas, contrary to what one might naively expect, the Weyl-covariantised Ricci scalar $\mathcal{R}$ would still be non-zero.

We will now relate the rest of the Weyl-covariant scalars, transverse vectors and symmetric, traceless transverse tensors listed above to the quantities appearing in the table 6.4.

There are six scalar/pseudo-scalar Weyl covariant combinations given by

$$\begin{aligned}
W_s^1 &\equiv \sigma_{\mu\nu}\sigma^{\mu\nu} = \mathrm{ST5}\\
W_s^2 &\equiv \omega_{\mu\nu}\omega^{\mu\nu} = \frac{1}{2}\mathrm{ST4}\\
W_s^3 &\equiv \mathcal{R} = 14\ \mathrm{ST1} + \frac{2}{3}\mathrm{ST3} - \mathrm{ST4} + 2\mathrm{ST5} - \frac{\mathrm{S3}}{m}\\
W_s^4 &\equiv n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n = \frac{1}{q}\left[\mathrm{QS2} - \frac{3q}{4m}\mathrm{S3} + 18\mathrm{qST1} + 5\mathrm{QS5}\right]\\
W_s^5 &\equiv n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\ \mathcal{D}_\nu n = \frac{1}{q^2}\left[\mathrm{QS4} + 6\mathrm{qQS5} + 9\mathrm{q}^2\mathrm{ST1}\right]\\
W_s^6 &\equiv l^\mu\mathcal{D}_\mu q = \mathrm{QS3} + 3\mathrm{qST2}.
\end{aligned} \tag{6.105}$$

and five vector/pseudo-vector Weyl covariant combinations given by

$$\begin{aligned}
(W_v)^1_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda{\sigma_\nu}^\lambda = \frac{5\mathrm{V4}}{9} + \frac{5\mathrm{V5}}{9} + \frac{5\mathrm{VT1}}{3} - \frac{5\mathrm{VT2}}{12} - \frac{11\mathrm{VT3}}{6}\\
(W_v)^2_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda{\omega_\nu}^\lambda = \frac{5\mathrm{V4}}{3} - \frac{\mathrm{V5}}{3} - \mathrm{VT1} - \frac{\mathrm{VT2}}{4} + \frac{\mathrm{VT3}}{2}\\
(W_v)^3_\mu &\equiv l^\lambda\sigma_{\mu\lambda} = \mathrm{VT5}\\
(W_v)^4_\mu &\equiv n^{-1}{\sigma_\mu}^\lambda\mathcal{D}_\lambda n = \frac{1}{q}\left[\mathrm{QV4} + 3\mathrm{qVT3}\right]\\
(W_v)^5_\mu &\equiv n^{-1}{\omega_\mu}^\lambda\mathcal{D}_\lambda n = \frac{1}{2q}\left[\mathrm{QV3} + 3\mathrm{qVT2}\right]
\end{aligned} \tag{6.106}$$

In the tensor sector, there are nine Weyl-covariant combinations

$$
\begin{aligned}
WT^{(1)}_{\mu\nu} &= u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} = TT1 + \frac{1}{3} TT4 + T3. \\
WT^{(2)}_{\mu\nu} &= -2 \left(\omega^{\mu}{}_{\lambda} \sigma^{\lambda\nu} + \omega^{\nu}{}_{\lambda} \sigma^{\lambda\mu} \right) = TT7. \\
WT^{(3)}_{\mu\nu} &= \sigma^{\mu}{}_{\lambda} \sigma_{\lambda\nu} - \frac{1}{3} P^{\mu\nu} \sigma^{\alpha\beta} \sigma_{\alpha\beta} = TT6. \\
WT^{(4)}_{\mu\nu} &= 4 \left(\omega^{\mu}{}_{\lambda} \omega_{\lambda\nu} + \frac{1}{3} P^{\mu\nu} \omega^{\alpha\beta} \omega_{\alpha\beta} \right) = TT5. \\
WT^{(5)}_{\mu\nu} &= n^{-1} \Pi^{\alpha\beta}_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}_\beta n \\
&= \frac{1}{q} \left[QT1 + 8QT4 + 15qTT1 + qTT4 + 3qT3 + 3qTT6 + \frac{3q}{4} TT5 \right] \\
WT^{(6)}_{\mu\nu} &= n^{-2} \Pi^{\alpha\beta}_{\mu\nu} \mathcal{D}_\alpha n \mathcal{D}_\beta n = \frac{1}{q^2} \left[QT3 + 6qQT4 + 9q^2 TT1 \right] \\
WT^{(7)}_{\mu\nu} &= \mathcal{D}_\mu l_\nu + \mathcal{D}_\nu l_\mu = 4TT2 + 2T2 - TT3. \\
WT^{(8)}_{\mu\nu} &= n^{-1} \Pi^{\alpha\beta}_{\mu\nu} l_\alpha \mathcal{D}_\beta n = \frac{1}{q} \left[QT2 + 3\ q\ TT2 \right]. \\
WT^{(9)}_{\mu\nu} &= n^{-1} \epsilon^{\alpha\beta\lambda}{}_{(\mu} \sigma_{\nu)\lambda} u_\alpha \mathcal{D}_\beta n = \frac{1}{q} \left[QT5 - \frac{3}{2}\ q\ TT2 + \frac{3}{2}\ q\ TT3 \right].
\end{aligned}
\tag{6.107}
$$

6.6.2 Source Terms in Scalar Sector: Second Order

There are three source terms in scalar sector at second order $S_k(r)$, $S_h(r)$ and $S_M(r)$. They are quite complicated functions. Here we provide the explicit form of these source terms in terms of weyl covariant quantities.

