String Theory and Quantum Field Theories in Three Dimensions

A Thesis

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by

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Sandip P. Trivedi, at the Tata Institute of Fundamental Research, Mumbai.

Shiroman Prakash

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Sandip P. Trivedi

Date

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Synopsis

Introduction

Recent developments in string theory and M-theory have led to increased interest and new progress in the study of three-dimensional quantum field theories. One of the main reasons for this is that conformal field theories in three dimensions (CFT_3's) involving $U(N)$ gauge groups are expected to be dual to quantum theories of gravity in 4-dimensional anti-de Sitter space (AdS_4), via the AdS/CFT (or holographic) duality [1]. The primary example of the AdS_4/CFT_3 correspondence is the ABJM theory discovered in 2008 [2] which is believed to describe the low-energy dynamics of multiple $M2$-branes in a Z_k orbifold, and hence is conjectured to be dual to M-theory on $AdS_4 \times S_7/Z_k$.

An important aspect of the AdS/CFT duality is that the classical limit of the four-dimensional quantum theory of gravity takes on a new role as a large-N saddle point to the three-dimensional conformal field theory in a particular strongly interacting limit. Various authors (e.g., [3, 4, 5, 6, 7]) have therefore been inspired to use gravitational systems as toy models for strongly interacting quantum systems near a quantum critical point.

Motivated by these developments, in this thesis we study some new examples of strongly interacting conformal quantum field theories in three dimensions in the large-N limit, using both holographic (i.e., gravitational) and quantum field theory techniques. This thesis is based on work communicated in [8], [9] and [10].

We first consider holography of charged dilaton black branes in AdS_4 in [8]. The most interesting feature of the solutions we study is that they have vanishing entropy at zero-temperature, in contrast with the Reissner-Nordström branes studied earlier. We then consider dyonic charged dilaton black branes in [9]. We are able to calculate various transport properties of the strongly-coupled field theories dual to these gravitational theories at finite chemical potential and in the presence of a magnetic field. Our work here builds on previous studies aimed at modelling strongly interacting quantum systems using holography including [3, 4, 5, 11, 12, 6, 13, 14, 7], as well as previous work on non-supersymmetric attractors [15].

In addition to exploiting this duality to generate and study new examples of strongly interacting conformal field theories in three dimensions using gravity; it is also of interest to find models (particularly non-supersymmetric ones) that can be solved directly from the field theory side at strong coupling in the large N limit, with a view towards gaining some intuition into the mechanism behind the duality itself. For this purpose, we conclude with a study [10] of $U(N)$ Chern-Simons theory coupled to fundamental fermions – a theory that turns out to be remarkably educational and (at least partially) exactly-solvable using traditional large N techniques.

We briefly highlight the main results of these studies below.

Holography of Charged Dilaton Black Branes

The earliest charged black hole solutions to low-energy string theory were found by Garfinkle, Horowitz and Strominger [16] (and had appeared earlier as part of a family of solutions in [17]). Those authors studied the Einstein-Maxwell action with the gauge coupling controlled by a scalar dilaton ϕ. These black holes, and generalisations thereof, were found to have thermodynamic properties very different from traditional charged black holes [18, 19]. Hence, with a view towards generating holographic models with thermodynamic behaviour different from the usual Reissner-Nordström black branes (which have the unpleasant feature of a large entropy at zero temperature) previously studied, it is natural to consider their AdS generalisations.

Gravity solution

We consider the following action,

$$S = \int d^4x \sqrt{-g} \left(R - 2(\nabla\phi)^2 - f(\phi) F^2 - 2\Lambda \right) \ . \tag{1}$$

which generalises the action studied in [16] to include a negative cosmological constant, and arbitrary dilaton coupling $f(\phi)$. We will focus on the specific case $f(\phi) = e^{2\alpha\phi}$, though in [8] we also briefly consider some other possibilities. The maximally symmetric vacuum solution for this action is AdS space with AdS scale L determined by $\Lambda = -\frac{3}{L^2}$. We will set $L = 1$ below.

We will look for electrically-charged black branes solutions of the above action, taking the metric to be of the form

$$ds^2 = -a(r)^2 \ dt^2 + a(r)^{-2} \ dr^2 + b(r)^2 \ (dx^2 + dy^2), \tag{2}$$

with gauge field

$$e^{2\alpha\phi} F = \frac{Q}{b(r)^2} \ dt \wedge dr. \tag{3}$$

We first look for a near horizon solution to the above action by considering the following scaling ansatz:

$$a = C_2 (r - r_h)^\gamma, \ b = C_1 (r - r_h)^\beta, \ \phi = - \ K \ \log(r - r_h) + C_3 \ , \tag{4}$$

We find that an *exact* solution is obtained if the exponents take the values

$$\gamma = 1, \ K = \frac{\frac{\alpha}{2}}{1 + (\frac{\alpha}{2})^2}, \ \beta = \frac{(\frac{\alpha}{2})^2}{1 + (\frac{\alpha}{2})^2} \ . \tag{5}$$

The constant C_2 is given by

$$C_2^2 = \frac{6}{(\beta+1)(2\beta+1)} \ . \tag{6}$$

By rescaling $(r - r_h)$, t, x and y, one can set the constant C_3 to zero and C_1 to unity. Q is then determined in terms of α by

$$Q^2 = \frac{6}{(\alpha^2+2)} \ . \tag{7}$$

In terms of $w = r - r_h$, the exact near-horizon solution is:

$$a = C_2 w, \ b = w^\beta, \ \phi = - \ K \ \log(w) \ . \tag{8}$$

Interestingly, this solution almost has a Lifshitz scaling symmetry under $w \to \lambda w$, $t \to \lambda t$, $x \to \lambda^z x$, and $y \to \lambda^z y$, with Lifshitz exponent $z = 1/\beta$; the symmetry is only broken by constant shifts in the dilaton ϕ.

The metric component g_{tt} has a second order zero at $w = 0$, and (8) is thus an extremal solution. A non-extremal generalisation is the following:

$$ds^2 = C_2^2 w^2 (1 - \frac{m}{w^{2\beta+1}}) dt^2 + \frac{dw^2}{C_2^2 w^2 (1 - \frac{m}{w^{2\beta+1}})} + w^{2\beta}(dx^2 + dy^2), \tag{9}$$

which depends on the parameter m.

A numerical asymptotically AdS_4 solution

In the extremal solution, the gauge coupling $g_{\mathrm{U}(1)} \sim e^{-\alpha\phi}$ becomes weak at the horizon, as $\phi \to \infty$. When $r \to \infty$ the gauge coupling becomes very strong; therefore, although exact, the solution (9) must be understood as a near-horizon geometry of a larger solution with different asymptotics. (The near-horizon solution appeared in a different coordinate system in the unpublished work [14], where it appeared unphysical due to this pathology.) In particular, we would like to find a generalisation of the extremal solution with a controlled and asymptotically constant dilaton as well as an asymptotically AdS_4 geometry.

The strategy for obtaining such a solution is to numerically integrate the equations of motion following from (1), starting near $w = 0$ with initial data taken from the near-horizon solution. However, the near-horizon solution is exact, so numerical integration using initial data drawn from it would simply reproduce the near-horizon solution unmodified. To numerically integrate to a solution that is asymptotically AdS_4, we must also take into account possible subleading corrections to the near-horizon solution.

We start with a fairly general ansatz for the modification to the metric:

$$\begin{aligned} a(w) &= C_2 w \,(1 + d_1 w^{\nu_1}) \\ b(w) &= w^\beta \,(1 + d_2 w^{\nu_2}) \end{aligned} \tag{10}$$

which vanishes as $w \to 0$. The form of the subleading correction to ϕ is then determined to be

$$\phi(w) = -K \log(w) + C_3 + d_3 w^{\nu_2} \tag{11}$$

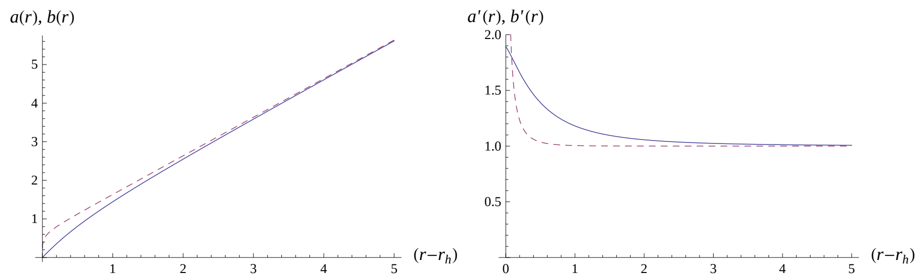

Figure 1: Numerical solution interpolating between the Lifshitz-like near horizon solution and AdS_4 for $\alpha = 1$ and $d_1 = -.514$. The second plot shows that $a'(r)$ and $b'(r)$ approach 1. Solid lines denote a, dashed lines denote b.

where $d_3 = \frac{2\beta+\nu_2-1}{2K} d_2$.

After some algebra we find the equations of motion imply that $\nu_1 = \nu_2$ and

$$d_1 = \left(\frac{2(1+\beta)(1+2\beta)}{(2\beta+2+\nu)(2\beta+1+\nu)} - 1 \right) d_2. \tag{12}$$

Numerically integrating Einstein's equations with initial data drawn from this solution (with $d_1 < 0$) yields a numerical solution that manifestly approaches AdS_4 with a constant dilaton as $r \to \infty$, as shown in Figure 1.

The extremal solution is labelled by two parameters – the total charge of the black brane Q and the asymptotic value of the dilaton ϕ_0. However, there is essentially only one numerical solution (for a given α). It is clear from the equations of motion that the metric and $(\phi - \phi_0)$ only depend on the quantity $Q^2 e^{-2\alpha\phi_0}$. Starting from the solution for $Q = 1, \phi_0 = 0$ one can rescale r via

$$r \to \lambda r, (t, x, y) \to \lambda^{-1}(t, x, y) \tag{13}$$

to obtain a solution for any given value of Q and then shift the dilaton to obtain a solution with any given value of ϕ_0.

Thermodynamics

Via the AdS/CFT dictionary, quantities calculated in the black hole background correspond to quantities calculated in a strongly interacting CFT at fixed temperature T and chemical potential μ. Thermodynamic relations between between the temperature, chemical potential, energy density ρ, entropy density s, pressure P, and the number density n can be extracted from the black brane solution.

An important subtlety in this analysis is that our solution is essentially numerical – we only have analytic expressions for the near-horizon solution and the asymptotic solution. However, as we will see below, this is enough to analytically calculate thermodynamics in the regime $T \ll \mu$, (which is when the solution (9) is valid.) Studying temperatures comparable to μ would require analysis of a finite-temperature numerical solution.

The horizon in the solution (9) is located at $w = w_h$, (or $r = r_h + w_h$):

$$w_h^{2\beta+1} = m. \tag{14}$$

The resulting temperature can be obtained as usual by continuing to Euclidean space [20] and comes out to be

$$T \sim w_h. \tag{15}$$

The entropy density of the slightly non-extremal black brane is proportional to the area element of the horizon and is

$$s = A_0 C T^{2\beta} \mu^{2-2\beta}, \tag{16}$$

where

$$C \sim L^2/G_N \tag{17}$$

is the central charge of the CFT dual to the AdS_4 background, as follows from substituting the near-horizon solution into the action (1). The numerical coefficient A_0 depends on the full numerical solution for the slightly non-extremal black hole. We note that the entropy vanishes at zero temperature.

Let us now consider the relations between charge, number density, and chemical potential the extremal limit, $T = 0$. It follows from the Einstein's equations that in the asymptotic AdS_4 region the solution must take the form

$$\begin{aligned} a^2 &= r^2\left(1 - e_1\frac{\rho}{r^3} + \frac{Q^2 e^{-2\alpha\phi_0}}{r^4} + \cdots\right) && (18)\\ b^2 &= r^2(1 + \cdots) && (19)\\ \phi &= \phi_0 + \frac{\phi_1}{r^3} + \cdots . && (20) \end{aligned}$$

e_1 is a constant which depends on L. Under the rescaling (13), $\rho \to \frac{\rho}{\lambda^3}, Q \to \frac{Q}{\lambda^2}$, which implies

$$\rho = D_1 (Q e^{-\alpha\phi_0})^{3/2}. \tag{21}$$

The coefficient D_1 is a numerical coefficient that is α dependent. A similar scaling argument tells us that

$$\mu = \int_{r_h}^{\infty} \frac{Q e^{-2\alpha\phi}}{b^2} dr \tag{22}$$

is given by

$$\mu = D_2 (Q e^{-\alpha\phi_0})^{1/2} e^{-\alpha\phi_0} \tag{23}$$

where D_2 is again an α dependent coefficient. This gives

$$\rho = D_3 C e^{3\alpha\phi_0} \mu^3. \tag{24}$$

The coefficient D_3 is a another α dependent numerical coefficient. From these relations it follows that the number density satisfies

$$n \propto Q. \tag{25}$$

Going to non-zero temperatures, we note that the specific heat is also positive

$$C_v = T(\frac{ds}{dT})_\mu = (2\beta) A_0 C T^{2\beta} \mu^{2-2\beta} \ . \tag{26}$$

The Gibbs-Duhem relation

$$sdT - dP + nd\mu = 0, \tag{27}$$

can be used to obtain the pressure. We find

$$P = \frac{1}{(2\beta+1)} A_0 C \mu^{2-2\beta} T^{2\beta+1} + \frac{1}{2} D_3 C e^{3\alpha\phi_0} \mu^3. \tag{28}$$

Substituting for P in (27) gives the number density,

$$n = \frac{(2-2\beta)}{(2\beta+1)} A_0 C T^{2\beta+1} \mu^{1-2\beta} + \frac{3}{2} D_3 C e^{3\alpha\phi_0} \mu^2. \tag{29}$$

From (29) we can also compute the susceptibility χ:

$$\chi \equiv (\frac{\partial n}{\partial \mu})_T = (1-2\beta)\frac{(2-2\beta)}{(2\beta+1)} A_0 C T^{2\beta+1} \mu^{1-2\beta} + 3 D_3 C \mu \ . \tag{30}$$

Finally the energy density can be obtained using the relation

$$\rho = sT + \mu n - p \tag{31}$$

which gives,

$$\rho = \frac{2}{(2\beta+1)} A_0 C \mu^{2-2\beta} T^{2\beta+1} + D_3 C e^{3\alpha\phi_0} \mu^3. \tag{32}$$

The near-extremal system a simple equation of state

$$P = \frac{1}{2}\rho \ . \tag{33}$$

It is clear from the calculations above that thermodynamic properties of the solution are essentially determined by the Lifshitz-like nature of the extremal near horizon geometry.

Conductivity

The power of the AdS/CFT correspondence is that it allows us to calculate properties of the dual quantum field theory beyond thermodynamics. The most natural quantities to study are transport coefficients. These are calculated by considering linearised fluctuations around the background gravitational solution presented above. Formally, via the AdS/CFT dictionary, this corresponds to coupling a conserved global $U(1)$ current operator in the dual CFT to a weak external source and calculating linear response, using the usual Kubo formula, as a two-point function of the current operator. Physically, this corresponds to studying the absorption and reflection of electromagnetic and/or gravitational waves by the black brane geometry.

Here, we calculate the optical conductivity $\sigma(\omega)$ of the extremal black brane background, via the AdS/CFT dictionary. Our approach is based on [21]. We first solve for the allowed fluctuations of the gauge field about our solution. Studying the gauge field equation of motion

$$\partial_\mu \left(\sqrt{-g} e^{2\alpha\phi} g^{\mu\lambda} F_{\lambda\sigma} g^{\sigma\nu}\right) = 0,$$

and the Einstein's equations, we find that the g_{tx} and A_x fluctuations mix. Luckily, however, one of the Einstein equations has just the right form to simplify the resulting coupled system, and we can obtain a second order differential equation for A_x. Defining the variable $\Psi = f(\phi)A_x$ we find

$$-\Psi'' + V(z)\Psi = \omega^2\Psi. \tag{34}$$

Here, $f(\phi) = 2\,e^{\alpha\phi}$ is the dilaton coupling, and primes denote derivatives with respect to the variable z, defined via

$$\frac{\partial}{\partial z} = a^2 \frac{\partial}{\partial r}. \tag{35}$$

The potential $V(z)$ is given by

$$V = \frac{f''}{f} + \frac{a^2 Q^2}{b^4 f^2}. \tag{36}$$

According to the AdS/CFT dictionary, we have to solve the fluctuation equation subject to in-going boundary conditions at the horizon. The solution to the fluctuation equation near the boundary then takes the form

$$A_x = A_x^{(0)} + A_x^{(1)}/r + \ldots.$$

The conductivity is then given by:

$$\sigma(\omega) = -\frac{i}{\omega}\frac{A_x^{(1)}}{A_x^{(0)}} \tag{37}$$

and is essentially determined by a reflection coefficient in the notation of the Schrodinger problem described above.

Solving for the conductivity at arbitrary frequencies is impossible because we only have an analytic expression for the near-horizon part of the solution; nevertheless, we are still able to extract the low-frequency behaviour of the conductivity using careful matching techniques and various properties of special functions (as will be detailed in the thesis). This illustrates an important principle – the near-horizon geometry effectively encodes all IR physics of the dual field theory.

Surprisingly, we find that the conductivity turns out to obey

$$\mathrm{Re}(\sigma) \sim \omega^2/\mu^2, \quad \mathrm{Im}(\sigma) \sim \frac{\mu}{\omega}. \tag{38}$$

at low frequencies. The frequency dependence is independent of the Lifshitz parameter of the near-horizon region. The reason for this is that the shape of the potential in the near-horizon region is $V(z) = \frac{2}{z^2}$, and does not depend on α. While thermodynamics depends non-trivially on the dilaton coupling α, the transport properties appear to be universal. We do not yet understand the physical origin of this intriguing result.

Dyonic Dilaton Black Branes

We next consider generalizing the analysis above to calculate transport properties in the presence of an external magnetic field [9]. By the AdS/CFT dictionary this corresponds to studying black branes with both electric and magnetic charge. In the most general setting in string theory, a dyonic solution is expected to source both a dilaton field as well as an axion field. An important class of axion-dilaton theories are those invariant under a $SL(2,R)$ electromagnetic duality. In particular, we consider the following theory[1]:

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - 2(\partial\phi)^2 - \frac{1}{2}e^{4\phi}(\partial a)^2 - e^{-2\phi}F^2 - aF\tilde{F}\right]. \tag{39}$$

which generalises the theory studied in [22] to AdS space.

To see the $SL(2,R)$ duality, we combine the axion a and dilaton as follows:

$$\lambda = \lambda_1 + i\lambda_2 = a + ie^{-2\phi}. \tag{40}$$

Then it is easy to see that under an $SL(2,R)$ transformation

$$M = \begin{pmatrix} \tilde{a} & b \\ c & d \end{pmatrix} \tag{41}$$

which takes

$$F_{\mu\nu} \to F'_{\mu\nu} = (c\lambda_1 + d)F_{\mu\nu} - c\lambda_2\tilde{F}_{\mu\nu} \tag{42}$$

and

$$\lambda \to \lambda' = \frac{\tilde{a}\lambda + b}{c\lambda + d} \tag{43}$$

while keeping the metric invariant, the equations of motion are left unchanged. (This is discussed in, e.g., [23].)

The electrically charged black brane solution to this action is nothing but the solution to (1) for $\alpha = 1$ discussed in the previous section. Starting with this purely electrically charged solution, we are able to generate solutions with arbitrary dyonic charges and boundary values of the dilaton-axion using $SL(2,R)$ duality. These solutions all exhibit a Lifshitz-like near-horizon geometry, and hence have vanishing entropy at extremality.

In [9], we consider the thermodynamics of these dyonic solutions as well as general thermo-electric linear response, which is characterized by the conductivity and two related transport coefficients, the thermoelectric coefficient α and the thermal conductivity κ. These are all tensorial and satisfy the relation

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \alpha\mathbf{T} & \kappa \end{pmatrix}\begin{pmatrix} \vec{E} \\ -\vec{\nabla}T \end{pmatrix} \tag{44}$$

where $\vec{E}$ is the electric field, $\vec{\nabla}T$ is the gradient of the temperature, $\vec{J}$ is the electric current and $\vec{Q}$ is the heat current.

[1] In our conventions $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\kappa}F_{\rho\kappa}$; $\epsilon^{\mu\nu\rho\sigma}$ has a factor of $\frac{1}{\sqrt{-g}}$ in its definition and $\epsilon_{trxy} > 0$.

Transport Coefficients

In the presence of an external magnetic field, conductivities are described via:

$$j_x = \sigma_{xx}F_{tx} + \sigma_{xy}F_{ty} \tag{45}$$
$$j_y = -\sigma_{xy}F_{tx} + \sigma_{xx}F_{ty}. \tag{46}$$

Upon working through the AdS/CFT dictionary, we find that the linear combinations

$$\sigma_+ = \sigma_1 + i\sigma_2,\ \sigma_- = \sigma_1 - i\sigma_2 \tag{47}$$

both transform in in a very nice way under the $SL(2,R)$ transformation (41):

$$\sigma_\pm \to \frac{\tilde{a}\sigma_\pm + b}{c\sigma_\pm + d} \tag{48}$$

An immediate and interesting consequence of these transformation rules is that conductivities σ_{xx} and σ_{xy} will in general obey "semi-circle" laws, as described in [24] and references therein.

Using these transformation laws, we are able to obtain the conductivity of a general dyonic black brane, starting from the conductivity of the purely electric brane in (38). The results are fairly complicated in general, so we do not present them in this synopsis. However, we note that the resulting transport fits in nicely with a magnetohydrodynamic formalism proposed in [5], which states that, at low frequencies, the conductivity of a relativistic plasma is given by:

$$\sigma_{xx} = \sigma_Q \frac{\omega\left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{(\omega + i\gamma)^2 - \omega_c^2} \tag{49}$$

and

$$\sigma_{xy} = -\left(\frac{n}{Q_m}\right)\frac{\gamma^2 + \omega_c^2 - 2i\gamma\omega}{(\omega + i\gamma)^2 - \omega_c^2}\ . \tag{50}$$

Here σ_Q, the damping frequency γ, and the cyclotron frequency ω_c depend on the magnetic field (or magnetic charge of the brane) Q_m, the temperature T and charge density n.

We find that

$$\sigma_Q \propto T^2,\ \ \gamma \propto (Q'_m)^2 T^2,\ \ \omega_c \propto Q'_m. \tag{51}$$

This qualitative behaviour is in agreement with the results of [11, 12] for the Reissner-Nordström black brane at small ω.

An interesting feature of the above result is that the DC Hall conductance is proportional to the attractor value of the axion at the horizon, which thus seems to act as a Chern-Simons coupling in a low-energy fixed-point effective field theory.

Large-N Chern-Simons Theories with Vector Fermion Matter

The characteristic feature of quantum field theories in three dimensions is the possibility of the a Chern-Simons kinetic term for the gauge field. As we shall see, this has important consequences for the dynamics of quantum field theories in three dimensions and their connections to string theory. This section is based on [10].

Consider a level k, $U(N)$ Chern-Simons theory coupled to a single fermion in any representation of the gauge group. The only gauge-invariant relevant or marginal terms possible in the Lagrangian of such a theory in addition to the Chern-Simons term are the fermion kinetic term and mass term. The resulting quantum field theory thus depends on the two integers k and N, and a single continuous parameter, the physical mass m of the fermionic field. At energies $E \gg m$, the dynamics of this theory is scale-invariant as well as nontrivial, due to the fact that the discrete Chern-Simons coupling is an integer and cannot run – this a conformal field theory can simply be obtained by tuning the physical mass of the fermion to zero.

Though the parameters k and N labeling the CFT are discrete, in the large-N and simultaneously large-k limit, the 't Hooft coupling $\lambda = \frac{N}{k}$, (which controls the strength of interactions), is effectively continuous (exactly as in ABJM theory [2]). For this reason the discretum of CFTs described by integer values of k and N coalesces into a line of fixed points in the large-N limit.

We emphasize that a variety of such non-supersymmetric fixed lines exist in three dimensions. Choosing the fermions that transform in, say, the adjoint representation of $U(N)$ gives one such fixed line of theories. Choosing fermions that transform in the bifundamental of $U(N) \times U(N)$ yields another example – one that can be thought of as a minimal, non-supersymmetric analog of the ABJM theory.

Such lines of fixed points are particularly important from the viewpoint of string theory and the AdS/CFT correspondence. While at small λ the theories are best described as weakly interacting quantum field theory; at large λ, when the field theory description becomes intractable, the hope is that a relatively simple classical four-dimensional gravitational description (such as the theories described in previous sections) could emerge.

In this section, we study what appears to be the simplest example (from the field theory perspective) of such non-supersymmetric fixed lines – the theory of a single fundamental fermion coupled to a $U(N)$ level k Chern-Simons theory. We are able to obtain several exact results for this theory, and comment on the nature of its holographic dual, as we summarize below. (We also mention the related work [25] on Chern-Simons theory coupled to fundamental scalars, appearing simultaneously.)

Exact Propagator and Free Energy in Light-Cone Gauge

In Euclidean space, the theory is described by the action:

$$S = \frac{ik}{4\pi} \int \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) + \int \bar{\psi} \gamma^\mu D_\mu \psi \ . \tag{52}$$

We work in an analog of light-cone gauge, defined using the condition

$$A_- = \frac{A^1 + iA^2}{\sqrt{2}} = 0.$$

This can be obtained from Wick rotation of the standard lightcone gauge in Lorentzian signature. Further details about conventions are in [10].

The choice of this gauge turns out to be crucial in allowing for the derivation of the free energy for the theory on R^2 at temperature $1/\beta$, (i.e., on $R^2 \times S^1$.)

We first derive the exact fermion propagator at zero temperature. After deriving the action for the theory in light-cone gauge (using the usual BRST formalism, and noting that ghosts decouple) we are able to derive the following integral Schwinger-Dyson equation for the exact fermion propagator in the large-N limit (see [26]):

$$\begin{aligned}
&\langle\psi_m(p)\bar{\psi}^n(p')\rangle \\
=&\frac{1}{ip_\mu\gamma^\mu}(2\pi)^3\delta^3(p'+p) \\
&-\frac{1}{ip_\mu\gamma^\mu}\frac{2\pi i}{k}\int\frac{d^3r}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{q^+}\gamma^+\left\langle\psi_a(p-q)\bar{\psi}^a(-r)\right\rangle\gamma^3\left\langle\psi_m(r+q)\bar{\psi}^n(p')\right\rangle \\
&+\frac{1}{ip_\mu\gamma^\mu}\frac{2\pi i}{k}\int\frac{d^3r}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{q^+}\gamma^3\left\langle\psi_a(p-q)\bar{\psi}^a(-r)\right\rangle\gamma^+\left\langle\psi_m(r+q)\bar{\psi}^n(p')\right\rangle \ .
\end{aligned}$$

This expression for the exact propagator is completely analogous to that for the exact propagator in 2d QCD obtained by 't Hooft [27].

Surprisingly, we are able to the solve this integral equation at zero temperature as well as finite temperature. Let us very briefly sketch the solution.

First define the self-energy, $\Sigma(p)=\Sigma=i\Sigma_\mu\gamma^\mu+\Sigma_I I-M_{bare}I$, of the fermion as follows

$$\langle\psi(p)_m\bar{\psi}(-q)^n\rangle=\delta_m^n\frac{1}{ip_\mu\gamma_\mu+M_{bare}+\Sigma}\times(2\pi)^3\delta(p-q). \tag{53}$$

The Schwinger-Dyson equation above can easily be translated into an integral equation for the Σ; from which it is easy to see that $\Sigma_-=\Sigma_3=0$. Further analysis also reveals that the remaining components of Σ must take the following form:

$$\begin{aligned}
\Sigma_I(p)&=f_0p_s \\
\Sigma_+(p)&=p_+g_0=p^-g_0
\end{aligned} \tag{54}$$

where

$$p_s=\sqrt{p_1^2+p_2^2}=\sqrt{2}|p^-|=\sqrt{2}|p^+| \tag{55}$$

and f_0 and g_0 are dimensionless functions of λ. The solution to the resulting integral equations for f_0 and g_0 turn out to be remarkably simple

$$\begin{aligned}
f_0&=\lambda \\
g_0&=-\lambda^2 \\
g_0+f_0^2&=0\ .
\end{aligned} \tag{56}$$

The zero-temperature, exact propagator is thus given by:

$$\langle\psi(p)_m\bar{\psi}(-q)^n\rangle=\delta_m^n\frac{1}{ip_3\gamma^3+ip_-\gamma^-+i(1-\lambda^2)p_+\gamma^++\lambda p_s}\times(2\pi)^3\delta(p-q). \tag{57}$$

The above calculation can be generalized to finite temperature. Furthermore it is also possible to express the free energy of the theory in terms of the exact fermion self-energy at finite energy. In the large-N saddle point, the partition function $Z = e^{-S_E}$ is given by:

$$-S_E = NV \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr} \left[\ln \left[i\gamma^\mu q_\mu + \Sigma(q) \right] - \frac{1}{2}\Sigma(q) \left(\frac{1}{i\gamma^\mu q_\mu + \Sigma(q)} \right) \right] . \tag{58}$$

Using this solution, we find that the free energy of our theory,

$$F = -\frac{1}{\beta} \ln Z,$$

as a function of temperature and λ, in a box of volume V_2 (which is taken to be very large) is given by

$$F = -\frac{NV_2T^3}{6\pi} \left[\tilde{c}^3 \frac{1-\lambda}{\lambda} + 6 \int_{\tilde{c}}^{\infty} dy \; y \ln \left(1 + e^{-y} \right) \right] \tag{59}$$

where $\tilde{c}$ is the unique real solution to the equation

$$\tilde{c} = 2\lambda \ln \left(2 \cosh \frac{\tilde{c}}{2} \right) . \tag{60}$$

(60) has no solutions for $|\lambda| > 1$; indeed, our fixed line of theories exists only in the interval $|\lambda| \in [0, 1)$.

Note that large-N counting (the fact that the leading contribution to the free energy comes from disc diagrams) implies that the free energy is proportional to N at leading order in the large N expansion. The nontrivial part of (59) is the function of λ that multiplies the factor $-V_2T^3N$. This function has an analytic expansion in even powers of λ, about $\lambda = 0$. As $|\lambda|$ increases in $[0, 1)$, it decreases monotonically from the free value $\frac{3}{4\pi}\zeta(3)$ to zero, and suggests a thinning of degrees of freedom at stronger coupling as $\lambda \to 1$

Nearly Conserved Currents

We next consider the holographic dual of the theory. Consider the free limit of the theory, obtained by setting $\lambda = 0$. It is well-known that free theories have an infinite number of conserved currents – one for each spin $s > 0$. The first few such currents take the following form:

$$\begin{aligned}
J_\mu &= \bar{\psi}\gamma_\mu\psi, \\
J_{\mu_1\mu_2} &= \bar{\psi}\gamma_{\mu_1} \left(\overrightarrow{\partial_{\mu_2}} - \overleftarrow{\partial_{\mu_2}} \right) \psi, \\
J_{\mu_1\mu_2\mu_3} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1} \left(3\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}} - 10\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 3\overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\eta_{\mu_2\mu_3} \right) \psi, \\
J_{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1} \Big(\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overleftarrow{\partial_{\mu_4}} - 7\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} + 7\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} - \overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}}, \\
&\quad +2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overleftarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4} - 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overrightarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4} \Big) \psi \; .
\end{aligned}$$

These currents, previously studied by [28], can be thought of as a basis for the set of all "single trace" operators, where by "single trace" we mean operators such as $\bar{\psi}^i\psi_i$, which are formed out of the contraction of a single fundamental fermionic index with a single antifundamental fermionic index.

Via the AdS/CFT dictionary, each conserved current corresponds to a gauge field in the gravitational dual. Hence, it has been conjectured in [29], extending [30], that the free theory admits a dual description as a Vasiliev higher-spin gauge theory (technically, the parity-preserving "type-B" theory in [31].)

In the interacting theory, the natural generalisation of these operators is to replace all derivatives by covariant derivatives, and to subtract a "multi-trace" term, so that the currents are traceless (with respect to Lorentz indices) and have a well defined spin. These first few of these currents are given by:

$$\begin{aligned}
\hat{J}^{(1)}_{\mu} &= \bar{\psi}\gamma_\mu\psi, \\
\hat{J}^{(2)}_{\mu_1\mu_2} &= \bar{\psi}\gamma_{\mu_1}\left(\overrightarrow{D_{\mu_2}} - \overleftarrow{D_{\mu_2}}\right)\psi, \\
\hat{J}^{(3)}_{\mu_1\mu_2\mu_3} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\left(3\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}} - 10\overleftarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 3\overrightarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 2(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\eta_{\mu_2\mu_3}\right)\psi, \\
\hat{J}^{(4)}_{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\Big(\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}}\overleftarrow{D_{\mu_4}} - 7\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}} + 7\overleftarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}} - \overrightarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}}, \\
&\quad +2\overleftarrow{D_{\mu_2}}(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\eta_{\mu_3\mu_4} - 2(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\overrightarrow{D_{\mu_2}}\eta_{\mu_3\mu_4}\Big)\psi\ .
\end{aligned}$$

While the divergence of currents $J^{(s)}_{\mu_1\ldots\mu_s}$ vanish in the free theory, at nonzero coupling the currents obey an equation of the schematic form

$$\partial\cdot J^{(s)} \sim \frac{1}{k}JJ + \frac{1}{k^2}JJJ\ . \tag{61}$$

Although the RHS of (61) is nonvanishing, it is a multitrace contribution, and so contributes only at subleading order in $\frac{1}{N}$ when inserted into a two point function. In other words the currents $J^{(s)}$ are effectively conserved, hence protected, within two point functions. The conservation fails at the first subleading order in $\frac{1}{N}$.

More specifically, we are able to use (61) to demonstrate that the scaling dimensions of currents $J^{(s)}$ in the interacting theory are not renormalized to leading order in $\frac{1}{N}$ even as we turn on interactions. As the anomalous dimension of the current operators are related to the mass of the corresponding gauge field in the bulk, we conclude that tower of higher-spin gauge fields remain massless as we turn on interactions. Therefore, even in the strongly interacting limit, the holographic dual to our theory must be a higher-spin gauge theory.

Discussion

In this thesis, we study various aspects of strongly-interacting quantum field theories in three dimensions in light of recent developments in string theory which imply that conformal quantum field theories in three-dimensions that admit a large-N limit effectively define quantum theories of gravity (holographic duals) in four dimensional anti-de Sitter space.

In [8] and [9], we illustrated how such holographic duals could be used to calculate thermodynamic properties as well as quantities such as transport in strongly interacting conformal field theories, at finite temperature, chemical potential and in the presence of a magnetic field. We incorporated dilatonic and axionic couplings to generate models with very realistic thermodynamic properties – in particular, a vanishing entropy at zero temperature – not present in the traditional gravitational theories without a dilaton.

From the field theory side, in [10], we considered the specific case of $U(N)$ Chern-Simons theory coupled to fundamental fermions in the large N limit. We were able to calculate the free energy of the theory on $R^2 \times S^1$ for all values of the 't Hooft coupling λ. We also studied the operator spectrum of the theory – the results suggest that the holographic dual is some sort of higher-spin gauge theory, even in the strongly interacting limit. It is, however, very interesting to note that a much wider class of Chern-Simons theories exist, that, although non-supersymmetric, are conformal and have a large N limit (e.g., the $U(N) \times U(N)$ Chern-Simons theory coupled to a massless fermion in the bifundamental representation mentioned earlier.) Some of these theories could well arise as conformal field theories describing the behaviour of a various of quantum systems near a quantum critical point, and it is eminently reasonable to conjecture that their dynamics at strong coupling is described by relatively simple gravitational duals based on traditional Einstein theories of gravity.

We are eagerly looking forward to future developments in this field.

List of Papers

Included on this thesis:

1. K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, "Holography of Charged Dilaton Black Holes," JHEP **1008**, 078 (2010) [arXiv:0911.3586 [hep-th]].

2. K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, "Holography of Dyonic Dilaton Black Branes," JHEP **1010**, 027 (2010) [arXiv:1007.2490 [hep-th]].

3. S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, "Chern-Simons Theory with Vector Fermion Matter," arXiv:1110.4386 [hep-th].

Not included on this thesis:

4. S. Giombi, S. Prakash and X. Yin, "A Note on CFT Correlators in Three Dimensions," arXiv:1104.4317 [hep-th].

Chapter 1

Introduction

Recent developments in string theory and M-theory have led to increased interest and new progress in the study of three-dimensional quantum field theories. One of the main reasons for this is that conformal field theories in three dimensions (CFT_3's) involving $U(N)$ gauge groups are expected to be dual to quantum theories of gravity in 4-dimensional anti-de Sitter space (AdS_4), via the AdS/CFT (or holographic) duality [1, 32, 33]. The primary example of the AdS_4/CFT_3 correspondence is the ABJM theory discovered in 2008 [2] which is believed to describe the low-energy dynamics of multiple $M2$-branes in a Z_k orbifold, and hence is conjectured to be dual to M-theory on $AdS_4 \times S_7/Z_k$.

Motivated by the AdS_4/CFT_3 correspondence, this thesis presents some new examples of strongly interacting conformal quantum field theories in three dimensions in the large-N limit, using both holographic (i.e., gravitational) and quantum field theory techniques.

This is thesis is based on [8], completed in collaboration with Kevin Goldstein, Shamit Kachru, and Sandip Trivedi; [9] completed in collaboration with Kevin Goldstein, Norihiro Iizuka, Shamit Kachru, Sandip Trivedi and Alexander Westphal; and [10] completed in collaboration with Simone Giombi, Shiraz Minwalla, Sandip Trivedi, Spenta Wadia, and Xi Yin. In this introduction, we explain the motivation and results of these works, as well as briefly review some relevant results already existing in the literature.

1.1 Holography of Dilaton Black Branes

Via the AdS/CFT duality, classical four-dimensional gravity in anti-de Sitter space takes on a new role as a large-N saddle point to a three-dimensional conformal field theory in a particular strongly interacting limit. Various authors (e.g., [3, 4, 5, 34, 6, 7, 35]) have been inspired to use gravitational systems as toy models for strongly interacting conformal field theories that arise in condensed matter (e.g., [36]). Here we sketch the general ideas underlying their approach. (For excellent reviews see [37, 38, 39, 40].)

A generic system of strongly interacting electrons near a quantum critical point would be described by an effective conformal field theory containing a conserved $U(1)$ current that can be coupled to an external electromagnetic field. Conductivity in the linear approximation is related – via a Kubo formula – to a two point function of the $U(1)$ current. Schematically, in the presence of a weak time-varying external field $E_j(t,x) = |E_j|e^{i\omega t}$, we find the expectation value of the current $\langle J^i(y)\rangle$ given by

$$\begin{aligned}\langle J^i(y)\rangle &= \int [D\psi]\, e^{-S_{eff} - \int d^3x J^\mu(x) A_\mu(x)}\; J^i(y) \\ &\approx \int [D\psi]\, e^{-S_{eff}} \left(1 - \int d^3x J^j(x) \left(\frac{i}{\omega}\right) E_j\right) J^i(y) \\ &= \frac{i}{\omega} \langle \int d^3x J^j(x) J^i(y)\rangle\, E_j(x) \equiv \int d^3x \sigma^{ji}(y,x) E_j(x)\ .\end{aligned}$$

This fits in very nicely with the AdS/CFT dictionary [32, 33] which enables us to calculate correlation functions of gauge invariant operators using the classical gravitational theory.