The source term S_k is given by

$$
S_C = \sum_{i=1}^{6} s_i^{(C)} W_s^i. \tag{6.108}
$$

The Weyl covariant terms W_s^i are given in §6.6.1. The functions $s_i^{(k)}$s are given by,

$$
\begin{aligned}
s_1^{(C)} &= \frac{r\left(4\left(m_0-3r^4\right)\left(r^2+rR+R^2\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R\left(m_0(r+R)-2R^3\left(r^2+rR+R^2\right)\right)\right)}{3R(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\\
s_2^{(C)} &= \frac{1}{3m_0^2r^7}\Big(-m_0^3\left(r^4+2r^2R^2+36R^4\kappa^2\right)+2m_0^2\left(18r^4R^4\kappa^2+r^2R^6+36R^8\kappa^2\right)\\
&\quad -36m_0R^8\kappa^2\left(2r^4+R^4\right)+36r^4R^{12}\kappa^2\Big)\\
s_3^{(C)} &= \frac{r}{3}\\
s_4^{(C)} &= \frac{2r^2\left(m_0-R^4\right)\left(rF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+6RF_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{R^6}\\
s_5^{(C)} &= -\frac{1}{2R^{16}\left(m_0-3R^4\right)}\Bigg(r^2\left(m_0-R^4\right)\Bigg(24R^4F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Big(r^3\Big(m_0^2-4m_0R^4\\
&\quad +\ 3R^8\Big)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+11r^2R\left(m_0^2-4m_0R^4+3R^8\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad +\ 6m_0R^7-4R^{11}\Big)+r\Big(r^2R^2\Big(m_0^2\left(25r^2-13R^2\right)+m_0\left(-25r^6-75r^2R^4+52R^6\right)\\
&\quad +\ 75r^6R^4-39R^{10}\Big)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Big(4R^9\left(m_0-R^4\right)\\
&\quad -\ r^3\left(m_0^2\left(R^2-r^2\right)+m_0\left(r^6+3r^2R^4-4R^6\right)+3R^4\left(R^6-r^6\right)\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Big)\\
&\quad +\ 2RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Big(-5r^3\Big(m_0^2\left(R^2-r^2\right)+m_0\left(r^6+3r^2R^4-4R^6\right)\\
&\quad +\ 3R^4\left(R^6-r^6\right)\Big)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+26m_0R^9-22R^{13}\Big)\\
&\quad +\ 16m_0R^6\left(m_0-R^4\right)F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Big)\Bigg)+96m_0R^7\left(m_0-R^4\right)F_1^{(0,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad +\ 288rR^6\left(m_0-3R^4\right)\left(m_0-R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2\Bigg)\Bigg)\\
s_6^{(C)} &= \frac{2\sqrt{3}\kappa\left(m_0-r^4\right)\left(R^4-m_0\right)\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{m_0R^7}.
\end{aligned}
\tag{6.109}
$$

The source term S_h is given by

$$
S_h=\sum_{i=1}^{6}s_i^{(h)}W_s^i, \tag{6.110}
$$

where the functions $s_i^{(h)}$'s are given by

$$
\begin{aligned}
s_1^{(h)} &= \frac{1}{3R(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2}\left(2r\left(2\left(m_0\left(4r^3+8r^2R+6rR^2+3R^3\right)\right.\right.\right. \\
&\quad \left.\left.\left. -3R^3\left(r^2+rR+R^2\right)^2\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)+r^2R\left(r^2+rR+R^2\right)^2\right)\right), \\
s_2^{(h)} &= \frac{2}{3r^7}\left(r^4-\frac{36R^4\kappa^2\left(m_0-R^4\right)^2}{m_0^2}\right), \\
s_3^{(h)} &= 0, \\
s_4^{(h)} &= 0, \\
s_5^{(h)} &= \frac{r^7\left(R^4-m_0\right)}{R^{16}}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)^2 \\
s_6^{(h)} &= \frac{4\sqrt{3}\kappa\left(R^4-m_0\right)}{m_0R^7}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right).
\end{aligned}
\tag{6.111}
$$

Finally the source term $S_M(r)$ is given by

$$
S_M(r) = \sum_{i=1}^{6} s_i^{(M)} W_s^i \ , \tag{6.112}
$$

with the functions $s_i^{(M)}$ being given by

$$\begin{aligned}
s_1^{(M)} &= \frac{4r\sqrt{R^2\left(m_0-R^4\right)}\left(r^2+rR+R^2\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\\
s_2^{(M)} &= -\frac{2\sqrt{R^2\left(m_0-R^4\right)}\left(m_0^2r^4+12R^2\kappa^2\left(m_0-R^4\right)\left(2m_0r^2+3m_0R^2-3R^6\right)\right)}{m_0^2r^7}\\
s_3^{(M)} &= 0\\
s_4^{(M)} &= -\frac{r^5\sqrt{R^2\left(m_0-R^4\right)}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{R^9}\\
s_5^{(M)} &= \frac{r^5\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{R^{17}\left(3R^4-m_0\right)}\left(r^2\left(-\left(6r\left(m_0-3R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^5\right)\right)F_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad -2\left(15r^2R\left(m_0-3R^4\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2+\left(3r\left(m_0-3R^4\right)\left(r^2F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\right.\right.\\
&\quad \left.\left.+35R^2F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+20R^7\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R\left(2m_0rRF_1^{(2,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad +r\left(39r\left(m_0-3R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+7R^5\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad \left.\left.\left.+10m_0R^2F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\right)\right)\\
s_6^{(M)} &= \frac{\sqrt{3}\kappa\left(R^4-m_0\right)}{m_0r^2R^7\sqrt{R^2\left(m_0-R^4\right)}}\left(-m_0r^4RF_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+r^4R^5F_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad +r^2R\left(20m_0r^2-17m_0R^2+17R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad \left.+r^3\left(4m_0r^2-7m_0R^2+7R^6\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+4R^9\right)
\end{aligned} \tag{6.113}$$

6.6.3 Source Terms in Vector Sector: Second Order

The source term in the vector sector at second order $S_E^{\text{vec}}(r)$ in (6.73) is given by

$$(S_E^{\text{vec}})_i(r) = \sum_{l=1}^{5} r_l^{(E)}(W_v)_i^l \tag{6.114}$$

where the Weyl covariant quantities W_v^i's are given in Appendix 6.6.1 and the functions $s_i^{(E)}$ are given by

$$
\begin{aligned}
r_1^{(E)} &= \frac{r^2+rR+R^2}{3(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
r_2^{(E)} &= \frac{1}{3r^3},\\
r_3^{(E)} &= \frac{\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}\left(m_0(r+2R)+3r\left(r^2+rR+R^2\right)^2\right)}{\sqrt{3}m_0r^3(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2},\\
r_4^{(E)} &= \frac{\left(m_0-R^4\right)}{3R^6(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2}\Bigg(-6r^2(r+R)\left(r^2+rR+R^2\right)(-m_0\\
&\quad +r^4+r^2R^2+R^4)\,F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)-6rR\left(3\left(r^2+rR+R^2\right)^2\left(r^3+2R^3\right)\right.\\
&\quad \left.-m_0\left(7r^3+14r^2R+12rR^2+6R^3\right)\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad -\frac{R^8\left(m_0(2r+R)+3R\left(r^2+rR+R^2\right)^2\right)}{m_0-3R^4}\Bigg),\\
r_5^{(E)} &= \frac{\left(R^4-m_0\right)\left(r\left(9RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+6R^2F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{r^2R^7}.
\end{aligned}
\tag{6.115}
$$

The other source term in the vector sector at second order $S_M^{\text{vec}}(r)$ in (6.74) is given by

$$
(S_M^{\text{vec}})_i(r)=\sum_{l=1}^{5}r_l^{(M)}(W_v)_i^l, \tag{6.116}
$$

where the coefficient functions $r_i^{(M)}$ are given by

$$
\begin{aligned}
r_1^{(M)} &= 0, \\
r_2^{(M)} &= \frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}\left(m_0 r^2+24R^2\kappa^2\left(R^4-m_0\right)\right)}{m_0 r^5}, \\
r_3^{(M)} &= \frac{6R\kappa\left(m_0-R^4\right)}{m_0 r^5 (r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\Bigg(r^2R\left(r\left(r^2+rR+R^2\right)\left(3r^3+R^3\right)\right. \\
&\quad \left.-m_0\left(3r^2+3rR+2R^2\right)\right)-8m_0(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Bigg), \\
r_4^{(M)} &= -\frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}}{R^6(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\Bigg(r^2(r+R)\left(-m_0+r^4\right. \\
&\quad \left.+r^2R^2+R^4\right)\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right) \\
&\quad +12rR\left(m_0-R^4\right)\left(r^2+rR+R^2\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^6\left(r^2+rR+R^2\right)\Bigg), \\
r_5^{(M)} &= \frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}}{m_0 r^5 R^6}\Bigg(6m_0r^3\left(m_0-R^4\right)\left(RF_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)-rF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right) \\
&\quad +m_0r^2R^6+24R^8\kappa^2\left(R^4-m_0\right)\Bigg).
\end{aligned}
\tag{6.117}
$$

6.6.4 Source Terms in Tensor Sector: Second Order

In this appendix we provide the source of the dynamical equation (6.86). We report the result in terms of the parameters M and R and the variable ρ defined in (6.6). The source $\mathbf{T}_{ij}(\rho)$ in (6.86) is given by

$$
\mathbf{T}_{ij}(r) = \sum_{l=1}^{9} \tau_l(r)\ WT_{ij}^{(l)}, \tag{6.118}
$$

where the weyl-covariant terms $WT^{(l)}_{ij}$ are defined in Appendix 6.6.1 in equation (6.107). The coefficient of the weyl-covariant terms in the above source is given by

$$
\begin{aligned}
\tau_1(r) =& \frac{3rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R}+\frac{m_0(r+R)-\left(r^2+rR+R^2\right)\left(3r^3+R^3\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
\tau_2(r) =& -\frac{1}{2R}\left(3rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)-\frac{2r^3R\left(r^2+rR+R^2\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}+R\right),\\
\tau_3(r) =& \frac{6rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R}+\frac{2\left(m_0(r+R)-2r^3\left(r^2+rR+R^2\right)\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
\tau_4(r) =& \frac{18R^4\kappa^2\left(m_0-R^4\right)^2\left(-m_0r^2+4m_0R^2+r^6-4R^6\right)}{m_0^2r^{10}}-\frac{m_0r^2+2m_0R^2+r^6-2R^6}{2r^6},\\
\tau_5(r) =& \frac{1}{R^6}\left(6r\left(R^4-m_0\right)\left(rF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+3RF_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\right),\\
\tau_6(r) =& \frac{3r\left(m_0-R^4\right)}{2R^{16}\left(m_0-3R^4\right)}\Bigg(r\Bigg(R^2F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\left(r^2\left(m_0-3R^4\right)\left(-25m_0r^2\right.\right.\\
&\left.\left.+37m_0R^2+25r^6-37R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+30m_0R^8-2R^{12}\right)\\
&-r^4\left(3R^4-m_0\right)(r-R)(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2\\
&+2rR\left(5r^2\left(3R^4-m_0\right)(R-r)(r+R)\left(-m_0+r^4\right.\right.\\
&\left.\left.+r^2R^2+R^4\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^8\left(m_0+R^4\right)\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&+16m_0R^6\left(m_0-R^4\right)F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Bigg)+48m_0R^7\left(m_0-R^4\right)F_1^{(0,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&+24R^{11}\left(3m_0-2R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Bigg)\\
\tau_7(r) =& \frac{3\sqrt{3}\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{2m_0r^5},\\
\tau_8(r) =& \frac{3\sqrt{3}\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{m_0r^5R^9}\left(2r^2\left(R\left(-m_0\left(5r^2+R^2\right)+5r^6+R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\right.\\
&\left.\left.+r\left(m_0\left(R^2-r^2\right)+r^6-R^6\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+R^9\right),\\
\tau_9(\rho) =& 0.
\end{aligned}
\tag{6.119}
$$

6.6.5 Comparison with Erdmenger et. al.

Firstly we shall present a dictionary of relations between the quantities defined in [18] and those in this chapter. To avoid confusion we shall use a subscript 'E' to denote the quantities in [18].