For each conserved current in the field theory, there is a corresponding field in the bulk – the conserved stress tensor corresponds to the bulk metric and the conserved $U(1)$ current corresponds to a Maxwell field. The simplest gravitational theory one can use to model a strongly interacting condensed matter system is therefore Einstein-Maxwell theory with a negative cosmological constant. A great deal of very interesting work has been done (e.g., [3, 4, 5, 6, 41, 7, 42, 35]) to model strongly interacting quantum critical points based on this minimalistic bulk theory. In these investigations and others, one is interested in studying the conformal field theory at finite temperature and chemical potential – which requires one to study charged black brane solutions in the bulk.

One potential source of concern with the program outlined above is that the charged black brane solutions (known as Reissner-Nordström black branes) of Einstein-Maxwell theory have a non-vanishing entropy at zero temperature, unlike most real world systems. While this is not necessarily a problem – for instance, instabilities caused by including a charged scalar field could allow for a phase transition to a zero-entropy state at low temperatures [34, 21] – it would be nice to enlarge the range of holographic models studied to include models with different thermodynamic properties.[1]

With these motivations in mind, in Chapter 3 (based on work published in [8]) we consider holography of charged dilaton black branes in AdS_4. Using a combination of analytic and numeric techniques, we study black brane solutions to the following theory:

$$S = \int d^4x \sqrt{-g}\left(R - 2(\nabla\phi)^2 - e^{2\alpha\phi}F^2 - 2\Lambda\right)\ . \tag{1.1}$$

The most interesting feature of the solutions we study is that they have vanishing entropy at zero-temperature, in contrast with the Reissner-Nordström black branes previously studied.

The near-horizon geometry of our solutions is Lifshitz-like [13] with a non-trivial dynamical exponent $1/\beta$ (where $\beta < 1$ is determined by the details of the dilaton coupling to the gauge field). Departures from extremality give rise to an entropy density s growing as a

[1] It has recently been suggested that the large entropy of the Reissner-Nordström brane can perhaps be interpreted as arising from some analogue of a "fractionalized Fermi liquid" phase in the boundary theory [43]. Some support for the existence of such a phase, at least in some AdS/CFT dual pairs, accrues from explicit lattice models with localized fermions in string constructions, where AdS_2 regions arise from bulk geometrization of the lattice spins [44]. Another, complementary approach to the entropy problem is developed in [45, 46].

power law $s \sim T^{2\beta}$, with a positive specific heat. Numerically, we are able to show that the Lifshitz-like near horizon geometries can be extended to asymptotically AdS_4 spacetimes.

After a detailed study of the thermodynamics of these solutions, we calculate the optical conductivity $\sigma(\omega)$ of the extremal black brane background, via the AdS/CFT dictionary. Our approach is based on [21], and is slightly subtle because we only have analytic expressions for the black brane background in the near-horizon and near-boundary regions. Surprisingly, we find that the conductivity turns out to obey

$$\mathrm{Re}(\sigma) \sim \omega^2/\mu^2, \quad \mathrm{Im}(\sigma) \sim \frac{\mu}{\omega} \ . \tag{1.2}$$

at low frequencies. The frequency dependence is independent of the Lifshitz parameter of the near-horizon region. While thermodynamics depends non-trivially on the dilaton coupling α, the transport properties appear to be universal. We do not yet understand the physical origin of this intriguing result.

1.2 Dyonic Dilaton Black Branes

We next consider generalizing the analysis above to calculate transport properties in the presence of an external magnetic field [9]. By the AdS/CFT dictionary this corresponds to studying black branes with both electric and magnetic charge. In the most general setting in string theory, a dyonic solution is expected to source both a dilaton field as well as an axion field. An important class of axion-dilaton theories are those invariant under an $SL(2,R)$ electromagnetic duality. In particular, we consider the following theory:

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - 2(\partial\phi)^2 - \frac{1}{2}e^{4\phi}(\partial a)^2 - e^{-2\phi}F^2 - aF\tilde{F}\right] \ . \tag{1.3}$$

which generalizes the theory studied in [22] to AdS space.

To see the $SL(2,R)$ duality, we combine the axion a and dilaton as follows:

$$\lambda = \lambda_1 + i\lambda_2 = a + ie^{-2\phi}. \tag{1.4}$$

Then it is easy to see that under an $SL(2,R)$ transformation

$$M = \begin{pmatrix} \tilde{a} & b \\ c & d \end{pmatrix} \tag{1.5}$$

which takes

$$F_{\mu\nu} \to F'_{\mu\nu} = (c\lambda_1 + d)F_{\mu\nu} - c\lambda_2\tilde{F}_{\mu\nu} \tag{1.6}$$

and

$$\lambda \to \lambda' = \frac{\tilde{a}\lambda + b}{c\lambda + d} \tag{1.7}$$

while keeping the metric invariant, the equations of motion are left unchanged. (This is discussed in, e.g., [23].)

The electrically charged black brane solution to this action is nothing but the solution to (1.1) for $\alpha = 1$ discussed in the previous section. Starting with this purely electrically charged solution, we are able to generate solutions with arbitrary dyonic charges and boundary values of the dilaton-axion using $SL(2,R)$ duality. These solutions all exhibit a Lifshitz-like near-horizon geometry, and hence have vanishing entropy at extremality.

Transport Coefficients

We consider the thermodynamics of these dyonic solutions as well as general thermo-electric linear response, which is characterized by the conductivity and two related transport coefficients, the thermoelectric coefficient α and the thermal conductivity κ. These are all tensorial and satisfy the relation

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \alpha \mathbf{T} & \kappa \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix} \tag{1.8}$$

where $\vec{E}$ is the electric field, $\vec{\nabla} T$ is the gradient of the temperature, $\vec{J}$ is the electric current and $\vec{Q}$ is the heat current.

In the presence of an external magnetic field, conductivities are described via:

$$j_x = \sigma_{xx} F_{tx} + \sigma_{xy} F_{ty} \tag{1.9}$$
$$j_y = -\sigma_{xy} F_{tx} + \sigma_{xx} F_{ty}. \tag{1.10}$$

Upon working through the AdS/CFT dictionary, we find that the linear combinations

$$\sigma_+ = \sigma_1 + i\sigma_2, \;\; \sigma_- = \sigma_1 - i\sigma_2 \tag{1.11}$$

both transform in in a very nice way under the $SL(2,R)$ transformation (41):

$$\sigma_\pm \to \frac{\tilde{a}\sigma_\pm + b}{c\sigma_\pm + d} \; . \tag{1.12}$$

An immediate and interesting consequence of these transformation rules is that conductivities σ_{xx} and σ_{xy} will in general obey "semi-circle" laws, as described in [24] and references therein.

Using these transformation laws, we are able to obtain the conductivity of a general dyonic black brane, starting from the conductivity of the purely electric brane in (1.2). The resulting transport fits in nicely with a magnetohydrodynamic formalism proposed in [5], which states that, at low frequencies, the conductivity of a relativistic plasma is given by:

$$\sigma_{xx} = \sigma_Q \frac{\omega \left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{(\omega + i\gamma)^2 - \omega_c^2} \tag{1.13}$$

and

$$\sigma_{xy} = -\left(\frac{n}{Q_m}\right) \frac{\gamma^2 + \omega_c^2 - 2i\gamma\omega}{(\omega + i\gamma)^2 - \omega_c^2} \; . \tag{1.14}$$

Here σ_Q, the damping frequency γ, and the cyclotron frequency ω_c depend on the magnetic field (or magnetic charge of the brane) Q_m, the temperature T and charge density n.

We find that

$$\sigma_Q \propto T^2, \quad \gamma \propto (Q'_m)^2 T^2, \quad \omega_c \propto Q'_m. \tag{1.15}$$

This qualitative behavior is in agreement with the results of [11, 12] for the Reissner-Nordström black brane at small ω.

An interesting feature of the above result is that the DC Hall conductance is proportional to the attractor value of the axion at the horizon, which thus seems to act as a Chern-Simons coupling in a low-energy fixed-point effective field theory.

1.3 A Chern-Simons Vector Model

In addition to exploiting this duality to generate and study new examples of strongly interacting conformal field theories in three dimensions using gravity; it is also of interest to find models (particularly non-supersymmetric ones) that can be solved directly from the field theory side at strong coupling in the large N limit, with a view towards gaining some intuition into the mechanism behind the duality itself. For this purpose, in Chapter 4 (based on [10]) we conclude with a study of level-k $U(N)$ Chern-Simons theory coupled to fundamental fermions – a theory that turns out to be remarkably educational and (at least partially) exactly-solvable using traditional large N techniques.

Chern-Simons theories are fascinating from several points of view. The attractive features of these theories, unique to three dimensions, had been recognized early on, notably in [47] and [48]. They arise, for example, in the study of knot invariants [49] and in the study of the quantum Hall effect [50, 51]. More recently, superconformal Chern-Simons theories (e.g., [52, 53, 54, 55, 56, 57, 58]) have been shown to play a crucial role in the AdS/CFT correspondence [2].

The Chern-Simons kinetic term for the gauge field takes the form:

$$S_{CS} = -\frac{k}{4\pi} \int \mathrm{tr}\, \left(A \wedge dA + \frac{2i}{3} A \wedge A \wedge A \right) . \tag{1.16}$$

Under a gauge transformation, the action changes by a total derivative. Single-valuedness of e^{iS} implies that the Chern-Simons level k must be an integer. The fact that k is an integer means that, unlike the Yang-Mills coupling, it cannot flow under the renormalization group.

Consider a level k, $U(N)$ Chern-Simons theory coupled to a single fermion in any representation of the gauge group. The only gauge invariant power counting relevant or marginal operators in such a theory are the fermion kinetic term and mass term. A continuum quantum theory built from such a Lagrangian depends on two discrete parameters, k and N, and a single continuous parameter, the physical mass m of the fermionic field. At energies $E \gg m$, the dynamics of this theory is scale invariant as well as non-trivial. Non-triviality is ensured by the fact that the discrete Chern-Simons coupling, which induces interactions among the fermions, cannot run and so is nonvanishing even at arbitrarily high energy. The nontrivial CFT that controls the high energy behavior of this system is most directly constructed by choosing the bare mass to set the physical mass m of this system to zero.

The parameters k and N labeling the CFT are discrete. However it is well known that the loop counting parameter in a $U(N)$ Chern-Simons theory is the 't Hooft coupling $\lambda = \frac{N}{k}$

whenever all matter fields transform in representations whose dimension does not grow faster with N than N^2. In the large N (and simultaneously large k) limit λ is effectively a continuous parameter (exactly as in ABJM theory [2]). For this reason the discretum of CFTs described by integer values of k and N coalesces into a fixed line in the large N limit.

Lines of fixed points parameterized by a coupling constant are especially interesting from the viewpoint of the AdS/CFT correspondence. Such lines of d dimensional CFTs have the potential of interpolating between a simple field theoretic description at weak coupling and a relatively simple bulk gravitational description at strong coupling, as demonstrated by the famous supersymmetric examples [1, 2].

Now when $d \geq 4$, lines of fixed points appear to be rather exotic. The examples we know of (like the large N Banks-Zaks fixed line of QCD) involve theories with a parametrically large number of flavors of matter fields. It is interesting, on the other hand, that effective fixed lines of large N Chern-Simons theories coupled to matter fields are plentiful and very easily constructed in 2+1 dimensions even with very simple matter content[2].

We emphasize that a variety of such non-supersymmetric fixed lines exist in three dimensions. Choosing the fermions that transform in, say, the adjoint representation of $U(N)$ gives one such fixed line of theories. Choosing fermions that transform in the bifundamental of $U(N) \times U(M)$ yields another example – one that preserves parity when $N = M$ and the two Chern-Simons levels are equal and opposite[3].

The study of fixed lines of large N Chern-Simons theories with matter and their bulk duals appears to be an interesting program. For examples with a large amount of supersymmetry this programme was spectacularly initiated by ABJM [2] and carried forward in several follow up papers [59, 60, 61]. In Chapter 4, we initiate a detailed study of perhaps the simplest of the non-supersymmetric fixed lines – the theory of a level k $U(N)$ Chern Simons theory coupled to a single fundamental fermion.

The Chern-Simons gauge field has N^2 components; superficially, the large N limit of these theories is governed by the summation over a complicated web of planar graphs. The complexity is illusory, as pure Chern-Simons theory has no propagating degrees of freedom – the only propagating degrees of freedom in our system are the fundamental fermions. Consequently, the theory we investigate is a vector model with N degrees of freedom. Large N limits of vector theories are much simpler than their matrix counterparts, and sometimes prove to be exactly solvable. Indeed, both in terms of diagrammatics and canonical structure, the theory we study bears a close resemblance to 't Hooft's solution of two-dimensional QCD in the large-N limit using light-cone gauge [27].

Exact Propagator and Free Energy in Light-Cone Gauge

In Euclidean space, the theory is described by the action:

$$S = \frac{ik}{4\pi} \int \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) + \int \bar{\psi} \gamma^\mu D_\mu \psi \ . \tag{1.17}$$

[2]We thank O. Aharony for discussions on this point.

[3]The ease of construction of conformal field theories in $d = 3$ intriguingly suggests that it is particularly easy to construct non supersymmetric quantum theories of gravity in $d = 4$.

We work in an analog of light-cone gauge, defined using the condition

$$A_- = \frac{A^1 + iA^2}{\sqrt{2}} = 0.$$

This can be obtained from Wick rotation of the standard lightcone gauge in Lorentzian signature. Further details about conventions are in Chapter 4.

The choice of this gauge turns out to be crucial in allowing for the derivation of the free energy of the theory on R^2 at temperature $1/\beta$, (i.e., on $R^2 \times S^1$).

We first derive the exact fermion propagator at zero temperature. After deriving the action for the theory in light-cone gauge (using the usual BRST formalism, and noting that ghosts decouple) we are able to derive the following integral Schwinger-Dyson equation for the exact fermion propagator in the large-N limit (see [26]):

$$\begin{aligned}
&\langle \psi_m(p)\bar{\psi}^n(p')\rangle \\
=&\frac{1}{ip_\mu\gamma^\mu}(2\pi)^3\delta^3(p'+p) \\
&-\frac{1}{ip_\mu\gamma^\mu}\frac{2\pi i}{k}\int\frac{d^3r}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{q^+}\gamma^+\langle\psi_a(p-q)\bar{\psi}^a(-r)\rangle\gamma^3\langle\psi_m(r+q)\bar{\psi}^n(p')\rangle \\
&+\frac{1}{ip_\mu\gamma^\mu}\frac{2\pi i}{k}\int\frac{d^3r}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{q^+}\gamma^3\langle\psi_a(p-q)\bar{\psi}^a(-r)\rangle\gamma^+\langle\psi_m(r+q)\bar{\psi}^n(p')\rangle\ .
\end{aligned}$$

This expression for the exact propagator is completely analogous to that for the exact propagator in 2d QCD obtained by 't Hooft [27].

Surprisingly, we are able to the solve this integral equation at zero temperature as well as finite temperature. Let us very briefly sketch the solution.

First define the self-energy, $\Sigma(p) = \Sigma = i\Sigma_\mu\gamma^\mu + \Sigma_I I - M_{bare}I$, of the fermion as follows

$$\langle\psi(p)_m\bar{\psi}(-q)^n\rangle = \delta_m^n\frac{1}{ip_\mu\gamma_\mu + M_{bare} + \Sigma}\times(2\pi)^3\delta(p-q). \tag{1.18}$$

The Schwinger-Dyson equation above can easily be translated into an integral equation for the Σ; from which it is easy to see that $\Sigma_- = \Sigma_3 = 0$. Further analysis also reveals that the remaining components of Σ must take the following form:

$$\begin{aligned}
\Sigma_I(p) &= f_0 p_s \\
\Sigma_+(p) &= p_+ g_0 = p^- g_0
\end{aligned} \tag{1.19}$$

where

$$p_s = \sqrt{p_1^2 + p_2^2} = \sqrt{2}|p^-| = \sqrt{2}|p^+| \tag{1.20}$$

and f_0 and g_0 are dimensionless functions of λ. The solution to the resulting integral equations for f_0 and g_0 turn out to be remarkably simple

$$\begin{aligned}
f_0 &= \lambda \\
g_0 &= -\lambda^2.
\end{aligned} \tag{1.21}$$

The zero-temperature, exact propagator is thus given by:

$$\langle \psi(p)_m \bar{\psi}(-q)^n \rangle = \delta_m^n \frac{1}{ip_3\gamma^3 + ip_-\gamma^- + i(1-\lambda^2)p_+\gamma^+ + \lambda p_s} \times (2\pi)^3 \delta(p-q). \tag{1.22}$$

The above calculation can be generalized to finite temperature. Furthermore it is also possible to express the free energy of the theory in terms of the exact fermion self-energy. In the large-N saddle point, the partition function $Z = e^{-S_E}$ is given by:

$$-S_E = NV \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr} \left[\ln\left[i\gamma^\mu q_\mu + \Sigma(q)\right] - \frac{1}{2}\Sigma(q) \left(\frac{1}{i\gamma^\mu q_\mu + \Sigma(q)} \right) \right] . \tag{1.23}$$

Using this solution, we find that the free energy of our theory,

$$F = -\frac{1}{\beta} \ln Z,$$

as a function of temperature and λ, in a box of volume V_2 (which is taken to be very large) is given by

$$F = -\frac{NV_2T^3}{6\pi} \left[\tilde{c}^3 \frac{1-\lambda}{\lambda} + 6 \int_{\tilde{c}}^{\infty} dy\; y \ln\left(1 + e^{-y}\right) \right] \tag{1.24}$$

where $\tilde{c}$ is the unique real solution to the equation

$$\tilde{c} = 2\lambda \ln\left(2\cosh\frac{\tilde{c}}{2} \right) . \tag{1.25}$$

(1.25) has no solutions for $|\lambda| > 1$; indeed, our fixed line of theories exists only in the interval $|\lambda| \in [0, 1)$.

Comment added: This result assumes that the holonomy of the gauge field around the thermal circle is the identity matrix; however, it is possible, and essentially straightforward, to generalize the calculation to nontrivial holonomy backgrounds. See the note added at the end of chapter 4.

Note that large-N counting (the fact that the leading contribution to the free energy comes from disk diagrams) implies that the free energy is proportional to N at leading order in the large N expansion. The nontrivial part of (1.24) is the function of λ that multiplies the factor $-V_2T^3N$. This function has an analytic expansion in even powers of λ, about $\lambda = 0$. As $|\lambda|$ increases in $[0, 1)$, it decreases monotonically from the free value $\frac{3}{4\pi}\zeta(3)$ to zero, and suggests a thinning of degrees of freedom at stronger coupling as $\lambda \to 1$

Nearly Conserved Currents

We next consider the holographic dual of the theory. Consider the free limit of the theory, obtained by setting $\lambda = 0$. It is well-known that free theories have an infinite number of conserved currents – one for each spin $s > 0$. In section 4.3.1, we present an explicit generating function for all currents with $s \geq 1$. The first few such currents take the following form:

$$\begin{aligned}
J_\mu &= \bar{\psi}\gamma_\mu\psi, \\
J_{\mu_1\mu_2} &= \bar{\psi}\gamma_{\mu_1}\left(\overrightarrow{\partial_{\mu_2}} - \overleftarrow{\partial_{\mu_2}}\right)\psi, \\
J_{\mu_1\mu_2\mu_3} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\left(3\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}} - 10\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 3\overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\eta_{\mu_2\mu_3}\right)\psi, \\
J_{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\Big(\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overleftarrow{\partial_{\mu_4}} - 7\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} + 7\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} - \overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}}, \\
&\quad +2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overleftarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4} - 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overrightarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4}\Big)\psi\ .
\end{aligned}$$

The set of all such currents are a basis for the set of all “single trace” operators, where by “single trace” we mean operators such as $\bar{\psi}^i\psi_i$, which are formed out of the contraction of a single fundamental fermionic index with a single antifundamental fermionic index.

Via the AdS/CFT dictionary, each conserved current corresponds to a gauge field in the gravitational dual. Hence, it has been conjectured in [29], extending [30], (see also [62] and [28]) that the free theory admits a dual description as a Vasiliev higher-spin gauge theory (technically, the parity-preserving “type-B” theory [63, 64, 31, 65]; see also [66, 67, 68]).

In the interacting theory, the natural generalization of these operators is to replace all derivatives by covariant derivatives (and to subtract a “multi-trace” term, so that the currents are traceless (with respect to Lorentz indices) and have a well defined spin.)

While the divergence of currents $J^{(s)}_{\mu_1\cdots\mu_s}$ vanish in the free theory, at nonzero coupling the currents obey an equation of the schematic form

$$\partial \cdot J^{(s)} \sim \frac{1}{k}JJ + \frac{1}{k^2}JJJ. \tag{1.26}$$

Although the RHS of (1.26) is nonvanishing, it is a multitrace contribution, and contributes only at subleading order in $\frac{1}{N}$ when inserted into a two point function. In other words the currents $J^{(s)}$ are effectively conserved, hence protected, within two point functions. The conservation fails at the first subleading order in $\frac{1}{N}$.

More precisely, we are able to use (1.26) to demonstrate that the scaling dimensions of currents $J^{(s)}$ in the interacting theory are not renormalized to leading order in $\frac{1}{N}$ even as we turn on interactions. As the anomalous dimensions of the current operators are related to the masses of the corresponding gauge fields in the bulk, we conclude that the tower of higher-spin gauge fields remain massless as we turn on interactions. The seems to imply that, even in the strongly interacting limit, the holographic dual to our theory must be a higher-spin gauge theory.

1.4 Summary

In this thesis, we study various aspects of strongly-interacting quantum field theories in three dimensions in the large-N limit.

We analyze charged dilatonic branes in considerable detail in Chapters 2 and 3, focusing on their thermodynamics and especially their transport properties. The dilaton case differs significantly from the Reissner-Nordström one in thermodynamics: while the Reissner-Nordström brane has a macroscopic ground-state entropy, the dilatonic black brane has vanishing entropy at zero temperature. Despite this significant difference in thermodynamics, our results show that many of the transport properties of dilaton black branes are quite similar to those of the Reissner-Nordström case. Overall, we feel that charged dilaton black branes in AdS provide a rich playground for studying black hole physics and holographic condensed matter.

In Chapter 4, we initiate the task of determining an exact solution to a simple but nontrivial fixed line of quantum field theories, namely $U(N)$ Chern-Simons theories coupled to fermionic fundamental matter. We have been able to exactly compute the finite temperature partition function and prove that the scaling dimensions of a large class of operators in the theory are protected in the large N limit, thereby allowing us to comment on the nature of the theory's holographic dual. It is interesting to note that our results imply that the theory (or any conformal vector model in three dimensions) cannot be dual to an Einstein theory of gravity – to obtain a field theory that may be dual to a traditional theory of gravity would require bifundamental or adjoint matter, and these theories are naturally much harder to solve from the field theory side than vector models. This illustrates the importance of the holographic techniques used in Chapters 2 and 3; they are expected to apply to precisely those theories (theories with matrix matter) that would otherwise have been the most difficult to solve within the traditional quantum field theory framework.

We hope the results of this thesis will, in combination with the work of others in the field, prove useful in the larger task of mapping out the range of physical behaviors possible in strongly interacting conformal field theories.

Chapter 2

Holography of Charged Dilaton Black Branes

This chapter is based on work presented in [8] and was completed in collaboration with Kevin Goldstein, Shamit Kachru, and Sandip Trivedi.

2.1 Introduction

Extremal black holes have been a fruitful source of theoretical questions and enigmas for several decades. The earliest charged black hole solutions to low-energy string theory were found by Garfinkle, Horowitz and Strominger [16] (and had appeared earlier as part of a family of solutions in [17]). Those authors studied the Einstein-Maxwell action with the gauge coupling controlled by a scalar dilaton ϕ.

$$S = \int d^4x \, \sqrt{-g} \left(-R + 2(\nabla\phi)^2 + e^{-2\phi} F^2 \right) \; . \tag{2.1}$$

This action admits remarkably simple extremal magnetically charged black hole solutions:

$$ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + (1 - \frac{2M}{r})^{-1} \, dr^2 + r(r - \frac{Q^2 e^{2\phi_0}}{M}) \, d\Omega^2 \; , \tag{2.2}$$

with dilaton profile

$$e^{-2\phi} = e^{-2\phi_0} - \frac{Q^2}{Mr} \tag{2.3}$$

and with gauge field

$$F = Q \, \sin(\theta) \, d\theta \wedge d\varphi \; . \tag{2.4}$$

Here, θ and φ are standard angular coordinates on the S^2 spatial slices in (2.2), and ϕ_0 is the asymptotic value of the dilaton. At extremality, $M = \sqrt{2} Q e_0^{\phi}$.

The action and the black hole solution are motivated by the low-energy α' expansion of heterotic string theory; one may obtain an equally simple electrically charged solution by exchanging $\phi \to -\phi$, while simultaneously exchanging the field strength F with its dual $\tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2\phi} \epsilon_{\mu\nu}{}^{\lambda\rho} F_{\lambda\rho}$.

The dilaton $e^{2\phi}$ blows up at the horizon of the extremal magnetically charged black hole, (2.2), (2.3), which lies at

$$r = \frac{Q^2 e^{2\phi_0}}{M} \ . \tag{2.5}$$

Therefore quantum loop corrections will become important close to the horizon. In the electrically charged case, although the dilaton vanishes at horizon, the string frame curvature blows up and therefore higher derivative corrections become important close to the horizon. This feature is quite common in dilatonic black holes – near the horizon, either string loop corrections or higher derivative corrections become significant and these corrections frustrate any attempt to calculate properties which depend sensitively on the near-horizon geometry. The simplest way to avoid this problem is to consider a slightly non-extremal black hole. For fixed charge and ϕ_0 the non-extremal hole has a horizon at a slightly larger value of r and as a result, the dilaton does not run to either infinity or zero value at the horizon. Adjusting the value of ϕ_0 appropriately, one can control the behaviour of the dilaton with a temperature much smaller than the charge and the properties of the resulting near-extremal black hole can be reliably calculated in the classical supergravity approximation.

These black holes, and generalizations thereof with general dilatonic coupling $e^{-2\alpha\phi}F^2$ for $\alpha \geq 0$, were found to have thermodynamic properties radically different from traditional charged black holes [18, 19]. The authors of [18, 19] found that the physics of extremal charged dilaton black holes (like that of extremal Reissner-Nordström black holes) exhibits a breakdown of the thermodynamic description. However, the conceptual reasons for the breakdown depend on details; and the relevant physics can be quite different for different values of α.

Why pursue their AdS generalization?

AdS/CFT may provide a powerful tool for studying strongly-coupled toy models of condensed matter systems (for excellent reviews see [37, 38, 39, 40]). In flat space, the extremal dilaton black holes exhibit features quite different from their extremal Reissner-Nordström cousins. Hence we expect that their AdS generalizations may widen the range of qualitative behaviors seen in simple and potentially relevant gravity models.

In particular, unlike the extremal Reissner-Nordström solutions, the solutions we study turn out to have vanishing entropy at extremality. Indeed, the large ground state degeneracy of the Reissner-Nordström AdS black branes is (at least superficially) in tension with the third law of thermodynamics, and suggests to some that they may be highly atypical holographic states of matter[1]. Since abelian gauge fields with dilatonic couplings parameterized by various values of α are fairly common in string compactifications, these may provide one generic class of simple bulk theories where the charged black holes do not have an undesired macroscopic ground-state entropy. (In the context of holographic superconductors [34, 6], another class of black branes with vanishing entropy at $T = 0$ was recently found in [70, 71, 21]).

[1] The large degeneracy is almost certainly an artifact of the large N approximation. S. Trivedi acknowledges discussions with A. Dabholkar and A. Sen in the course of preparing [69], and also with T. Senthil, in this regard.

Another motivation for this study is that the phenomena discussed in [19, 18] for flat-space charged dilaton black holes strongly suggested that, at least for some values of the parameters, charged dilaton black branes in AdS could provide novel holographic duals of insulators. While the bulk theory clearly has excitations at arbitrarily low-energy (it has uncharged Schwarzschild black brane solutions,) in the sector with non-trivial charge density, one may have expected a gap in the charged excitations in analogy with [19, 18]. We will find that this is not so, at least in the absence of a dilaton potential and for the well-motivated simple forms of the gauge coupling function that we consider..

A final motivation for this investigation comes from the study of extremal black hole/brane solutions. It is now well known that these extremal configurations exhibit the attractor mechanism regardless of supersymmetry – their near-horizon geometry is universal and independent of the asymptotic values taken by the moduli. Different kinds of attractors correspond to different kinds of universal behaviour. In the context of AdS/CFT characterizing the different kinds of attractors tells us about the different kinds of IR behaviour which can arise in the dual CFT which is at zero temperature but is now deformed by the addition of a chemical potential (or charge). This is clearly of interest as we develop the AdS/CFT dictionary further. From the point of view of possible connections to condensed matter physics, an early and important paper on the subject, [4], noted that the optical conductivity of the dual CFT in $2+1$ dimensions at finite temperature (and zero chemical potential) is actually independent of frequency and temperature and thus very universal. The attractor mechanism tells us that there should be some considerable universality as we deform the CFT along the chemical potential direction instead of temperature as well. And the extent of allowed variation should be determined by the different classes of attractors. Understanding the different classes of attractors is therefore of interest from this new point of view. A brief comment on the literature: There is by now considerable literature on the attractor mechanism. The seminal paper is [72]. For a recent review with a good collection of references, see [73]. Some references on attractors without supersymmetry are [74, 75, 76, 15, 77].

In this chapter, we will study extremal and non-extremal black branes in dilatonic gravity. We first analytically find the form of the near-horizon geometry for electrically charged extremal dilaton black branes as a function of α and then numerically extend our near-horizon solutions of §2 to provide full black brane solutions with AdS asymptotics. We then use the near-horizon geometry to compute the entropy and specific heat as a function of α, and see that the extremal branes have vanishing entropy and positive specific heat for all $\alpha \geq 0$; we also discuss other thermodynamic quantities. In section 2.5, we compute the conductivity in a controlled approximation as one approaches the extremal limit, using techniques similar to those in [21]. We find that somewhat surprisingly, in the simple cases we check (including the electrically charged black branes for all values of α), the result is that $\sigma(\omega) \sim \omega^2$ at $T = 0$ and low frequency.

While the results in the chapter were being readied for submission to [8], the papers [78, 79], which have some overlap in motivations with this work, appeared.

2.2 Gravity solution

We consider the following action,

$$S = \int d^4x \sqrt{-g} \left(R - 2(\nabla\phi)^2 - e^{2\alpha\phi} F^2 - 2\Lambda \right) \ . \tag{2.6}$$

which generalizes the action studied in [16] to include a negative cosmological constant, and a dilaton coupling parameterized by α. The maximally symmetric vacuum solution for this action is AdS space with AdS scale L given by $\Lambda = -\frac{3}{L^2}$. We will set $L = 1$ below.

We look for electrically-charged black branes solutions of the above action[2]. Our metric ansatz is of the form

$$ds^2 = -a(r)^2 \ dt^2 + a(r)^{-2} \ dr^2 + b(r)^2 \ (dx^2 + dy^2), \tag{2.7}$$

with corresponding gauge field

$$e^{2\alpha\phi} F = \frac{Q}{b(r)^2} \ dt \wedge dr. \tag{2.8}$$

Extremizing (2.6) implies the equations of motion for $a(r)$, $b(r)$ and $\phi(r)$:

$$(a^2 b^2)'' = -4\Lambda b^2 \tag{2.9}$$

$$\frac{b''}{b} = -(\partial_r \phi)^2 \tag{2.10}$$

$$\partial_r(a^2 b^2 \partial_r \phi) = -\alpha e^{-2\alpha\phi} \frac{Q^2}{b^2} \ , \tag{2.11}$$

as well as a first order constraint,

$$a^2 b'^2 + \frac{1}{2} a^{2\prime} b^{2\prime} = \phi'^2 a^2 b^2 - e^{-2\alpha\phi} \frac{Q^2}{b^2} - b^2 \Lambda \ . \tag{2.12}$$

Here and below, primes denote derivatives with respect to r.

We first look for a near horizon solution to the above action by considering the following scaling ansatz:

$$a = C_2 (r - r_h)^\gamma, \ b = C_1 (r - r_h)^\beta, \ \phi = - \ K \ \log(r - r_h) + C_3 \ , \tag{2.13}$$

We find that an *exact* solution is obtained if the exponents take the values

$$\gamma = 1, \ K = \frac{\frac{\alpha}{2}}{1 + (\frac{\alpha}{2})^2}, \ \beta = \frac{(\frac{\alpha}{2})^2}{1 + (\frac{\alpha}{2})^2} \ . \tag{2.14}$$

The constant C_2 is given by

$$C_2^2 = \frac{6}{(\beta + 1)(2\beta + 1)} \ . \tag{2.15}$$

[2]Related work finding charged black brane solutions in dilaton gravity with a Liouville-type potential appears in e.g. [80, 81].

By rescaling $(r - r_h)$, t, x and y, we can set the constant C_3 to zero and C_1 to unity. Q is then determined in terms of α by

$$Q^2 = \frac{6}{(\alpha^2 + 2)} \ . \tag{2.16}$$

In terms of $w = r - r_h$, the exact near-horizon solution is:

$$a = C_2 w, \ b = w^\beta, \ \phi = - \ K \ \log(w) \ . \tag{2.17}$$

Interestingly, this solution almost has a Lifshitz scaling symmetry under $w \to \lambda w$, $t \to \lambda t$, $x \to \lambda^z x$, and $y \to \lambda^z y$, with Lifshitz exponent $z = 1/\beta$. The symmetry is only broken by the fact that the dilaton ϕ shifts by a constant under such a rescaling.

The metric component g_{tt} has a second order zero at $w = 0$, and (2.17) is thus an extremal solution. A non-extremal generalization is the following:

$$ds^2 = C_2^2 w^2 (1 - \frac{m}{w^{2\beta+1}}) dt^2 + \frac{dw^2}{C_2^2 w^2 (1 - \frac{m}{w^{2\beta+1}})} + w^{2\beta}(dx^2 + dy^2), \tag{2.18}$$

which depends on the parameter m, with unchanged gauge field and dilaton. [14]

Note that the solution (2.17) is mildly singular; though all curvature invariants are finite, it is geodesically incomplete at $w = 0$ (see [38]). By restricting our attention to slightly non-extremal black branes, this singularity would be hidden. (See [82].)

2.2.1 A numerical asymptotically AdS_4 solution

In the extremal solution, the gauge coupling $g_{\mathrm{U}(1)} \sim e^{-\alpha\phi}$ becomes weak at the horizon, as $\phi \to \infty$. When $r \to \infty$ the gauge coupling becomes very strong; therefore, although exact, the solution (2.18) must be understood as a near-horizon geometry of a larger solution with different asymptotics. In particular, we would like to find a generalization of the extremal solution with a controlled and asymptotically constant dilaton as well as an asymptotically AdS_4 geometry.

The strategy for obtaining such a solution is to numerically integrate the equations of motion following from (2.6), starting near $w = 0$ with initial data taken from the near-horizon solution. However, the near-horizon solution is exact, so numerical integration using initial data drawn from it would simply reproduce the near-horizon solution unmodified. To numerically integrate to a solution that is asymptotically AdS_4, we must also take into account possible subleading corrections to the near-horizon solution.

This is completely analogous to the case of the Reissner Nordström black brane. The near horizon geometry $AdS_2 \times R^2$ is an exact solution to the equations of motion; however corrections (which are subleading in the near-horizon limit) are permitted which allow it to be embedded in an asymptotically AdS_4 spacetime. (This is analogous to the case of an extremal Reissner-Nordström black hole – the near horizon-solution is an exact solution to the equations of motion, however, subleading near-horizon corrections are permitted and allow the black hole to be embedded in asymptotically flat spacetime.)

We start with a fairly general ansatz for the modification to the metric:

$$\begin{aligned} a(w) &= C_2 w \, (1 + d_1 w^{\nu_1}) \\ b(w) &= w^\beta \, (1 + d_2 w^{\nu_2}) \end{aligned} \tag{2.19}$$

which vanishes as $w \to 0$. The form of the perturbation of ϕ is determined from the ansatz for b by the equation of motion (2.10):

$$\phi(w) = -K \log(w) + C_3 + d_3 w^{\nu_2} \tag{2.20}$$

where $d_3 = \frac{2\beta+\nu_2-1}{2K} d_2$.

We first note that $\nu_1 = \nu_2$. This can be seen by substitution of the ansatz into (2.9). Since we require both ν_1 and ν_2 to be positive, the two terms proportional to w^{ν_1} and w^{ν_2} cannot separately cancel, so $\nu_1 = \nu_2 \equiv \nu$. (Even if we allow negative solutions, it turns out that allowing $\nu_1 \neq \nu_2$ yields only one consistent perturbation: $\nu_1 = \frac{-4+3\alpha^2}{4+\alpha^2}$ for which $d_2 = 0$. This solution can also be obtained considering the perturbation with $\nu_1 = \nu_2$.)

We now substitute the ansatz into (2.9) and (2.11), which we solve to leading order in w. We will use one of the equations to solve for d_1 in terms of d_2 and ν. Substituting into the remaining equation will result in a quartic equation for ν. There will be no constraint on d_2, which is a free parameter that determines the strength of the perturbation. (The structure is similar to that of an eigenvalue problem: We are looking for vectors $(d_1\ d_2)$ in the kernel of some 2×2 matrix. The matrix depends on ν and ν^2, hence we expect the condition that the determinant of the matrix vanishes to yield a quartic equation for ν.)

Substituting the ansatz into (2.9) implies

$$d_1 = \left(\frac{2(1+\beta)(1+2\beta)}{(2\beta+2+\nu)(2\beta+1+\nu)} - 1 \right) d_2 \ . \tag{2.21}$$

Using this expression for d_1, (2.11) is satisfied to leading order if ν satisfies the following quartic equation:

$$(\nu+1)(4\beta+\nu)\left(-4\beta^2 + (2\beta+1)\nu - 6\beta + \nu^2 - 2\right) = 0 \ . \tag{2.22}$$

The only positive root is

$$\begin{aligned} \nu &= \frac{1}{2}\left(-2\beta + \sqrt{(2\beta+1)(10\beta+9)} - 1\right) \\ &= \frac{-3\alpha^2 + \sqrt{57\alpha^4 + 184\alpha^2 + 144} - 4}{2\left(\alpha^2+4\right)} \ . \end{aligned} \tag{2.23}$$

To find all allowed perturbations we must also consider values of ν for which (2.21) is singular – either $d_1 = 0$ or $d_2 = 0$. This happens if $(2\beta+2+\nu)(2\beta+1+\nu) = 0$. Both these roots are negative, so they do not concern us here. However, for reference we note that the finite temperature solution (2.26) is obtained from choosing $\nu = -2\beta - 1 = -\frac{3\alpha^2+4}{\alpha^2+4}$ for which $d_2 = 0$. We have not explored what happens when we consider the other negative values of ν, perhaps they also give rise to interesting solutions.