The charge and mass of the black brane in the two papers are related by

$$\begin{aligned}(Q)_E &= -q \\ (b)_E^4 &= \frac{1}{m}.\end{aligned} \tag{6.120}$$

Also the gauge field in [18] is twice the gauge field in this chapter

$$(A_\mu)_E = 2A_\mu. \tag{6.121}$$

We list the relation between several other quantities in the two papers

$$\begin{aligned}(r_+)_E &= R \\ (r_-)_E &= R\sqrt{\left(Q^2+\frac{1}{4}\right)^{\frac{1}{2}}-\frac{1}{2}} \\ \mu_E &= -\frac{\sqrt{3}q}{R^2} = -2\mu \\ T_E &= \frac{R}{2\pi}(3-M) = \frac{R}{2\pi}(2-Q^2) \\ N_E^2 &= \frac{\pi}{2G_5} \\ (\sigma_{\mu\nu})_E &= 2\sigma_{\mu\nu} \\ (l_\mu)_E &= -l_\mu.\end{aligned} \tag{6.122}$$

Finally the various functions that go into the first order metric and the gauge field are related by

$$\begin{aligned}(F(r))_E &= \frac{1}{R}F_2(\rho, M) \\ (j^\kappa(r))_E &= \frac{\sqrt{3}Q(2-Q^2)^3}{2\pi R(2+3Q^2+Q^4)}F_1(\rho, M) \\ (a^\kappa(r))_E &= -\frac{\rho^5(2-Q^2)^3}{4\pi(2+3Q^2+Q^4)}F_1^{(1,0)}(\rho, M).\end{aligned} \tag{6.123}$$

These statements are true only up to zeroth order in the expansion of R in terms of the boundary derivatives. Further for the tensor sector matching we have to use the following relations

$$\begin{aligned}\mathcal{D}_i\left(\frac{\mu}{T}\right) &= \frac{2\pi\sqrt{3}(2+3Q^2+Q^4)}{R^3(2-Q^2)^3}\mathcal{D}_i q \\ \mathcal{D}_i\mathcal{D}_j\left(\frac{\mu}{T}\right) &= \frac{2\pi\sqrt{3}(2+3Q^2+Q^4)}{R^3(2-Q^2)^3}\mathcal{D}_i\mathcal{D}_j q + \frac{2\pi\sqrt{3}Q(1+Q^2)(60+40Q^2+Q^4)}{R^6(2-Q^2)^5}\mathcal{D}_i q\mathcal{D}_j q,\end{aligned} \tag{6.124}$$

where,

$$\mu = \frac{\sqrt{3}q}{2R^2}; \qquad T = \frac{R}{2\pi}(2-Q^2),$$

are respectively the chemical potential and the temperature in our notation.

Using this dictionary our stress tensor and charge current matches perfectly with [18].

7 Conclusion

As a part of this thesis, we have elaborated on the rich interplay between gravity in AdS spacetimes on one hand and on the other hand, the hydrodynamics that arises out of scale invariant field theories. We have noted through a variety of examples, how this correspondence works in detail, how physical mathematical structures natural to one side of this correspondence make their appearence on the other side. This detailed dictionary is a part of the broader AdS/CFT dictionary which seeks to establish a complete dictionary between various questions in quantum gravity to questions in quatum field theory.

The fluid-gravity correspondence that we have described in this thesis is among the few methods which allow us to go beyond supersymmetric and time-independent states in AdS/CFT dictionary. Such questions are just beginning to be explored. Their importance for a broader understanding of how field theory and gravity are related can hardly be overstated.

7.1 Acknowledgements

I would like to thank all the students and the faculty of the theory department for their support, guidance and encouragement for all these years. I am especially grateful to my collaborators in TIFR and outside who have taught me a lot over these years. Several of the calculations quoted in this thesis were performed using the excellent Mathematica package `diffgeo.m` developed by Matthew Headrick.

None of this would have been possible but for Prof. Shiraz Minwalla with his enthusiastic support, guidance, encouragement and everything else. Since I do not know of an appropriate way to thank him, let me follow the advice of Ludwig Wittgenstein who said

"Where (or of what) one cannot speak, one must pass over in silence."

Finally, I would like to acknowledge my debt to all those who have generously supported and encouraged the pursuit of science in India.

References

[1] P. Di Vecchia, *Large N gauge theories and AdS / CFT correspondence*, hep-th/9908148.

[2] R. Loganayagam, *Entropy Current in Conformal Hydrodynamics*, *JHEP* **0805** (2008) 087, [arXiv:0801.3701].

[3] G. S. Hall, *Weyl manifolds and connections*, *Journal of Mathematical Physics* **33** (July, 1992) 2633–2638.

[4] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv.Theor.Math.Phys.* **2** (1998) 231–252, [hep-th/9711200].

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys.Rept.* **323** (2000) 183–386, [hep-th/9905111].

[6] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **0802** (2008) 045, [arXiv:0712.2456].

[7] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, *Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions*, *JHEP* **0812** (2008) 116, [arXiv:0809.4272].

[8] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance, and holography*, *JHEP* **0804** (2008) 100, [arXiv:0712.2451].

[9] M. Van Raamsdonk, *Black Hole Dynamics From Atmospheric Science*, *JHEP* **0805** (2008) 106, [arXiv:0802.3224].

[10] M. Natsuume and T. Okamura, *A Note on causal hydrodynamics for M-theory branes*, *Prog.Theor.Phys.* **120** (2008) 1217–1222, [arXiv:0801.1797].

[11] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi, *et. al.*, *Forced Fluid Dynamics from Gravity*, *JHEP* **0902** (2009) 018, [arXiv:0806.0006].

[12] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*, *JHEP* **0810** (2008) 063, [arXiv:0806.4602].

[13] M. Natsuume, *Causal hydrodynamics and the membrane paradigm*, *Phys.Rev.* **D78** (2008) 066010, [arXiv:0807.1392].

[14] M. Haack and A. Yarom, *Universality of second order transport coefficients from the gauge-string duality*, *Nucl.Phys.* **B813** (2009) 140–155, [arXiv:0811.1794].

[15] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, *JHEP* **0809** (2008) 054, [arXiv:0708.1770].

[16] G. Gibbons, H. Lu, D. N. Page, and C. Pope, *The General Kerr-de Sitter metrics in all dimensions*, *J.Geom.Phys.* **53** (2005) 49–73, [hep-th/0404008].