Finally, we observe that the constraint (2.12) is satisfied for ν given by (2.23) and d_1 given by (2.21), as required.

The final form of the perturbed solution is (2.19) and (2.20) with ν given by (2.23), and d_1 related to d_2 according to (2.21).

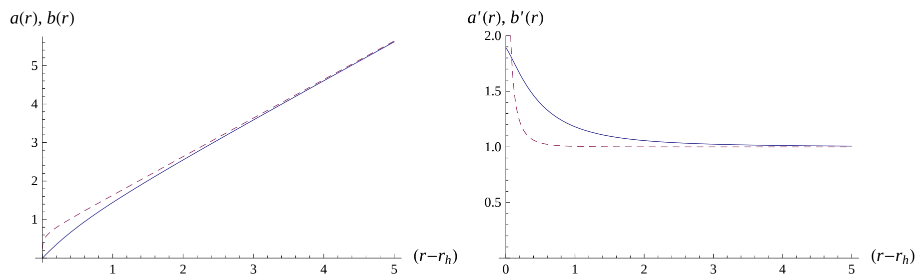

Figure 2.1: Numerical solution interpolating between the Lifshitz-like near horizon solution and AdS_4 for $\alpha = 1$ and $d_1 = -.514$. The second plot shows that $a'(r)$ and $b'(r)$ approach 1. Solid lines denote a, dashed lines denote b.

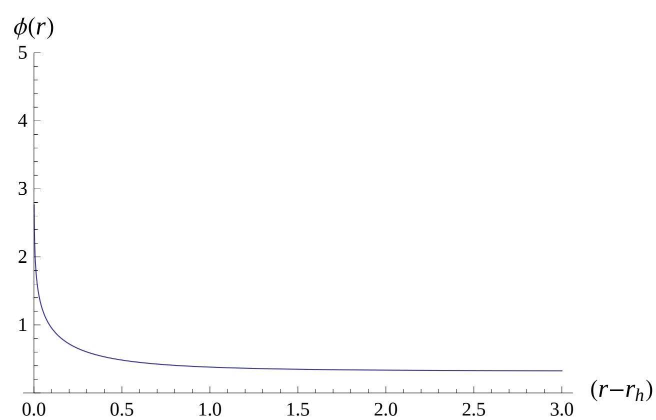

Figure 2.2: Numerical solution for ϕ, for $\alpha = 1$.

Numerically integrating Einstein's equations with initial data drawn from this solution (with $d_1 < 0$) yields a numerical solution that manifestly approaches AdS_4 with a constant dilaton as $r \to \infty$, as shown in Figures 2.1 and 2.2.

Initial data for numerical integration is taken from the modified near-horizon solutions (2.19) and (2.20). Figures 2.1 and 2.2 show the resulting solution for $\alpha = 1$. The strength of the perturbation was chosen to be $d_1 = -.514219$, so that the solution meets the condition that $b'(r) \to 1$ as $r \to \infty$. (For other negative values of d_1, $b'(r)$ approaches a constant which is different from one, a coordinate transformation then brings the solution back to a form with the standard asymptotics of AdS_4 space. For positive d_1 the numerical solution becomes singular.) The numerical integration above is shown for $\alpha = 1$ but a similar numerical solution can also obtained for other values of α.

The extremal solution is labelled by two parameters – the total charge of the black brane Q and the asymptotic value of the dilaton ϕ_0. However, there is essentially only one numerical solution (for a given α) – the numerical solution above corresponds to a particular value of the two parameters Q and ϕ_0, determined by the choice of gauge for the near-horizon solution (2.17). It is clear from the equations of motion that the metric and $(\phi - \phi_0)$ only depend on the quantity $Q^2 e^{-2\alpha\phi_0}$. Starting from the numerical solution, we can rescale r via

$$r \to \lambda r, (t, x, y) \to \lambda^{-1}(t, x, y) \tag{2.24}$$

to obtain a solution for any given value of Q and then shift the dilaton to obtain a solution with any given value of ϕ_0.

2.3 Thermodynamics

Via the AdS/CFT dictionary, quantities calculated in the black hole background correspond to quantities calculated in a strongly interacting CFT at fixed temperature T and chemical potential μ. Thermodynamic relations between between the temperature, chemical potential, energy density ρ, entropy density s, pressure P, and the number density n can be extracted from the black brane solution.

An important subtlety in this analysis is that our solution is essentially numerical – we only have analytic expressions for the near-horizon solution and the asymptotic solution. However, as we will see below, this is enough to analytically calculate thermodynamics in the regime $T \ll \mu$, (which is when the solution (2.18) is valid.) Studying temperatures comparable to μ would require analysis of a finite-temperature numerical solution.

The horizon in the solution (2.18) is located at $w = w_h$, (or $r = r_h + w_h$):

$$w_h^{2\beta+1} = m \ . \tag{2.25}$$

The resulting temperature can be obtained as usual by continuing to Euclidean space [20] and comes out to be

$$T \sim w_h \ . \tag{2.26}$$

The entropy density of the slightly non-extremal black brane is proportional to the area element of the horizon and is

$$s = A_0 C T^{2\beta} \mu^{2-2\beta}, \tag{2.27}$$

where

$$C \sim L^2/G_N \tag{2.28}$$

is the central charge of the CFT dual to the AdS_4 background, as follows from substituting the near-horizon solution into the action (2.6). The numerical coefficient A_0 depends on the full numerical solution for the slightly non-extremal black hole. We note that the entropy vanishes at zero temperature.

Let us now consider the relations between charge, number density, and chemical potential the extremal limit, $T = 0$. It follows from the Einstein's equations that in the asymptotic AdS_4 region the solution must take the form

$$\begin{aligned} a^2 &= r^2(1 - e_1\frac{\rho}{r^3} + \frac{Q^2 e^{-2\alpha\phi_0}}{r^4} + \cdots) && (2.29)\\ b^2 &= r^2(1 + \cdots) && (2.30)\\ \phi &= \phi_0 + \frac{\phi_1}{r^3} + \cdots . && (2.31) \end{aligned}$$

e_1 is a constant which depends on L. Under the rescaling (2.24), $\rho \to \frac{\rho}{\lambda^3}, Q \to \frac{Q}{\lambda^2}$, which implies

$$\rho = D_1(Qe^{-\alpha\phi_0})^{3/2} . \tag{2.32}$$

The coefficient D_1 is a numerical coefficient that is α dependent. A similar scaling argument tells us that

$$\mu = \int_{r_h}^{\infty} \frac{Qe^{-2\alpha\phi}}{b^2} dr \tag{2.33}$$

is given by

$$\mu = D_2(Qe^{-\alpha\phi_0})^{1/2} e^{-\alpha\phi_0} \tag{2.34}$$

where D_2 is again an α dependent coefficient. This gives

$$\rho = D_3 C e^{3\alpha\phi_0} \mu^3 . \tag{2.35}$$

D_3 is an α dependent numerical coefficient (that can be determined in terms of D_1 and D_2). From these relations it follows that the number density satisfies

$$n \propto Q . \tag{2.36}$$

Going to non-zero temperatures, we note that the specific heat is also positive

$$C_v = T(\frac{ds}{dT})_\mu = (2\beta) A_0 C T^{2\beta} \mu^{2-2\beta} . \tag{2.37}$$

The Gibbs-Duhem relation

$$sdT - dP + nd\mu = 0, \tag{2.38}$$

can be used to obtain the pressure. We find (using (2.42) below to fix the temperature-independent part)

$$P = \frac{1}{(2\beta+1)} A_0 C \mu^{2-2\beta} T^{2\beta+1} + \frac{1}{2} D_3 C e^{3\alpha\phi_0} \mu^3 . \tag{2.39}$$

Substituting for P in (2.38) gives the number density,

$$n = \frac{(2-2\beta)}{(2\beta+1)} A_0 C T^{2\beta+1} \mu^{1-2\beta} + \frac{3}{2} D_3 C e^{3\alpha\phi_0} \mu^2 \ . \tag{2.40}$$

From (2.40) we can also compute the susceptibility χ:

$$\chi \equiv (\frac{\partial n}{\partial \mu})_T = (1-2\beta)\frac{(2-2\beta)}{(2\beta+1)} A_0 C T^{2\beta+1} \mu^{1-2\beta} + 3 D_3 C \mu \ . \tag{2.41}$$

Finally the energy density can be obtained using the relation

$$\rho = sT + \mu n - p \tag{2.42}$$

which gives,

$$\rho = \frac{2}{(2\beta+1)} A_0 C \mu^{2-2\beta} T^{2\beta+1} + D_3 C e^{3\alpha\phi_0} \mu^3 \ . \tag{2.43}$$

The near-extremal system a simple equation of state

$$P = \frac{1}{2}\rho \ . \tag{2.44}$$

It is clear from the calculations above that thermodynamic properties of the solution are essentially determined by the Lifshitz-like nature of the extremal near horizon geometry.

2.4 Attractor Behavior

Consider a more general theory with a Lagrangian of the form,

$$S = \int d^4x \sqrt{-g}\big(R - 2\Lambda - 2(\partial\phi_i)^2 - f_{ab}(\phi) F^a F^b - \frac{1}{2}\tilde{f}_{ab}\epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma}\big), \tag{2.45}$$

which has $i = 1 \cdots N$ scalars with standard kinetic energy terms and $a = 1, \cdots M$ gauge fields.

Let us search for an extremal black brane solution to this action with both electric and magnetic charge. In terms of the metric ansatz, (2.7); the gauge fields are given by

$$F^a = f^{ab}(Q_{eb} - \tilde{f}_{bc} Q^c_m)\frac{1}{b^2} dt \wedge dr + Q^a_m dx \wedge dy, \tag{2.46}$$

where Q^a_m and Q_{ea} determine the electric and magnetic charges of the system and f^{ab} is the inverse of the gauge coupling function f_{ab}.

The three equations of motion for the metric and scalars, corresponding to (2.9), (2.10), and (2.11), can be obtained by varying the one-dimensional action,

$$S = \int dr\big(-2a^2 b b'' - 2a^2 b^2 (\partial_r \phi_i)^2 - 2\frac{V_{eff}}{b^2} + \frac{6b^2}{L^2}\big) \tag{2.47}$$

with the effective potential V_{eff} given by

$$V_{eff} = f^{ab}(Q_{ea} - \tilde{f}_{ac}Q_m^c)(Q_{eb} - \tilde{f}_{bd}Q_m^d) + f_{ab}Q_m^a Q_m^b. \tag{2.48}$$

For the special case we studied above, (2.6), $V_{eff} = e^{-2\alpha\phi}Q^2$. We must also impose the following equation as a constraint, corresponding to (2.12):

$$a^2 b'^2 + \frac{1}{2}a^{2\prime}b^{2\prime} = (\phi_i')^2 a^2 b^2 - \frac{V_{eff}}{b^2} + 3\frac{b^2}{L^2} \ . \tag{2.49}$$

In a generic attractor situation, V_{eff} would have a critical point at some finite point in moduli space $\phi_{i*} \neq \infty$. The resulting extremal black brane has a horizon where the scalars are drawn to their critical values, ϕ_{i*}, regardless of their asymptotic values at infinity. $g_{tt} = a^2$ has a second order zero at the horizon, while the metric component b^2 has a non-zero value b_h^2 at the horizon. As a result the near-horizon geometry is of the form $AdS_2 \times R^2$ with an $SO(2,1)$ isometry arising from the AdS_2 factor. The entropy density of the black brane $s \sim b_h^2$. From the constraint (2.49), it follows that b_h^2 is determined by the value of effective potential at the critical point,

$$b_h^4 = \frac{L^2 V_{eff}(\phi_{i*})}{3} \ . \tag{2.50}$$

In contrast in the situation we have studied above, the effective potential is of "run-away" form, $V_{eff} = e^{-2\alpha\phi}Q^2$, with a critical point which lies at infinity. At the critical point, the effective potential vanishes. As a consequence, the entropy of the solution vanishes and the near horizon geometry is Lifshitz-like, where the scalar runs towards its infinite critical value and where the entropy vanishes. It is easy to see that from the equations of motion, (2.9)-(2.12) that the full solution for the metric only depends on the combination $Q^2 e^{-2\alpha\phi_0}$, where ϕ_0 is the asymptotic value of the dilaton. But the attractor mechanism implies that the near-horizon metric is independent of the dilaton ϕ_0. It then also implies that the scaling solution is independent of the charge parameter Q, which is consistent with our previous result (2.17).

This Lifshitz-like near horizon geometry can arise as an attractor in more general situations. As an example, consider a situation with several scalars, and a run-away potential which depends on a linear combination of these scalars,

$$V_{eff} = V_0 e^{-\alpha_i \phi_i} \ . \tag{2.51}$$

After a field redefinition this maps to the single scalar case above. As another example consider an effective potential which has a critical point at a finite value in field space for all but one of the scalars. The one remaining scalar runs away to infinity driving the potential to zero in an exponentially rapid fashion. In this case the same attractor geometry, with appropriate scaling exponents, will arise. Finally consider a situation where there are several scalars with a potential

$$V_{eff} = \sum_i V_i e^{-2\alpha_i \phi_i} \tag{2.52}$$

with all $V_i > 0$. Again the near horizon region is Lifshitz-like:

$$a \sim w, b \sim w^\beta, \phi_i \sim -K_i \log(w) + C_i \ , \tag{2.53}$$

now with

$$\beta = \frac{1}{(1 + 4\sum \frac{1}{\alpha_i^2})} \tag{2.54}$$

$$K_i = \frac{2\beta}{\alpha_i} \; . \tag{2.55}$$

Other kinds of attractors

To complete this discussion let us also consider other possible attractor solutions which can arise. We list some of these possibilities below. We work in the coordinate system (2.7) below with $w = r - r_h$. In all the cases we consider below $a \sim w$ in the near horizon region, so that the g_{tt} component has a second order zero at $w = 0$. Though the scaling solution (2.13) is an exact solution to the equations of motion, the solutions we write down below are not exact solutions; they only describe the leading near-horizon behaviour for the metric and dilaton in these cases.

1) Suppose the effective potential takes the form,

$$V_{eff} = V_0 + V_1 e^{-2\alpha\phi} \tag{2.56}$$

where $V_0, V_1 > 0$. This results in a run-away situation, but the critical value of V is now V_0 and does not vanish. In this case we find that

$$b = b_h + \frac{C_1}{\log(w)} \tag{2.57}$$

$$b_h^4 = \frac{L^2 V_0}{3} \tag{2.58}$$

$$\phi = \frac{1}{2\alpha} \log(-\log(w)). \tag{2.59}$$

Since $a \sim w$ and $b_h \neq 0$ the near horizon geometry is $AdS_2 \times R^2$. Interestingly though the scalar does not become a constant in the near-horizon region going instead to infinity as $w \to 0$.

2) Next consider the case where the potential vanishes at the critical point but as a power-law rather than an exponential:

$$V_{eff} = \frac{V_0}{\phi^p}. \tag{2.60}$$

we find

$$b \sim \frac{1}{\log(w)^{\frac{p}{8}}} \tag{2.61}$$

$$\phi \sim (-\log(w))^{1/2} \; . \tag{2.62}$$

The power law nature of the potential results in b and ϕ varying more slowly than in the Lifshitz-like solution (2.13).

3) We can also contrast this with a potential that vanishes more rapidly than an exponential:

$$V = Q^2 e^{-A(e^{\alpha\phi})}. \tag{2.63}$$

We find

$$b \sim w + \frac{C_1 w}{\log(w)} \tag{2.64}$$

$$\phi \sim \frac{1}{\alpha} \log\left(-\frac{1}{A}\log\left(h(w)\right)\right) \tag{2.65}$$

$$h(w) = \frac{3}{2Q^2\alpha^2} w^4(-\log(w))^{-2} + \dots\ . \tag{2.66}$$

The metric component b is almost linear in w resulting in the near horizon geometry being approximately AdS_4, with a very slowly varying scalar.

4) Finally we can consider a situation where the four-dimensional theory contains a potential which depends on the scalar field. We write the effective one-dimensional Lagrangian as

$$S = \int dr \left(-2a^2bb'' - 2a^2b^2(\partial\phi)^2 - 2\frac{V_{eff}}{b^2} + 6\frac{b^2}{L^2} - 2V_1(\phi)b^2\right) \tag{2.67}$$

where $V_1(\phi)$ is the extra field-dependent potential. There are now several possibilities. Let us only discuss one of these here. If b_h and ϕ_* can be found which solve the two equations, (as would generically be the case)

$$\frac{\partial_\phi V_{eff}(\phi_*)}{b_h^2} + \partial_\phi V_1(\phi_*)b_h^2 = 0 \tag{2.68}$$

and

$$(\frac{3}{L^2} - V_1(\phi_*))b_h^4 = V_{eff}(\phi_*). \tag{2.69}$$

then it is easy to see that an $AdS_2 \times R^2$ near-horizon solution arises, where the metric component b^2, (2.7), takes a constant value b_h^2 and the scalar takes a constant value ϕ_*.

2.5 Conductivity

The power of the AdS/CFT correspondence is that it allows us to calculate properties of the dual quantum field theory beyond thermodynamics. The most natural quantities to study are transport coefficients. These are calculated by considering linearized fluctuations around the background gravitational solution presented above. Formally, via the AdS/CFT dictionary, this corresponds to coupling a conserved global $U(1)$ current operator in the dual CFT to a weak external source and calculating linear response, using the usual Kubo formula, as a two-point function of the current operator. Physically, this corresponds to studying the absorption and reflection of electromagnetic and/or gravitational waves by the black brane geometry.

Here, we calculate the optical conductivity $\sigma(\omega)$ of the extremal black brane background, via the AdS/CFT dictionary. Our approach is based on the conductivity calculation in [21], and we crucially rely on some generalizations of various formula in [83, 84]..

2.5.1 Fluctuations

We first solve for the allowed fluctuations of the gauge field about our solution. Studying the gauge field equation of motion

$$\partial_\mu \left(\sqrt{-g} e^{2\alpha\phi} g^{\mu\lambda} F_{\lambda\sigma} g^{\sigma\nu} \right) = 0, \tag{2.70}$$

and the Einstein's equations, we find that the g_{tx} and A_x fluctuations mix:

$$\partial_r \left(e^{2\alpha\phi} a^2 \partial_r \delta A_x \right) - e^{2\alpha\phi} b^2 F^{(0)}_{rt} \partial_r (b^{-2} \delta g_{tx}) + \omega^2 e^{2\alpha\phi} a^{-2} \delta A_x = 0 \tag{2.71}$$

where $F^{(0)}_{rt}$ is the background gauge field (2.8). Luckily, as was the case in [83, 84, 21], one of the Einstein equations has just the right form to simplify the resulting coupled system – the xt component of the trace-reversed Einstein equations gives

$$R_{xr} = 2e^{2\alpha\phi} \left(-F_{rt} F_{tx} g^{tt} + F_{rx} F_{xt} g^{xt} \right) = 2i\omega e^{2\alpha\phi} F^{(0)}_{rt} a^{-2} \delta A_x \tag{2.72}$$

with

$$R_{xr} = -i\omega \frac{\partial_r \left(g^{xx} \delta g_{tx} \right)}{2 g_{tt} g^{xx}} = -\frac{1}{2} i\omega a^{-2} b^2 \partial_r \left(b^{-2} \delta g_{tx} \right) \tag{2.73}$$

in our coordinate system. Substituting into (2.72) into (2.71) we can obtain a second order differential equation for A_x:

$$a^2 \partial_r \left(a^2 e^{2\alpha\phi} \partial_r \delta A_x \right) = 4a^2 e^{4\alpha\phi} F^2_{rt} \delta A_x - \omega^2 e^{2\alpha\phi} \delta A_x \ . \tag{2.74}$$

Defining the variable $\Psi = f(\phi) \delta A_x$ we find

$$-\Psi'' + V(z)\Psi = \omega^2 \Psi \ . \tag{2.75}$$

Here, $f(\phi) = 2\, e^{\alpha\phi}$ is the dilaton coupling, and primes denote derivatives with respect to the variable z, defined via

$$\frac{\partial}{\partial z} = a^2 \frac{\partial}{\partial r}. \tag{2.76}$$

The potential $V(z)$ is given by

$$V = \frac{f''}{f} + \frac{a^2 Q^2}{b^4 f^2}. \tag{2.77}$$

Let us now use the AdS/CFT dictionary to a formula for the conductivity, in the presence of a weak external electric field. The electromagnetic part of the bulk action is

$$S_{em} = \int d^4x \sqrt{-g} \frac{1}{4} f^2(\phi) F_{\mu\nu} F^{\mu\nu} \tag{2.78}$$

and the metric, in terms of the coordinate z, is given by

$$ds^2 = a^2(-dt^2 + dz^2) + b^2(dx^2 + dy^2) \ . \tag{2.79}$$

Asymptotically (as $z \to 0$), the metric approaches AdS_4 and $a^2 = b^2 = z^{-2}$. The horizon is located at $z = -\infty$.

In the boundary theory, the current $\langle j_x \rangle$ can be obtained by

$$\langle j_x \rangle = \frac{\delta \log(Z)}{\delta A_x} \ . \tag{2.80}$$

The standard AdS/CFT dictionary then tells us that in the bulk,

$$\langle j_x \rangle = f^2(\phi) F_{zx} \Big|_{z \to 0} \ . \tag{2.81}$$

The electric field at the boundary is $E_x = F_{tx}\Big|_{z\to 0}$. Thus studying solutions of the Einstein's equations subject to the boundary condition $F_{tx}\Big|_{z\to 0} = E_x$ allows us to determine the response of the system to an external electric field. Solving the linearized fluctuation equation for the gauge field around our background solution (taking into account the backreaction) gives the (linear) response of the system to a weak external field.

According to the now well-established procedure, e.g., [21], we have to solve the fluctuation equation subject to in-going boundary conditions at the horizon. The solution to the fluctuation equation near the boundary then takes the form

$$A_x = A_x^{(0)} + A_x^{(1)} z + \dots$$

The conductivity (defined via $\sigma = \langle j_x \rangle / E_x$) is then given by:

$$\sigma(\omega) = \frac{i}{\omega} f^2(\phi_0) \frac{A_x^{(1)}}{A_x^{(0)}} \tag{2.82}$$

and is essentially determined by a reflection coefficient in the notation of the Schrödinger problem described above

$$\sigma = \frac{1-\mathcal{R}}{1+\mathcal{R}} \ . \tag{2.83}$$

2.5.2 Conductivity for $\omega \ll \mu$

Solving for the conductivity at arbitrary frequencies analytically is impossible because we only have an analytic expression for the near-horizon part of the solution. Nevertheless, posessing only the near horizon solution, we are still able to extract the low-frequency behaviour of the conductivity using careful matching techniques. This illustrates an important principle – the near-horizon geometry effectively encodes all IR physics of the dual field theory. Our approach is similar in spirit to [7].

We find, using the solution for $e^{\alpha\phi}$ in section 2.2, that in the near-horizon region:

$$V(z) = \frac{c}{z^2} \ . \tag{2.84}$$

It further transpires that the constant c is *independent* of the value of α:

$$c = 2 \ . \tag{2.85}$$

This is similar to the universality seen, in a different context, in §5.3 of [21]. We will nevertheless continue our discussion with arbitrary values of c, for possible use in related problems.

The change of variables from z to w takes the near-horizon region to $z = -\infty$ and the boundary to $z = 0$. Therefore, the scattering problem we wish to study has incoming plane waves with energy ω^2 at 0.

Because the potential has a $\frac{1}{z^2}$ form, the WKB approximation does not apply. However, we can still solve this problem using the method of matched asymptotics, developed in roughly this context in [83]. Define χ via

$$\Psi = \sqrt{\frac{-\pi\omega z}{2}}\chi \ . \tag{2.86}$$

Then the Schrödinger equation satisfied by Ψ becomes

$$z^2\frac{\partial^2\chi}{\partial z^2} + z\frac{\partial\chi}{\partial z} + (z^2\omega^2 - \nu^2)\chi \ = 0 \ , \quad \nu^2 = c + \frac{1}{4} \ . \tag{2.87}$$

Ingoing modes

The AdS/CFT correspondence instructs us to choose modes which are purely ingoing at the horizon. The solutions to the differential equation (2.87) are Hankel functions; the *purely ingoing* solution, at the horizon, is given by

$$\chi = H_\nu^{(1)}(-\omega z) \sim \sqrt{\frac{2}{-\pi\omega z}}\exp\left(-i\omega z - i(\nu + \frac{1}{2})\frac{\pi}{2}\right) \tag{2.88}$$

where after $\sim$ we give the behaviour as $z \to -\infty$. Including time dependence, this yields a wavefunction

$$\psi \sim \exp\left(-i\omega(t+z) - i(\nu + \frac{1}{2})\frac{\pi}{2}\right) \ . \tag{2.89}$$

This is the desired, purely ingoing, mode at the horizon.

Matched Asymptotics

We now solve for the ω-dependence of the resistivity by a method of matched asymptotics in the regime $\omega \ll \mu$. Our strategy is as follows: For small ω, the ω^2 term in the Schrödinger equation is only relevant in the regions where the potential can be made arbitrarily small. This happens at $z \to -\infty$ (the horizon) and at the boundary $z \to 0$. Therefore, *away* from these regions, we can neglect the ω-dependent terms entirely, and use the approximation that $V(z)$ is the dominant term in the Schrödinger equation. We therefore start with the solution of the ω-dominated equation near the boundary, and show that we can continue it (still using a near-boundary approximation) to a solution in the potential-dominated region – potential domination happens even very near the boundary, for sufficiently small ω. We then continue the resulting solution all the way to the near-horizon, while remaining in the potential dominated region – which introduces no new ω dependence. Finally, using the exact near-horizon solution (2.89) and the conservation of flux between infinity and the horizon, we are able to determine the ω-dependence of the coefficient of the reflected wave.

Step 1: Near-boundary analysis

For $r \gg r_{\text{horizon}}$, we have

$$a^2 \sim r^2, \quad z \sim -\frac{1}{r}, \quad V(z) \sim c_2 z^2 \ . \tag{2.90}$$

We can therefore neglect the potential in the Schrödinger equation if we satisfy

$$c_2 z^2 \ll \omega^2 \to r \gg \frac{1}{\omega} \ . \tag{2.91}$$

This close to the AdS boundary, the equation has solutions

$$\Psi(z) = D_1 e^{-i\omega z} \ + \ D_2 e^{i\omega z} \ . \tag{2.92}$$

Now, choose a point z_1 with $z_1^2 \ll \omega^2$. It follows that in the small frequency limit, we also have $|z_1| \ll \frac{1}{\omega}$. Therefore we can Taylor expand our wavefunction (2.92), yielding:

$$\Psi \sim (D_1 + D_2) + i\omega(-D_1 + D_2)z \ . \tag{2.93}$$

Now, still in the near-boundary region, we can choose a point z_2 where the *potential* now dominates the ω^2 term in the Schrödinger equation. This only requires $z_2^2 \gg \omega^2$, and is possible in the near-boundary region for small ω. Let us suppose that while $V(z) \gg \omega^2$ in the vicinity of this point, in truth *both* terms are negligible in the Schrödinger equation. The conditions for this to hold are that:

$$V(z)(\Delta z)^2 \ll 1, \quad \omega^2(\Delta z)^2 \ll 1, \quad \Delta z \equiv |z_2 - z_1| \ . \tag{2.94}$$

These conditions can be satisfied for small frequency; they simply require that $\omega \ll |z_2| \ll 1$.

Then in reaching z_2 from z_1, we can neglect both potential and frequency terms in the Schrödinger equation, which means we can simply use linear extrapolation from z_1 to z_2. Hence:

$$\Psi \sim E_1 + E_2 z, \tag{2.95}$$

and matching with (2.92) we find that

$$E_1 = D_1 + D_2, \quad E_2 = i\omega(D_2 - D_1) \ . \tag{2.96}$$

Step 2: Near-horizon analysis

We will momentarily try to match the wavefunction extrapolated from z_2, to a wavefunction in the near-horizon region. What is the appropriate wave-function there? For z approaching $-\infty$, as we have already discussed,

$$V(z) \sim \frac{c}{z^2}, \ z \sim \frac{-1}{r - r_h} \ . \tag{2.97}$$

We can find points in the near-horizon region where $V(z)$ dominates ω^2; this simply requires $|\omega z| \ll 1$, and is true for arbitrarily large $|z|$ for small enough ω. Let us choose such a point, z_3. In this region, $|\omega z_3| \ll 1$, the Hankel function reduces to

$$\begin{aligned}\Psi \sim \sqrt{-\frac{\pi}{2}\omega z} H_\nu^{(1)}(-\omega z) \quad &\sim \quad \sqrt{-\frac{\pi}{2}\omega z}\left(J_\nu(-\omega z) + i N_\nu(-\omega z)\right) \\ &\sim \quad \sqrt{-\frac{\pi}{2}\omega z} i \frac{-(\nu-1)!}{\pi}\left(\frac{-2}{\omega z}\right)^\nu \; . \end{aligned} \tag{2.98}$$

Step 3: Matching

Now, we need to match the wavefunction (2.95) with coefficients (2.96) to the wavefunction (2.98). This involves using the Schrödinger equation to integrate from the point z_2 (near the boundary) to the point z_3 (near the horizon). The key point, however, is that *in the entire intermediate region, we can neglect the frequency dependence* in the Schrödinger equation. Therefore, the frequency dependence of E_1 and E_2 in (2.96) can be determined from the frequency dependence we see in (2.98). This yields

$$E_1,\ E_2 \sim \omega^{\frac{1}{2}-\nu} \ \rightarrow \ D_1 + D_2 \sim \omega^{\frac{1}{2}-\nu},\ D_2 - D_1 \sim \omega^{-\frac{1}{2}-\nu} \; . \tag{2.99}$$

Next, how do we determine the conductivity σ? The key point, as observed in [21] following [83], is that the exact Schrödinger equation has a conserved flux

$$\mathcal{F} = i\left(\Psi^* \partial_z \Psi - \Psi \partial_z \Psi^*\right) \; . \tag{2.100}$$

Evaluating the frequency dependence close to the horizon, we find

$$\mathcal{F} \sim \omega \; . \tag{2.101}$$

Now at the boundary, we can write [21]

$$\mathcal{F} \sim |D_1 + D_2|^2 \, \omega \left(\mathrm{Re}(\sigma)\right) \; . \tag{2.102}$$

This immediately fixes

$$\mathrm{Re}(\sigma) \sim \omega^{2\nu-1} \; . \tag{2.103}$$

Finally, noting that for all values of α, $\nu = 3/2$, we find

$$\mathrm{Re}(\sigma)|_{\text{dilaton black hole}} \sim \omega^2 \; . \tag{2.104}$$

The ω^2 behaviour of the conductivity, independent of the value of α, is intriguing. We do not have a good understanding for this universal result. Mathematically it arises because the coefficient c in the near-horizon potential always takes the value 2, (2.85). However one gets the feeling that something deeper is at work here which merits further understanding. The same behaviour of the conductivity was also obtained in some cases in [21], and has been proved to hold in general for black branes with a near-horizon AdS_2 geometry in [85, 86]. However, here we find that this behaviour emerges under far more general circumstances.

More general attractors

One might suspect that the universal low-frequency behaviour of the conductivity found above continues to be true even for some of the other classes of attractors considered above. Here we present some additional evidence in support of this, leaving a more detailed analysis for the future.

Consider as an example case 3) which is a limiting situation where the potential vanishes very rapidly. In this case the effective potential takes the form

$$V_{eff}(\phi) = Q^2 \ \mathrm{Exp}\left(-Ae^{\alpha\phi}\right) \ , \tag{2.105}$$

which would arise in a theory where the gauge coupling function f^2 is characterized by

$$\frac{1}{4}f(\phi)^2 = \mathrm{Exp}\left(Ae^{\alpha\phi}\right) \ . \tag{2.106}$$

Expressing everything in terms of z, and substituting the solution (2.64) into the effective Schrödinger potential (2.77), we find

$$V(z) = \frac{2}{z^2}\left(1 + \frac{3}{\log(-z)} + \dots\right) \ . \tag{2.107}$$

The leading order potential near the horizon, where $|z| \to \infty$, is therefore still $2/z^2$. As a result we expect the low frequency conductivity to behave as $\sigma \sim \omega^2$ in this case as well.

2.6 Summary and Discussion

In this chapter, we have shown that charged dilaton black branes in AdS provide a rich playground for studying black hole physics and holographic condensed matter physics. The basic features for the standard charged dilaton branes with gauge-coupling function $f^2 \sim e^{2\alpha\phi}$ are:

• The near-horizon metric of the black holes has a Lifshitz-like symmetry in the metric, with a dynamical critical exponent z that depends on α, although the full background solution breaks the symmetry. This near-horizon structure is universal for black holes of arbitrary charge and asymptotic coupling, at fixed α; this represents a generalization of the attractor mechanism to this class of (much less symmetric) black branes.

• The ground state at finite charge density has vanishing entropy at extremality, and positive specific heat, as expected for a garden-variety condensed matter system. At finite but low temperature, there are plentiful low-energy degrees of freedom. Compared to a 2+1 dimensional CFT where the entropy density (and shear viscosity [3]) scale like T^2 at low temperature, our system has $s \sim T^{2\beta}$ (with a similar behaviour for the viscosity). Since $\beta < 1$, there are *more* low-energy degrees of freedom in these states with finite charge density than would be present in a CFT.

• In the range, $T \ll \omega \ll \mu$ AC conductivity exhibits an intriguing universality and behaves (apart from a delta function at zero frequency) as $\sigma(\omega) \sim c(\alpha)\ \omega^2$ for all values of α.

Chapter 3

Holography of Dyonic Dilaton Black Branes

This chapter is based on work published in [9], completed in collaboration with Kevin Goldstein, Norihiro Iizuka, Shamit Kachru, Sandip Trivedi, and Alexander Westphal.

3.1 Introduction

In this chapter, we continue the study of dilatonic black branes, expanding our discussion to dyonic branes in a more general class of theories involving an axion field as well as a dilaton.

We begin by completing our study of the electrically charged black brane solutions of the previous chapter by calculating the temperature dependence of their AC conductivities at low temperatures (in the range $\omega \ll T \ll \mu$). In this regime, we find that the conductivity is Re $(\sigma) \sim T^2$ (with an additional delta function at $\omega = 0$).

The field theory dual to the theories we are studying has a global conserved $U(1)$ current – the conductivity we calculate is the linear response of the theory, in the presence of a chemical potential μ, to an external electric field coupled to the $U(1)$ current. To further characterize the field theory it is natural to consider linear response in the presence of not only a chemical potential but also a background magnetic field. This also corresponds to turning on a magnetic field in the gravity dual, which is achieved by considering black brane solutions posessing not only electric but also magnetic charge. See, e.g., [5, 11].

Once we allow for a bulk magnetic field it is natural to consider bulk theories containing not only a dilaton but also an axion (which will be denoted as $\lambda_1(x)$)[1]. A particularly interesting case is when the bulk theory has an $SL(2, R)$ symmetry.[2] Here, the behavior of a system carrying both electric and magnetic charges is related to the purely electric case by an $SL(2, R)$ transformation. Under an $SL(2, R)$ transformation the dilaton-axion, $\lambda = \lambda_1 + ie^{-2\phi}$, transforms like

$$\lambda \to \frac{\tilde{a}\lambda + b}{c\lambda + d}.$$

[1]It is reasonable to believe that varying the boundary value of the axion corresponds to adjusting the value of a Chern-Simons coupling in the boundary theory [87]; we briefly expand on this comment below.

[2]This symmetry is expected to only be approximate and would receive corrections beyond the classical gravity approximation; for instance, in many quantum string theories, it is broken to $SL(2, Z)$ non-perturbatively.

We find that the two complex combinations of the conductivity[3] $\sigma_\pm = \sigma_{yx} \pm i\sigma_{xx}$ also transform in exactly the same way,

$$\sigma_\pm \to \frac{\tilde{a}\sigma_\pm + b}{c\sigma_\pm + d}$$

. An interesting feature of our results is that the DC Hall conductivity agrees with the attractor value of the axion, in accord with expectations that the axion determines the coefficient of the Chern-Simons coupling in the boundary theory (which in turn determines the Hall conductivity.)

We also calculate the thermoelectric and thermal conductivity for a general system carrying both electric and magnetic charges. These are related to the electric conductivity by Weidemann-Franz type relations which are very analogous to those obtained in the non-dilatonic case [5, 11].

Finally we consider a more general class of bulk theories containing a dilaton-axion but without $SL(2, R)$ symmetry. For some range of parameters we find that the deep infra-red geometry is an attractor and changing the asymptotic value of the axion does not lead to a change in this geometry. Outside this parametric range, however, the attractor behavior appears to be lost and a small change in the asymptotic value of the axion results in a solution which becomes increasingly different in the infrared.

3.2 Review of Earlier Results

Here we very briefly summarize some of the results of the previous chapter and [8]. Consider a four-dimensional system consisting of a dilaton coupled to a gauge field and gravity with action

$$S = \int d^4x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - e^{2\alpha\phi}F^2 - 2\Lambda\right) \ . \tag{3.1}$$

$\Lambda = -\frac{3}{L^2}$ is the cosmological constant. We will often set $L = 1$ in the discussion below.

The metric of a black brane has the form

$$ds^2 = -a(r)^2\ dt^2 + a(r)^{-2}\ dr^2 + b(r)^2\ (dx^2 + dy^2)\ . \tag{3.2}$$

For an electrically charged brane the gauge field is

$$e^{2\alpha\phi}F = \frac{Q}{b(r)^2}\ dt \wedge dr. \tag{3.3}$$

The extremal black brane is asymptotically AdS_4 and characterized by two parameters, the charge Q and ϕ_0 - the asymptotically constant value of the dilaton. In the extremal case, the near-horizon region is universal and independent of both these parameters, due to the attractor mechanism.[4] The metric is of the Lifshitz form [13][5]

$$ds^2 = -(C_2 r)^2 dt^2 + \frac{dr^2}{(C_2 r)^2} + r^{2\beta}(dx^2 + dy^2), \tag{3.4}$$

[3] Note that the conductivities σ_{xx}, σ_{yx}, are frequency dependent and hence complex so $\sigma_\pm$ are not complex conjugates of each other.

[4] The curvature scale in the near-horizon region is set by the cosmological constant Λ.

[5] See also [88].

with dynamical exponent

$$z = \frac{1}{\beta}. \tag{3.5}$$

The near-horizon solution is valid when

$$r \ll \mu \tag{3.6}$$

where $\mu \propto \sqrt{Q}$ is the chemical potential.

The dilaton in the near-horizon region is

$$\phi = -K \log(r) \ . \tag{3.7}$$

The constants which appear in the metric and dilaton above are given in terms of α, the coefficient in the dilaton coupling (3.1):

$$C_2^2 = \frac{6}{(\beta+1)(2\beta+1)}, \ \beta = \frac{(\frac{\alpha}{2})^2}{1+(\frac{\alpha}{2})^2}, \quad K = \frac{\frac{\alpha}{2}}{1+(\frac{\alpha}{2})^2}. \tag{3.8}$$

This class of solutions, but with different asymptotics than those of interest to us, was discussed in [14] (the solutions there were asymptotically Lifshitz, and have strong coupling at infinity; for other asymptotically Lifshitz black hole solutions, see [89, 90, 91, 92, 93, 94, 95, 96, 97, 98]).

The entropy of the extremal black brane vanishes. For a near-extremal black brane the temperature dependence of entropy and other thermodynamic quantities is essentially determined by the near-horizon region. (For a careful discussion of how the global embedding affects the thermodynamics, see appendix A of [99]; see also the recent paper [100] for a discussion of how the non-extremal branes embed into AdS.).

The bulk theory above is dual to a $2+1$ dimensional boundary theory which is a CFT with a globally conserved $U(1)$ symmetry. The electrically charged black brane is dual to the boundary theory in a state with constant charge density determined by Q.

The black brane geometry can be used to calculate transport coefficients in the boundary theory. In particular, the real part of the longitudinal electric conductivity (Re $(\sigma) \equiv \sigma_{xx} = \sigma_{yy}$) at zero temperature and small frequency[6] is found to be

$$\text{Re } (\sigma) = C \frac{\omega^2}{\mu^2}. \tag{3.9}$$

Here C is a constant which depends on α and ϕ_0. We note that the frequency dependence of Re (σ) is universal and is independent of α. The conductivity is dimensionless in $2+1$ dimensions. This fixes the dependence on μ - the chemical potential- once the dependence on ω is known.

More generally, at finite temperature and frequency, σ is a function of two dimensionless variables $\sigma(\frac{T}{\mu}, \frac{\omega}{\mu})$. Eq.(3.9) gives the leading dependence when $T \ll \omega \ll \mu$. We also note that in the purely electric case the Hall conductivity σ_{xy} vanishes.

[6]There is a delta function Drude peak at $\omega = 0$ in addition which we have subtracted.

3.3 The DC Conductivity

In this section we calculate the leading behavior of the conductivity, σ, when

$$\omega \ll T \ll \mu. \tag{3.10}$$

Our analysis will closely follow the discussion in the previous chapter and section 3 of [8] (which itself used heavily the results of [21]). We consider a perturbation in A_x, which mixes with the metric component g_{xt}, impose in-going boundary conditions at the horizon, and then carry out a matched asymptotic expansion which determines the behavior near the boundary and hence the conductivity. We skip some of the details here and emphasize only the central points.[7]

The leading behavior of the conductivity in the parametric range (3.10) will turn out to be

$$\text{Re }(\sigma) = C'\frac{T^2}{\mu^2}\ . \tag{3.11}$$

This is independent of ω. The DC conductivity defined as the limit $\omega \to 0$ of the above formula then just gives (3.11) as the result. Actually there is an additional delta function contribution at $\omega = 0$; we will comment on this more in the following subsection. C' in (3.11) is a constant that depends on ϕ_0.

We begin by observing that the variable

$$\Psi = f(\phi)A_x \tag{3.12}$$

satisfies a Schrödinger equation,

$$-\Psi'' + V(z)\Psi = \omega^2\Psi. \tag{3.13}$$

Here, $f(\phi) = 2\,e^{\alpha\phi}$ is the dilaton coupling, as discussed in eq.(3.10) of [8], and primes denote derivatives with respect to the variable z, defined as

$$\frac{\partial}{\partial z} = a^2\frac{\partial}{\partial r}. \tag{3.14}$$

The potential $V(z)$ is given by

$$V = \frac{f''}{f} + \frac{a^2Q^2}{b^4f^2}. \tag{3.15}$$

In the near-boundary region, Ψ takes the form

$$\Psi = (D_1 + D_2) + i\omega(-D_1 + D_2)z. \tag{3.16}$$

The resulting flux is

$$\mathcal{F} \sim |D_1 + D_2|^2\omega\text{Re }(\sigma). \tag{3.17}$$

[7] The result of this section has also been obtained in [101], which appeared while our paper was being readied for publication. Other related papers which appeared recently include [102, 103, 104, 105, 106, 107].

We are interested here in a slightly non-extremal black brane. This has a near-horizon metric

$$ds^2 = -C_2^2 r^2 \left(1 - (\frac{r_h}{r})^{2\beta+1}\right) dt^2 + \frac{dr^2}{C_2^2 r^2 \left(1 - (\frac{r_h}{r})^{2\beta+1}\right)} + r^{2\beta}(dx^2 + dy^2). \tag{3.18}$$

The temperature is

$$T \sim r_h. \tag{3.19}$$

The dilaton is the same as in the extremal case. The near-horizon form of the metric above is valid for $r \ll \mu$. The temperature dependence of the conductivity is essentially determined by the near-horizon region, as long as $\frac{T}{\mu} \ll 1$. This is similar to what happens for the frequency dependence when $\frac{\omega}{\mu} \ll 1$.

In the near-horizon region r_h is the only scale, as we can see from (3.18). It is therefore convenient, in the discussion below, to rescale variables by appropriate powers of r_h. We define

$$\hat{r} = \frac{r}{r_h} \tag{3.20}$$

$$\hat{a}^2 \equiv \frac{a^2}{r_h^2} = C_2^2 \hat{r}^2 \left(1 - \frac{1}{\hat{r}^{2\beta+1}}\right) \tag{3.21}$$

and

$$\frac{\partial}{\partial \hat{z}} \equiv \frac{1}{r_h}\frac{\partial}{\partial z} = \hat{a}^2 \frac{\partial}{\partial \hat{r}} \ . \tag{3.22}$$

The Schrödinger equation then becomes,

$$-\frac{d^2\Psi}{d\hat{z}^2} + \hat{V}\Psi = \frac{\omega^2}{r_h^2}\Psi \tag{3.23}$$

where the rescaled potential, $\hat{V}$, is dependent on the rescaled variable $\hat{z}$ alone without any additional dependence on r_h .

Very close to the horizon, $\hat{V}$ goes to zero and we have

$$\psi \sim e^{-i\omega(t+z)} = e^{-i\omega t} e^{-i(\frac{\omega}{r_h}\hat{z})} \tag{3.24}$$

resulting in the flux

$$\mathcal{F} \sim \omega. \tag{3.25}$$

From (3.17), (3.25) we see that the conductivity is

$$\text{Re } (\sigma) \sim \frac{1}{|D_1 + D_2|^2}. \tag{3.26}$$

Now, consider the region of the near-horizon geometry where

$$\frac{\mu}{T} \gg \hat{r} \gg 1 \ . \tag{3.27}$$

Since the temperature is small (3.10), these conditions are compatible. In this region the temperature dependent terms in the metric are subdominant and $a^2 \simeq C_2^2 r^2$. Eq.(3.22) then leads to

$$\hat{z} = -\frac{1}{C_2^2 \hat{r}} \tag{3.28}$$

and (3.15) to a potential,

$$\hat{V} = \frac{c}{\hat{z}^2}, \tag{3.29}$$

with the constant

$$c = 2. \tag{3.30}$$

Now since the frequency is even smaller than the temperature, (3.10), $\omega/T \ll 1$ and (3.27) and (3.19) imply that

$$\hat{r} \gg \frac{\omega}{r_h}. \tag{3.31}$$

In terms of z this becomes

$$\frac{1}{\hat{z}^2} \gg \left(\frac{\omega}{r_h}\right)^2. \tag{3.32}$$

It follows that the frequency term in the Schrödinger equation (3.23) is subdominant compared to the potential term in this region. The resulting solution becomes

$$\Psi \simeq \hat{z}^{1/2} \left(\frac{a_1}{\hat{z}^\nu} + b_1 \hat{z}^\nu\right) \tag{3.33}$$

with

$$\nu = \sqrt{c + \frac{1}{4}}. \tag{3.34}$$

From the condition $\hat{r} \gg 1$ and (3.28) we see that in this region

$$|\hat{z}| \ll 1. \tag{3.35}$$

As a result, the first term on the RHS of (3.33) dominates[8] giving

$$\Psi \sim a_1 (r_h z)^{\frac{1}{2}-\nu} \ . \tag{3.36}$$

Here we have used the fact that $\hat{z} = r_h z$.

We have seen above that once r lies in the region which meets the condition (3.27) both the temperature and frequency effects can be neglected. Moving outwards towards the boundary this continues to be true all the way till the near boundary region. This region is described in Step 1 of section 2.5.2 in the previous chapter. As a result, one gets

$$D_1 \sim D_2 \sim r_h^{\frac{1}{2}-\nu} \ . \tag{3.37}$$

[8]This would not be true if a_1 was suppressed compared to b_1 by a power of ω. However, this does not happen, as we discuss further in Appendix A of this chapter.

From (3.26), (3.19), (3.30) and (3.34), this gives

$$\text{Re }(\sigma) \sim r_h^{2\nu-1} \sim T^{2\nu-1} \sim T^2. \tag{3.38}$$

The dependence on μ then follows from dimensional analysis, leading to (3.11).

Finally we note that it is simple to see that the Hall conductivity continues to vanish at finite temperature as well.

3.3.1 The pole in Im (σ) and related delta function in Re (σ)

The real part of σ has a delta function contribution at $\omega = 0$, which arises because the system has a net charge and it is transported in a momentum conserving manner. A Kramers-Kronig relation relates the delta function to a pole in the imaginary part of σ. It will be important to keep track of this pole and the related delta function when we turn to the discussion of the system in a magnetic field, so let us discuss it in some more detail here.

As discussed in section 2.5 of the previous chapter, following [21], the conductivity is given in terms of the reflection coefficient $\mathcal{R}$ by

$$\sigma = \frac{1-\mathcal{R}}{1+\mathcal{R}} \tag{3.39}$$

(the extra term in eq.(3.12) of [8] drops out since $f'(0)$ vanishes like z^3 towards the boundary).

Now in the notation of section 3.2 of [8] close to the boundary Ψ is

$$\Psi = D_1 e^{-i\omega(t+z)} + D_2 e^{-i\omega(t-z)}, \tag{3.40}$$

giving

$$\sigma = \frac{D_1 - D_2}{D_1 + D_2}. \tag{3.41}$$

The coefficients D_1, D_2 can be related to E_1, E_2 which govern the solution in the not-so near boundary region. This region is defined in Step 1 of section 2.5.2 in the previous chapter. and corresponds to taking $|\omega| \ll z \ll 1$. The coefficients E_1, E_2 are defined by

$$D_1 + D_2 = E_1, \quad D_1 - D_2 = i\frac{E_2}{\omega}, \tag{3.42}$$

giving from (3.41)

$$\sigma = i\frac{E_2}{E_1}\frac{1}{\omega}. \tag{3.43}$$

E_1 and E_2 are obtained by starting from the near horizon region where in-going boundary conditions are imposed and integrating out towards the boundary. The zero temperature, leading order solution in the near-horizon region, is of the form $\psi = Cz^{1/2-\nu}$, as discussed in section 2.5. Integrating this out towards the boundary gives E_2/E_1 to be real and of order unity in units of the chemical potential. Similarly, at non-zero temperature in the parametric range (3.10), the solution in the near-horizon region (3.27) is given by (3.36). Integrating out towards the boundary again gives E_2/E_1 to be real and of order unity. Thus we learn that near $\omega = 0$

$$\text{Im}\ (\sigma) = C'' \frac{\mu}{\omega} \tag{3.44}$$

where C is a coefficient of order unity and we have restored the μ dependence on dimensional grounds. As a result, there is a pole at $\omega = 0$ in Im (σ) and hence a delta function in Re (σ) at $\omega = 0$.

In the presence of disorder the frequency dependence changes, $\omega \to \omega + i/\tau_{imp}$ [5], and the pole acquires an imaginary part. The delta function peak in Re (σ) is therefore broadened out, as will be discussed further in section 3.6.

3.4 Purely Magnetic Case

Next, as a warm-up for general dyonic branes, we consider the case of a black brane which carries only magnetic charge. The action is given by (3.1), but we are now interested in the case where the gauge field strength is

$$F = Q_m dx \wedge dy. \tag{3.45}$$

It is easy to see that the equations of motion for the system are invariant under a duality transformation which keeps the metric invariant (this is the Einstein frame metric) and takes

$$\phi \to -\phi, \quad F_{\mu\nu} \to e^{2\alpha\phi} \tilde{F}_{\mu\nu}. \tag{3.46}$$

Here

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{3.47}$$

So we see that the duality transformation takes us from the purely electric case (3.3), to the purely magnetic one (3.45). The value of Q_m is

$$Q_m = Q. \tag{3.48}$$

As a result, the metric for the extremal magnetic case in the near horizon region is still of the Lifshitz form (3.4). To avoid confusion we denote the dilaton after duality by ϕ' in the subsequent discussion; it is given by

$$\phi' = K \log r \tag{3.49}$$

where the constants which appear in the metric and in the dilaton continue to be given by (3.8). The gauge coupling is $(g')^2 = e^{-2\alpha\phi'}$. From (3.49), (3.8) we see that the theory now gets driven to strong coupling, $(g')^2 \to \infty$, near the horizon, and if a string embedding is possible this would mean that quantum loop effects would get important near the horizon. By considering a slightly non-extremal black brane such effects can be controlled.

The behavior of the dilaton can also be understood in terms of the effective potential [15]. In general, with electric and magnetic charges the effective potential is (from section 2.4 of the previous chapter and eq.(2.19) of [8]):

$$V_{eff} = e^{-2\alpha\phi} Q_e^2 + e^{2\alpha\phi} Q_m^2 \ . \tag{3.50}$$

Since after duality, $Q_e = 0, Q_m = Q$, we get,

$$V_{eff} = Q^2 e^{2\alpha\phi'} \tag{3.51}$$

so that the minimum does indeed lie at $e^{2\alpha\phi'} \to 0$, or equivalently $e^{-2\alpha\phi'} \to \infty$.

In mapping the magnetic case to the boundary theory it is best to think of weakly gauging the global $U(1)$ symmetry of the boundary theory. Then the magnetic case corresponds to turning on a constant magnetic field in the boundary theory. The electric-magnetic duality therefore has an interesting consequence. In the electric case, the electric field is a normalizable mode and corresponds to a state in the boundary theory at constant number density or chemical potential. In contrast, in the magnetic case, the magnetic field is a non-normalizable mode and corresponds to changing the Lagrangian of the boundary theory.

The metric in the slightly non-extremal case is also unchanged by duality and hence given in the near-horizon region by (3.18). We now elaborate on the resulting thermodynamics.

3.4.1 Thermodynamics

Let us begin by briefly reviewing the purely electric case. From the Maxwell term in the action

$$S_{em} = -\int d^4x\sqrt{-g}e^{2\alpha\phi}F_{\mu\nu}F^{\mu\nu} \tag{3.52}$$

using standard techniques in AdS/CFT and the definition of Q, (3.3), we learn that the the charge density n in the boundary theory is

$$n = 4Q. \tag{3.53}$$

A purely electric system satisfies the thermodynamic relation

$$TdS = dE + pdV - \mu dN \ . \tag{3.54}$$

From this relation, using electric-magnetic duality, one can obtain the thermodynamic quantities in the magnetic case. For this purpose it is convenient to take the independent thermodynamic variables in the electric case to be (E, V, T, n), since these can be mapped directly to the independent variables (E, V, T, Q_m) in the magnetic case. Here Q_m is the magnetic field. (The magnetic field is usually denoted by H or B, but Q_m is more natural for us in view of the duality transformation.) Since the Einstein frame action is duality invariant (E, V, T) are left unchanged in going from the electric to the magnetic case. And from (3.53) and (3.48) it follows that $n \to 4Q_m$. Thus, the four independent variables can be easily mapped to one another.

Expressing the number $N = nV = 4QV = 4Q_mV$ we get from (3.54) in the electric case that

$$TdS = dE + (p - 4\mu Q)dV - 4\mu V dQ_m. \tag{3.55}$$

Comparing (3.55) with the standard thermodynamic relation in the purely magnetic case (as discussed in e.g. Reif, *Fundamentals of Statistical and Thermal Physics*, 11.1.7)

$$TdS = dE + pdV + MdH \ , \tag{3.56}$$

and noting that the magnetic field is Q_m in our notation, we get that the magnetization is

$$M = -4\mu V \tag{3.57}$$

and the pressure in the magnetic case is

$$p_{mag} = p_{el} - 4\mu Q = p_{el} + \frac{MH}{V}. \tag{3.58}$$

In the electric case the chemical potential is a function of the energy density ρ, T, n, $\mu(\rho, T, n)$. In the formulae above for the magnetic case, (3.57), (3.58), the chemical potential should now be interpreted as a function of ρ, T, Q_m given by $\mu(\rho, T, 4Q_m)$.

It is worth discussing the extremal situation in the magnetic case further. The energy density (see eq.(2.52) of [8]) is given by

$$\rho = CQ^{3/2}e^{-3\alpha\phi_0/2} = C(V_{eff0})^{3/4} \tag{3.59}$$

where we have used the definition of the effective potential in (3.50). The subscript "0" on V_{eff} indicates that it is to be evaluated at ∞, where the dilaton takes value ϕ_0.

The chemical potential is

$$\mu = \frac{\partial\rho}{\partial n} = \frac{3}{8}CQ^{1/2}e^{-3\alpha\phi_0/2} = \frac{3}{8}C(Q_m)^{1/2}e^{3\alpha\phi_0'/2} \tag{3.60}$$

where we have used (3.48) and (3.46). We see from (3.57) that the magnetization is opposite to the magnetic field. As a result, the susceptibility for this system is negative, and the theory is diamagnetic.

Using $p_{el} = \rho/2$, ([8] eq.(2.53)), the pressure in the magnetic case is

$$p_{mag} = -\rho = -CH^{3/2}e^{3\alpha\phi_0'/2}\ . \tag{3.61}$$

It seems puzzling at first that that this is negative, since one would expect the boundary theory to be stable. This turns out to be a familiar situation in magnetohydrodynamics, see the discussion around eq.(3.10) in [5]. In the presence of a magnetic field the pressure and spatial components of stress energy are different and related by

$$T^{xx} = T^{yy} = p_{mag} - \frac{MH}{V}. \tag{3.62}$$

Stability really depends on the sign of T^{xx}, which determines the force acting on the system. From (3.58), we see that $T^{xx} = p_{el}$, and is thus positive.[9]

3.4.2 Controlling the flow to strong coupling

We saw above, (3.51), that for the magnetic case $e^{2\alpha\phi'} \to 0$ and thus the gauge coupling $g^2 = e^{-2\alpha\phi'}$ gets driven to strong coupling at the horizon. In a string theory embedding one would expect the string coupling to become large and thus quantum corrections to become

[9]In fact this had to be true since T^{xx}, T^{yy} are duality invariant and in the electric case $p_{el} = T^{xx} = T^{yy}$.

important near the horizon. To control these corrections one can consider turning on a small temperature and dealing with the near-extremal brane instead. From (3.48), (3.19), and (3.8) we see that if the temperature is $T \sim r_h$ the coupling at the horizon is

$$e^{-2\alpha\phi'} \sim \frac{1}{T^{4\beta}} \; . \tag{3.63}$$

The only other dimensionful quantity in the boundary theory is the magnetic field, so the dependence on magnetic field can be fixed by dimensional analysis. An explicit bulk analysis also shows that this dependence is correct. In addition there is a dependence on the asymptotic value of the dilaton ϕ'_0. It is easy to see that ϕ'_0 only enters in the combination $Q_m e^{\alpha\phi'_0}$ with the magnetic field and as $(\phi' - \phi'_0)$ with the varying dilaton. This is enough to fixed the ϕ'_0 dependence of (3.63) and we get

$$e^{-2\alpha\phi'} \sim e^{-2\alpha\phi'_0} \left(\frac{Q_m e^{\alpha\phi'_0}}{T^2} \right)^{2\beta} . \tag{3.64}$$

For the temperature to be small and the brane to be near-extremal,

$$T^2 \ll Q_m e^{\alpha\phi'_0}. \tag{3.65}$$

Thus to make $e^{-2\alpha\phi'} \ll 1$ we need to adjust the asymptotic value of dilaton and start with a theory which is at very weak coupling

$$e^{-2\alpha\phi'_0} \ll \left(\frac{T^2}{Q_m e^{\alpha\phi'_0}} \right)^{2\beta} . \tag{3.66}$$

Once this is done the coupling will continue to be small all the way to the horizon.

3.4.3 Dyonic case with only dilaton

Most of this section has dealt with the purely magnetic case. Below we will turn to a dyonic system with an axion. Before doing so though let us briefly discuss the dyonic case in the presence of only a dilaton without an axion. From (3.50) we see that the dilaton now has the attractor value ϕ_* with,

$$e^{2\alpha\phi_*} = \left| \frac{Q_e}{Q_m} \right| . \tag{3.67}$$

From the equations of motion it then follows that the metric component b^2, (3.2), at the horizon is

$$b_h^2 \sim \sqrt{V_{eff}(\phi_*)} \sim \sqrt{|Q_e Q_m|}. \tag{3.68}$$

The resulting entropy is then

$$s \propto b_h^2 / G_N \sim C\sqrt{|Q_e Q_m|} \tag{3.69}$$

where $C \sim L^2/G_N$ is the central charge of the AdS_4. As has been discussed above the purely electric case has no ground state degeneracy. Once a magnetic field is also turned on we see that such a degeneracy does arise. By itself this is not surprising. However, the resulting entropy formula, (3.69), is quite intriguing and understanding it better should provide important clues for the microscopic dual of the purely dilatonic case.

3.5 The $SL(2,R)$ Invariant Case

In this section we discuss a theory which has $SL(2,R)$ duality symmetry, in the presence of an axion, with action[10]

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - 2(\partial\phi)^2 - \frac{1}{2}e^{4\phi}(\partial\lambda_1)^2 - e^{-2\phi}F^2 - \lambda_1 F\tilde{F}\right]. \tag{3.70}$$

Comparing with (3.1) we see that the gauge coupling function here corresponds to taking $\alpha = -1$. We will mostly follow the notation of [22] below (see also [23]) and denote the complexified dilaton-axion by

$$\lambda = \lambda_1 + i\lambda_2 = \lambda_1 + ie^{-2\phi}. \tag{3.71}$$

It is easy to see that under an $SL(2,R)$ transformation

$$M = \begin{pmatrix} \tilde{a} & b \\ c & d \end{pmatrix} \tag{3.72}$$

which takes

$$F_{\mu\nu} \to F'_{\mu\nu} = (c\lambda_1 + d)F_{\mu\nu} - c\lambda_2\tilde{F}_{\mu\nu} \tag{3.73}$$

and

$$\lambda \to \lambda' = \frac{\tilde{a}\lambda + b}{c\lambda + d} \tag{3.74}$$

while keeping the metric invariant, the equations of motion are left unchanged. (This is discussed for example in [23] around eq.(18) with $(ML)_{ab} \to -1$). Note that we are denoting $M_{11} = \tilde{a}$ (as the axion field is often called a in the literature) and the axion by λ_1 to avoid confusion. (The metric element $b(r) = g_{xx}$ will always contain an explicit r-dependence, so no confusion should arise with the matrix element $M_{12} = b$.) Also, since M is an element of $SL(2,R)$

$$\tilde{a}d - bc = 1. \tag{3.75}$$

Thus starting from the purely electric case where only the dilaton is non-trivial and carrying out a general duality transformation, we can obtain a dyonic brane with both axion and dilaton excited. In the discussion below we will follow the conventions established above of referring to parameters obtained after duality with a prime superscript.

The starting electric brane is characterized by four parameters: a mass M, a charge Q, and asymptotic values of the dilaton and axion, $\lambda_{20} \equiv e^{-2\phi_0}$, $\lambda_{10} \equiv \lambda_1(\infty)$. The axion is radially constant. The $SL(2,R)$ transformation adds three additional parameters[11], resulting in a 7 parameter set of solutions. Two of these parameters are redundant, though, since the general dyonic brane solution only has only 5 independent parameters: $M', Q'_e, Q'_m, \lambda'_{20}, \lambda'_{10}$. This redundancy can be removed by setting $\lambda_{10} = 0$ in the electric case, and also setting $Q = 1$.[12] In the discussion below we will set $\lambda_{10} = 0$, but not necessarily set $Q = 1$.

[10] In our conventions $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\kappa}F_{\rho\kappa}$ and $\epsilon^{\mu\nu\rho\sigma}$ has a factor of $\frac{1}{\sqrt{-g}}$ in its definition, thereby making the axionic coupling independent of the metric. We have chosen conventions $\epsilon_{trxy} > 0$.

[11] $\det(M) = 1$ so there is one constraint among the 4 matrix elements.

[12] More correctly the scaling symmetry allows one to set $|Q| = 1$.

The gauge field can be written in terms of the electric and magnetic charges as follows

$$F' = \frac{(Q'_e - Q'_m \lambda'_1)}{b(r)^2} (\lambda'_2)^{-1} dt \wedge dr + Q'_m dx \wedge dy \; . \tag{3.76}$$

It can be seen that Q'_e, Q'_m being constant solves the gauge field equations of motion and Bianchi identities. From (3.76) we see that

$$F'_{xy} = Q'_m. \tag{3.77}$$

Using (3.73) this gives,

$$Q'_m = -c\lambda_2 \tilde{F}_{xy} = cQ. \tag{3.78}$$

Similarly from (3.76) we see that

$$\lambda'_2 F'_{tr} = \frac{(Q'_e - \lambda'_1 Q'_m)}{b(r)^2}. \tag{3.79}$$

And (3.73) now gives

$$F'_{tr} = (c\lambda_1 + d)F_{tr} - c\lambda_2 \tilde{F}_{tr} = dF_{tr} = d\frac{Q}{\lambda_2 b(r)^2} \tag{3.80}$$

where we have used (3.78) and the fact that $\lambda_1 = 0$ and $\tilde{F}_{tr} = 0$ in the electric case. Together these imply

$$Q'_e = \left(\frac{\lambda'_2}{\lambda_2} d + \lambda'_1 c \right) Q. \tag{3.81}$$

Using (3.74), and relation $\tilde{a}d - bc = 1$ then gives

$$Q'_e = \tilde{a}Q. \tag{3.82}$$

It is now easy to see that the effective potential, which is given by

$$V'_{eff} = (Q'_e - Q'_m \lambda'_1)^2 (\lambda'_2)^{-1} + (Q'_m)^2 \lambda'_2 \; , \tag{3.83}$$

is in fact duality invariant and thus equal to its value in the purely electric frame,

$$V_{eff} = \frac{Q^2}{\lambda_2}. \tag{3.84}$$

Thermodynamic quantities of a system carrying electric charge in a magnetic field satisfy the relation

$$TdS = dE + pdV - \mu dN + MdQ_m \; . \tag{3.85}$$

We will be particularly interested in the extremal case where the TdS term vanishes. Writing $E = \rho V, N = nV$ we get in this case,

$$(d\rho - \mu dn + \frac{M}{V} dQ_m)V + (\rho - \mu n + p)dV = 0 \; . \tag{3.86}$$

From this it follows that both,

$$d\rho - \mu dn + \frac{M}{V} dQ_m = 0 \tag{3.87}$$

and

$$\rho - \mu n + p = 0. \tag{3.88}$$

We are interested in applying these relations to the dyonic case obtained after duality. The energy density is duality invariant, since it can be extracted from the Einstein frame metric which is duality invariant. Thus we get,

$$\rho' = \rho = C(V_{eff0})^{3/4} = C\left[(Q'_e - Q'_m\lambda'_{10})^2(\lambda'_{20})^{-1} + (Q'_m)^2\lambda'_{20}\right]^{3/4} \ . \tag{3.89}$$

The subscript "0" on V_{eff} and the moduli indicates that the effective potential must be evaluated at $r = \infty$ where the moduli take values $\lambda'_{20} \equiv e^{-2\phi'_0}, \lambda'_{10}$. Straightforward manipulations then give us that

$$\mu' = \frac{1}{4}\frac{\partial\rho'}{\partial Q'_e} = \frac{3C}{8}(V_{eff0})^{-1/4}\left(\frac{Q'_e - \lambda'_{10}Q'_m}{\lambda'_{20}}\right) \tag{3.90}$$

where we have used the fact that $n' = 4Q'_e$. The magnetization per unit volume is

$$\frac{M'}{V} = -\frac{\partial\rho'}{\partial Q'_m} = -\frac{3C}{2(V_{eff0})^{1/4}\lambda'_{20}}\left[Q'_m(\lambda'^2_{20} + \lambda'^2_{10}) - \lambda'_{10}Q'_e\right] \tag{3.91}$$

and the pressure is

$$p' = \mu' n' - \rho' = -\frac{C}{(V_{eff0})^{1/4}\lambda'_{20}}\left[(Q'_m)^2(\lambda'^2_{20} + \lambda'^2_{10}) - \frac{1}{2}(Q'^2_e + \lambda'_{10}Q'_eQ'_m)\right] \ . \tag{3.92}$$

In (3.90)-(3.92) the moduli take their values at infinity. From (3.91) it follows that the susceptibility is negative, and thus the system is diamagnetic. From (3.92) we see that the pressure can be positive or negative. The stress energy tensor component $T^{xx} = T^{yy} = \rho/2$ and is always positive.

Finally, we discuss the compressibility of this system. This is defined to be

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_{TQ_mN} \ . \tag{3.93}$$

The partial derivative on the RHS is to be evaluated at constant temperature T, magnetic field Q_m and total number $N = Vn$. For a system of fermions which has precisely enough particles to fill an integer number of Landau levels, reducing the volume while keeping the magnetic field Q_m fixed would change the available number of states in the occupied Landau levels. But since the total number of fermions is not being changed in the process, and there is a large gap to the next available Landau level, this cannot happen without significant energetic cost, and as a result the compressibility vanishes. This happens for example in quantum Hall systems. For our case, from (3.87) (3.88) we have that

$$\frac{\partial p}{\partial V}\Big|_{TQ_mN} = n\frac{\partial\mu}{\partial V}\Big|_{TQ_mN} = n\frac{\partial\mu}{\partial n}\Big|_{TQ_m}\frac{\partial n}{\partial V}\Big|_N. \tag{3.94}$$

This gives

$$\kappa = \frac{1}{n^2}\left(\frac{\partial n}{\partial \mu}\right)\Big|_{TQ_m}. \tag{3.95}$$

From the expression for μ' (3.90) it is easy to see that $(\frac{\partial \mu'}{\partial n'})\Big|_{TQ'_m}$ cannot go to infinity for finite V_{eff} and non-vanishing λ_{20}, and thus the compressibility cannot vanish except in extreme limits. So the system at hand cannot become incompressible, except when $V_{eff} \to 0$ and/or $e^{-2\phi} \to 0$. We will see that some of the natural attractor flows in $SL(2,R)$ invariant theories do result in incompressible states of holographic matter.

3.6 Conductivity in the $SL(2,R)$ Invariant Case

We now turn to calculating the conductivity in the $SL(2,R)$ invariant case discussed in the previous section. The conductivity is defined as follows

$$j_x = \sigma_{xx}F_{tx} + \sigma_{xy}F_{ty} \tag{3.96}$$

$$j_y = \sigma_{yx}F_{tx} + \sigma_{yy}F_{ty}. \tag{3.97}$$

Under a rotation by $\pi/2$, which is a symmetry of the system, $(x,y) \to (y,-x)$. Transforming all quantities appropriately in the above equations we learn that

$$\sigma_{xx} = \sigma_{yy}, \quad \sigma_{xy} = -\sigma_{yx}. \tag{3.98}$$

Thus there are two independent components in the conductivity tensor. In the discussion below we will use the notation

$$\sigma_1 = \frac{\sigma_{yx}}{4}, \quad \sigma_2 = \frac{\sigma_{xx}}{4}. \tag{3.99}$$

Below we will use the bulk description to calculate j_x, j_y, in terms of the boundary value of gauge fields. From the resulting equations we will find that the two complex combinations

$$\sigma_+ = \sigma_1 + i\sigma_2 \tag{3.100}$$

$$\sigma_- = \sigma_1 - i\sigma_2 \tag{3.101}$$

both transform in the same way as the axion dilaton under an $SL(2,R)$ transformation. Namely

$$\sigma_\pm \to \frac{\tilde{a}\sigma_\pm + b}{c\sigma_\pm + d} \tag{3.102}$$

under the transformation (3.72). Note that the conductivity components σ_{xx}, σ_{yx} are in general complex. Thus σ_+ and σ_- are not complex conjugates of each other. Starting from the purely electric case, for which the conductivity has already been obtained above, and using the transformation properties, (3.102), we can then easily obtain the conductivity for a general dyonic case.

The electromagnetic part of the bulk action is

$$S_{em} = \int d^4x\sqrt{-g}\left[\lambda_2 F_{\mu\nu}F^{\mu\nu} - \lambda_1 F\tilde{F}\right]. \tag{3.103}$$

In the subsequent discussion it is useful to work in a coordinate system where the metric takes the form

$$ds^2 = a^2(-dt^2 + dz^2) + b^2(dx^2 + dy^2) \ . \tag{3.104}$$

Asymptotically, the metric approaches AdS_4 and $a^2 = b^2 = z^{-2}$. In the boundary theory, the current $\langle j_x \rangle$ can be obtained by

$$\langle j_x \rangle = \frac{\delta \log(Z)}{\delta A_x} \ . \tag{3.105}$$

The standard AdS/CFT dictionary then tells us that in the bulk,

$$\langle j_x \rangle = 4 \left[\lambda_2 F_{zx} - \lambda_1 F_{ty}\right]_{z \to 0} \tag{3.106}$$

(here we have chosen conventions so that $\epsilon_{tzxy} > 0$). Similarly,

$$\langle j_y \rangle = 4 \left[\lambda_2 F_{zy} + \lambda_1 F_{ty}\right]_{z \to 0} . \tag{3.107}$$

In this section we will be mainly concerned with using these formula to calculate the conductivity. For ease of notation in the subsequent discussion we will not specify that the moduli and field strengths which appear are to evaluated at the boundary, $z \to 0$.

From (3.106), (3.107), (3.96), (3.97) and (3.100) we get

$$\begin{aligned} \lambda_2 F_{zx} - \lambda_1 F_{ty} &= \sigma_2 F_{tx} - \sigma_1 F_{ty} & (3.108) \\ \lambda_2 F_{zy} + \lambda_1 F_{tx} &= \sigma_2 F_{ty} + \sigma_1 F_{tx}. & (3.109) \end{aligned}$$

A general $SL(2,R)$ transformation can be obtained by a product of two kinds of $SL(2,R)$ elements. The first, which we denote as T_b, is of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ . \tag{3.110}$$

And the second, which we denote by S, is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.111}$$

To show that (3.108), (3.109) transform in a covariant way under a general $SL(2,R)$ transformation when $\sigma_\pm$ transform as given in (3.102), it is enough to show this for the transformations T_b, S.

Under T_b the field strength $F_{\mu\nu}$ does not change, (3.73). The dilaton-axion transform as $\lambda_1 \to \lambda_1 + b$, (3.74), and $\sigma_1 \to \sigma_1 + b$, (3.102). So we see that (3.108), (3.109) are left unchanged. The LHS of (3.108) can be written as,

$$[\lambda_2 F_{zx} - \lambda_1 F_{ty}] = -\frac{\lambda}{2}(F_+)_{ty} - \frac{\bar{\lambda}}{2}(F_-)_{ty} \tag{3.112}$$

where $F_\pm = F \pm i\tilde{F}$. Under a general $SL(2R)$ transformation

$$\begin{aligned} F_+ &\to F'_+ = (c\lambda + d)F_+ & (3.113) \\ F_- &\to F'_- = (c\bar{\lambda} + d)F_-. & (3.114) \end{aligned}$$

From this it follows that under S the LHS of (3.108) goes to

$$[\lambda_2 F_{zx} - \lambda_1 F_{ty}] \to F_{ty}. \tag{3.115}$$

The RHS of (3.108) can be written as

$$\text{RHS} = \sigma_2 F_{tx} - \sigma_1 F_{ty} = \frac{1}{2i}\left[\sigma_+ (F_{tx} - iF_{ty}) - \sigma_- (F_{tx} + iF_{ty})\right]. \tag{3.116}$$

Under a general $SL(2,R)$ transformation this becomes

$$\begin{aligned}\text{RHS} \quad \to \quad & \frac{1}{2i}\Big[\left(\frac{\tilde{a}\sigma_+ + b}{c\sigma_+ + d}\right)\left\{(c\lambda_1 + d)(F_{tx} - iF_{ty}) - c\lambda_2\left(\tilde{F}_{tx} - i\tilde{F}_{ty}\right)\right\} \\ & - \left(\frac{\tilde{a}\sigma_- + b}{c\sigma_- + d}\right)\left\{(c\lambda_1 + d)(F_{tx} + iF_{ty}) - c\lambda_2\left(\tilde{F}_{tx} + i\tilde{F}_{ty}\right)\right\}\Big] . \end{aligned} \tag{3.117}$$

From (3.102) after some algebra it then follows that under S

$$\text{RHS} \to \frac{1}{\sigma_+\sigma_-}\left[\sigma_2(\lambda_1 F_{tx} + \lambda_2 F_{zy}) + \sigma_1(\lambda_1 F_{ty} - \lambda_2 F_{zx})\right] . \tag{3.118}$$

Using (3.108), (3.109) this becomes,

$$\text{RHS} \to \frac{1}{\sigma_+\sigma_-}\left[\sigma_2(\sigma_1 F_{tx} + \sigma_2 F_{ty}) + \sigma_1(\sigma_1 F_{ty} - \sigma_2 F_{tx})\right] = F_{ty} . \tag{3.119}$$

Thus the LHS and RHS of (3.108) transform the same way if the conductivity transforms as given in (3.102). A similar result can be obtained for (3.109) thereby establishing that (3.102) is the correct transformation law for $\sigma_\pm$.

Similarly, some algebra shows that if σ transforms as in (3.102) the RHS of (3.108) becomes,

$$\sigma_2 F_{tx} - \sigma_1 F_{ty} \to \frac{1}{\sigma_1^2 + \sigma_2^2}\left[\sigma_2(\lambda_1 F_{tx} + \lambda_2 F_{zy}) - \sigma_1(\lambda_2 F_{zx} - \lambda_1 F_{ty})\right] . \tag{3.120}$$

Upon using (3.108) this gives

$$\sigma_2 F_{tx} - \sigma_1 F_{ty} \to F_{ty} \tag{3.121}$$

which is indeed equal to the transformation of LHS, as seen in (3.115). Similarly (3.109) can also be shown to be covariant under S. This proves that (3.108), (3.109) transform in a covariant manner under $SL(2,R)$.

Since a general dyonic system can be obtained by starting from a purely electric one and carrying out an $SL(2,R)$ transformation, we can now obtain the conductivity for the general dyonic case using (3.102). We will follow the conventions of the previous section and refer to quantities in the electric frame without a prime superscript and in the dyonic frame with a prime superscript. In the purely electric case we have $\sigma_{xy} = \sigma_{yx} = 0$. Thus $\sigma = i\sigma_{xx}/4$. Also, it is enough to consider the case with the axion set to zero, $\lambda_1 = 0$, in the electric frame. Thus $\lambda = i\lambda_2$. Then using (3.102) we get

$$\sigma'_{xx} = \frac{\sigma_{xx}}{d^2 + c^2\left(\frac{\sigma_{xx}}{4}\right)^2} \tag{3.122}$$

and

$$\sigma'_{yx} = 4\frac{\tilde{a}c(\frac{\sigma_{xx}}{4})^2 + bd}{d^2 + c^2(\frac{\sigma_{xx}}{4})^2}\,. \tag{3.123}$$

To complete the analysis one would like to express the $SL(2,R)$ matrix elements which appear on the RHS of (3.122), (3.123) in terms of parameters in the dyonic frame.