[17] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, *et. al.*, *Hydrodynamics from charged black branes*, *JHEP* **1101** (2011) 094, [arXiv:0809.2596].

[18] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **0901** (2009) 055, [arXiv:0809.2488].

[19] Z.-W. Chong, M. Cvetic, H. Lu, and C. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys.Rev.Lett.* **95** (2005) 161301, [hep-th/0506029].

[20] D. T. Son and P. Surowka, *Hydrodynamics with Triangle Anomalies*, *Phys.Rev.Lett.* **103** (2009) 191601, [arXiv:0906.5044].

[21] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*. Course of theoretical physics, Oxford: Pergamon Press, 1959.

[22] P. Kovtun, D. Son, and A. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys.Rev.Lett.* **94** (2005) 111601, [hep-th/0405231].

[23] R. Maartens, *Causal thermodynamics in relativity*, astro-ph/9609119.

[24] N. Andersson and G. Comer, *Relativistic fluid dynamics: Physics for many different scales*, *Living Rev.Rel.* **10** (2005) 1, [gr-qc/0605010].

[25] C. Eckart, *The Thermodynamics of irreversible processes. 3.. Relativistic theory of the simple fluid*, *Phys.Rev.* **58** (1940) 919–924.

[26] W. Israel, *Nonstationary irreversible thermodynamics: A Causal relativistic theory*, *Annals Phys.* **100** (1976) 310–331.

[27] W. Israel and J. Stewart, *Transient relativistic thermodynamics and kinetic theory*, *Annals Phys.* **118** (1979) 341–372.

[28] W. A. Hiscock and L. Lindblom, *Stability and causality in dissipative relativistic fluids.*, *Annals of Physics* **151** (1983) 466–496.

[29] R. P. Geroch, *On hyperbolic 'theories' of relativistic dissipative fluids*, gr-qc/0103112.

[30] A. M. Anile, D. Pavon, and V. Romano, *The Case for hyperbolic theories of dissipation in relativistic fluids*, gr-qc/9810014.

[31] L. Herrera and D. Pavon, *Hyperbolic theories of dissipation: Why and when do we need them?*, *Physica* **A307** (2002) 121–130, [gr-qc/0111112].

[32] R. P. Geroch and L. Lindblom, *Dissipative relativistic fluid theories of divergence type*, *Phys.Rev.* **D41** (1990) 1855.

[33] I. Muller, *Speeds of propagation in classical and relativistic extended thermodynamics*, *Living Rev.Rel.* **2** (1999) 1.

[34] A. Muronga, *Second order dissipative fluid dynamics for ultrarelativistic nuclear collisions*, *Phys.Rev.Lett.* **88** (2002) 062302, [nucl-th/0104064].

[35] A. Muronga, *Causal theories of dissipative relativistic fluid dynamics for nuclear collisions*, *Phys.Rev.* **C69** (2004) 034903, [nucl-th/0309055].

[36] U. W. Heinz, H. Song, and A. K. Chaudhuri, *Dissipative hydrodynamics for viscous relativistic fluids*, *Phys.Rev.* **C73** (2006) 034904, [nucl-th/0510014].

[37] R. Baier and P. Romatschke, *Causal viscous hydrodynamics for central heavy-ion collisions*, *Eur.Phys.J.* **C51** (2007) 677–687, [nucl-th/0610108].

[38] P. Romatschke, *Causal viscous hydrodynamics for central heavy-ion collisions. II. Meson spectra and HBT radii*, *Eur.Phys.J.* **C52** (2007) 203–209, [nucl-th/0701032].

[39] H. Song and U. W. Heinz, *Causal viscous hydrodynamics in 2+1 dimensions for relativistic heavy-ion collisions*, *Phys.Rev.* **C77** (2008) 064901, [arXiv:0712.3715].

[40] R. Bhalerao and S. Gupta, *Aspects of causal viscous hydrodynamics*, *Phys.Rev.* **C77** (2008) 014902, [arXiv:0706.3428].

[41] E. Witten, *Anti-de Sitter space and holography*, *Adv.Theor.Math.Phys.* **2** (1998) 253–291, [hep-th/9802150].

[42] P. Kraus, F. Larsen, and R. Siebelink, *The gravitational action in asymptotically AdS and flat space-times*, *Nucl.Phys.* **B563** (1999) 259–278, [hep-th/9906127].

[43] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **9807** (1998) 023, [hep-th/9806087].

[44] S. de Haro, S. N. Solodukhin, and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun.Math.Phys.* **217** (2001) 595–622, [hep-th/0002230].

[45] K. Skenderis, *Asymptotically Anti-de Sitter space-times and their stress energy tensor*, *Int.J.Mod.Phys.* **A16** (2001) 740–749, [hep-th/0010138].

[46] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **0508** (2005) 004, [hep-th/0505190].

[47] M. C. Cheng and K. Skenderis, *Positivity of energy for asymptotically locally AdS spacetimes*, *JHEP* **0508** (2005) 107, [hep-th/0506123].

[48] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **0506** (2005) 023, [hep-th/0504233].

[49] R. Olea, *Regularization of odd-dimensional AdS gravity: Kounterterms*, *JHEP* **0704** (2007) 073, [hep-th/0610230].

[50] K. S. Thorne, R. Price, and D. Macdonald, *BLACK HOLES: THE MEMBRANE PARADIGM*, .

[51] V. Cardoso, O. J. Dias, and L. Gualtieri, *The Return of the membrane paradigm? Black holes and strings in the water tap*, *Int.J.Mod.Phys.* **D17** (2008) 505–511, [arXiv:0705.2777].

[52] S. Hawking, C. Hunter, and M. Taylor, *Rotation and the AdS / CFT correspondence*, *Phys.Rev.* **D59** (1999) 064005, [hep-th/9811056].

[53] G. Gibbons, M. Perry, and C. Pope, *The First law of thermodynamics for Kerr-anti-de Sitter black holes*, *Class.Quant.Grav.* **22** (2005) 1503–1526, [hep-th/0408217].