As discussed in the previous section, the most general dyonic case can be obtained by starting with a purely electric case with axion set to zero and $Q = 1$. From (3.82), (3.78) we see that with $Q = 1$

$$Q'_e = \tilde{a}, Q'_m = c. \tag{3.124}$$

The invariance of the effective potential gives, from (3.83), (3.84),

$$\lambda_{20}^{-1} = (Q'_e - Q'_m\lambda'_{10})^2(\lambda'_{20})^{-1} + (Q'_m)^2\lambda'_{20}. \tag{3.125}$$

This allows the asymptotic value of the dilaton in the electric frame to be expressed in terms of quantities in the dyonic frame. Using this and (3.81) we learn that d is

$$d = \frac{Q'_e - \lambda'_{10}Q'_m}{(Q'_e - \lambda'_{10}Q'_m)^2 + (Q'_m)^2(\lambda'_{20})^2}\,. \tag{3.126}$$

And then, finally, using the relation $\tilde{a}d - bc = 1$ gives

$$b = \frac{\lambda'_{10}Q'_e - Q'_m(\lambda'^2_{10} + \lambda'^2_{20})}{(Q'_e - \lambda'_{10}Q'_m)^2 + (Q'_m)^2(\lambda'_{20})^2}\,. \tag{3.127}$$

3.6.1 More on the conductivity

The formulae obtained for the conductivity (3.122) (3.123) are valid in general. Let us discuss the resulting behavior of the conductivity at small frequencies and temperatures in the parametric range (3.10) more explicitly.

To start it is useful to state the parametric range (3.10) in a duality invariant manner. The $SL(2,R)$ transformation with $b = c = 0, \tilde{a} = 1/d$ is a scaling transformation. Starting with the purely electric case, this SL(2,R) transformation yields $Q'_e = Q_e/d, Q'_m = 0$. From (3.90), (3.74), it follows that the chemical potential and dilaton transform as

$$\mu' = \mu d, \quad \sqrt{\lambda'_2} = \sqrt{\lambda_2}/d, \tag{3.128}$$

so that $\mu\sqrt{\lambda_2}$ is invariant under the rescaling. This combination can in fact be expressed in terms of the effective potential, which is duality invariant, as $\mu\sqrt{\lambda_2} \sim (V_{eff0})^{1/4}$. The frequency ω and temperature T are duality invariant.[13] Thus the duality invariant way to state the parametric range of interest is

$$\omega \ll T \ll (V_{eff0})^{1/4}. \tag{3.129}$$

[13] The duality invariance of the temperature follows from that of the Einstein frame metric.

In the purely electric case, the conductivity to leading order is

$$\sigma_{xx} = C' \frac{T^2}{\mu^2} + iC'' \frac{\mu}{\omega} \ . \tag{3.130}$$

Under the rescaling discussed in the previous paragraph, $\sigma'_{xx} = \sigma_{xx}/d^2$. From this and (3.128) it follows that C' is independent of ϕ_0 while $C'' \propto (\lambda_2)^{3/2}$. Both Re (σ_{xx}) and Im (σ_{xx}) have corrections, which result in a fractional change of order ω^2,

$$\text{Re}\ (\sigma_{xx}) = C' \frac{T^2}{\mu^2} \left(1 + O(\omega^2)\right), \quad \text{Im}\ (\sigma_{xx}) = C'' \frac{\mu}{\omega} \left(1 + O(\omega^2)\right). \tag{3.131}$$

Plugging (3.130) into the transformation laws (3.122), (3.123), gives the conductivity for the general dyonic case.

Let us consider the Hall conductance first. When the magnetic field is non-zero, $c \neq 0$ and the pole in the imaginary part of σ_{xx} will dominate the low frequency behavior. As a result, we get

$$\sigma'_{yx} = 4\frac{\tilde{a}}{c} + \mathcal{O}(\omega^2) \ . \tag{3.132}$$

From (3.124), (3.53) we see that the leading behavior is

$$\sigma'_{yx} = \frac{n'}{Q'_m} \tag{3.133}$$

where n', Q'_m are the charge density and the magnetic field respectively. This result in fact just follows from Lorentz invariance.

Intuitively, one would expect that the DC value of the Hall conductivity agrees with the coefficient of the Chern-Simons term of the dual field theory in the far infra-red, which in turn should be given by the value of the axion close to the horizon in the bulk. From (3.74) it follows that the axion after the duality transformation is given by

$$\lambda'_1 = \frac{\tilde{a} c \lambda_2^2 + bd}{c^2 \lambda_2^2 + d^2} \ . \tag{3.134}$$

Near the horizon in the electric case $\lambda_2 \to \infty$; thus, the attractor value of the axion is

$$\lambda'_{1*} = \frac{\tilde{a}}{c} \tag{3.135}$$

which is indeed proportional to the value of the Hall conductance (3.133) (the factor of 4, which is the proportionality constant, follows from (3.108), (3.109)).

Actually, it turns out that the $\mathcal{O}(\omega^2)$ terms in (3.132) can also be calculated reliably in terms of C', C''. From (3.123) and (3.130) we get that

$$\sigma'_{yx} = \frac{n'}{Q'_m} \left[1 + \omega^2 \left(-4 \left(\frac{T^2 C'}{C'' \mu^3} \right)^2 + \frac{64 d}{\mu^2 n' (Q'_m)^2 (C'')^2} \right) + \mathcal{O}(\omega^4) \right] \ . \tag{3.136}$$

Next let us consider the longitudinal conductivity. From (3.122) we get,

$$\sigma'_{xx} = -i\frac{16}{(Q'_m)^2}\frac{\omega}{C''\mu}\left[1 + i\frac{C'}{C''}\frac{\omega T^2}{\mu^3} + \mathcal{O}(\omega^2)\right] \ . \tag{3.137}$$

Here C', C'' are the coefficients as given in (3.130) and μ is the chemical potential in the electric theory. We see that the longitudinal conductivity vanishes as $\omega \to 0$. This result also follows from Lorentz invariance in the presence of a magnetic field. We also see that the imaginary part does not have a pole after the duality transformation; this shows that there is no delta function at zero frequency in the real part of σ_{xx}. The absence of this delta function again is to be expected on general grounds, since in the presence of the background magnetic field, momentum is not conserved.

It is worth comparing our results with the general discussion of conductivity for a relativistic plasma in [5]. From general reasoning based on linear response in magnetohydrodynamics it was argued in [5] (see also [11]) that at small frequency

$$\sigma_{xx} = \sigma_Q \frac{\omega\left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{(\omega + i\gamma)^2 - \omega_c^2} \tag{3.138}$$

and

$$\sigma_{xy} = -\left(\frac{n'}{Q'_m}\right)\frac{\gamma^2 + \omega_c^2 - 2i\gamma\omega}{(\omega + i\gamma)^2 - \omega_c^2} \ . \tag{3.139}$$

Here $\sigma_Q, \gamma, \omega_c$ depend on the magnetic field Q'_m, T and charge density n'. γ is the damping frequency and ω_c is the cyclotron frequency. Expanding in a power series for small ω gives

$$\sigma_{xx} = -i\frac{\sigma_Q\omega}{\gamma}\left[1 + \frac{i\gamma\omega}{\gamma^2 + \omega_c^2} + \mathcal{O}(\omega^2)\right] \tag{3.140}$$

and

$$\sigma_{xy} = \frac{n'}{Q'_m}\left[1 + \frac{\omega^2}{\gamma^2 + \omega_c^2}\right] \ . \tag{3.141}$$

Comparing with (3.136), (3.137) we see that[14]

$$\begin{aligned}
\frac{\gamma}{\gamma^2 + \omega_c^2} &= \frac{C'T^2}{C''\mu^3} \\
\frac{1}{\gamma^2 + \omega_c^2} &= \frac{64d}{n'Q'^2_m C''} - 4\left(\frac{T^2C'}{C''\mu^3}\right)^2 \\
\frac{\sigma_Q}{\gamma} &= \frac{16}{(Q'_m)^2 C''\mu}
\end{aligned} \tag{3.142}$$

These three relations determine $\sigma_Q, \gamma, \omega_c$ in terms of the parameters of our calculations. To express the answer in terms of the dyonic duality frame variables we should bear in mind that d is given in terms of the charges etc in (3.126), $\mu\sqrt{\lambda_{20}} \sim (V_{eff0})^{1/4}$, and λ_{20} is given in (3.125). Also while C' is independent of λ_{20}, $C'' \propto \lambda_{20}^{3/2}$.

[14] Our convention for σ_{xy} differs from that of [11] by a sign.

The equations in (3.142) are valid for small temperature (3.129) and arbitrary n', Q'_m. It is easy to solve them and obtain σ_Q, γ and ω_c in a small T expansion. While we do not present the results in detail, let us note that one finds at small T and also small magnetic field Q'_m that $\sigma_Q, \gamma, \omega_c$ scale as,

$$\sigma_Q \propto T^2, \quad \gamma \propto (Q'_m)^2 T^2, \quad \omega_c \propto Q'_m. \tag{3.143}$$

This qualitative behavior is in agreement with the results of [11, 12] for the Reissner-Nordström black brane at small ω and Q'_m.

3.6.2 Thermal and thermoelectric conductivity

There are two transport coefficients related to the conductivity, the thermoelectric coefficient α and the thermal conductivity κ. Both should be thought of as tensors. These are defined by the relations,

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \alpha \mathbf{T} & \kappa \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix} \tag{3.144}$$

where $\vec{E}$ is the electric field, $\vec{\nabla} T$ is the gradient of the temperature, $\vec{J}$ is the electric current and $\vec{Q}$ is the heat current.

It is easy to see, using the second law, that Q^i is given by[15]

$$Q^i = T^{ti} - \mu J^i \tag{3.145}$$

where T^{ti} is a component of the stress energy tensor and $\vec{J}$ is the electric current.[16]

In AdS/CFT the source term corresponding to the electric field is a non-normalizable mode of the bulk gauge field A_i, while the source corresponding to a thermal gradient $\nabla_i T$ corresponds, to a combination of the non-normalizable mode for the metric component g_{it} and A_i. By turning these on and calculating the response we can calculate the thermoelectric and thermal conductivities.

The thermoelectric conductivity

The thermoelectric coefficient α can be determined by calculating the heat current $\vec{Q}$ generated in response to an electric field in the absence of a temperature gradient. In AdS/CFT we turn on a non-normalizable mode for A_i and calculate the resulting value for Q_i. We will take the time dependence to be of the form $e^{-i\omega t}$ throughout. To begin we consider the $SL(2, R)$ case (3.70) but in fact our results will be quite general and we comment on this at the end of the subsection.

[15] Ambiguities in the definition of the heat current can arise because entropy is not conserved. However they enter in higher orders and are not important in linear response theory.

[16] Some of the literature, e.g., [5], defines transport coefficients in terms of currents where a magnetization dependent term is subtracted out. It is straightforward to relate our answers to those obtained after such a subtraction.

For a metric

$$ds^2 = -a^2dt^2 + \frac{dr^2}{a^2} + b^2(dx^2 + dy^2) + 2g_{xt}dxdt + 2g_{yt}dydt \tag{3.146}$$

and with action given by (3.70) we find that the xt component of the trace-reversed Einstein equations gives

$$R_{xr} = 2\lambda_2(-F_{rt}F_{tx}g^{tt} + F_{ry}F_{xy}g^{yy} + F_{rx}F_{xt}g^{xt} + F_{rt}F_{xt}g^{yt} + F_{rt}F_{xy}g^{yt}) \tag{3.147}$$

with

$$R_{xr} = -i\omega\frac{\partial_r(g^{xx}g_{tx})}{2g_{tt}g^{xx}}. \tag{3.148}$$

The standard procedure to calculate the stress tensor in terms of the extrinsic curvature [108, 109] gives

$$T_{tx} = [a\partial_r g_{tx} - 2g_{tx}] \tag{3.149}$$

where the right hand side is to be evaluated close to the boundary as $r \to \infty$.

While we skip some of the steps in the analysis below, it is easy to see that close to the boundary, the leading behavior on the RHS of (3.147) comes from the first two terms. Thus, we get close to the boundary from (3.148), (3.147)

$$-i\omega\frac{\partial_r(g^{xx}g_{tx})}{2g_{tt}g^{xx}} \simeq 2\lambda_2\left(-F_{rt}F_{tx}g^{tt} + F_{ry}F_{xy}g^{yy}\right) . \tag{3.150}$$

Substituting (3.76) for the field strength then yields,

$$T_{tx} = \frac{2}{i\omega}\left[-2\frac{(Q'_e - \lambda'_{10}Q'_m)}{a}E'_x + 2\lambda'_2 Q'_m F'_{ry}a\right] . \tag{3.151}$$

Some of the notation we have adopted here is potentially confusing. The superscript prime here denotes a dyonic configuration with both electric and magnetic charge as in the previous sections. In particular, the variable λ'_{10} denotes the asymptotic axion in the system with both electric and magnetic charge. The variable a in the equation above stands for the redshift factor in the metric.

Using the relation between the variable r used above and z used in (3.104) we see that

$$\lambda'_2 F'_{ry} = -\frac{1}{a^2}\lambda'_2 F'_{zy} = -\frac{1}{a^2}\left(\frac{j'_y}{4} - \lambda'_{10}E'_y\right) \tag{3.152}$$

where on the RHS we have also used (3.107).

To complete the calculation we need to express T_{tx} in terms of boundary theory coordinates. This requires us to multiply the RHS of (3.151) by a factor of a. After doing this we get in the boundary theory

$$T_{tx} = \frac{1}{i\omega}\left[-4\left(Q'_e - \lambda'_{10}Q'_m\right)E'_x - j'_y Q'_m + 4\lambda'_{10}Q'_m E'_y\right] . \tag{3.153}$$

Finally using the relation

$$Q_x = T^{tx} - \mu J^x = -T_{tx} - \mu J_x = T\alpha_{xx}E_x + T\alpha_{yx}E_y \tag{3.154}$$

gives

$$\alpha'_{xx} = \frac{(n' - 4\lambda'_{10}Q'_m)}{i\omega T} + \frac{Q'_m}{i\omega T}\sigma'_{yx} - \frac{\mu'}{T}\sigma'_{xx} \tag{3.155}$$

$$\alpha'_{xy} = \frac{1}{i\omega T}[\sigma'_{yy}Q'_m - 4\lambda'_{10}Q'_m] - \frac{\mu'}{T}\sigma'_{xy} \tag{3.156}$$

where we have used the relation $n' = 4Q'_e$. By symmetries $\alpha'_{yy} = \alpha'_{xx}, \alpha'_{yx} = -\alpha'_{xy}$.

We have considered the action (3.70) in the analysis above, but it is easy to see that the relations (3.155), (3.156) stay the same for the more general case

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - 2(\partial\phi)^2 - h(\phi)(\partial\lambda_1)^2 - \lambda_2 F^2 - \lambda_1 F\tilde{F}\right], \tag{3.157}$$

with $h(\phi)$ and λ_2 being general functions of ϕ.

The results above are quite analogous with those in [11], which studied transport properties in the AdS Reissner-Nordström case. It is instructive to compare the cases with and without a dilaton-axion. Consider first the purely electric case. We have seen earlier that the thermodynamics in the extremal limit for the cases with and without a dilaton are quite different, since the entropy vanishes in the presence of a dilaton. Despite this difference, we have also seen that the electric conductivity at both small and large frequency and small and large temperature qualitatively agree. In this subsection, we find that the relation between the thermoelectric and electric conductivities is essentially the same in the two cases. Thus, the thermoelectric conductivity also agrees qualitatively in the two cases. Once a magnetic field is turned on, in the presence of an axion the thermodynamics of the extremal situation continues to behave differently from the extremal Reissner-Nordström case, with vanishing entropy, while we saw in the previous subsection that the electrical conductivity is still quite similar. Here we see that the thermoelectric conductivity gets additional contributions due to the presence of the axion, but these only affect the imaginary part and not the dissipative real part at non-zero frequency. Thus, the thermoelectric conductivity continues to be quite similar.

Thermal conductivity

Next we turn to the thermal conductivity. It is easy to see using a Kubo formula that the thermal conductivity κ_{ij} is related to the retarded two-point function of the heat current Q_i [11],

$$\kappa_{ij} = -\frac{\langle Q_i, Q_i\rangle}{i\omega T}. \tag{3.158}$$

Using the definition of Q_i (3.145) one then gets

$$\langle Q_i, Q_j\rangle = \left\langle (T^t_i - \mu J_i), -\mu J_j)\right\rangle + \langle T^t_i, T^t_j\rangle - \mu\left\langle J_i, (T^t_j - \mu J_j)\right\rangle - \mu^2\left\langle J_i, J_j\right\rangle . \tag{3.159}$$

Now it is easy to see from the rules of AdS/CFT that

$$\left\langle \left(T_i^t - \mu J_i\right), J_j \right\rangle = \left\langle J_j, \left(T_i^t - \mu J_i\right) \right\rangle$$

so that the first and third terms on the RHS can be related to each other. Further using the definition of thermoelectric and electric conductivity,

$$\left\langle \left(T_i^t - \mu J_i\right), J_j \right\rangle = (-i\omega T)\alpha_{ij}, \quad \langle J_i, J_j \rangle = (-i\omega)\sigma_{ij} \tag{3.160}$$

then gives

$$\langle Q_i, Q_j \rangle = i\omega\mu T(\alpha_{ij} + \alpha_{ji}) + i\omega\mu^2\sigma_{ij} + \langle T_i^t, T_j^t \rangle. \tag{3.161}$$

As we will see in Appendix B

$$\langle T_i^t, T_j^t \rangle = \frac{\rho}{2}\delta_{ij} \tag{3.162}$$

where ρ is the energy density. Substituting the last few equations in (3.158) then finally gives the relation

$$\kappa_{ij} = -\mu(\alpha_{ij} + \alpha_{ji}) - \frac{\mu^2}{T}\sigma_{ij} + \frac{i}{2\omega T}\rho\delta_{ij}. \tag{3.163}$$

We note that this relation follows essentially from the Kubo formula and is valid in general. For the case where there is no magnetic field we get from (3.155) and (3.163)

$$\text{Re } (\kappa_{xx}) = \frac{\mu^2}{T}\text{Re } (\sigma_{xx}). \tag{3.164}$$

This is a Weidemann-Franz like relation, and is analogous to those obtained in the non-dilatonic case studied in [5, 11]. At low temperature and frequency, we have seen in section 3 that Re $(\sigma)_{xx} \sim \frac{T^2}{\mu^2}$, leading to a linear behavior of thermal conductivity

$$\text{Re } (\kappa_{xx}) \sim T. \tag{3.165}$$

The derivation of (3.162) is discussed in Appendix B. We note that the result in (3.162) is independent of momentum, and is therefore a contact term. Often in AdS/CFT calculations such contact terms are simply discarded. We do not delve into this issue here any further except to note that [11] discusses it and does subtract this term from the final answer.

3.6.3 Disorder and power-law temperature dependence of resistivity

So far we have neglected the effects of disorder. Disorder can be incorporated in a phenomenological way by adding a small imaginary part to the frequency, following [5], $\omega \to \omega + i/\tau$. We focus on the resulting effects on electric conductivity in the discussion below.

To begin, consider the purely electric case. The conductivity, at small frequency, is given by (3.130)

$$\sigma_{xx} = \frac{C'T^2}{\mu^2} + iC''\frac{\mu}{(\omega + i/\tau)}, \tag{3.166}$$

with $\sigma_{xy} = 0$. For very small frequencies, $\omega \ll 1/\tau$ the disorder will dominate the imaginary part of σ_{xx} and we get,

$$\sigma_{xx} \simeq C''\mu\tau + \frac{C'T^2}{\mu^2}. \tag{3.167}$$

The first term on the RHS is a Drude-like contribution to the conductivity which is proportional to the relaxation time τ. For small disorder, $\mu\tau \gg 1$ and we see that first term on the RHS of (3.167) is large[17]. In the theory without disorder Im (σ_{xx}) has a pole and Re (σ_{xx}) has a corresponding delta function at $\omega = 0$. We see from (3.167) that after adding disorder, the pole and the delta function have both disappeared as expected, leaving a large, but finite, Drude-like contribution in Re (σ_{xx}).

Now consider the purely magnetic case obtained by carrying out an S transformation, (3.111) on the purely electric case. Since $\tilde{a} = d = 0$ we see from (3.123) that $\sigma'_{yx} = 0$ and since $c = 1$ from (3.122) that the resistivity,

$$\rho'_{xx} = \frac{1}{\sigma'_{xx}} = \frac{\sigma_{xx}}{16}. \tag{3.168}$$

Thus the large Drude-like contribution in σ_{xx} discussed above turns into a large resistivity in the magnetic case, scaling with the relaxation time τ. In addition we see that the resistivity now grows as T^2 with increasing temperature.

The S duality transformation is also a symmetry of the dilaton theory without an axion for all values of the coupling α defined in (3.1). Thus our results apply to these cases as well. More generally, see e.g. [101], once an additional potential is added for the dilaton-axion, one expects that the conductivity in the purely electric case can vary with temperature in ways different from the T^2 dependence we have found. This will then result in a different dependence for the resistivity in the purely magnetic case. In particular, we expect that one can obtain a linear dependence $\rho_{xx} \sim T$ reminiscent of strange metal behavior in this manner.

3.6.4 $SL(2,R)$ and $SL(2,Z)$ in the boundary theory

It is natural to ask how the $SL(2,R)$ symmetry is implemented in the boundary theory. The gauge symmetry in the bulk corresponds to a global symmetry in the boundary. To implement the $SL(2,R)$ in the boundary one needs to gauge this global symmetry [110]. This is because, starting with a state which carries only electric charge in the bulk, one gets after a general $SL(2,R)$ transformation a system with both electric charge and a magnetic field. Now, the magnetic field corresponds to a non-normalizable deformation and therefore requires a change in the boundary Lagrangian. Once the global symmetry is gauged in the boundary theory, there is a boundary gauge field a^μ, and the required change in the boundary Lagrangian can be identified as turning on a background magnetic field.

T_b

The $SL(2,R)$ symmetry is generated by the two elements T_b and S discussed in (3.110) and (3.111). Under T_b the axion shifts, $\lambda_1 \to \lambda_1 + b$. It is natural to identify this with a

[17] C'' which is dimensionless is $O(1)$.

change in the coefficient of the Chern-Simons term for the gauge field in the boundary theory [87]. In fact, this cannot be the whole story. The reason is that, even for abelian gauge fields, the Chern-Simons term must appear with a quantized coefficient [110]. In defining the Chern-Simons term on a three-manifold Σ_3, one chooses an extension of the gauge field to a four-manifold Σ_4 with $\partial\Sigma_4 = \Sigma_3$, and writes

$$\int_{\Sigma_3} A \wedge dA = \int_{\Sigma_4} F \wedge F \ . \tag{3.169}$$

Of course, to avoid arbitrariness in the definition, (3.169) must yield an answer which is independent of the choice of Σ_4 and the extension of the gauge field – or more precisely, the action $S(A)$ should depend on this choice only up to shifts by integer multiples of 2π, so that e^{iS} is invariant. This condition leads to a precise quantization of the coefficient of the Chern-Simons term.

Now, this poses a mystery, because in our system the Hall conductance takes arbitrary rational values (once we relax the full $SL(2,R)$ symmetry to the more realistic $SL(2,Z)$). However, this does not require violation of the quantization condition. Rather, consider a (toy, boundary) Lagrangian of the form

$$S = \frac{1}{4\pi}\int d^3x \left(k\ \epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho \ - \ \frac{1}{2\pi}a^\mu\epsilon_{\mu\nu\rho}\partial_\nu A_\rho \right) \ . \tag{3.170}$$

This is the sort of Lagrangian that one finds in effective field theory descriptions of the quantum Hall effect; A^μ is to be identified with the "emergent" gauge field (so the electromagnetic current is $J_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial^\nu A^\rho$) and a^μ is the external electromagnetic field. Integrating out A^μ, one finds an effective Lagrangian for a^μ which gives fractional Hall conductance, and is roughly a Chern-Simons theory at level $1/k$ [111]. Identifying J^μ with the global current in our boundary theory, and a_μ with the boundary gauge field, we see how "effective" fractional Hall conductances can arise in a theory with well-quantized Chern-Simons terms. The generalization to describe arbitrary fractional quantum Hall states is discussed in, for instance, [111].

S

The S transformation is more complicated. It action in the boundary theory has been discussed in [110]. In $2+1$ dimensions (at least in the absence of charged matter) the gauge field a^μ is dual to a scalar ϕ. The dual scalar theory has a global symmetry, $\phi \to \phi + c$. The S transformation requires gauging this global symmetry and turning on a magnetic field for the resulting dual gauge field. This prescription for implementing S also roughly agrees with the discussion in [5] in which the S duality acts by turning electrically charged particles into vortices. Electrically charged particles of the gauge field a^μ are vortices under the global symmetry for ϕ. Gauging the global symmetry corresponds to turning on a gauge field which couples (via local couplings) to these vortices.

In the bulk, $SL(2,R)$ invariance means that the theory comes back to itself with a different electric and magnetic field and altered dilaton-axion. This means in the boundary, starting with the gauge theory containing the gauge field a^μ and carrying out the $SL(2,R)$

transformation should give back the same gauge theory with the new magnetic field and couplings corresponding to the new dilaton-axion and in a state with the new charge.

$SL(2,R)$ **vs** $SL(2,Z)$

In string theory, one does not expect that the $SL(2,R)$ symmetry is exact. Instead it will be broken to an $SL(2,Z)$ subgroup generated by the elements $T_{b=1}, S$. It is this $SL(2,Z)$ subgroup which should be a symmetry (in the sense described above) of the boundary theory as well. The breaking of $SL(2,R)$ to $SL(2,Z)$ occurs due to stringy or quantum corrections in the supergravity action; it can also be understood as being related to charge quantization. In any case, at the level of bulk solutions, if the supergravity approximation we are working with here is good, at large values of the charges the supergravity will have an approximate $SL(2,R)$ symmetry and the approximation we make discussing the full $SL(2,R)$ is a good one. This means our conductivity and thermodynamic calculations using the $SL(2,R)$ to relate situations with different electric and magnetic charges should be accurate, and the $SL(2,R)$ transformations in the boundary theory should be approximately valid. One can always restrict consideration to $SL(2,Z)$ transformations acting on the electrically charged brane with minimal charge, to get a more accurate picture.

3.7 Attractor Behavior

3.7.1 Attractor behavior in systems with $SL(2,Z)$ symmetry

In this section, we discuss the structure of attractor flows in the dilaton-axion plane. We will only discuss the flows governed by the action (3.70), which has a classical $SL(2,R)$ symmetry, though quantum effects and/or an explicit potential for the dilaton-axion can break the symmetry to $Sl(2,Z)$. The main feature of interest here is that the $SL(2,Z)$ symmetry acts to relate different attractor flows to one another; in the field theory, this would mean that different RG trajectories are related by the modular group. In the system without a potential, the endpoints of the flows have rational σ_{xy} and vanishing longitudinal conductivity.

In addition to the intrinsic interest of the subject, we are motivated to point out the action of $SL(2,Z)$ on these flows because $SL(2,Z)$ (or more properly, its subgroup $\Gamma_0(2)$) has been argued to organize the phase diagrams of real systems of charged particles in background magnetic fields. Discussions in the context of the fractional quantum Hall system can be found in [112, 113, 114, 115], and a nice review appears in [116]. Needless to say, it would be very interesting to modify our system to give incompressible phases and analogues of Hall plateaux, but we do not pursue this here. Discussions of holography and the quantum Hall system can be found in [87, 117, 118, 119, 120, 121].

Before proceeding, we should emphasize that there is an obvious difficulty with controlling the RG flows of greatest interest in our system. With a magnetic field turned on, the IR-attractor lies along the real axis in λ, at strong coupling. To the extent that one can trust the analysis it is attractive for both the dilaton and axion directions. More correctly, close enough to the fixed point, supergravity breaks down and corrections would have to be included to study the nature of the RG flow in more detail. In this section, we will simply take the attractor flows at face value.

One wide class of attractor flows in the $SL(2,R)$ invariant case are easily determined, as follows. The flows in the original electric solutions of [8] are extremely simple, involving logarithmic variation of the dilaton (running to weak coupling at the horizon). Using the $SL(2,R)$ transformation properties of the dilaton-axion (5.5), one can translate these dilaton trajectories into more non-trivial dilaton-axion trajectories, governing the flow to dyonic black holes in the extremal limit. By the general properties of $SL(2,R)$ transformations, it is easy to see that these trajectories must form semi-circles in the λ_1-λ_2 plane. It is also clear from the nature of the $SL(2,R)$ duality, which relates the axion to $e^{-2\phi}$, that the axion is attracted to its fixed-point value in a power-law manner.

All of the fixed points in this case lie on the real λ axis, with rational values of λ_1 (and hence σ_{xy}) and vanishing σ_{xx}. Because of the extreme value of the dilaton at infinity, these states are also incompressible. This is happily rather similar to the flows in the quantum Hall system, but the underlying physics of our charged fluid is perhaps quite different.

3.7.2 Attractor behavior in more general system without $SL(2,R)$ symmetry

In this section, we study a more general theory which does not have $SL(2,R)$ symmetry. The action we study has one parameter $\alpha \neq -1$,

$$S = \int d^4x\sqrt{g}\left(R - 2\Lambda - 2\left(\partial\phi\right)^2 - \frac{1}{2}e^{4\phi}\left(\partial\lambda_1\right)^2 - e^{2\alpha\phi}F^2 - \lambda_1 F\tilde{F}\right) . \tag{3.171}$$

We will analyze the attractor mechanism for dyonic black branes in this theory. Of course other parameters in the action (3.171) could have also been varied from their values in the $SL(2,R)$-invariant case. We do not carry out a full analysis of the resulting set of theories here, but the limited class we do study already exhibit rather interesting phenomena.

The effective potential is now given by

$$V_{eff}(\phi,\lambda_1) = e^{-2\alpha\phi}\left(Q_e - \lambda_1 Q_m\right)^2 + e^{2\alpha\phi}Q_m^2, \tag{3.172}$$

where Q_e, Q_m are the electric and magnetic charges. The extremum of the potential arises at

$$\lambda_1 = \lambda_{1*} = \frac{Q_e}{Q_m}, \quad e^{2\alpha\phi} \to -\infty. \tag{3.173}$$

We work in the coordinate system (3.2) below. If the axion takes its attractor value λ_{1*} at $r \to \infty$, it is constant everywhere and the resulting solution is that of a purely magnetically charged dilatonic brane. This has a near horizon metric given in (3.4) and the near-horizon dilaton

$$\phi = K\log(r), \tag{3.174}$$

with the constants C_2, β, K taking values given in (3.8).

To investigate if this magnetic solution is an attractor, we take the asymptotic value of the axion at infinity to be slightly different from its attractor value and study the resulting solution. As we will see below, in the ranges $\alpha > 0$ and $\alpha \leq -1$ we find attractor behavior, with the axion settling down to its attractor value exponentially rapidly in r (except for the special case $\alpha = -1$ discussed in section 3.6 and 3.7.1, where the attractor is power-law in nature). In the range $-1 < \alpha < 0$ we find that there is no attractor behavior. Instead, starting with a value for the axion at infinity which is slightly different from its attractor value, one finds that the solution increasingly deviates from the purely magnetic case for small enough r. We have not been able to find the end point of the attractor flow in this case.

Attractor behavior for $\alpha > 0$, $\alpha < -1$

The axion equation of motion is

$$\partial_r \left(e^{4\phi} a^2 b^2 \partial_r \lambda_1 \right) = \frac{4e^{-2\alpha\phi} Q_m^2}{b^2} \left(\lambda_1 - \lambda_{1*} \right) \ . \tag{3.175}$$

Putting in the solution for ϕ, a^2, b^2 in the near horizon region of the purely magnetic case gives

$$\partial_r \left(r^{4K+2\beta+2} \partial_r \lambda_1 \right) = \frac{D}{r^{2\beta+2\alpha K}} \left(\lambda_1 - \lambda_{1*} \right) \tag{3.176}$$

where $D > 0$ is a constant.

Define the variable x as

$$x = \frac{1}{|4K + 1 + 2\beta|} \frac{1}{r^{4K+1+2\beta}} \ . \tag{3.177}$$

In terms of x (3.176) becomes a Schrödinger-type equation,

$$\partial_x^2 \lambda_1 = \tilde{D} x^{-P} \left(\lambda_1 - \lambda_{1*} \right) \tag{3.178}$$

where $\tilde{D} > 0$ is a constant and

$$P = \frac{4K - 2\alpha K + 2}{4K + 1 + 2\beta}. \tag{3.179}$$

By rescaling x the constant $\tilde{D}$ can be set to unity[18]. To avoid notational clutter, we continue to refer to this rescaled variable as x below. Also, to simplify things, we henceforth take $(\lambda_1 - \lambda_{1*}) \to \lambda_1$, i.e., from now on we use λ_1 to denote the deviation of the axion from its attractor value. This gives

$$\partial_x^2 \lambda_1 = x^{-P} \lambda_1 \ . \tag{3.180}$$

There are two separate cases of interest.

[18]This does not work for the case $P = 2$ which arises when $\alpha = 0, -1$. The $\alpha = -1$ case has $SL(2, R)$ invariance and has been extensively discussed above. The $\alpha = 0$ case needs to be dealt with separately because here the dilaton does not enter in the gauge kinetic energy or the effective potential.

Case A

The first case arises when

$$4K + 1 + 2\beta > 0 \ . \tag{3.181}$$

Here we see from (3.177) that $x \to \infty$ as $r \to 0$. When

$$P < 2 \tag{3.182}$$

a solution to (3.180) can be found in the WKB approximation. It is of the form

$$\lambda_1 \sim e^{-S}, \tag{3.183}$$

with

$$S = \frac{x^{1-P/2}}{1 - P/2}. \tag{3.184}$$

We see that as $x \to \infty, S \to \infty$ and $\lambda_1 \to 0$, so the axion goes to its attractor value in the near horizon region exponentially rapidly. In finding the solution we have neglected the backreaction of the axion on the other fields; this is now seen to be a self-consistent approximation. Since the other fields vary in a power law fashion with r, the backreaction of the axion on them is small.

Substituting for the constants from (3.8) in the conditions (3.181) (3.182), we find that the solution (3.183) is valid in the range

$$\alpha > 0, \quad \text{or} \quad \alpha < -2. \tag{3.185}$$

Case B

The second case arises when

$$4K + 2 + 2\beta < 0 \ . \tag{3.186}$$

Now the variable $x \to 0$ as $r \to 0$.

A solution to (3.180) can be found in the WKB approximation when

$$P > 2. \tag{3.187}$$

It is again of the form given in (3.183), with S being

$$S = \frac{x^{1-P/2}}{P/2 - 1}. \tag{3.188}$$

The conditions, (3.186), (3.187) are valid when α lies in the range

$$-2 < \alpha < -1. \tag{3.189}$$

No attractor when $-1 < \alpha < 0$

Our discussion above left out the region $-1 < \alpha < 0$. In this region, we will see below that there is no attractor behavior.

First, consider the case when $4K + 1 + 2\beta > 0$ and $P > 2$, which corresponds to $-2/3 < \alpha < 0$. In this case, we see from (3.177) that $x \to \infty$ in the near horizon region where $r \to 0$. The equation for the axion (3.180) has two solutions in the near-horizon region where $x \to \infty$. Both solutions can be expressed as a power series in x. The first is

$$\lambda_1 = c_1 x + c_2 x^{\alpha} + \cdots \, . \tag{3.190}$$

For the second term on RHS to be subdominant compared to the first when $x \to \infty$

$$\alpha < 1. \tag{3.191}$$

Substituting (3.190) in (3.180) and equating powers of x gives,

$$\alpha = 3 - P. \tag{3.192}$$

Requiring that condition (3.191) is met gives,

$$P > 2 \tag{3.193}$$

which is indeed true. This solution blows up as $x \to \infty$.

The second solution to (3.180) is

$$\lambda_1 = c_0 + c_1 x^{\alpha} + \cdots \, , \tag{3.194}$$

with the condition,

$$\alpha < 0. \tag{3.195}$$

Substituting in (3.180) and equating powers of x gives

$$\alpha = 2 - P, \tag{3.196}$$

so that (3.195) is again met. Equating coefficients determines c_1 in terms of c_0.

In summary we learn that for the axion to be non-zero (i.e. away from its attractor value) and for it to not blow up at the horizon, it must be of the form (3.194) with c_0 non-vanishing. Thus, λ_1 does not vanish as $x \to \infty$ and we do not get attractor behavior in this case.

Next consider the case when $4K + 1 + 2\beta < 0$ and $P < 2$, which corresponds to $-1 < \alpha < -2/3$. Here $x \to 0$, when $r \to 0$. In this case there is a solution in which the axion attains its attractor value as $x \to 0$. A straightforward analysis shows that this takes the form, for small x,

$$a = c_1 x + c_2 x^p, p > 1 \; . \tag{3.197}$$

However, since the approach to the attractor value is a power-law in x and thus in r, one now finds that the resulting back-reaction of the axion in the equations of motion for the other fields cannot be neglected, and in fact in some cases dominates over the other contributions.

Thus, again, the resulting solution will deviate significantly from the purely magnetic case, leading to a loss of attractor behavior.

In this last case especially, one might hope to find a fully corrected solution which represents the end point of the attractor flow, in which all fields behave in a power-law fashion near the horizon, and in which the back-reaction of the axion is completely incorporated. However, a reasonably thorough analysis failed to find any purely power-law solution of this kind.

3.7.3 Comments

In the cases where we did get attractor behavior above, we saw that the axion approached its attractor value exponentially rapidly in the near-horizon region. This exponential behavior is intriguing from the point of view of a dual field theory. The radial direction r is roughly the RG scale in the boundary theory and a power-law dependence on r of a field in the bulk is related to the anomalous dimension of the corresponding operator in the boundary. In contrast an exponential dependence, of the kind we find here, leads to a beta function for the dual operator in the boundary in which the RG scale appears explicitly.

The exponentially rapid approach also means that in cases where we do get attractor behavior, the black brane in the near-horizon region can be taken to be the purely magnetic dilatonic brane up to small corrections. This means the behavior of the dyonic black brane at small temperature and frequency in these cases is given by that of the dyonic brane with the asymptotic axion set to its attractor value, up to small corrections. For example, from (3.126) we see that when $\lambda_{1\infty} = \lambda_{1*}$ the $SL(2,R)$ matrix element d vanishes. The conductivity can then be read off from (3.136), (3.137) keeping this in mind. Similarly the thermoelectric and thermal transport coefficients can also be found easily from (3.155), (3.156).