[54] M. Cvetic, H. Lu, and C. Pope, *Charged rotating black holes in five dimensional U(1)**3 gauged N=2 supergravity*, *Phys.Rev.* **D70** (2004) 081502, [hep-th/0407058].

[55] Z. Chong, M. Cvetic, H. Lu, and C. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys.Rev.* **D72** (2005) 041901, [hep-th/0505112].

[56] Z. Chong, M. Cvetic, H. Lu, and C. Pope, *Non-extremal rotating black holes in five-dimensional gauged supergravity*, *Phys.Lett.* **B644** (2007) 192–197, [hep-th/0606213].

[57] Z.-W. Chong, M. Cvetic, H. Lu, and C. Pope, *Non-extremal charged rotating black holes in seven-dimensional gauged supergravity*, *Phys.Lett.* **B626** (2005) 215–222, [hep-th/0412094].

[58] Z.-W. Chong, M. Cvetic, H. Lu, and C. Pope, *Charged rotating black holes in four-dimensional gauged and ungauged supergravities*, *Nucl.Phys.* **B717** (2005) 246–271, [hep-th/0411045].

[59] M. Cvetic, G. Gibbons, H. Lu, and C. Pope, *Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines*, hep-th/0504080.

[60] D. T. Son and A. O. Starinets, *Viscosity, Black Holes, and Quantum Field Theory*, *Ann.Rev.Nucl.Part.Sci.* **57** (2007) 95–118, [arXiv:0704.0240].

[61] G. Policastro, D. Son, and A. Starinets, *The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma*, *Phys.Rev.Lett.* **87** (2001) 081601, [hep-th/0104066].

[62] O. Aharony, S. Minwalla, and T. Wiseman, *Plasma-balls in large N gauge theories and localized black holes*, *Class.Quant.Grav.* **23** (2006) 2171–2210, [hep-th/0507219].

[63] S. Lahiri and S. Minwalla, *Plasmarings as dual black rings*, *JHEP* **0805** (2008) 001, [arXiv:0705.3404].

[64] D. T. Son and A. O. Starinets, *Hydrodynamics of r-charged black holes*, *JHEP* **0603** (2006) 052, [hep-th/0601157].

[65] S. S. Gubser and I. Mitra, *Instability of charged black holes in Anti-de Sitter space*, hep-th/0009126.

[66] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **0509** (2005) 038, [hep-th/0506177].

[67] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, *Non-supersymmetric attractors*, *Phys.Rev.* **D72** (2005) 124021, [hep-th/0507096].

[68] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, *Rotating attractors*, *JHEP* **0610** (2006) 058, [hep-th/0606244].

[69] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun.Math.Phys.* **275** (2007) 209–254, [hep-th/0510251].

[70] T. Harmark and N. Obers, *Thermodynamics of spinning branes and their dual field theories*, *JHEP* **0001** (2000) 008, [hep-th/9910036].

[71] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories*, *Adv.Theor.Math.Phys.* **8** (2004) 603–696, [hep-th/0310285].

[72] D. S. Berman and M. K. Parikh, *Holography and rotating AdS black holes*, *Phys.Lett.* **B463** (1999) 168–173, [hep-th/9907003].

[73] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, *et. al.*, *Local Fluid Dynamical Entropy from Gravity*, *JHEP* **0806** (2008) 055, [arXiv:0803.2526].

[74] S. Dutta, *Higher Derivative Corrections to Locally Black Brane Metrics*, *JHEP* **0805** (2008) 082, [arXiv:0804.2453].

[75] R. A. Janik and R. B. Peschanski, *Asymptotic perfect fluid dynamics as a consequence of Ads/CFT*, *Phys.Rev.* **D73** (2006) 045013, [hep-th/0512162].

[76] R. A. Janik, *Viscous plasma evolution from gravity using AdS/CFT*, *Phys.Rev.Lett.* **98** (2007) 022302, [hep-th/0610144].

[77] M. P. Heller and R. A. Janik, *Viscous hydrodynamics relaxation time from AdS/CFT*, *Phys.Rev.* **D76** (2007) 025027, [hep-th/0703243].

[78] M. Natsuume and T. Okamura, *Comment on "Viscous hydrodynamics relaxation time from AdS/CFT correspondence"*, arXiv:0712.2917.

[79] A. Buchel, *Shear viscosity of boost invariant plasma at finite coupling*, *Nucl.Phys.* **B802** (2008) 281–306, [arXiv:0801.4421].

[80] A. Buchel, R. C. Myers, M. F. Paulos, and A. Sinha, *Universal holographic hydrodynamics at finite coupling*, *Phys.Lett.* **B669** (2008) 364–370, [arXiv:0808.1837].

[81] A. Buchel and M. Paulos, *Second order hydrodynamics of a CFT plasma from boost invariant expansion*, *Nucl.Phys.* **B810** (2009) 40–65, [arXiv:0808.1601].

[82] A. Buchel and M. Paulos, *Relaxation time of a CFT plasma at finite coupling*, *Nucl.Phys.* **B805** (2008) 59–71, [arXiv:0806.0788].

[83] H. S. Reall, *Private communication*, .

[84] A. Chamblin, S. Hawking, and H. Reall, *Brane world black holes*, *Phys.Rev.* **D61** (2000) 065007, [hep-th/9909205].

[85] P. Benincasa, A. Buchel, M. P. Heller, and R. A. Janik, *On the supergravity description of boost invariant conformal plasma at strong coupling*, *Phys.Rev.* **D77** (2008) 046006, [arXiv:0712.2025].

[86] M. P. Heller, P. Surowka, R. Loganayagam, M. Spalinski, and S. E. Vazquez, *Consistent Holographic Description of Boost-Invariant Plasma*, *Phys.Rev.Lett.* **102** (2009) 041601, [arXiv:0805.3774].

[87] S. Kinoshita, S. Mukohyama, S. Nakamura, and K.-y. Oda, *A Holographic Dual of Bjorken Flow*, *Prog.Theor.Phys.* **121** (2009) 121–164, [arXiv:0807.3797].

[88] S. Hawking and G. Ellis, *The Large scale structure of space-time*, .

[89] R. M. Wald, *General relativity.* Chicago, University of Chicago Press, 1984, 504 p., 1984.

[90] A. Krolak, *Definitions of black holes without use of the boundary at infinity*, *General Relativity and Gravitation* **14** (Aug., 1982) 793–801.

[91] A. Krolak, *Black holes and the strong cosmic censorship*, *General Relativity and Gravitation* **16** (Feb., 1984) 121–130.

[92] A. Krolak, *Black holes and the weak cosmic censorship*, *General Relativity and Gravitation* **16** (Apr., 1984) 365–373.

[93] P. T. Chrusciel, E. Delay, G. J. Galloway, and R. Howard, *The Area theorem*, *Annales Henri Poincare* **2** (2001) 109–178, [gr-qc/0001003].

[94] J. R. David, G. Mandal, S. Vaidya, and S. R. Wadia, *Point mass geometries, spectral flow and AdS(3) - CFT(2) correspondence*, *Nucl.Phys.* **B564** (2000) 128–141, [hep-th/9906112].

[95] K. Kajantie, J. Louko, and T. Tahkokallio, *The Gravity dual of 1+1 dimensional Bjorken expansion*, *Phys.Rev.* **D76** (2007) 106006, [arXiv:0705.1791].

[96] M. Cvetic and S. S. Gubser, *Thermodynamic stability and phases of general spinning branes*, *JHEP* **9907** (1999) 010, [hep-th/9903132].

[97] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys.Rev.* **D60** (1999) 064018, [hep-th/9902170].

[98] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys.Rev.* **D60** (1999) 104026, [hep-th/9904197].

[99] D. Astefanesei, N. Banerjee, and S. Dutta, *(Un)attractor black holes in higher derivative AdS gravity*, *JHEP* **0811** (2008) 070, [arXiv:0806.1334].

[100] D. Astefanesei, H. Nastase, H. Yavartanoo, and S. Yun, *Moduli flow and non-supersymmetric AdS attractors*, *JHEP* **0804** (2008) 074, [arXiv:0711.0036].

[101] R.-G. Cai and A. Wang, *Thermodynamics and stability of hyperbolic charged black holes*, *Phys.Rev.* **D70** (2004) 064013, [hep-th/0406057].

[102] R.-G. Cai and K.-S. Soh, *Critical behavior in the rotating D-branes*, *Mod.Phys.Lett.* **A14** (1999) 1895–1908, [hep-th/9812121].

[103] M. Cvetic, M. Duff, P. Hoxha, J. T. Liu, H. Lu, *et. al.*, *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, *Nucl.Phys.* **B558** (1999) 96–126, [hep-th/9903214].

[104] J. T. Liu and W. Sabra, *Mass in anti-de Sitter spaces*, *Phys.Rev.* **D72** (2005) 064021, [hep-th/0405171].

[105] M. Cvetic and S. S. Gubser, *Phases of R charged black holes, spinning branes and strongly coupled gauge theories*, *JHEP* **9904** (1999) 024, [hep-th/9902195].

[106] S. S. Gubser, *Thermodynamics of spinning D3-branes*, *Nucl.Phys.* **B551** (1999) 667–684, [hep-th/9810225].

[107] N. Banerjee and S. Dutta, *Phase transition of electrically charged Ricci-flat black holes*, *JHEP* **0707** (2007) 047, [arXiv:0705.2682].

[108] D. Marolf, *Chern-Simons terms and the three notions of charge*, hep-th/0006117.

[109] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun.Math.Phys.* **208** (1999) 413–428, [hep-th/9902121].

[110] D. T. Son and A. O. Starinets, *Minkowski space correlators in AdS / CFT correspondence: Recipe and applications*, *JHEP* **0209** (2002) 042, [hep-th/0205051].

[111] G. Policastro, D. T. Son, and A. O. Starinets, *From AdS / CFT correspondence to hydrodynamics*, *JHEP* **0209** (2002) 043, [hep-th/0205052].

[112] C. P. Herzog, *The Hydrodynamics of M theory*, *JHEP* **0212** (2002) 026, [hep-th/0210126].

[113] G. Policastro, D. T. Son, and A. O. Starinets, *From AdS / CFT correspondence to hydrodynamics. 2. Sound waves*, *JHEP* **0212** (2002) 054, [hep-th/0210220].

[114] C. Herzog and D. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence*, *JHEP* **0303** (2003) 046, [hep-th/0212072].

[115] C. P. Herzog, *The Sound of M theory*, *Phys.Rev.* **D68** (2003) 024013, [hep-th/0302086].

[116] P. Kovtun and L. G. Yaffe, *Hydrodynamic fluctuations, long time tails, and supersymmetry*, *Phys.Rev.* **D68** (2003) 025007, [hep-th/0303010].

[117] A. Buchel and J. T. Liu, *Thermodynamics of the N=2* flow*, *JHEP* **0311** (2003) 031, [hep-th/0305064].

[118] P. Kovtun, D. T. Son, and A. O. Starinets, *Holography and hydrodynamics: Diffusion on stretched horizons*, *JHEP* **0310** (2003) 064, [hep-th/0309213].

[119] A. Buchel and J. T. Liu, *Universality of the shear viscosity in supergravity*, *Phys.Rev.Lett.* **93** (2004) 090602, [hep-th/0311175].

[120] A. Buchel, *N=2* hydrodynamics*, *Nucl.Phys.* **B708** (2005) 451–466, [hep-th/0406200].

[121] A. Buchel, J. T. Liu, and A. O. Starinets, *Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory*, *Nucl.Phys.* **B707** (2005) 56–68, [hep-th/0406264].

[122] A. Buchel, *On universality of stress-energy tensor correlation functions in supergravity*, *Phys.Lett.* **B609** (2005) 392–401, [hep-th/0408095].

[123] P. K. Kovtun and A. O. Starinets, *Quasinormal modes and holography*, *Phys.Rev.* **D72** (2005) 086009, [hep-th/0506184].

[124] A. O. Starinets, *Transport coefficients of strongly coupled gauge theories: Insights from string theory*, *Eur.Phys.J.* **A29** (2006) 77–81, [nucl-th/0511073].