3.8 Discussion

We have analyzed charged dilatonic branes in considerable detail in this paper, focusing on their thermodynamics and especially their transport properties. Our results show that many of the transport properties are quite similar to those of the Reissner-Nordström case. This is true despite the fact that the Reissner-Nordström and dilaton cases differ significantly in their thermodynamics: while the Reissner-Nordström brane has a macroscopic ground-state entropy, the dilatonic black brane has vanishing entropy at extremality.

More concretely, in [8] it was already noted that the optical conductivity at zero temperature and small frequency has the behavior Re $(\sigma) \sim \omega^2$, and this behavior is independent of the parameter α which governs the dilaton coupling, (3.1). In particular, it is the same as in the Reissner-Nordström case which has $\alpha = 0$ [85, 86]. In this paper we find something analogous for the DC conductivity at small temperature, which goes like[19] Re $(\sigma) \sim T^2$, and is independent of α again. In the presence of a magnetic field, the DC Hall conductivity is $\sigma_{yx} = \frac{n}{B}$, where n, B are the electric charge density and the magnetic field, while the DC

[19] There is an additional delta function strictly at $\omega = 0$.

longitudinal conductivity vanishes, as required by Lorentz invariance. The DC Hall conductance is related to the attractor value of the axion. In more detail, the frequency dependence fits the form derived from general considerations of relativistic magnetohydrodynamics in [5]. These features in the presence of a magnetic field, being general in their origin, also agree with the Reissner-Nordström case. We also found that the thermoelectric and the thermal conductivities of the dyonic case satisfy Weidemann-Franz like relations which relate them to their electrical conductivity. In this respect too then the dyonic system behaves in a manner quite analogous to the Reissner-Nordström case. It is worth pointing out that, in contrast to these similarities, the viscosity of a near-extremal dilaton-axion system is much smaller than in the Reissner-Nordström case. In both cases the famous relation $\eta/s = 1/4\pi$ [122] is satisfied. However, the vanishing entropy of the extremal dilaton-axion system makes its viscosity much smaller.

The overall picture is that the charged dilatonic brane behaves like a charged plasma. The electrical conductivity, which is suppressed at small temperature and grows like T^2, suggests that strong repulsion prevents the transmission of electric currents in this system. The spectrum is not gapped in the conventional sense above the ground state, since this would lead to a conductivity vanishing exponentially quickly at small temperature. Rather, the system has a "soft" gap, resulting in a power-law vanishing as $T \to 0$.[20] It should be pointed out that the entropy density s also scales in a power-law fashion as $s \sim T^{2\beta}$, and since $\beta < 1$, it decreases more slowly near extremality (as $T \to 0$) than the charge conductivity. This makes physical sense: only some fraction of all the degrees of freedom can carry charge and contribute to electrical conductivity.

A case we investigated in considerable detail was the one with an $SL(2, R)$ symmetry. Here, the complex conductivities $\sigma_\pm$ transform like the dilaton-axion under an $SL(2, R)$ transformation. Once quantum corrections to the bulk action are included (or charge quantization is imposed), one expects this symmetry to be broken to an $SL(2, Z)$ subgroup. The transformation law for $\sigma_\pm$ is an elegant result, and one has the feeling that its full power has not been exploited in the discussion above. Perhaps suitable modifications of the bulk theory, with an additional potential for the dilaton-axion preserving the $SL(2, Z)$ symmetry and/or with disorder put in, might prove interesting in this respect. These modifications might lead to similarities with systems exhibiting the quantum Hall effect, and the transformation law of the conductivity could then tie in with some of the existing discussion in this subject on RG flows between different fixed points characterized by the various subgroups of $SL(2, Z)$ [112, 113, 114, 115, 116].

We have not shown that the dilaton-axion theories considered here can arise in string theory. However, the Lagrangians we consider are quite simple and generic, and as discussed above many of our results are quite robust. These facts suggest that an embedding in string theory should be possible. String embeddings of Lifshitz solutions have been described in [98, 123, 124], and simple generalisations of those ideas may well suffice to capture our geometries as well (since the near-horizon physics is governed by a Lifshitz-like metric).

We are not aware, at the moment, of condensed matter systems or model Hamiltonians

[20] Strictly speaking, our calculations break down at extremality, so these comments apply for temperatures much smaller than the chemical potential, but not very close to zero. The precise condition can be obtained using reasoning analogous to (3.66) in the magnetic case.

which give rise to such a power-law behavior in the conductivity. (The systems we are considering here do not have any disorder. In the presence of disorder such power laws are well known to arise [125]. We thank P. Raichaudhuri and N. Trivedi for related discussions.) It would be quite interesting to construct or find such examples, and attempt to relate their behavior to the kinds of gravitational systems studied here.

Appendix A

In this appendix we carry out a more careful examination of the Schrödinger equation (3.23) and show that the coefficient a_1 in (3.33) is of order unity and not suppressed by a power of ω.

The potential $V(z)$ is given by (3.15). In the scaling region where $r \ll \mu$, after a suitable rescaling the metric and dilaton are given by (3.18), (3.7), with coefficients given in (3.8). The constant Q^2 which appears in the potential takes the value ((2.12) of [8])

$$Q^2 = \frac{6}{\alpha^2 + 2}. \tag{3.198}$$

We use the notation

$$\hat{\omega} = \frac{\omega}{r_h} \tag{3.199}$$

below.

At the horizon, where a^2 vanishes, the potential has a first order zero and for

$$\hat{r} - 1 \ll 1 \tag{3.200}$$

it takes the form

$$V = A(\hat{r} - 1), \tag{3.201}$$

where A is a coefficient of order unity. Also in this region the variable $\hat{z}$ (3.22) is given by

$$\hat{z} = \int \frac{d\hat{r}}{\hat{a}^2} \simeq \frac{1}{B} \ln(\hat{r} - 1) \tag{3.202}$$

where B is again a coefficient of order unity.

We begin in the very near horizon region where

$$|\hat{r} - 1| \ll \frac{\hat{\omega}^2}{A}. \tag{3.203}$$

In this region the potential is subdominant compared to the frequency in the Schrödinger equation and as a result, the solution with the correct normalization to obtain the required flux is (3.24)

$$\psi = e^{-i\hat{\omega}\hat{z}} \tag{3.204}$$

(there is an additional e^{-iwt} factor but it will not be important in the discussion of this section and we will omit it below).

Now suppose one is close enough to the horizon so that (3.203) is met, but not too close, so that

$$|\hat{w}\hat{z}| \simeq \left|\hat{\omega}\frac{\ln(\hat{r}-1)}{B}\right| \ll 1. \tag{3.205}$$

Then the exponential in (3.204) can be expanded and the solution in this region becomes

$$\psi \simeq 1 - i\hat{w}\hat{z}. \tag{3.206}$$

The condition (3.205) is

$$\hat{r} - 1 \gg e^{-\frac{B}{\hat{\omega}}} \tag{3.207}$$

which is compatible with (3.203) for $\hat{\omega} \ll 1$.

Next consider the region

$$1 \gg \hat{r} - 1 \gg \frac{\hat{w}^2}{A}. \tag{3.208}$$

In this region the frequency term in the Schrödinger equation is now subdominant compared to the potential term. Moving even further away from the horizon the frequency will continue to be unimportant all the way to the region $\mu \gg \hat{r} \gg 1$ where the coefficient a_1 is defined. So it is enough to understand the solution in the region (3.208) for establishing that the coefficient a_1 is unsuppressed by further powers of ω.

By carrying out a change of variables

$$x \equiv e^{\frac{B\hat{z}}{2}}\sqrt{\frac{4A}{B^2}} = \sqrt{\frac{(\hat{r}-1)4A}{B^2}}, \tag{3.209}$$

where in obtaining the last equality we have used the relation (3.202), we can recast the Schrödinger equation in the region (3.208) in the form

$$-x^2\frac{d^2\psi}{dx^2} - x\frac{d\psi}{dx} + x^2\psi = 0. \tag{3.210}$$

This is closely related to the standard Bessel equation. From (3.209) and (3.208) we see that in this region

$$x \ll 1. \tag{3.211}$$

The solution to (3.210) then takes the form,

$$\psi = C_0 + C_1\ln(x) = C_0 + \tilde{C}_1\hat{z}. \tag{3.212}$$

Now notice that (3.206) and (3.212) are of the same form. There is in fact a good reason for this. As we will see below we can extend the solution from the region (3.207) where (3.206) is valid to the region (3.208) where (3.212) is valid by neglecting both the potential and the frequency dependent terms in the Schrodinger equation. Neglecting these terms gives a free Schrödinger equation at zero energy,

$$\frac{d^2\psi}{d\hat{z}^2} = 0, \tag{3.213}$$

with the solution which agrees with (3.206), (3.212).

The coefficients C_0 and $\tilde{C}_1$ can therefore be fixed by equating (3.206) and (3.212) giving

$$C_0 = 1, \tilde{C}_1 = -i\hat{\omega} \ . \tag{3.214}$$

In the region (3.208) it follows from (3.202) that

$$|\tilde{C}_1 z| \sim |\hat{\omega} \ln(\hat{r} - 1)| \leq |\hat{\omega} \ln(\hat{\omega})| \ll 1, \tag{3.215}$$

where the last inequality follows from the fact that $\hat{\omega} \ll 1$. Thus to good approximation we can take

$$\psi = C_0 = 1 \tag{3.216}$$

in this region.

We see therefore that the solution is of order unity in this region (without any power law suppression by a factor of $\hat{\omega}$). And it follows then that going further away from the horizon to the region where $\mu/T \gg \hat{r} \gg 1$ the coefficient a_1 will also be of order unity.

To complete the argument let us discuss how to extend the solution from the region (3.207) to (3.208). Choose a point with coordinate

$$\hat{r}_1 - 1 = c_1 \frac{\hat{\omega}^2}{A}. \tag{3.217}$$

Here c_1 is a constant which does not scale with $\hat{\omega}$ and meets the condition $c_1 \ll 1$ so that the condition (3.203) is met. Since $\hat{\omega} \ll 1$ and c_1 does not scale with $\hat{\omega}$ we see that (3.207) is also met and this point lies in the region (3.207). Next choose a second point with coordinate

$$\hat{r}_2 - 1 = c_2 \frac{\hat{\omega}^2}{A}, c_2 \gg 1 \tag{3.218}$$

such that $\hat{r}_2 \ll 1$. This point lies in the region (3.208). Using (3.202) we see that the change in $\hat{z}$ in going from $\hat{r}_1$ to $\hat{r}_2$ is

$$\delta \hat{z} = \frac{1}{B} \ln(\frac{c_2}{c_1}) \tag{3.219}$$

and is independent of $\hat{\omega}$.

For the frequency dependent term in the Schrodinger equation to be neglected in the process of continuing the solution from $\hat{r}_1$ to $\hat{r}_2$, the condition

$$\omega^2 (\delta \hat{z})^2 \ll 1 \tag{3.220}$$

must be met. Since $\hat{\omega} \ll 1$ we see that this is true. Similarly for the potential dependent term to be negligible the condition

$$V(z)(\delta \hat{z})^2 \sim (\hat{r} - 1)(\delta \hat{z})^2 \sim \hat{\omega}^2 (\delta \hat{z})^2 \ll 1 \tag{3.221}$$

must be met. This condition is also true, thereby completing the argument.

Appendix B

Here we discuss how (3.162) is obtained. In AdS/CFT the metric is dual to the boundary stress tensor. So (3.162) is obtained by doing a bulk path integral with a fixed boundary metric and then obtaining the two-point function from it. It is well known that after using the equations of motion, the resulting answer is obtained in terms of the extrinsic curvature of the boundary. In the $SL(2,R)$ invariant case we are dealing with here, this calculation is particularly simple since the metric is invariant under $SL(2,R)$. Thus one can work in the purely electric case which is a considerable simplification. This gives the result (3.162) as we will see shortly. Transforming to the dyonic frame then keeps the result unchanged since the energy density is invariant.

To calculate (3.162) in the purely electric case we go back to (3.149) but now are more careful since a non-normalizable mode for g_{tx} is also turned on. This requires the first subleading corrections in a^2, b^2 to be kept,

$$a^2 = r^2\left(1-\frac{\kappa^2\rho}{r^3}\right) \tag{3.222}$$

$$b^2 = r^2+\cdots\ . \tag{3.223}$$

Here we have reinstated the factors of κ^2; the action (3.70) has an overall factor of $2\kappa^2$ in front of it. We are also working in units where radius of AdS space is set to unity $L=1$. The ellipses on the RHS of the equation for b^2 indicate corrections which fall sufficiently fast and can be neglected in the calculation below. Keeping these corrections in (3.149) leads to

$$\langle T_{tx}\rangle = \left(\frac{1}{2\kappa^2}\right)\left[a^3\partial_r\left(\frac{g_{tx}}{a^2}\right)+2g_{tx}(\partial_r a-1)\right] \tag{3.224}$$

$$= \left(\frac{1}{2\kappa^2}\right)\left[a^3\partial_r\left(\frac{g_{tx}}{a^2}\right)+2\frac{g_{tx}\kappa^2\rho}{r^3}\right]\ . \tag{3.225}$$

Eq.(3.150) then becomes

$$\partial_r\left(\frac{g_{tx}}{a^2}\right)\frac{a^2}{b^2}+\frac{g_{tx}}{a^2}\left(\frac{a^2}{b^2}\right)' = \frac{2}{i\omega}\left(\frac{a^2}{b^2}\right)\left[2\lambda_2-F_{rt}F_{tx}g^{tt}\right]\ . \tag{3.226}$$

Leading to

$$\langle T_{tx}\rangle = -\frac{\rho g_{tx}}{2r^3}-\frac{4}{i\omega}\left(\frac{Q'_eE'_x}{a}\right)\ . \tag{3.227}$$

Now differentiating with respect to g_{tx} and converting to gauge theory variables gives (3.162) for $i=j=x$. In the absence of a magnetic field there is no cross-talk between the g_{xt} and g_{yt} perturbations so $\langle T_{tx}, T_{ty}\rangle = 0$, which is the second relation contained in (3.162).

Chapter 4

A Chern-Simons Vector Model

This chapter is based on [10], which was completed in collaboration with Simone Giombi, Shiraz Minwalla, Sandip Trivedi, Spenta Wadia and Xi Yin.

4.1 Introduction

The characteristic feature of quantum field theories in three dimensions is the possibility of a Chern-Simons kinetic term for the gauge field. As has been well studied, (e.g., [48, 49, 50, 51, 52, 53, 54, 56, 57, 55, 58, 2]) this has important consequences for the dynamics of quantum field theories in three dimensions and their connections to string theory.

Consider a level k, $U(N)$ Chern-Simons theory coupled to a single fermion in any representation of the gauge group. The only gauge-invariant relevant or marginal terms possible in the Lagrangian of such a theory in addition to the Chern-Simons term are the fermion kinetic term and mass term. The resulting quantum field theory thus depends on the two integers k and N, and a single continuous parameter, the physical mass m of the fermionic field. At energies $E \gg m$, the dynamics of this theory is scale-invariant as well as nontrivial, due to the fact that the discrete Chern-Simons coupling is an integer and cannot run – this conformal field theory can simply be obtained by tuning the physical mass of the fermion to zero.

Though the parameters k and N labeling the CFT are discrete, in the simultaneous large-N and large-k limit, the 't Hooft coupling $\lambda = \frac{N}{k}$, (which controls the strength of interactions), is effectively continuous (exactly as in ABJM theory [2]). For this reason the discretum of CFTs described by integer values of k and N coalesces into a line of fixed points in the large-N limit.

We emphasize that a variety of such non-supersymmetric fixed lines exist in three dimensions. Choosing the fermions that transform in, say, the adjoint representation of $U(N)$ gives one such fixed line of theories. Choosing fermions that transform in the bifundamental of $U(N) \times U(N)$ yields another example – one that can be thought of as a minimal, non-supersymmetric analog of the ABJM theory.

Such lines of fixed points are particularly important from the viewpoint of string theory and the AdS/CFT correspondence [1, 2]. While at small λ the theories are best described as weakly interacting quantum field theory; at large λ, when the field theory description becomes intractable, the hope is that a relatively simple classical four-dimensional gravitational description could emerge.

In this chapter, we study what appears to be the simplest example (from the field theory perspective) of such non-supersymmetric fixed lines – the theory of a single fundamental fermion coupled to a $U(N)$ level k Chern-Simons theory. The field theory turns out to be (at least, to some extent) exactly solvable. In section 4.2 we calculate the free energy of the theory on R^2 at finite temperature (that is, the free energy on $R^2 \times S^1$), exactly in the large N limit, for all values of λ. To do this, we use light-cone gauge and a dimensional regularization scheme, detailed in section 4.2. Our results imply that the theory does not exist for $\lambda > 1$, we comment on this below.

We then study the operator content of the theory. From the AdS/CFT dictionary, conserved currents in the field theory correspond to gauge fields in the bulk.[1] Our theory, in the free limit, is a theory of free fermions (in the singlet sector) and has an infinite tower of conserved currents – one for each spin $s \geq 1$ – and was therefore conjectured [28] to be dual to a particular higher-spin gauge theory [63, 64, 31, 65], containing an infinite tower of gauge fields – one for each spin $s \geq 1$. To address the nature of the bulk dual at finite λ, we study the non-conservation of these currents as we turn on interactions. More precisely, the masses of the bulk fields are directly related to the anomalous scaling dimensions of the corresponding operators in the dual field theory. In section 4.3 we will explicitly construct the tower of spin-s current operators in the field theory. We then present a simple argument based on conformal representation theory that the scaling dimensions of these operators must be protected in the large-N limit. The non-conservation of these currents is subleading in N; their divergences calculated in the classical theory translate directly into an expression for the order λ^2 contribution to the $\frac{1}{N}$ correction to the scaling dimensions.

The fact that our theory contains an infinite tower of higher spin currents whose scaling dimensions are protected in the large-N limit seems to be enough to conclude that the bulk dual to our theory at finite λ, should it exist, must also be some higher-spin gauge theory. Because the anomalous dimensions of the nearly-conserved currents vanish in the large-N limit even at finite λ means that the corresponding bulk gauge fields must be locally massless in the classical limit, even at finite λ.

4.2 Free Energy on $\mathbb{R}^2$ at Finite Temperature

In this section we will evaluate the free energy of our theory (a single species of massless fundamental fermions coupled to a Chern-Simons gauge field) in the t' Hooft large N limit. Our theory is taken to be at temperature T, and lives on a spatial $\mathbb{R}^2$ whose regulated volume we denote as V_2.

[1] Of course, there may be several descriptions of the bulk theory with varying amounts gauge symmetry. (Gauge symmetry is, after all, a redundancy in description.) So the situation may be a bit more delicate than the statement "conserved currents in the field theory correspond to gauge fields in the bulk" seems to imply, especially for theories with an infinite number of gauge fields. We hope future investigations shed more light on this issue.

In order to evaluate the free energy of our theory, we first fix a gauge. We work in the lightcone gauge $A_- = 0$. This gauge is defined in terms of an analytic continuation from Lorentzian space. For this reason $A_- = 0$ does not imply $A_+ = 0$. Alternatively one can work in Lorentzian space and analytically continue to Euclidean space (after integrating out the gauge fields) in (4.2.13). The gauge boson self interaction term vanishes in this gauge, a feature that enormously simplifies analysis. [2] Computations in pure Chern-Simons theory on $\mathbb{R}^3$ in this Euclidean continuation of the light-cone gauge were done previously in [126], [127].

In our analysis below we will encounter divergent integrals that need to be regulated. We choose to regulate all integrals in the scheme of dimensional reduction. More specifically we evaluate all integrals as follows. We evaluate γ traces, ϵ contraction etc in $d = 3$. This process leaves us with a set of scalar integrals. We then evaluate the resultant integrals by analytic continuation from $d = 3-\epsilon$ dimensions. This regularization scheme is widely employed in the previous literature on Chern-Simons matter theories (see e.g. [128, 129, 130, 131, 52, 56]). It is manifestly Lorentz invariant, and also respects gauge invariance at least up to two-loops (see [131]). We will assume without proof in what follows that our regularization scheme is indeed gauge invariant. If this is indeed the case then the theory defined by this regularization scheme must be Lorentz invariant even though we work in a gauge that breaks Lorentz invariance. We will find some evidence for the Lorentz invariance of our final results giving some a posteriori evidence for our assumption that our regularization scheme respects gauge invariance.

As we will explain below, the finite temperature free energy of our theory is completely determined by the fermion self energy on $\mathbb{R}^2 \times S^1$ (see (4.2.62)). In order to evaluate the free energy we proceed as follows. As a preliminary step to our analysis, we first determine the exact fermion propagator of our theory on $\mathbb{R}^3$. We then determine the exact fermion propagator on $\mathbb{R}^2 \times S^1$. Finally, we proceed to use this result to determine the free energy of our theory.

Comment: The calculation below assumes that the holonomy of the gauge field around the thermal circle is the identity matrix; however, it is possible to generalize the calculation of this section to nontrivial holonomy backgrounds. See the note added at the end of this chapter.

4.2.1 Conventions for Propagators and Gauge Conditions

The Euclidean action for our theory is

$$S = \frac{ik}{4\pi} \int \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) + \int \bar{\psi} \gamma^\mu D_\mu \psi, \tag{4.2.1}$$

where

$$D_\mu \psi = \partial_\mu \psi - i A^a T^a \psi.$$

[2] This feature is also true of the more straightforward "temporal" or axial gauge $A_3 = 0$. Feynman diagrams in this gauge are, however, plagued by logarithmic divergences that we have found difficult to interpret and deal with. In contrast the divergences in the lightcone gauge employed in this subsection are relatively tame, and are easy to interpret and deal with. We thank S. Bhattacharyya and J. Bhattacharya for extensive discussions on perturbation theory and its divergences in Axial gauge.

We have

$$\begin{aligned}\epsilon_{123} &= \epsilon^{123} = 1, \\ \gamma^i &= \sigma^i, \quad (i = 1 \dots 3) \\ \bar{\psi}^\alpha &= (\psi_\alpha)^*,\end{aligned} \tag{4.2.2}$$

where σ^i are the ordinary Pauli matrices. Note then that all γ^μ are Hermitian. This implies that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ are real, while $\int dx \bar{\psi}\gamma^\mu\partial_\mu\psi$ is imaginary. The gauge field will be taken to be $A^a T_a$ where T_a is the fundamental generator normalized so that $TrT_a^2 = \frac{1}{2}$. Note that

$$\sum_a (T^a)^n_m (T^a)^q_p = \frac{1}{2}\delta^q_m \delta^n_p. \tag{4.2.3}$$

Lightcone gauge

Let us define

$$\begin{aligned}x^\pm &= \frac{x^1 \pm i x^2}{\sqrt{2}}, \\ A^\pm &= A_\mp = \frac{A^1 \pm i A^2}{\sqrt{2}}, \\ p^\pm &= p_\mp = \frac{p^1 \pm i p^2}{\sqrt{2}}, \\ p_s^2 &\equiv p_1^2 + p_2^2 = 2p^+p^-.\end{aligned} \tag{4.2.4}$$

We define the lightcone gauge in Euclidean signature by the condition $A_- = 0$. This can be obtained from Wick rotation of the standard lightcone gauge in Lorentzian signature. However, in this paper we often think of x^1, x^2 as purely spatial coordinates. In particular, in the finite temperature calculation, the thermal time direction is orthogonal to the complex lightcone direction.

Note that under a rotation in the 12 plane $A_- \to e^{i\alpha}A_-$. Consequently rotations in the 12 plane commute with the condition $A_- = 0$.

Defining the momentum space fields

$$\begin{aligned}A_\mu(x) &= \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} A_\mu(p), \\ \psi_\alpha(x) &= \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \psi_\mu(p), \\ \bar{\psi}^\alpha(p) &= (\psi_\alpha(-p))^*,\end{aligned} \tag{4.2.5}$$

the momentum space action in lightcone gauge is

$$\begin{aligned}S = \frac{-ik}{2\pi} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr} A_3(-p) p_- A_+(p) + \int \frac{d^3p}{(2\pi)^3} i\bar{\psi}(-p) \left(\gamma^\mu p_\mu + M_{bare}\right) \psi(p) \\ - i \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \bar{\psi}(-p)(\gamma^+ A_+(-q) + \gamma^3 A_3(-q))\psi(p+q),\end{aligned} \tag{4.2.6}$$

where $A = A^a T_a$.

It follows from this action that[3]

$$\begin{aligned}\langle \psi(p)_n \bar{\psi}^m(-q)\rangle &= \delta_n^m \frac{-i\gamma^\mu p_\mu}{p^2} \times (2\pi)^3 \delta(p-q),\\ \langle A_3^a(p) A_+^b(-q)\rangle &= -\langle A_+^b(p) A_3^a(-q)\rangle = -\frac{4\pi i}{k}\frac{1}{p^+} \times (2\pi)^3\delta(p-q)\delta^{ab}.\end{aligned} \tag{4.2.7}$$

Adopting the notation

$$\langle A_\mu^a(p) A_\nu^b(-q)\rangle = (2\pi)^3 \delta(p-q) G_{\mu\nu}(p)\delta^{ab},$$

we have

$$G_{+3}(p) = -G_{3+}(p) = \frac{4\pi i}{kp^+}. \tag{4.2.8}$$

Temporal gauge

The temporal gauge is defined by the condition $A_3 = 0$ (Wick rotating the Lorentzian temporal gauge $A_0 = 0$). In this gauge, the gauge field propagator is written in position space as

$$\langle A_i(x) A_j(0)\rangle = \frac{2\pi i}{k}\epsilon_{ij}\mathrm{sign}(x^3)\delta^2(\vec{x}), \tag{4.2.9}$$

and in momentum space,

$$\begin{aligned}\langle A_i(p) A_j(-q)\rangle &= \frac{2\pi}{k}\epsilon_{ij}\left[\frac{1}{p^3+i\epsilon} + \frac{1}{p^3 - i\epsilon}\right](2\pi)^3\delta^3(p-q)\\ &= \frac{4\pi}{k}\epsilon_{ij}\frac{p^3}{(p^3)^2+\epsilon^2}(2\pi)^3\delta^3(p-q).\end{aligned} \tag{4.2.10}$$

Feynman gauge

We may add to the action a covariant gauge fixing term of the form

$$S_F = \frac{k}{4\pi}\int d^3x\, \xi \mathrm{Tr}(\partial_\mu A^\mu)^2. \tag{4.2.11}$$

The Feynman gauge is obtained in the limit $\xi \to \infty$, in which case the propagator for A_μ becomes simply

$$\langle A_\mu^a(p) A_\nu^b(-q)\rangle = -\frac{4\pi}{k}\delta^{ab}\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}(2\pi)^3\delta^3(p-q). \tag{4.2.12}$$

[3]The propagator for a theory whose Euclidean action is

$$S = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\phi_a(-p)Q^{ab}(p)\phi_b(p)$$

is given by

$$\langle \phi_a(p)\phi_b(-q)\rangle = (2\pi)^3\delta(p-q)Q^{-1}_{ab}(p).$$

This rule is correct for both bosons as well as fermions. In the case of the gauge field we have $Q^{3+} = -Q^{+3} = \frac{-ikp_-}{4\pi}$. In the case of the fermionic field we have $Q^{\bar{\psi}\psi} = ip_\mu\gamma^\mu + M_{bare}$ and $Q^{\psi\bar{\psi}} = ip_\mu(\gamma^\mu)^T + M_{bare}$.

4.2.2 Exact fermion propagator on $\mathbb{R}^3$

The starting point of our analysis is the path integral representation of the partition function

$$Z = \int D\psi D\bar{\psi} DA_\mu e^{-S}$$

where the action S is the Euclidean space lightcone gauge action for our field theory, listed explicitly in (4.2.6). The gauge fields appear quadratically in in (4.2.6) and may be integrated out. Integrating out the gauge field from (4.2.6) yields the path integral[4]

$$Z = \int D\psi D\bar{\psi} e^{-S}$$

where the action S is now given by

$$\begin{aligned} S = & i \int \frac{d^3p}{(2\pi)^3} \bar{\psi}(-p)\gamma^\mu p_\mu \psi(p) \\ & + \frac{2\pi i}{k} \int \frac{d^3p}{(2\pi)^3} \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{q^+} \bar{\psi}^m(-p)\gamma^+\psi_n(p-q)\bar{\psi}^n(-r)\gamma^3\psi_m(r+q) \ . \end{aligned} \tag{4.2.13}$$

Starting from the bilocal action (4.2.13), one can conveniently derive the Schwinger-Dyson equation for the fermion self-energy (as in [26]) via

$$0 = \int D\psi D\bar{\psi} \frac{\delta}{\delta\bar{\psi}^m(-p)} \left(e^{-S} \bar{\psi}^n(p') \right) \tag{4.2.14}$$

$$= \int D\psi D\bar{\psi} \left(\delta^n_m \delta^3(p'+p) - \frac{\delta S}{\delta\bar{\psi}^m(-p)} \bar{\psi}^n(p') \right) e^{-S} \tag{4.2.15}$$

which gives the following relation involving the exact fermion propagator and four-point functions,

$$\begin{aligned} (2\pi)^3\delta^3(p'+p) = & i p_\mu\gamma^\mu \langle \psi_m(p)\bar{\psi}^n(p') \rangle \\ & + \frac{2\pi i}{k} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{q^+} \gamma^+ \langle \psi_a(p-q)\bar{\psi}^a(-r)\gamma^3\psi_m(r+q)\bar{\psi}^n(p') \rangle \\ & - \frac{2\pi i}{k} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{q^+} \gamma^3 \langle \psi_a(p-q)\bar{\psi}^a(-r)\gamma^+\psi_m(r+q)\bar{\psi}^n(p') \rangle \ . \end{aligned} \tag{4.2.16}$$

In the large-N limit, this factorizes to yield

$$\begin{aligned} \langle \psi_m(p)\bar{\psi}^n(p') \rangle = & \frac{1}{i p_\mu\gamma^\mu} (2\pi)^3\delta^3(p'+p) \\ & - \frac{1}{i p_\mu\gamma^\mu} \frac{2\pi i}{k} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{q^+} \gamma^+ \langle \psi_a(p-q)\bar{\psi}^a(-r) \rangle \langle \gamma^3\psi_m(r+q)\bar{\psi}^n(p') \rangle \\ & + \frac{1}{i p_\mu\gamma^\mu} \frac{2\pi i}{k} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{q^+} \gamma^3 \langle \psi_a(p-q)\bar{\psi}^a(-r) \rangle \gamma^+ \langle \psi_m(r+q)\bar{\psi}^n(p') \rangle \ . \end{aligned} \tag{4.2.17}$$

[4] In integrating out the gauge field, we absorb the factor coming from the determinant of the gauge field kinetic operator in the normalization of the path integral. In other words, we normalize the pure Chern-Simons partition function to 1.

Let the exact fermion propagator be given by

$$\langle \psi(p)_m \bar{\psi}(-q)^n \rangle = \delta_m^n \frac{1}{ip_\mu \gamma_\mu + M_{bare} + \Sigma} \times (2\pi)^3 \delta(p-q). \tag{4.2.18}$$

Then

$$\Sigma(p) = \frac{N}{2} \int \frac{d^3q}{(2\pi)^3} \left(\gamma^\mu \frac{1}{i\gamma^\alpha q_\alpha + M_{bare} + \Sigma(q)} \gamma^\nu \right) G_{\mu\nu}(p-q), \tag{4.2.19}$$

where

$$\langle A_\mu^a(p) A_\nu^b(-q) \rangle = (2\pi)^3 \delta(p-q) G_{\mu\nu}(p) \delta^{ab}$$

is the bare gluon propagator.[5] Here M_{bare} is the mass term that appears in the bare Lagrangian. In what follows we will adjust M_{bare} to ensure that the physical fermion mass vanishes (this choice corresponds to tuning the theory to the conformality).

The equation (4.2.19) applies in fact in any ghost free gauge in which all gluon interactions vanish. In this section we will solve (4.2.19) in the lightcone gauge.

Exact solution of the gap equation

In the this subsection we will solve the gap equation (4.2.19) in light-cone gauge:

$$\Sigma(p) = -i2\pi\lambda \int \frac{d^3q}{(2\pi)^3} \left(\gamma^3 \frac{1}{i\gamma^\mu q_\mu + M_{bare} + \Sigma(q)} \gamma^+ - \gamma^+ \frac{1}{i\gamma^\mu q_\mu + M_{bare} + \Sigma(q)} \gamma^3 \right) \frac{1}{(p-q)^+}. \tag{4.2.20}$$

Let us first better understand the matrix structure of the self energy Σ. Let A represent an arbitrary 2×2 matrix

$$A = A_I I + A_+ \gamma^+ + A_- \gamma^- + A_3 \gamma^3.$$

For use below and in later sections we define the matrix valued functions of A, $H_+(A)$ and $H_-(A)$ as

$$H_+(A) \equiv \gamma^3 A \gamma^+ - \gamma^+ A \gamma^3 = 2\left(A_I \gamma^+ - A_-\, I\right), \tag{4.2.21}$$

$$H_-(A) \equiv \gamma^3 A \gamma^- - \gamma^- A \gamma^3 = 2\left(-A_I \gamma^- + A_+ I\right)\ . \tag{4.2.22}$$

The gap equation (4.2.20) may be rewritten as

$$\Sigma(p) = -i2\pi\lambda \int \frac{d^3q}{(2\pi)^3} \left(H_+ \left[\frac{1}{i\gamma^\mu q_\mu + M_{bare} + \Sigma(q)} \right] \frac{1}{(p-q)^+} \right). \tag{4.2.23}$$

In the discussion which follows we will sometime abbreviate the notation $\Sigma(p)$ to Σ for brevity. Now let

$$\Sigma = i\Sigma_\mu \gamma^\mu + \Sigma_I I - M_{bare} I. \tag{4.2.24}$$

Using

$$\frac{1}{i\gamma^\mu(q_\mu + \Sigma_\mu) + \Sigma_I} = \frac{-i\gamma^\mu(p_\mu + \Sigma_\mu) + \Sigma_I}{(p+\Sigma)^2 + \Sigma_I^2}$$

[5]The factor of $\frac{1}{2}$ in (4.2.19) has its origin in the $\frac{1}{2}$ on the RHS of (4.2.3).

together with (4.2.21), we may rewrite (4.2.23) as

$$\Sigma(p) = -i4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{\gamma^+\Sigma_I + iI(q+\Sigma(q))_-}{(q_\mu + \Sigma_\mu(q))(q^\mu + \Sigma^\mu(q)) + \Sigma_I(q)^2} \frac{1}{(p-q)^+} \ . \tag{4.2.25}$$

Plugging (4.2.24) into the LHS of (4.2.25) and equating the coefficients of linearly independent matrices, it follows immediately that

$$\Sigma_- = \Sigma_3 = 0, \tag{4.2.26}$$

and that

$$\begin{aligned} \Sigma_+(p) &= -4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{\Sigma_I}{((q+\Sigma(q))^2 + \Sigma_I(q)^2)} \frac{1}{(p-q)^+} \\ \Sigma_I(p) - M_{bare} &= 4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{q_-}{((q+\Sigma(q))^2 + \Sigma_I(q)^2)} \frac{1}{(p-q)^+} \ . \end{aligned} \tag{4.2.27}$$

What can we say about the dependence of $\Sigma_+(p)$ and $\Sigma_I(p)$ on p? First note that the that the RHS of the two equations in (4.2.27) is independent of p^3. It follows that Σ is a function only of the in plane momenta p^1 and p^2 but is independent of p^3. Rotational invariance in the 12 plane and requirement of conformality (i.e. the requirement that no mass scale enter the physical propagator) then together completely fix the momentum dependence of Σ_+ and Σ_I:

$$\begin{aligned} \Sigma_I(p) &= f_0 p_s \\ \Sigma_+(p) &= p_+ g_0 = p^- g_0, \end{aligned} \tag{4.2.28}$$

where

$$p_s = \sqrt{p_1^2 + p_2^2} = \sqrt{2}|p^-| = \sqrt{2}|p^+|, \tag{4.2.29}$$

and f_0 and g_0 are dimensionless numbers (that are functions of the coupling constant λ).

Plugging (4.2.29) and (4.2.26) into (4.2.27) we find

$$\begin{aligned} g_0 &= -\frac{4\pi\lambda}{p^-} \int \frac{d^3q}{(2\pi)^3} \frac{q_s f_0}{q_3^2 + q_s^2(1 + g_0 + f_0^2)} \frac{1}{(p-q)^+} \\ f_0|p| - M_{bare} &= 4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{q^+}{q_3^2 + q_s^2(1 + g_0 + f_0^2)} \frac{1}{(p-q)^+} \ . \end{aligned} \tag{4.2.30}$$

We will now proceed to determine the numbers g_0 and f_0 as functions of λ.

We first note that the integrals on the RHS of (4.2.30) are (power counting) linearly divergent. In order to proceed we need to regulate these divergences. We adopt a regulator that is manifestly Lorentz invariant as well as plausibly gauge invariant. Our regularization procedure is simply the following: we analytically continue all diagrams to $3-\epsilon$ dimensions. Two of these dimensions span the 1-2 plane. We integrate over the remaining $1-\epsilon$ dimensions using the formula

$$\int_{-\infty}^{\infty} \frac{d^{1-\epsilon}x}{a^2 + x^2} = \frac{A^\epsilon \pi}{|a|^{1+\epsilon}} \ . \tag{4.2.31}$$

Here A is a number of order unity which is easily computed. However, as we will see below, none of the integrals we compute in this paper have a $\frac{1}{\epsilon}$ type divergence (this corresponds to an absence of logarithmic divergences, were we to use a momentum cutoff). The only effect of the dimensional regularization cut off procedure, employed in our paper, is to discard linear (and below also cubic) divergences in a gauge and Lorentz invariant manner. For that reason, when we take $\epsilon \to 0$ at the end of the calculation, we effectively set A to unity. Hence we immediately set A to unity instead of carrying it around in our computation. The regularization procedure we employ here is essentially the dimensional reduction scheme used in [131], adapted to our light-cone gauge.

Using (4.2.31) in (4.2.30) yields

$$\begin{aligned} g_0 &= -\frac{2\pi\lambda}{p^-}\frac{f_0}{\sqrt{1+g_0+f_0^2}}\int \frac{d^2q}{(2\pi)^2}\frac{1}{q_s^\epsilon (p-q)^+} \\ f_0 p_s - M_{bare} &= 2\pi\lambda \frac{1}{\sqrt{1+g_0+f_0^2}}\int \frac{d^2q}{(2\pi)^2}\frac{q^+}{q_s^{1+\epsilon}(p-q)^+} \ . \end{aligned} \tag{4.2.32}$$

In order to do the integrals we move to polar coordinates in the 12 plane. The integrals we need to evaluate are

$$\begin{aligned} \int \frac{d^2q}{(2\pi)^2}\frac{q^+}{q_s^{1+\epsilon}(p-q)^+} &= \frac{1}{(2\pi)^2}\int_0^\infty q_s^{1-\epsilon}dq \int_0^{2\pi} d\theta \frac{p_s\cos\theta - q_s}{q_s^2+p_s^2-2p_sq_s\cos\theta} \\ \int \frac{d^2q}{(2\pi)^2 p^-}\frac{q_s^{1-\epsilon}}{q_s^\epsilon (p-q)^+} &= \frac{2}{(2\pi)^2 p_s}\int_0^\infty q_s^{1-\epsilon}dq \int_0^{2\pi} d\theta \frac{p_s - q_s\cos\theta}{q_s^2+p_s^2-2p_sq_s\cos\theta} \ . \end{aligned} \tag{4.2.33}$$

Using contour techniques it is not difficult to verify that for $q > p$

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{q^2+p^2-2pq\cos\theta} &= \frac{2\pi}{q^2-p^2} \\ \int_0^{2\pi} \frac{d\theta\cos\theta}{q^2+p^2-2pq\cos\theta} &= \frac{p}{q}\frac{2\pi}{q^2-p^2} \ . \end{aligned} \tag{4.2.34}$$

It follows that

$$\begin{aligned} \int_0^{2\pi} d\theta \frac{p_sq_s\cos\theta - q_s^2}{q_s^2+p_s^2-2p_sq_s\cos\theta} &= 0 \quad (q_s < p_s) \\ \int_0^{2\pi} d\theta \frac{p_sq_s\cos\theta - q_s^2}{q_s^2+p_s^2-2p_sq_s\cos\theta} &= -2\pi \quad (q_s > p_s) \\ \int_0^{2\pi} d\theta \frac{p_sq_s - q_s^2\cos\theta}{q_s^2+p_s^2-2p_sq_s\cos\theta} &= 2\pi \quad (q_s < p_s) \\ \int_0^{2\pi} d\theta \frac{p_sq_s - q_s^2\cos\theta}{q_s^2+p_s^2-2p_sq_s\cos\theta} &= 0 \quad (q_s > p_s). \end{aligned} \tag{4.2.35}$$

It follows that (4.2.32) reduces to

$$\begin{aligned} g_0 &= -\frac{\lambda f_0}{\sqrt{1+g_0+f_0^2}}, \\ f_0 p_s - M_{bare} &= -\frac{\lambda}{\sqrt{1+g_0+f_0^2}}\int_{p_s}^\infty q^{-\epsilon}dq = \frac{\lambda}{\sqrt{1+g_0+f_0^2}}p_s \ . \end{aligned} \tag{4.2.36}$$

It follows from (4.2.36) that $M_{bare} = 0$ (this is a consequence of our use of dimensional regularization; M_{bare} is linearly divergent in a cut off regulator, as is clear from the second of (4.2.36)). The remaining equations reduce to

$$\begin{aligned} g_0 &= -\frac{\lambda f_0}{\sqrt{1+g_0+f_0^2}}, \\ f_0 &= \frac{\lambda}{\sqrt{1+g_0+f_0^2}} \ . \end{aligned} \tag{4.2.37}$$

The solution to (4.2.37) is remarkably simple

$$\begin{aligned} f_0 &= \lambda, \\ g_0 &= -\lambda^2, \\ g_0 + f_0^2 &= 0 \ . \end{aligned} \tag{4.2.38}$$

In other words, the self energy receives contributions from one and two loop graphs but not at any higher order in perturbation theory! This completes our solution of the gap equation.

We emphasize that our final result (4.2.38) depends in a crucial way on our choice of regularization scheme. Were we, for instance to regulate the integrals (4.2.30) by modifying the gauge boson propagator with a Gaussian damping factor $e^{-\frac{(p-q)^2}{2\Lambda^2}}$, then we would have found $g_0 = 0$ (from the integral over the angle in the vector $p-q$). It is of course quite clear that a crude cut off on the gauge boson momentum does not preserve either gauge or Lorentz invariance. We hope on the other hand that the more sophisticated dimensional regularization scheme preserves both these symmetries. This assumption, which is as yet unproved, is the main weakness in the analysis presented in this section. We will return to this point at the end of the section.

In summary, the exact fermion propagator, at leading order in large N, is given by

$$\langle \psi(p)_m \bar{\psi}(-q)^n \rangle = \delta_m^n \frac{1}{ip_3\gamma^3 + ip_-\gamma^- + i(1-\lambda^2)p_+\gamma^+ + \lambda p_s} \times (2\pi)^3 \delta(p-q). \tag{4.2.39}$$

(Here m, n are color indices and we have suppressed the spinor indices).

Rewriting the field theory as a path integral over singlet fields

In this subsection we will reformulate the path integral that evaluates the partition function of our field theory as a path integral over singlest fields. The new path integral is weakly coupled in the large N limit (the action in terms of the new variables is proportional to N). The gap equation (4.2.20) follows as the classical equation of motion of this large N action.

While the work of this subsection is considerably more complicated than that of the previous subsection, it has one significant advantage; it reveals how the solution of the gap equation is related to the value the partition function of the theory. While the value of the partition function is of no physical significance for the theory on $\mathbb{R}^3$, it is of great significance on $\mathbb{R}^2 \times S^1$ (as it determines the thermal partition function of the theory on $\mathbb{R}^2$). For this reason the results of this subsection will prove very useful in our discussion of the finite temperature partition function in the next section.

We now introduce some convenient shorthand notation. Let

$$M(P,q) = \frac{1}{N}\int \frac{dq^3}{2\pi}\psi_m(\frac{P}{2}+q)\bar{\psi}^m(\frac{P}{2}-q)\,. \tag{4.2.40}$$

M is a 2×2 matrix in spinor space but a singlet in color space. While one of its arguments, P, is a 3 momentum, its second argument q is a 2 momentum (in integral on the RHS of (4.2.40) is over the 3 component of q). (4.2.13) may be rewritten as

$$\begin{aligned}
S &= i\int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\gamma^\mu p_\mu \psi(p) \\
&- \frac{2\pi i N^2}{k}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\frac{1}{(q-q')^+}Tr\left(M(P,q)\gamma^+ M(-P,q')\gamma^3\right) \\
&= i\int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\gamma^\mu p_\mu \psi(p) \\
&- \frac{\pi i N^2}{k}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\frac{1}{(q-q')^+}Tr\left[\left(M(P,q)\gamma^+ M(-P,q')\gamma^3\right) - \left(M(P,q)\gamma^3 M(-P,q')\gamma^+\right)\right]
\end{aligned} \tag{4.2.41}$$

(the flip in sign is due to the fact that we had to take one fermionic field through three others). Expanding the matrix M in a complete basis of 2×2 matrices

$$M = M_+\gamma^+ + M_-\gamma^- + M_3\gamma^3 + M_I I, \tag{4.2.42}$$

we find that (4.2.41) reduces to

$$\begin{aligned}
S &= i\int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\gamma^\mu p_\mu \psi(p) \\
&+ \frac{8\pi i N^2}{k}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\;\frac{1}{(q-q')^+}\; M_-(P,q)M_I(-P,q'),
\end{aligned} \tag{4.2.43}$$

where we have used

$$Tr\left([\gamma^-,\gamma^+]\gamma^3\right) = -4. \tag{4.2.44}$$

Note in particular that M_+ and M_3 drop out of this expression.

We will now rewrite the interaction term (the term quadratic in M) in (4.2.43) in terms of a Lagrange multiplier field

$$\Sigma = \Sigma_+\gamma^+ + \Sigma_I I. \tag{4.2.45}$$

where Σ will turn out to be the self energy of the fermion field. To this end we define the "inverse" Greens function $G^{-1}(p)$ by the requirement that

$$\int \frac{d^2q}{(2\pi)^2}G^{-1}(p-q)\frac{1}{(q-r)^+} = (2\pi)^2\delta^2(p-r)\ . \tag{4.2.46}$$

Note that G^{-1} is an odd function of its argument. Note also that

$$\int \frac{d^2r}{(2\pi)^2}\frac{1}{(q-r)^+}G^{-1}(r-p) = (2\pi)^2\delta^2(q-p)\ . \tag{4.2.47}$$

These are the only properties of G^{-1} that we will need in this paper; in particular we will never need the explicit form of the function G^{-1}.

Now it is obvious that

$$Z = \frac{\int D\psi D\Sigma_- D\Sigma_I e^{-(S+E)}}{\int D\Sigma_- D\Sigma_I e^{-E}}, \tag{4.2.48}$$

where we have chosen

$$\begin{aligned} E = 2 \times \frac{N}{4\pi i\lambda}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2} \\ \left[\left(\Sigma_+(P,q) - 4\pi i\lambda\int \frac{d^2r}{(2\pi)^2}M_I(P,r)\frac{1}{(r-q)^+}\right) \times G^{-1}(q-q')\right. \\ \left.\times\left(\Sigma_I(-P,q') - 4\pi i\lambda\int \frac{d^2r'}{(2\pi)^2}\frac{1}{(q'-r')^+}M_-(-P,r')\right)\right]\ . \end{aligned} \tag{4.2.49}$$

Note that E is a function of the two new Lagrange multiplier fields Σ_- and Σ_I. The path integral in the denominator in (4.2.48) is simply a number of order unity and we will omit to write it in the equations that follow. The effective action in the numerator, $S+E$, evaluates to

$$\begin{aligned} S = i\int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\gamma^\mu p_\mu\psi(p) \\ + \int \frac{d^3P}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\bar{\psi}(\frac{P}{2}-q)\Sigma(-P,q)\psi(\frac{P}{2}+q) \\ + \frac{N}{2\pi i\lambda}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\Sigma_+(P,q)G^{-1}(q-q')\Sigma_I(-P,q')\ . \end{aligned} \tag{4.2.50}$$

In the second line above we have used the fact that

$$-2(\Sigma_+M_- + \Sigma_I M_I) = -Tr\Sigma M = -\frac{1}{N}\int \frac{dq_3}{2\pi}Tr\Sigma\psi\bar{\psi} = \frac{1}{N}\int \frac{dq_3}{2\pi}\bar{\psi}\Sigma\psi$$

Using (4.2.44) the last line in (4.2.43) may be rewritten as a trace, yielding

$$\begin{aligned} S = i\int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\gamma^\mu p_\mu\psi(p) \\ + \int \frac{d^3P}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\bar{\psi}(\frac{P}{2}-q)\Sigma(-P,q)\psi(\frac{P}{2}+q) \\ - \frac{N}{8\pi i\lambda}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(\gamma^-\Sigma(P,q)\gamma^3\Sigma(-P,q')\right)\ . \end{aligned} \tag{4.2.51}$$

The action (4.2.51) may be rewritten as

$$\begin{aligned} S = & \int \frac{d^3P}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\bar{\psi}(\frac{P}{2}-q)\left((2\pi)^3\delta^3(P)i\gamma^\mu q_\mu + \Sigma(-P,q)\right)\psi(\frac{P}{2}+q) \\ & - \frac{N}{8\pi i\lambda}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(\gamma^-\Sigma(P,q)\gamma^3\Sigma(-P,q')\right) \ . \end{aligned} \tag{4.2.52}$$

The dependence of (4.2.52) on fermionic fields is quadratic, so the later may be integrated out. Performing this operation yields

$$Z = \int D\Sigma_- D\Sigma_I e^{-S} \tag{4.2.53}$$

where

$$\begin{aligned} S = & -N\mathrm{Tr}\ln\left((2\pi)^3\delta^3(P)i\gamma^\mu q_\mu + \Sigma(-P,q)\right) \\ & - \frac{N}{8\pi i\lambda}\int \frac{d^3P}{(2\pi)^3}\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(\gamma^-\Sigma(P,q)\gamma^3\Sigma(-P,q')\right) \ . \end{aligned} \tag{4.2.54}$$

Notice that (4.2.54) is written purely in terms of singlet fields, and is multiplied by an overall factor of N. (4.2.54) represents an exact rewriting of the partition function of the original theory as a partition function over the singlet fields Σ; this path integral is weakly coupled in the large N limit.

The action, (4.2.54), is somewhat formal, as it is written in terms of a determinant over an infinite dimensional matrix. However the equivalent of (4.2.54) is much simpler for translationally invariant Σ configurations of the form

$$\Sigma(P,q) = (2\pi)^3\delta(P)\Sigma(q).$$

The form of this action is perhaps most clearly obtained by retreating to (4.2.52), which reduces, for translationally Σ configurations to

$$S = \int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\left[i\gamma^\mu p_\mu + \Sigma(p)\right]\psi(p) - \frac{NV}{8\pi i\lambda}\int \frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(\gamma^-\Sigma(q)\gamma^3\Sigma(q')\right) \ . \tag{4.2.55}$$

V here is a factor of the volume of spacetime, and we have used

$$\left[(2\pi)^3\delta(P)\right]^2 = V(2\pi)^3\delta(P)$$

in the last term of the last line.

Integrating out the fermions in (4.2.55) yields a very explicit special case of (4.2.54)

$$S = -NV\int \frac{d^3q}{(2\pi)^3}\mathrm{Tr}\ln\left(i\gamma^\mu q_\mu + \Sigma(p)\right) - \frac{NV}{8\pi i\lambda}\int \frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(\gamma^-\Sigma(q)\gamma^3\Sigma(q')\right) \ . \tag{4.2.56}$$

While all terms in the action in (4.2.56) are proportional to N, the fields in that action are gauge singlets. At leading order in the large N expansion it follows that the free energy for our theory may evaluated simply by minimizing (4.2.56) w.r.t. Σ. The variational equation we encounter in this minimization process is

$$\int \mathrm{Tr}\left[\frac{d^2q}{(2\pi)^2}\delta\Sigma[q]\int\left(\frac{d^2q'}{(2\pi)^2}\frac{-1}{8\pi i\lambda}G^{-1}(q-q')\left(\gamma^3\Sigma(q')\gamma^- - \gamma^-\Sigma(q')\gamma^3\right) - \int\frac{dq^3}{2\pi}\frac{1}{i\gamma^\mu q_\mu+\Sigma}\right)\right] = 0\,. \tag{4.2.57}$$

In terms of the function H_- defined (4.2.22)

$$\int \mathrm{Tr}\left[\frac{d^2q}{(2\pi)^2}\delta\Sigma[q]\int\left(\frac{d^2q'}{(2\pi)^2}\frac{-1}{8\pi i\lambda}G^{-1}(q-q')\left(H_-\left(\Sigma(q')\right)\right) - \int\frac{dq^3}{2\pi}\frac{1}{i\gamma^\mu q_\mu+\Sigma}\right)\right] = 0\,. \tag{4.2.58}$$

The equation (4.2.58) is of the form

$$\int \mathrm{Tr}\left[\frac{d^2q}{(2\pi)^2}\delta\Sigma[q]B(q)\right] = 0, \tag{4.2.59}$$

where

$$B(q) = \int\left(\frac{d^2q'}{(2\pi)^2}\frac{-1}{8\pi i\lambda}G^{-1}(q-q')\left(H_-\left(\Sigma(q')\right)\right) - \int\frac{dq^3}{2\pi}\frac{1}{i\gamma^\mu q_\mu+\Sigma}\right)\,.$$

As $\delta\Sigma$ is an arbitrary matrix of the form (4.2.45) it follows that

$$B_-(q) = B_I(q) = 0,$$

i.e., that

$$H_+(B) = 0.$$

(see (4.2.21)) Using the fact that

$$H_+(H_-(\Sigma)) = 4\Sigma,$$

and integrating both sides of (4.2.58) against the kernel $\frac{1}{(p-q)^+}$ and using the defining property of the function G^{-1}, it follows from $H_+(B))=0$ that

$$\Sigma(p) = -2\pi i\lambda\int\frac{d^3q}{(2\pi)^3}\left(\gamma^3\frac{1}{i\gamma^\mu q_\mu+\Sigma}\gamma^+ - \gamma^+\frac{1}{i\gamma^\mu q_\mu+\Sigma}\gamma^3\right)\frac{1}{(p-q)^+}, \tag{4.2.60}$$

in precise agreement with (4.2.20).

The value of the Euclidean action on the saddle point, (4.2.56), may be rewritten as

$$\begin{aligned}
S &= -NV\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}\ln\left(i\gamma^\mu q_\mu+\Sigma(q)\right) + \frac{NV}{16\pi i\lambda}\int\frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}G^{-1}(q-q')Tr\left(H_-[\Sigma(q)]\Sigma(q')\right)\\
&= -NV\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}\left[\ln\left(i\gamma^\mu q_\mu+\Sigma(q)\right) + \frac{1}{8}H_-(\Sigma(q))H_+\left(\left(\frac{1}{i\gamma^\mu q_\mu+\Sigma(q)}\right)\right)\right]\\
&= -NV\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}\left[\ln\left[i\gamma^\mu q_\mu+\Sigma(q)\right] - \frac{1}{2}\Sigma(q)\left(\frac{1}{i\gamma^\mu q_\mu+\Sigma(q)}\right)\right],
\end{aligned} \tag{4.2.61}$$

where we have used the equation of motion in going from first to the second line. In going from the second to the third line we have used the fact that for an arbitrary matrix A

$$\mathrm{Tr}\left(H_-(\Sigma)H_+(A)\right) = -4\mathrm{Tr}\left(\Sigma A\right).$$

4.2.3 The finite temperature theory

In this section we study the logarithm of the path integral of our system on $R^2 \times S^1$ where the circumference of the S^1 is taken to be β. This path integral determines the free energy of the field theory at temperature $T = \beta^{-1}$.

The formulas that determine the path integral on $\mathbb{R}^2 \times S^1$ are straightforward generalizations of the formulas on $\mathbb{R}^3$. Every equation in subsection 4.2.2 carries through with the replacement

$$\int \frac{dp_3}{(2\pi)} f(p_3) \to \frac{1}{\beta} \sum_{n \in Z + \frac{1}{2}} f(\frac{2\pi n}{\beta}),$$

$$V \to V_2 \beta$$

so that

$$V \int \frac{d^3p}{(2\pi)^3} \to V_2 \int \frac{d^2p}{(2\pi)^2} \sum_n .$$

In particular the Euclidean action is given by

$$S = N V_2 \sum_n \int \frac{d^2q}{(2\pi)^2} \mathrm{Tr} \left[\ln \left[i\gamma^\mu q_\mu + \Sigma_T(q) \right] - \frac{1}{2} \Sigma_T(q) \left(\frac{1}{i\gamma^\mu q_\mu + \Sigma_T(q)} \right) \right], \qquad (4.2.62)$$

where T, the temperature is β^{-1} and the function $\Sigma_T(q)$ obeys the gap equation

$$\Sigma_T(p) = -2\pi i \lambda \, \frac{1}{\beta} \sum_n \int \frac{d^2q}{(2\pi)^2} \left(\gamma^3 \frac{1}{i\gamma^\mu q_\mu + \Sigma_T(q)} \gamma^+ - \gamma^+ \frac{1}{i\gamma^\mu q_\mu + \Sigma_T(q)} \gamma^3 \right) \frac{1}{(p-q)^+}, \qquad (4.2.63)$$

where n is an integer and

$$q^3 = \frac{2\pi(n + \frac{1}{2})}{\beta}.$$

In order to determine the free energy at finite temperature T, we need to solve the gap equation (4.2.63) and plug the solution into (4.2.62). We take up these exercises in turn.

The finite temperature gap equation

As in subsection 4.2.2 it follows immediately that Σ_T is a linear combination of γ^+ and I, and that it is independent of p_3. Rotational symmetry and the constraints of conformality then imply

$$\Sigma_T(p) + M_{bare} I = f(\beta p_s) p_s I + i g(\beta p_s) p^- \gamma^+$$

for some as yet unknown functions $f(\beta p_s)$ and $g(\beta p_s)$. Note that the new dimensionful scale β, now allows f and g to be functions of p_s, generalizing the pure numbers f_0 and g_0 of the previous section. The zero temperature results of the previous subsection imply that

$$\begin{aligned} \lim_{y \to \infty} f(y) &= f_0 = \lambda, \\ \lim_{y \to \infty} g(y) &= g_0 = -\lambda^2 \ . \end{aligned} \qquad (4.2.64)$$

The analogue of (4.2.27) is

$$p_s f(p_s\beta) - M_{bare} = 4\pi\frac{\lambda}{\beta}\int\sum_n \frac{d^{2+\epsilon}q}{(2\pi)^2}\frac{q^+}{\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2 + q_s^2(1+g(q_s\beta)+|f(q_s\beta)|^2)}\frac{1}{(p-q)^+}, \tag{4.2.65}$$

$$g(p_s\beta)p^- = -4\pi\frac{\lambda}{\beta}\int\sum_n \frac{d^{2+\epsilon}q}{(2\pi)^2}q_s\frac{f(q_S\beta)}{\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2 + q_s^2(1+g+f^2(q_s\beta))}\frac{1}{(p-q)^+}\ . \tag{4.2.66}$$

The summations in these equations are easily carried out using the formula

$$\int dq^\epsilon \sum_{n=-\infty}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2 + a^2 + q^2} = \frac{\pi}{|a|^{1-\epsilon}}\tanh(\pi|a|), \tag{4.2.67}$$

(this is the analogue of (4.2.31) in the previous section - as in the previous section we have set ϵ to zero in every place where it will be inessential for regularization) yielding

$$fp_s - M_{bare} = 2\pi\lambda\int\frac{d^2qq_s^{-\epsilon}}{(2\pi)^2}\tanh\left(\frac{\beta q_s}{2}\sqrt{1+g+f^2}\right)\frac{q^+}{q_s(p-q)^+\sqrt{1+g+f^2}}, \tag{4.2.68}$$

$$gp^- = -2\pi\lambda\int\frac{d^2qq^{-\epsilon}}{(2\pi)^2}\tanh\left(\frac{\beta q_s}{2}\sqrt{1+g+f^2}\right)\frac{f}{\sqrt{1+g+f^2}}\frac{1}{(p-q)^+}, \tag{4.2.69}$$

where we have left implicit the fact that the f and g are functions of p_s on the LHS of (4.2.68)(4.2.69), but are functions of q_s on the RHS of the same equations.

In each of (4.2.68) and (4.2.69) we move to polar coordinates and use use (4.2.34) to perform the angular integrals to obtain

$$fp_s = -\lambda\int_p^\infty q^{-\epsilon}dq_s\tanh\left(\frac{\beta q}{2}\sqrt{1+g+f^2}\right)\frac{1}{\sqrt{1+g+f^2}}, \tag{4.2.70}$$

$$g = -2\lambda\int_0^p\frac{q_s^{1-\epsilon}dq_s}{p_s^2}\tanh\left(\frac{\beta q_s}{2}\sqrt{1+g+f^2}\right)\frac{f}{\sqrt{1+g+f^2}}\ . \tag{4.2.71}$$

Adding and subtracting $q^{-\epsilon}$ from the integrand of (4.2.70) and doing the integral on the trivial piece we find

$$f = \lambda - \lambda\int_p^\infty\frac{dq_s}{p_s}\left(\frac{\tanh\left(\frac{\beta q_s}{2}\sqrt{1+g+f^2}\right)}{\sqrt{1+g+f^2}} - 1\right), \tag{4.2.72}$$

$$g = -2\lambda\int_0^p\frac{q_s dq_s}{p_s^2}\frac{\tanh\left(\frac{\beta q_s}{2}\sqrt{1+g+f^2}\right)f}{\sqrt{1+g+f^2}}\ . \tag{4.2.73}$$

In terms of the variable

$$x = \frac{q}{p},$$

$$f(y) = \lambda - \lambda \int_1^\infty dx \left(\frac{\tanh\left(\frac{yx}{2}\sqrt{1+g(yx)+f(yx)^2}\right)}{\sqrt{1+g(yx)+f(yx)^2}} - 1 \right), \tag{4.2.74}$$

$$g(y) = -2\lambda \int_0^1 xdx \left(\frac{\tanh\left(\frac{yx}{2}\sqrt{1+g(yx)+f(yx)^2}\right) f(yx)}{\sqrt{1+g(yx)+f(yx)^2}} \right), \tag{4.2.75}$$

where the variable

$$y = p\beta.$$

Equivalently

$$\begin{aligned} f(y) &= \lambda - \frac{\lambda}{y} \int_y^\infty dx \left(\frac{\tanh\left(\frac{x}{2}\sqrt{1+g(x)+f(x)^2}\right)}{\sqrt{1+g(x)+f(x)^2}} - 1 \right) \\ g(y) &= -2\frac{\lambda}{y^2} \int_0^y xdx \frac{\tanh\left(\frac{x}{2}\sqrt{1+g(x)+f(x)^2}\right) f(x)}{\sqrt{1+g(x)+f(x)^2}} . \end{aligned} \tag{4.2.76}$$

The exact solution

Quite remarkably it is possible to find the exact solution to (4.2.76). We start with (4.2.76) written in the form

$$\begin{aligned} y\left(f(y) - f_0\right) &= -\lambda \int_y^\infty dx \left(\frac{\tanh\left(\frac{x}{2}\sqrt{1+g(x)+f(x)^2}\right)}{\sqrt{1+g(x)+f(x)^2}} - 1 \right) \\ y^2 g(y) &= -2\lambda \int_0^y xdx \frac{\tanh\left(\frac{x}{2}\sqrt{1+g(x)+f(x)^2}\right) f(x)}{\sqrt{1+g(x)+f(x)^2}}. \end{aligned} \tag{4.2.77}$$

Differentiating both equations w.r.t. y we obtain

$$\begin{aligned} yf'(y) + f(y) &= \lambda \frac{\tanh(\frac{y}{2}\sqrt{1+g(y)+f(y)^2})}{\sqrt{1+g+f^2}}, \\ yg'(y) + 2g(y) &= -2\lambda f(y) \frac{\tanh(\frac{y}{2}\sqrt{1+g(y)+f(y)^2})}{\sqrt{1+g+f^2}}. \end{aligned} \tag{4.2.78}$$

Multiplying the first equation by $2f$ and add it to the second equation, we cancel the RHS and obtain

$$y\frac{d}{dy}(g+f^2) + 2(g+f^2) = 0. \tag{4.2.79}$$

From this we solve

$$g(y) + f(y)^2 = \frac{c}{y^2}, \tag{4.2.80}$$

where c is a constant. Now the first equation in (4.2.78) becomes simply

$$yf'(y) + f(y) = \lambda \frac{\tanh(\frac{y}{2}\sqrt{1+\frac{c}{y^2}})}{\sqrt{1+\frac{c}{y^2}}} \ . \tag{4.2.81}$$

Integrating we have

$$\begin{aligned} f(y) &= \frac{\lambda}{y}\int_0^y dz \frac{\tanh(\frac{z}{2}\sqrt{1+\frac{c}{z^2}})}{\sqrt{1+\frac{c}{z^2}}} + \frac{\tilde{c}}{y} = \frac{2\lambda}{y}\ln\left(\frac{\cosh\left[\frac{1}{2}\sqrt{c+y^2}\right]}{\cosh\left[\frac{\sqrt{c}}{2}\right]}\right) + \frac{\tilde{c}}{y} \\ g(y) &= \frac{c}{y^2} - f(y)^2. \end{aligned} \tag{4.2.82}$$

Here $\tilde{c}$ is another integration constant.

While the solution to a differential equation depends on integration constants, the solution to an integral equation is unique (it does not have undetermined integration constants). The appearance of c and $\tilde{c}$ in our solutions above is an artifact of our having solved by converting the integral equation into a differential equation. The integral equations (4.2.76) are actually solved by (4.2.82) only for a particular choice of c and $\tilde{c}$.

We first note that the function f must tend to f_0 at large y. This is automatic in all our solutions (it does not impose any constraints on c or $\tilde{c}$). However a further requirement is that the expansion of f about this constant value (at large y) should start at $\frac{1}{y^2}$ rather than $\frac{1}{y}$ (this follows immediately upon plugging (4.2.80) into the RHS of the first of (4.2.77); the RHS of that equation is manifestly $\propto \frac{1}{y}$). The requirement that

$$f(y) = f_0 + \mathcal{O}(1/y^2) \tag{4.2.83}$$

determines

$$\tilde{c} = 2\lambda \ln\left(2\cosh\frac{\sqrt{c}}{2}\right). \tag{4.2.84}$$

Let us now turn to the small y behavior of f and g. At small y

$$\begin{aligned} f(y) &= \frac{\tilde{c}}{y} + \mathcal{O}(y), \\ g(y) &= \frac{c-\tilde{c}^2}{y^2} + \mathcal{O}(y^0). \end{aligned} \tag{4.2.85}$$

Plugging (4.2.80) into the RHS of the second of (4.2.77), however, we find that the RHS evaluates to $\mathcal{O}(y^2)$ at small y implying that $g(y) = \mathcal{O}(y^0)$ at small y. It follows that

$$c = \tilde{c}^2. \tag{4.2.86}$$

Plugging this relation into (4.2.84) yields the following equation for $\tilde{c}$

$$\frac{\tilde{c}}{2\lambda} = \ln\left(2\cosh\frac{\tilde{c}}{2}\right). \tag{4.2.87}$$

Note that $\tilde{c}$ is an odd function of λ. $|\tilde{c}|$ is a monotonically increasing function of λ, which diverges at $|\lambda| = 1$. (4.2.87) has no solution for $|\lambda| > 1$, indicating that the theory does not exist for $|k| > N$. At leading order in at small λ we have

$$\tilde{c} = 2\lambda \ln 2 + \mathcal{O}(\lambda^3). \tag{4.2.88}$$

As λ approaches unity we have

$$\tilde{c} = \ln \frac{2}{(1-\lambda)} - \ln \left(\ln \frac{2}{1-\lambda} \right) + \mathcal{O}\left(\ln \ln \ln \frac{2}{1-\lambda} \right). \tag{4.2.89}$$

In order to physically interpret the divergence of $\tilde{c}$ as $\lambda \to 1$ note that the exact thermal propagator has a pole whenever

$$p^2 + \tilde{c}^2 T^2 = 0.$$

In other words $\tilde{c}$ has a simple physical interpretation; it is the thermal mass of the field ψ in units of the temperature. It follows that the fermion thermal mass diverges in this limit $\lambda \to 1$.

Using (4.2.87) and (4.2.86) we may rewrite our solutions for f and g as

$$\begin{aligned} f(y) &= \frac{2\lambda}{y} \ln \left(2 \cosh \frac{\sqrt{\tilde{c}^2 + y^2}}{2} \right), \\ g(y) &= \frac{\tilde{c}^2}{y^2} - f(y)^2. \end{aligned} \tag{4.2.90}$$

with $\tilde{c}$ given by (4.2.87). In the large y limit

$$\begin{aligned} f(y) &= \lambda \sqrt{1 + \frac{\tilde{c}^2}{y^2}} + \mathcal{O}(e^{-y}), \\ g(y) &= -\lambda^2 + \frac{\tilde{c}^2 \left(1 - \lambda^2\right)}{y^2} + \mathcal{O}(e^{-y}), \end{aligned} \tag{4.2.91}$$

while at small y

$$\begin{aligned} f(y) &= \frac{\tilde{c}}{y} + \frac{\lambda y}{2\tilde{c}} \tanh \left(\frac{\tilde{c}}{2} \right) + \mathcal{O}(y^3), \\ g(y) &= -\lambda \tanh \left(\frac{\tilde{c}}{2} \right) + \mathcal{O}(y^2). \end{aligned} \tag{4.2.92}$$

Free energy as a function of temperature

As we have explained above, the path integral of our theory on the manifold $\mathbb{R}^2 \times S^1$, with the circumference of the S^1 equal to β, is given by e^{-S_T} where

$$S_T = N V_2 \sum_n \int \frac{d^2 q}{(2\pi)^2} \mathrm{Tr} \left[\ln \left[i\gamma^\mu q_\mu + \Sigma_T(q) \right] - \frac{1}{2} \Sigma_T(q) \left(\frac{1}{i\gamma^\mu q_\mu + \Sigma_T(q)} \right) \right]. \tag{4.2.93}$$

Let us define

$$S_0 = -NV_2\beta \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[\ln\left[i\gamma^\mu q_\mu + \Sigma(q)\right] - \frac{1}{2}\Sigma(q)\left(\frac{1}{i\gamma^\mu q_\mu + \Sigma(q)}\right)\right]. \tag{4.2.94}$$

Then the partition function

$$Z = \mathrm{Tr} e^{-\beta H}$$

of our system in a flat spatial box of volume V_2 is given by

$$\ln Z = S_0 - S_T,$$

so that the finite temperature free energy, $F(T)$, of the theory is given by

$$F(T) = \frac{S_T - S_0}{\beta}.$$

We will now proceed to use our exact solution to the finite temperature gap equation to compute $S_T - S_0$.

Explicit evaluation of the free energy

In order to find an explicit expression for the free energy, we find it convenient to use the expressions in the second line of (4.2.93) and (4.2.94). $S_T - S_0$ may be written as

$$\begin{aligned} S_T - S_0 = &-NV_2 \sum_n \int \frac{d^2q}{(2\pi)^2} \mathrm{Tr}\ln\left(i\gamma^\mu q_\mu + \Sigma_T(q)\right) + NV_2 \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\ln\left(i\gamma^\mu q_\mu + \Sigma_T(q)\right) \\ &- NV_2\beta \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\ln\frac{(i\gamma^\mu q_\mu + \Sigma_T(q))}{(i\gamma^\mu q_\mu + \Sigma(q))} \\ &+ \frac{NV_2}{2} \sum_n \int \frac{d^2q}{(2\pi)^2} \mathrm{Tr}\left(\frac{q_s f(\beta q_s) + ig(\beta q_s) q_+ \gamma^+}{i\gamma^\mu q_\mu + \Sigma_T(q)}\right) - \frac{NV_2\beta}{2} \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left(\frac{f_0 q_s + ig_0 p_+ \gamma^+}{i\gamma^\mu q_\mu + \Sigma(q)}\right). \end{aligned} \tag{4.2.95}$$

The integral on the first line of (4.2.95) is convergent and evaluates to

$$\begin{aligned} &- NV_2 \sum_n \int \frac{d^2q}{(2\pi)^2} \ln\left(q_3^2 + q_s^2 + \tilde{c}^2T^2\right) + NV_2\beta \int \frac{d^3q}{(2\pi)^3} \ln\left(q_3^2 + q_s^2 + \tilde{c}^2T^2\right) \\ &= -2NV_2 \int \frac{d^2q}{(2\pi)^2} \ln\left(1 + e^{-\beta\sqrt{\tilde{c}^2T^2 + q_s^2}}\right) \\ &= -2NT^2V_2 \int \frac{d^2x}{(2\pi)^2} \ln\left(1 + e^{-\sqrt{\tilde{c}^2 + x_s^2}}\right) \\ &= -\frac{2NT^2V_2}{2\pi} \int_{|\tilde{c}|}^{\infty} dy\, y \ln\left(1 + e^{-y}\right). \end{aligned} \tag{4.2.96}$$

The second line of (4.2.95) has a linear divergence, which however disappears in our dimensional reduction scheme

$$\begin{aligned}
&- NV_2\beta \int \frac{d^{3-\epsilon}q}{(2\pi)^3} \ln \frac{\left(q^2 + \tilde{c}^2 T^2\right)}{(q^2)} \\
&= -NV_2 T^2 |\tilde{c}|^3 \int \frac{d^{3-\epsilon}y}{(2\pi)^3} \ln \frac{y^2+1}{y^2} \\
&= -\frac{NV_2 T^2 |\tilde{c}|^3}{2\pi^2} \int_0^\infty dy y^{2-\epsilon} \ln\left(1 + \frac{1}{y^2}\right) = \frac{NV_2 T^2 |\tilde{c}|^3}{6\pi},
\end{aligned} \tag{4.2.97}$$

where in the last step we integrated by parts and used

$$\int_0^\infty \frac{y^{2-\epsilon}}{1+y^2} = -\frac{\pi}{2}$$

in the limit of small ϵ. Let us now turn to the third line of (4.2.95). The second term in the third line simply vanishes under dimensional regularization. The first term in the third line may be evaluated as follows

$$\begin{aligned}
&\frac{NV_2}{2} \sum_n \int \frac{d^2q}{(2\pi)^2} \frac{2q_s^2 f^2(\beta q_s) + q_s^2 g(\beta q_s)}{q_3^2 + q_s^2 + T^2\tilde{c}^2} \\
&= \frac{NV_2}{4} \int \frac{d^2q}{(2\pi)^2} \frac{\left(2q_s^2 f^2(\beta q_s) + q_s^2 g(\beta q_s)\right) \tanh \frac{\sqrt{q^2+\tilde{c}^2}}{2}}{\sqrt{q_s^2 + T^2\tilde{c}^2}} \\
&= \frac{NV_2 T^2}{4} \left[\int \frac{d^2q}{(2\pi)^2} \frac{\left(2q_s^2 f^2(q_s) + q_s^2 g(q_s)\right) \tanh \frac{\sqrt{q^2+\tilde{c}^2}}{2} - \lambda^2\left(\tilde{c}^2 + q^2\right) - \tilde{c}^2}{\sqrt{q_s^2 + \tilde{c}^2}} \right. \\
&\quad \left. + \int \frac{d^2q}{(2\pi)^2} \frac{\lambda^2\left(\tilde{c}^2 + q^2\right) + \tilde{c}^2}{\sqrt{q_s^2 + \tilde{c}^2}} \right].
\end{aligned} \tag{4.2.98}$$

The first term in (4.2.98) is finite and evaluates to

$$\frac{NV_2 T^2}{4\pi} |\tilde{c}|^3 \left(\frac{\lambda^2}{6} - \frac{1}{6|\lambda|} - \frac{1}{2|\lambda|} + \frac{1}{2} \right). \tag{4.2.99}$$

The second term in (4.2.98) is divergent. In dimensional regularization it evaluates to

$$-\frac{NV_2 T^2 |\tilde{c}|^3}{4\pi} \left(\frac{\lambda^2}{6} + \frac{1}{2} \right). \tag{4.2.100}$$

The third line of (4.2.95) is given by the sum of (4.2.99) and (4.2.100) and evaluates to

$$-\frac{NV_2 T^2 |\tilde{c}|^3}{6\pi|\lambda|}. \tag{4.2.101}$$

Putting it all together we find that the free energy is given by

$$F = -\frac{NV_2T^3}{6\pi}\left[|\tilde{c}|^3\frac{1-|\lambda|}{|\lambda|} + 6\int_{|\tilde{c}|}^{\infty} dy\ y\ln\left(1+e^{-y}\right)\right]. \tag{4.2.102}$$

Although it is not manifest, the free energy can also be written in the form

$$F = -\frac{NV_2T^3}{6\pi}\left[\tilde{c}^3\frac{1-\lambda}{\lambda} + 6\int_{\tilde{c}}^{\infty} dy\ y\ln\left(1+e^{-y}\right)\right]. \tag{4.2.103}$$

and is an analytic function of λ in the interval $(-1,1)$.[6] To see this more explicitly, note that we can write the term in square brackets in (4.2.102) as

$$\begin{aligned}&\frac{|\tilde{c}|^3}{|\lambda|} - |\tilde{c}|^3 + 6\int_0^{\infty} dy\ y\ln\left(1+e^{-y}\right) + 6\int_0^{|\tilde{c}|} dy\left(\frac{y^2}{2} - y\ln\left(2\cosh\frac{y}{2}\right)\right)\\ &= \frac{9}{2}\zeta(3) + \frac{\tilde{c}^3}{\lambda} - 6\int_0^{|\tilde{c}|} dy\ y\ln\left(2\cosh\frac{y}{2}\right),\end{aligned} \tag{4.2.104}$$

where in the last line we used $|\tilde{c}|^3/|\lambda| = \tilde{c}^3/\lambda$, as follows from (4.2.87). Note that the integral in the last term is clearly an even function of $|\tilde{c}|$, and so the absolute value can be omitted. This shows that one may effectively rewrite (4.2.102) as in (4.2.103), which is manifestly analytic. Indeed its small λ expansion is given by

$$F = -NV_2T^3\left[\frac{3\zeta(3)}{4\pi} - \frac{2(\log 2)^3}{3\pi}\lambda^2 - \frac{(\log 2)^4}{2\pi}\lambda^4 + \mathcal{O}(\lambda^6)\right]. \tag{4.2.105}$$

Note that it contains only even powers of λ, consistently with parity. A plot of the free energy as a function of λ is given in Fig. 4.1. We see that $-F$ decreases monotonically from the free field value $-F = \frac{3NV_2T^3}{4\pi}\zeta(3)$ to zero at $\lambda = 1$.