[125] J. Mas, *Shear viscosity from R-charged AdS black holes*, *JHEP* **0603** (2006) 016, [hep-th/0601144].

[126] O. Saremi, *The Viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential*, *JHEP* **0610** (2006) 083, [hep-th/0601159].

[127] K. Maeda, M. Natsuume, and T. Okamura, *Viscosity of gauge theory plasma with a chemical potential from AdS/CFT*, *Phys.Rev.* **D73** (2006) 066013, [hep-th/0602010].

[128] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled N=4 Yang-Mills*, *Phys.Rev.* **D74** (2006) 085012, [hep-ph/0605199].

[129] S. S. Gubser, *Drag force in AdS/CFT*, *Phys.Rev.* **D74** (2006) 126005, [hep-th/0605182].

[130] R. A. Janik and R. B. Peschanski, *Gauge/gravity duality and thermalization of a boost-invariant perfect fluid*, *Phys.Rev.* **D74** (2006) 046007, [hep-th/0606149].

[131] H. Liu, K. Rajagopal, and U. A. Wiedemann, *An AdS/CFT Calculation of Screening in a Hot Wind*, *Phys.Rev.Lett.* **98** (2007) 182301, [hep-ph/0607062].

[132] S. Nakamura and S.-J. Sin, *A Holographic dual of hydrodynamics*, *JHEP* **0609** (2006) 020, [hep-th/0607123].

[133] P. Benincasa, A. Buchel, and R. Naryshkin, *The Shear viscosity of gauge theory plasma with chemical potentials*, *Phys.Lett.* **B645** (2007) 309–313, [hep-th/0610145].

[134] S. Lin and E. Shuryak, *Toward the AdS/CFT gravity dual for High Energy Collisions: I.Falling into the AdS*, *Phys.Rev.* **D77** (2008) 085013, [hep-ph/0610168].

[135] O. Saremi, *Shear waves, sound waves on a shimmering horizon*, hep-th/0703170.

[136] Y. V. Kovchegov and A. Taliotis, *Early Time Dynamics in Heavy Ion Collisions from AdS/CFT Correspondence*, *Phys.Rev.* **C76** (2007) 014905, [arXiv:0705.1234].

[137] R. C. Myers, A. O. Starinets, and R. M. Thomson, *Holographic spectral functions and diffusion constants for fundamental matter*, *JHEP* **0711** (2007) 091, [arXiv:0706.0162].

[138] S. Lin and E. Shuryak, *Toward the AdS/CFT gravity dual for high energy collisions. 2. The stress tensor on the boundary*, *Phys.Rev.* **D77** (2008) 085014, [arXiv:0711.0736].

[139] P. M. Chesler and L. G. Yaffe, *The Stress-energy tensor of a quark moving through a strongly-coupled N=4 supersymmetric Yang-Mills plasma: Comparing hydrodynamics and AdS/CFT*, *Phys.Rev.* **D78** (2008) 045013, [arXiv:0712.0050].

[140] Q. J. Ejaz, T. Faulkner, H. Liu, K. Rajagopal, and U. A. Wiedemann, *A Limiting velocity for quarkonium propagation in a strongly coupled plasma via AdS/CFT*, *JHEP* **0804** (2008) 089, [arXiv:0712.0590].

[141] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, *JHEP* **0901** (2009) 044, [arXiv:0712.0743].

[142] M. Natsuume and T. Okamura, *Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality*, *Phys.Rev.* **D77** (2008) 066014, [arXiv:0712.2916].

[143] K. Kajantie, J. Louko, and T. Tahkokallio, *Gravity dual of conformal matter collisions in 1+1 dimensions*, *Phys.Rev.* **D77** (2008) 066001, [arXiv:0801.0198].

[144] J. Choi, S. Lee, and S. Lee, *Near Horizon Analysis of Extremal AdS(5) Black Holes*, *JHEP* **0805** (2008) 002, [arXiv:0802.3330].

[145] S. S. Gubser and A. Yarom, *Linearized hydrodynamics from probe-sources in the gauge-string duality*, *Nucl.Phys.* **B813** (2009) 188–219, [arXiv:0803.0081].

[146] M. Chernicoff and A. Guijosa, *Acceleration, Energy Loss and Screening in Strongly-Coupled Gauge Theories*, *JHEP* **0806** (2008) 005, [arXiv:0803.3070].

[147] A. Buchel, *On SUGRA description of boost-invariant conformal plasma at strong coupling*, *AIP Conf.Proc.* **1031** (2008) 196–206, [arXiv:0803.3421].

[148] R. Myers and S. Vazquez, *Quark Soup al dente: Applied Superstring Theory*, *Class.Quant.Grav.* **25** (2008) 114008, [arXiv:0804.2423].

[149] G. Siopsis, *Analytic calculation of quasi-normal modes*, *Lect.Notes Phys.* **769** (2009) 471–508, [arXiv:0804.2713].

[150] S. S. Gubser, S. S. Pufu, and A. Yarom, *Entropy production in collisions of gravitational shock waves and of heavy ions*, *Phys.Rev.* **D78** (2008) 066014, [arXiv:0805.1551].

[151] R. C. Myers, M. F. Paulos, and A. Sinha, *Quantum corrections to eta/s*, *Phys.Rev.* **D79** (2009) 041901, [arXiv:0806.2156].

[152] A. O. Starinets, *Quasinormal spectrum and the black hole membrane paradigm*, *Phys.Lett.* **B670** (2009) 442–445, [arXiv:0806.3797].

[153] C. P. Herzog, M. Rangamani, and S. F. Ross, *Heating up Galilean holography*, *JHEP* **0811** (2008) 080, [arXiv:0807.1099].

[154] M. Natsuume, *String theory implications on causal hydrodynamics*, *Prog.Theor.Phys.Suppl.* **174** (2008) 286–297, [arXiv:0807.1394].

[155] A. S. Miranda, J. Morgan, and V. T. Zanchin, *Quasinormal modes of plane-symmetric black holes according to the AdS/CFT correspondence*, *JHEP* **0811** (2008) 030, [arXiv:0809.0297].