In the limit that $|\lambda| \to 1$, $|\tilde{c}|$ is large and the integral in (4.2.102) may be approximated by

$$\int_{|\tilde{c}|}^{\infty} dy\ y\ln\left(1+e^{-y}\right) = |\tilde{c}|e^{-|\tilde{c}|} + e^{-|\tilde{c}|} + \mathcal{O}(e^{-2\tilde{c}|}).$$

At leading nontrivial order this evaluates to

$$\tilde{c}^2\frac{1-|\lambda|}{2},$$

and the free energy is given by

$$F = -\frac{NV_2T^3(1-|\lambda|)}{6\pi}\left[|\tilde{c}|^3 + 3\tilde{c}^2 + \mathcal{O}(\tilde{c})\right], \qquad |\lambda| \to 1, \tag{4.2.106}$$

where $\tilde{c}$ is given by (4.2.89).

[6]Theories with massless bosons usually do not have an analytic expansion of their free energy in terms of the coupling constant. Such non-analytic behavior in Chern-Simons-matter theories with massless bosons was observed explicitly e.g. in [56], [132]. This non-analyticity has its origin in infrared divergences which are absent in our theory because the only propagating degrees of freedom are the fermions which lack a zero-mode along the thermal circle.

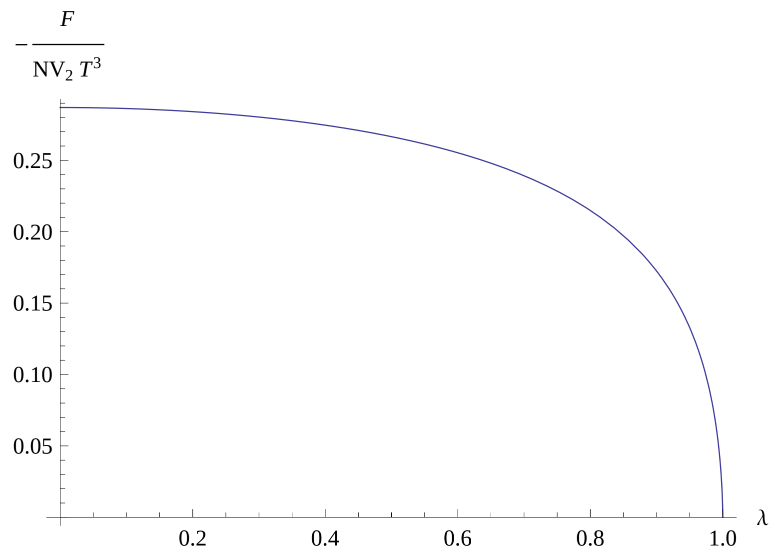

Figure 4.1: The free energy on $S^1 \times \mathbb{R}^2$ as a function of λ.

4.2.4 Consistency of our gauge and regularization scheme

As we have emphasized on multiple occasions, we have obtained the beautiful result (4.2.102) by employing a rather unusual gauge (a lightcone like gauge in Euclidean space) and the regularization scheme of dimensional reduction, which we have assumed preserves the gauge invariance of our theory. In this section we list the independent evidence that the procedure employed in this section defines a sensible and Lorentz invariant theory.

1. The exact fermion propagator (4.2.39) develops poles only when $p^2 = 0$, a condition that is Lorentz invariant. This is a necessary condition for the Lorentz invariance of fermion scattering processes in our theory. [7] The Lorentz invariance of the poles of the propagator is far from automatic, and is in fact violated in several regularization schemes. In, for instance, the regularization scheme described just below (4.2.38) we find $g_0 = 0$ but $f_0 \neq 0$ (in fact f_0 obeys the equation $f_0 = -\frac{\lambda}{\sqrt{1+f_0^2}}$ with this regulator). The poles of the fermion propagator with this regulator occur at $p_3^2 + p_s^2(1 + f_0)$ and are not Lorentz invariant.

2. In Appendix C of [10] we have demonstrated that the expectation value, at one loop, of the gauge invariant Wilson line operator is Lorentz invariant and moreover agrees with the result obtained in the manifestly Lorentz invariant Feynman gauge.

[7] Particle scattering is, strictly speaking, not well defined in our theory (or any conformal theory) due to infrared divergences. We could cure these divergences in our theory softly breaking conformal invariance with a fermion mass. Scattering processes in this deformed theory are presumably well defined. The poles of the mass deformed theory are clearly physical and must be Lorentz invariant. As our computation of the propagator of massless theory, presented in this paper, exhibits no IR divergences, it must equal the zero mass limit of the mass deformed propagator, and hence must be Lorentz invariant.

3. Our results above indicate that our theory is infinitely strongly coupled at $\lambda = 1$, and that the theory does not exist (at least as a conformally invariant theory) for $\lambda > 1$. There is in fact a very simple interpretation of this result. It is well-known that the bare Chern-Simons level can acquire a finite shift at 1-loop order in perturbation theory. This effect is regularization dependent. It was demonstrated by [131] that the Chern-Simons matter theory defined using the dimensional reduction scheme, employed in this paper, does not acquire a 1-loop shift of k. On the other hand, if the theory is regulated by the addition of a small Yang Mills term, the bare level, which we denote k_{YM}, gets shifted by $sign(k_{YM})\, N$. Therefore the results of [131] imply that the two regularizations yield the same physical theory provided

$$|k| = |k_{YM}| + N.$$

Let us define $\lambda = \frac{N}{k}$ as we have done in this paper, and $|\lambda_{YM}| = \frac{N}{|k_{YM}|}$. It follows that

$$|\lambda_{YM}| = \frac{|\lambda|}{1 - |\lambda|}$$

and

$$|\lambda| = \frac{|\lambda_{YM}|}{1 + |\lambda_{YM}|}\ .$$

In particular as $|\lambda| \to 1$, we have $|\lambda_{YM}| \to \infty$. Now a Chern-Simons theory regulated by the addition of a small Yang Mills term clearly exists at all values of $|\lambda_{YM}|$. However it follows from the discussion above that this only requires the dimensionally regulated theory to exist for $|\lambda| \leq 1$. Moreover the limit $|\lambda| \to 1$ should be interpreted as the approach to strong coupling. Our exact result for the finite temperature free energy is perfectly consistent with this interpretation. The theory ceases to exist for $|\lambda| > 1$. Moreover the limit $|\lambda| \to 1$ displays the extreme thinning of degrees of freedom that is plausible in an extreme strong coupling limit.[8]

4. In Appendix F of [10], we have explicitly computed the two loop anomalous dimension, at first subleading order in $\frac{1}{N}$, of the scalar operator $\bar{\psi}\psi$. The result so obtained agrees perfectly with the result of the same computation performed in Feynman gauge.

The points listed above make it at least plausible that our gauge choice is free of problems, and that our regularization scheme preserves gauge invariance and so defines a Lorentz invariant theory. While the direct evidence for these claims is substantial, it is not overwhelming. Significant additional evidence for the Lorentz invariance of our final theory could be obtained by demonstrating the Lorentz invariance of four Fermi scattering at one loop. We will not, however, attempt that consistency check in this paper.

[8]Note that (4.2.106) expressed in terms of k_{YM} takes the form

$$F = -\frac{|k_{YM}| V_2 T^3}{6\pi} \left(\ln \frac{N}{|k_{YM}|} \right)^3 + \cdots\ , \qquad \lambda \to 1.$$

This behavior in the $\lambda \to 1$ limit could be consistent with a weakly coupled boson description (see [56]).

4.3 Operator Spectrum

We now study the spectrum of gauge-invariant operators in our theory. We focus to operators whose dimension is held fixed as N is taken to infinity. By large-N factorization, all such operators are products of *single trace* operators, which, in the context of the vector models we are considering, are naturally defined as those constructed by contracting a single operator in the fundamental representation of $U(N)$ with a single operator in the anti-fundamental representation of $U(N)$ – i.e., fermion bi-linears such as $\bar{\psi}^i \gamma_\mu \psi_i$.

We demonstrate below that the spectrum of single-trace operators in our theory includes one scalar and one current of spin s for $s = 1 \ldots \infty$. A simple argument based on conformal representation theory shows that the scaling dimension of these operators are protected in the large N limit. For all values of λ, the scaling dimension of the scalar operator is $2 + \mathcal{O}(1/N)$ and the dimension of the spin s-current $J^{(s)}$ is given by $\Delta_s = s + 1 + \mathcal{O}(1/N)$. We will also see that the currents $J^{(s)}$ are "almost" conserved; more precisely that these currents obey the anomalous conservation law listed schematically in (4.3.10).

These results immediately allow us to make some very general comments about the possible holographic dual to the theory. In particular, the free theory (obtained in the limit $\lambda \to 0$) has been conjectured [28] to be dual to a particular higher-spin gauge theory constructed by Vasiliev, containing an infinite tower of gauge fields – one for each conserved current $J^{(s)}$. By the AdS/CFT dictionary the masses of these gauge fields in the bulk are directly related to the anomalous dimensions of the corresponding operators; the fact that the higher spin currents have vanishing anomalous dimensions in the large-N limit implies that the bulk gauge fields (should a bulk dual description exist) must remain locally massless even for finite λ.

4.3.1 The free limit

In this subsection we review several well-known properties of Chern-Simons theory coupled to fundamental fermions in the free ($\lambda \to 0$) limit. In this limit our theory reduces to a theory of free fundamental fermions subject to a $U(N)$ singlet Gauss law constraint; this limit has, of course, been studied in the previous literature (see e.g. [28]). In this section we gather those results that will help us study the theory at finite λ.

"Single trace" conformal representation content of the free theory

In this subsection we compute the single-trace operator content of the theory of free fermions. The results of this subsection were already presented in [28]; however we rederive them here for completeness[9].

Let Δ represent the scaling dimension and s the z component of the angular momentum of any operator. We now compute the partition function

$$\mathrm{Tr} x^{\Delta} \mu^{s} \tag{4.3.1}$$

over all single trace operators in our theory. As "single traces" are obtained by multiplying an arbitrary fundamental field with an arbitrary anti-fundamental, the trace over "single traces"

[9]This was worked out in collaboration with J. Bhattacharya.

is simply the product of the partition function for elementary fundamental fields with the partition function for elementary antifundamental fields.

The partition function over elementary fundamental fermionic fields (including arbitrary numbers of derivatives but modulo equations of motion) is simply the character of the $(1, \frac{1}{2})$ representation of the conformal group and is given by

$$F_F(x, \mu) = \frac{x(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})}{(1 - \mu x)(1 - \mu^{-1}x)} \ . \tag{4.3.2}$$

The partition function over antifundamentals is given by the same formula, and so the "single trace" partition function is given by F_F^2.

It is not difficult to decompose this partition function into the contribution of primary operators and their descendents. In order to accomplish this we note that

$$\begin{aligned}
&\left[\frac{x(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})}{(1 - \mu x)(1 - \mu^{-1}x)}\right]^2 \\
&= \frac{1}{(1 - \mu x)(1 - \mu^{-1}x)(1 - x)} \left[\frac{x^2(1 - x)(2 + \mu + \mu^{-1})}{(1 - \mu x)(1 - \mu^{-1}x)}\right] \\
&= \frac{1}{(1 - \mu x)(1 - \mu^{-1}x)(1 - x)} \left[x^2 + \sum_{s=1}^{\infty} \left(x^{s+1}\chi_s(\mu) - x^{s+2}\chi_{s-1}(\mu)\right)\right] \\
&= \chi_{2,0}(x, \mu) + \sum_{s=1}^{\infty} \chi_{s+1,s}(x, \mu) \ .
\end{aligned} \tag{4.3.3}$$

where $\chi_{\Delta,s}(x, \mu)$ is the character of a representation of the conformal algebra with dimension Δ and spin s and we have used the character formulae listed in section 4.3.8. It follows that single trace operators are given by primaries that transform in the representations

$$(2, 0) + \sum_{j=1}^{\infty}(j, j + 1)$$

together with their descendents.

It will be important below that the only long operator that appears in this decomposition is $(2, 0)$ (see section 4.3.8); all other operators in the list above appear in short representations of the conformal algebra.

Explicit form of the primary operators

As we have explained above, the primary field content of the free fermion theory is given by the $(2, 0)$ operator $\bar{\psi}\psi$ plus symmetric traceless currents $J^{(s)}_{\mu_1 \cdots \mu_s}$ for all $s \geq 1$. As we have seen above, the currents are primaries that head short representations of the conformal algebra; the shortening condition is simply the statement that the currents obey the conservation equation

$$\partial^\mu J^{(s)}_{\mu\mu_1 \cdots \mu_{s-1}} = 0 \ . \tag{4.3.4}$$

Single trace currents $J^{(s)}$ that have dimension $s+1$, spin s and are conserved are unique up to a choice of scale in the free fermion theory. We have found explicit expressions for each of the currents $J^{(s)}$. Following [66] we find it convenient to package these expressions in the form of a generating function $O(x,\epsilon)$ defined by

$$\mathcal{O}(x;\epsilon) = \sum J^{(s)}_{\mu_1\mu_2\ldots\mu_s}\epsilon^{\mu_1}\ldots\epsilon^{\mu_s} \tag{4.3.5}$$

where ϵ^μ is an arbitrary vector. As all currents are bilinear in the fermions, the generating function is given by an expression of the form

$$\mathcal{O}(x;\epsilon) = \bar\psi F(\vec\gamma, \overrightarrow{\partial_\mu}, \overleftarrow{\partial_\mu}, \vec\epsilon)\psi \ . \tag{4.3.6}$$

Making a convenient choice for the overall scale of each $J^{(s)}$ we find, (see section 4.3.9 for the derivation),

$$F = \vec\gamma.\vec\epsilon f(\overrightarrow{\partial_\mu}, \overleftarrow{\partial_\mu}, \vec\epsilon) \tag{4.3.7}$$

where

$$f(\vec u, \vec v, \vec\epsilon) = \frac{\exp\left(\vec u\cdot\vec\epsilon - \vec v\cdot\vec\epsilon\right)\sinh\sqrt{2\vec u\cdot\vec v\vec\epsilon\cdot\vec\epsilon - 4\vec u\cdot\vec\epsilon\vec v\cdot\vec\epsilon}}{\sqrt{2\vec u\cdot\vec v\vec\epsilon\cdot\vec\epsilon - 4\vec u\cdot\vec\epsilon\vec v\cdot\vec\epsilon}} \ . \tag{4.3.8}$$

The Taylor expansion

$$\begin{aligned} f &= 1+\epsilon(u-v)+\frac{1}{6}\epsilon^2\left(3u^2-10uv+3v^2+2w\right) \\ &\quad +\epsilon^3\left(\frac{u^3}{6}-\frac{7u^2v}{6}+\frac{7uv^2}{6}+\frac{uw}{3}-\frac{v^3}{6}-\frac{vw}{3}\right) \\ &\quad +\frac{1}{120}\epsilon^4\left(10(u-v)^2(2w-4uv)+(4uv-2w)^2+5(u-v)^4\right)+\mathcal{O}(\epsilon^5) \end{aligned}$$

(above, $w=\vec u\cdot\vec v$, $u=\vec u\cdot\vec\epsilon$, $v=\vec v\cdot\vec\epsilon$.) implies the following explicit expressions for the first four currents

$$\begin{aligned} J_\mu &= \bar\psi\gamma_\mu\psi, \\ J_{\mu_1\mu_2} &= \bar\psi\gamma_{\mu_1}\left(\overrightarrow{\partial_{\mu_2}}-\overleftarrow{\partial_{\mu_2}}\right)\psi, \\ J_{\mu_1\mu_2\mu_3} &= \frac{1}{6}\bar\psi\gamma_{\mu_1}\left(3\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}-10\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}+3\overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}+2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\eta_{\mu_2\mu_3}\right)\psi, \\ J_{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{6}\bar\psi\gamma_{\mu_1}\Big(\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overleftarrow{\partial_{\mu_4}}-7\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}}+7\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}}-\overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}}, \\ &\quad +2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overleftarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4}-2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overrightarrow{\partial_{\mu_2}}\eta_{\mu_3\mu_4}\Big)\psi \end{aligned}$$

where all indices above are understood to be symmetrized.

The existence of spin s conserved currents gives rise to a large symmetry algebra, called the higher spin algebra, of the free fermion theory.

4.3.2 Non-renormalization of the scaling dimension of the current operators

As we have explained above, the single trace operator content, in the free limit, is given in terms of representations of the conformal algebra by

$$(2,0)+\sum_{j=1}^{\infty}(j+1,j).$$

With the exception of the scalar operator, these representations are all of the form $(s+1,s)$ and are short. Operators that transform in these representations can only develop anomalous dimensions after combining with a long representation with quantum numbers $(s+2,s-1)$ (see (4.3.29)).

The operators in the $(2,1)$ and $(3,2)$ conformal representations are the conserved currents for the $U(1)$ fermion flavor symmetry and the fermionic stress tensor respectively. These operators are exactly conserved currents so cannot develop anomalous dimensions at any value of λ. Let us now consider the operator that transforms in the $(4,3)$ representation of the conformal algebra in the free theory. This operator can develop an anomalous dimension only upon combining with an operator in the representation $(5,2)$. However the only spin two single trace operator, the stress tensor, transforms as $(3,2)$ for all values of λ.

We now make a key assumption that we justify in more detail below: we assume that, as far as the analysis of leading large N scaling dimensions is concerned, we can simply ignore all mixing of single-trace and multi-trace operators. It follows that, at leading order in $\frac{1}{N}$, the operator in the $(4,3)$ representation cannot develop an anomalous dimension at any value of λ, simply because at no value of λ can there exist a single-trace operator in the $(5,2)$ representation with which the $(4,3)$ operator can combine to form a long representation.

This argument can now be repeated recursively, to demonstrate that the scaling dimensions of all "single trace" spin s operators are exactly protected (at leading order in N) at all values of λ. The only way this argument could fail is if the theory underwent a severe phase transition at a finite value of λ, across which the spectrum of the theory was not continuous.

Note that the non-renormalization theorem relies on the assumption of non-mixing of single and multi trace operators (which we will justify in detail below) and the the sparsity of single trace operators in our system. As we will see explicitly below, our non-renormalization theorem will be violated by $\frac{1}{N}$ corrections.

Finally note that nothing in the argument we have presented so far prevents the operator $\bar{\psi}\psi$ from developing an anomalous dimension. However we will proceed to argue below that this is also impossible; the scaling dimension of this operator is also protected (as a function of λ) in the interacting theory.

4.3.3 Explicit form of the current operators

In subsection 4.3.1 above we have determined the explicit form of the primary operators that transform in the $(s+1,s)$ representation of the conformal algebra. In the previous subsection we have argued that the interacting theory has primaries with the same quantum numbers at all values of λ. In this subsection we will determine the explicit form of these primary operators, in the interacting theory, in terms of the bare fermionic fields ψ.

In the interacting theory at finite λ let $\hat{J}^{(s)}$ denote the currents obtained from the following procedure: replace all derivatives in the expression for $J^{(s)}$ in the free theory (see subsection 4.3.1) with covariant derivatives. Explicitly, for $s = 1 \ldots 4$ we have

$$\begin{aligned}
\hat{J}^{(1)}_{\mu} &= \bar{\psi}\gamma_\mu\psi, \\
\hat{J}^{(2)}_{\mu_1\mu_2} &= \bar{\psi}\gamma_{\mu_1}\left(\overrightarrow{D_{\mu_2}} - \overleftarrow{D_{\mu_2}}\right)\psi, \\
\hat{J}^{(3)}_{\mu_1\mu_2\mu_3} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\left(3\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}} - 10\overleftarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 3\overrightarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 2(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\eta_{\mu_2\mu_3}\right)\psi \\
\hat{J}^{(4)}_{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\Big(\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}}\overleftarrow{D_{\mu_4}} - 7\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}} + 7\overleftarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}} - \overrightarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}}\overrightarrow{D_{\mu_4}}, \\
&\quad +2\overleftarrow{D_{\mu_2}}(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\eta_{\mu_3\mu_4} - 2(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\overrightarrow{D_{\mu_2}}\eta_{\mu_3\mu_4}\Big)\psi
\end{aligned}$$

where all indices on the RHS are understood to be symmetrized.

The currents $\hat{J}^{(s)}$ are not automatically tracelessness in their vector indices. The reason for this lack of tracelessness is that covariant derivatives, unlike their ordinary counterparts, do not commute. However the commutator of two covariant derivatives is a field strength. Now according to the classical Chern-Simons equation of motion

$$(F_{\mu\nu})^i_j = \frac{\pi}{k}\epsilon_{\mu\nu\rho}\bar{\psi}^i\gamma^\rho\psi_j \tag{4.3.9}$$

where the i and j indices are color indices. Now consider a factor of the field strength F inserted inside a "single trace" fermion bilinear. By the equation of motion cited above[10], this insertion splits the "single trace" into a "double trace" operator divided by k. Further factors of F inside any of the new resultant "traces" repeats this operation. It follows that the spin $s-2$, $s-4$ etc components of the current $\hat{J}^{(s)}$ is given (via the equations of motion) by 'multi trace' operators that are schematically take the form $\frac{s^m}{k^{m-1}}$ where s represents any "single trace" operator so s^m stands for an m trace operator.

The fact that the operators $\hat{J}^{(s)}$ are not traceless means that these currents are not of definite spin; they include components of spin s, spin $s-2$ etc. Let us define the interacting currents $J^{(s)}$ as the projection of $\hat{J}^{(s)}$ to its spin s component, i.e. the projection that removes all traces from $\hat{J}^{(s)}$. $J^{(s)}$ is, by definition, a spin s current, and is of power counting dimension $s+1$. As we have explained above, $J^{(s)}$ and $\hat{J}^{(s)}$ differ only by "multi trace" expressions.

Now the primary operator that transforms in the $(s+1, s)$ representation of the conformal algebra is necessarily an expression of spin s and of power counting dimension $s+1$ (we use a renormalization scheme in which operators can mix only if they have equal classical dimension). The full set of such operators is easily enumerated; in the free theory it consists of those descendents of the primaries $(j+1, j)$ that are of dimension $s+1$ and spin s (clearly this requires $j \leq s$). Note that (in the free theory) these are all single trace operators. All spin s multitrace operators in the free theory have dimension greater than or equal to $s+2$.

[10]While the equation (4.3.9) was derived classically, we believe it also applies quantum mechanically, in an appropriate regulator scheme, as the current $J^{(1)}$ is the unique dimension two, spin one operator in the theory.

In the interacting theory, the full set of operators of spin s and classical dimension $s+1$, is given by replacing all derivatives by covariant derivatives in the free answer and then projecting onto the spin s component (i.e. removing all traces). As above, the projection onto spin s leaves the "single trace" part of the operator untouched, but adds "multi trace" operators to the mix.

Now let us compute the divergence of the most general possible spin s, classical dimension $s+1$ current listed above in the interacting theory. The computation of this divergence differs from the same computation in the free theory in three ways. First covariant derivatives do not commute, and that results in extra factors of the field strength; as we have explained above such factors modify the multi trace parts of the answer but leave the "single trace" part of the answer untouched. Second, as explained above, the current $J^{(s)}$ has extra multitrace operators as compared to the free current. This additional complication also affects only the multi trace parts of the answer. Finally the fermion equation of motion could be quantum mechanically modified, but any such modification is necessarily in terms of 'multi trace' operators.

In other words the single trace part of the divergence of the general spin s current in the interacting theory is identical to the result of the same computation in the free theory (after replacing derivatives with covariant derivatives). However the interacting divergence includes, in addition, several multi trace contributions that are absent in the free theory.

Now in subsection 4.3.1 we computed the unique operator of dimension $s+1$ and spin s, in the free theory, that is also conserved. The interacting theory possesses no exactly conserved operator of this dimension. The operator $J^{(s)}$ comes closest to a conserved current, in that it is the unique current that obeys the schematic equation

$$\partial \cdot J^{(s)} \sim \frac{1}{k} JJ + \frac{1}{k^2} JJJ \tag{4.3.10}$$

(on the RHS of the equation above the symbol J refers either to a current or to a descendent of a current; the important point in this equation is that the RHS contains no single trace pieces). [11] In other words $J^{(s)}$ is the unique spin s and classical dimension $s+1$ field in the interacting theory whose divergence has no single trace component.

As we will see below, the operator $J^{(s)}$ constructed above, may be identified with unique dimension $s+1$ spin s primary of the interacting theory. Before proceeding we will first, however, give an example of how our abstract and exact construction of $J^{(s)}$, (as the projector onto the spin s sector of the current $\tilde{J}^s$) may actually be practically implemented at leading order in λ, for the special case of the current $J^{(3)}$. For this purpose in the next subsection we evaluate the current $J^{(3)}$ and its divergence using the classical (but interacting) field equations. In the subsequent subsection we return to the demonstration that the dimension of $J^{(s)}$ is $s+1$.

[11] The absence of higher trace operators on the RHS of this equation follows from a consideration of quantum numbers. The is of spin $s-1$ and classical dimension $s+2$. Recall that $\Delta - s \geq 1$ for all single trace operators. It follows that $\Delta - s \geq m$ for m trace operators, and so $m \leq 3$.

$J^{(3)}$ as an example

The trace of $\hat{J}^{(3)}$ is given by

$$\hat{J}^{(3)\nu}_{\mu\nu} = \frac{1}{6}\left[\bar{\psi}\gamma_\mu(\overrightarrow{D}^2 + \overleftarrow{D}^2)\psi + \bar{\psi}\gamma^\nu([\overrightarrow{D}_\nu, \overrightarrow{D}_\mu] + [\overleftarrow{D}_\mu, \overleftarrow{D}_\nu])\psi\right]. \tag{4.3.11}$$

Using the identities listed in Appendix 4.3.9

$$\begin{aligned}\hat{J}^{(3)\nu}_{\mu\nu} &= -\frac{1}{12}\bar{\psi}F^{\rho\sigma}\epsilon_{\rho\sigma\nu}(\gamma^\nu\gamma_\mu + \gamma_\mu\gamma^\nu)\psi \\ &= -\frac{1}{6}\bar{\psi}F^{\rho\sigma}\epsilon_{\rho\sigma\mu}\psi \\ &= \frac{\pi}{3k}(\bar{\psi}\psi)(\bar{\psi}\gamma^\mu\psi).\end{aligned}$$

In the last line we used the equation of motion for the field strength F (see Appendix 4.3.9). Upon projecting out the trace we find

$$J^{(3)}{}_{\mu_1\mu_2\mu_3} - \hat{J}^{(3)}_{\mu_1\mu_2\mu_3} = -\frac{\pi}{5k}\eta_{(\mu_1\mu_2}(\bar{\psi}\psi)(\bar{\psi}\gamma_{\mu_3)}\psi). \tag{4.3.12}$$

As we have explained, the currents $J^{(s)}$ are not expected to be exactly divergence free for $s \geq 3$. On the other hand the divergence of the currents $J^{(1)}$ and $J^{(2)}$ vanishes even in the interacting theory (this is obvious for $J^{(1)}$ and is also true for the stress tensor $T^{\mu\nu}$ as it can be explicitly checked). In Appendix 4.3.9 we have also explicitly computed the divergence of $J^{(3)}$ in the classical but interacting fermion theory. Our result is (see Appendix 4.3.9)

$$\partial^\mu J^{(3)}_{\mu\nu_1\nu_2} = -\frac{16\pi}{5k}\left[\eta_{\nu_1\nu_2}\left(\partial^\mu J^{(0)}\right)J^{(1)}_\mu - 3\left(\partial_{(\nu_1}J^{(0)}\right)J^{(1)}_{\nu_2)} + 2J^{(0)}\partial_{(\nu_1}J^{(1)}_{\nu_2)}\right]. \tag{4.3.13}$$

We have presented our result in terms of the scalar "current" $J^{(0)} = \bar{\psi}\psi$ and the vector current $J^{(1)}_\mu = \bar{\psi}\gamma_\mu\psi$.

The equation (4.3.13) has been derived classically; it could certainly receive quantum corrections. However all such corrections are necessarily of higher order in λ; at small λ (4.3.13) gives the leading order contribution to the divergence of $J^{(3)}$.

4.3.4 Anomalous dimensions of the current operators

We will now use the fact that the operators $J^{(s)}$ obey equations of the form (4.3.10) to argue that the scaling dimension of the operator $J^{(s)}$ is $s+1$ up to corrections of order $\frac{1}{N}$. We will also explain how the knowledge of the precise form of the RHS of (4.3.10) can immediately be converted into a computation of the anomalous dimensions of $J^{(s)}$ at leading order in $\frac{1}{N}$, and systematically in the $\frac{1}{N}$ expansion.

Let us first review how it follows that a conserved spin s current has $\Delta = s+1$. For this purpose we use the state operator map, and denote the state dual to the operator O by $|O\rangle$. The standard argument takes the schematic form

$$\langle\partial J|\partial J\rangle = \langle J|[K,P]|J\rangle = (\Delta - s - 1)\langle J||J\rangle$$

(see below for the argument with all indices in place). If $\langle \partial J | \partial J \rangle$ vanishes then it follows immediately that $\Delta = s+1$. In the our theory $\langle \partial J | \partial J \rangle$ does not quite vanish (see (4.3.10)). However as we will argue below, (4.3.10) determines it to be a $\frac{1}{N}$ times a product of known two point functions.

In order to all see this in detail we start with some preliminaries. In radial quantization, $J^{(s)}_{\underline{\mu}}$ inserted at the origin corresponds to a state $|J^{(s)}_{\underline{\mu}}\rangle$ on the sphere. For convenience we introduce a set of differently normalized currents, $j^{(s)}_{\underline{\mu}}$, whose corresponding states $|j^{(s)}_{\underline{\mu}}\rangle$ obey

$$\langle j^{(s)}_{\underline{\mu}} | j^{(s')}_{\underline{\nu}} \rangle = \delta_{ss'} \delta_{\underline{\mu},\underline{\nu}} \tag{4.3.14}$$

where $\delta_{\underline{\mu},\underline{\nu}}$ is 1 if $\underline{\mu},\underline{\nu}$ are the same set of indices up to permutation and 0 otherwise. $J^{(s)}$ and $j^{(s)}$ are related by $J^{(s)}_{\underline{\mu}} = a_s j^{(s)}_{\underline{\mu}}$, where the tree level result for the normalization constant a_s is computed in section 4.3.9. Using the inversion $x'_\mu = x_\mu / x^2$, the two point function in position space is related by

$$\begin{aligned}\langle j^{(s)}_{\underline{\mu}}(x) j^{(s)}_{\underline{\nu}}(0) \rangle &= (-)^s x^{-2-2\delta_s} \frac{\partial x'^{\sigma_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\sigma_s}}{\partial x^{\mu_s}} \langle j^{(s)}_{\underline{\sigma}}(x') | j^{(s)}_{\underline{\nu}}(0) \rangle \\ &= (-)^s x^{-2s-2-2\delta_s} \prod_{i=1}^{s} (\delta^{\sigma_i}_{\mu_i} - \frac{2x^{\sigma_i} x^{\mu_i}}{x^2}) \cdot \langle j^{(s)}_{\underline{\sigma}} | j^{(s)}_{\underline{\nu}} \rangle .\end{aligned} \tag{4.3.15}$$

where δ_s is a possible anomalous dimension for $J^{(s)}$. In the second step we moved $j^{(s)}_{\underline{\sigma}}(x')$ to $j^{(s)}_{\underline{\sigma}}(0)$ at no cost, since the difference is a conformal descendant, which is orthogonal to $|j(0)\rangle$. One can analogously work out the momentum space two point function.

We can also translate (4.3.10) into the language of states in radial quantization, schematically of the form

$$P^\mu | j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} \rangle = \frac{1}{\sqrt{N}} A | jj \rangle + \frac{1}{N} B | jjj \rangle . \tag{4.3.16}$$

Taking its norm (using that $P^\dagger = K$), we have

$$\langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | K^\mu P^\nu | j^{(s)}_{\nu\nu_1\cdots\nu_{s-1}} \rangle = \frac{1}{N} A^2 \langle jj | jj \rangle + \mathcal{O}(\frac{1}{N^2}) \, . \tag{4.3.17}$$

Since $|j^{(s)}_{\nu\nu_1\cdots\nu_{s-1}}\rangle$ is a conformal primary, it is annihilated by K_μ. Using the conformal algebra, the LHS of (4.3.17) is equal to

$$\begin{aligned}&\langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | [K^\mu, P^\nu] | j^{(s)}_{\nu\nu_1\cdots\nu_{s-1}} \rangle = 2 \langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | (\delta^{\mu\nu} D + M^{\mu\nu}) | j^{(s)}_{\nu\nu_1\cdots\nu_{s-1}} \rangle \\ &= 2 \Big[(s+1+\delta_s) \langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | j^{(s)}_{\mu\nu_1\cdots\nu_{s-1}} \rangle + (\delta_{\mu\nu}\delta_{\nu\rho} - \delta_{\nu\nu}\delta_{\mu\rho}) \langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | j^{(s)}_{\rho\nu_1\cdots\nu_{s-1}} \rangle \\ &\quad + \sum_{i=1}^{s-1} (\delta_{\mu\nu_i}\delta_{\nu\rho} - \delta_{\nu\nu_i}\delta_{\mu\rho}) \langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | j^{(s)}_{\nu\nu_1\cdots\nu_{i-1}\rho\nu_{i+1}\cdots\nu_{s-1}} \rangle \Big] \\ &= 2\delta_s \langle j^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} | j^{(s)}_{\mu\nu_1\cdots\nu_{s-1}} \rangle .\end{aligned} \tag{4.3.18}$$

In the last step we used the fact that $j^{(s)}_{\mu_1\cdots\mu_s}$ is traceless. The order $1/N$ contribution to the RHS of (4.3.17) comes from the disconnected four-point function, i.e. schematically

$$\langle jj|jj\rangle = \langle j|j\rangle\langle j|j\rangle + \mathcal{O}(\frac{1}{N}). \tag{4.3.19}$$

This relates the anomalous dimension δ_s at order $1/N$ to the product of two two-point functions of J of lower spins in the *free* theory. Note that this method cannot be used in the $s=0$ case, which we handle separately below.

As an explicit example, let us consider the spin 3 current $J^{(3)}_{\mu\nu\rho}$, which obeys the anomalous current conservation relation (4.3.13). In terms of states in radial quantization, we have the (tree level) relation

$$|\psi_{\nu_1\nu_2}\rangle \equiv P^\mu|j^{(3)}_{\mu\nu_1\nu_2}\rangle = c\left[\delta_{\nu_1\nu_2}P^\mu|j^{(0)}\rangle\otimes|j^{(1)}_\mu\rangle - 3P_{(\nu_1}|j^{(0)}\rangle\otimes|j^{(1)}_{\nu_2)}\rangle + 2|j^{(0)}\rangle\otimes P_{(\nu_1}|j^{(1)}_{\nu_2)}\rangle\right] \tag{4.3.20}$$

where $c=-\frac{16\pi}{5k}\frac{a_0a_1}{a_3}$. Taking its norm and using the conformal algebra commutation relations, we find, in particular,

$$\begin{aligned}\langle\psi_{\mu\nu}|\psi_{\mu\nu}\rangle &= 252|c|^2\\ &= 2\delta_3\langle j^{(3)}_{\mu\nu\rho}|j^{(3)}_{\mu\nu\rho}\rangle = 20\delta_3,\end{aligned} \tag{4.3.21}$$

and so

$$\delta_3 = \frac{63}{5}|c|^2 = \frac{252}{625}\frac{\lambda^2}{N}, \tag{4.3.22}$$

to the leading nontrivial order, i.e. two-loop order. There are both planar and non-planar corrections at higher orders in λ and in $1/N$.

A relation of the form (4.3.17) holds to higher order in $1/N$ as well; we would however need the connected four-point functions etc., as well as potential $1/N$ corrections to the relation (4.3.10), in order to compute the next order contribution to δ_s in $1/N$.

4.3.5 Anomalous current conservation relations within correlation functions

As we have explained above, the primary operators that transform in the $(s+1,s)$ representation (in the large N limit) obey anomalous conservation equations of the form (4.3.10). The equation obeyed, in particular, by $J^{(3)}$ (in the classical interacting theory) is listed in (4.3.13). The nonlinear equations of form (4.3.13) carry a lot of information, as we will explore in this subsection.

Let us first investigate the implication of anomalous conservation equations on three point functions of spin $J^{(s)}$ operators at leading order in N. [12]. We find the schematic equation

$$\partial^\mu\langle J_{\mu\ldots}JJ\rangle = \frac{1}{k}\langle JJJJ\rangle + \frac{1}{k^2}\langle JJJJJ\rangle\ . \tag{4.3.23}$$

While the leading behavior of the second term in (4.3.23) (which comes from factorizing the 5 pt function into the product of a 2 and 3 pt function) is $\mathcal{O}(1)$ (and so subleading in the

[12] We have seen in the previous section that the operators $J^{(s)}$ are effectively conserved within two point functions at leading order in N

large N limit) the leading behavior of the first term on the RHS (this comes from factoring the 4 pt function into a product of two point functions) is $\mathcal{O}(N)$ and so of leading order. It follows that our current operators are not, in general, conserved within three point functions even at leading order in the $\mathcal{O}(1/N)$ expansion. By equating scaling dimensions on the LHS and RHS of (4.3.23) it follows immediately that the RHS of (4.3.23) can be nonzero only when $s \geq s_1 + s_2$. In other words the three point function

$$\langle J^{(s)} J^{(s_1)} J^{(s_2)} \rangle \tag{4.3.24}$$

can obey anomalous conservation equations (rather than genuine conservation equations) only if s_1, s_2 and s_3 violate or saturate the triangle inequality. In all the explicit computations we perform below, it will turn out that the RHS of (4.3.23) is nonzero only when the triangle inequality is explicitly violated (i.e. it vanishes when $s_1 = s_2 + s_3$). It is possible that this is a general exact result that could perhaps be proved by an analysis of allowed structures for 3 point functions, but we will not attempt such an analysis here.

4.3.6 Non-renormalization of the scalar operator $J^{(0)}$

Let us now apply the arguments of the previous subsection to the three point correlator $\langle J^{(3)}(x) J^{(1)}(y) J^{(0)}(z) \rangle$. According to the arguments of the previous subsection, and (4.3.13), the divergence of this three point function w.r.t the variable x is schematically proportional (in the large N limit) to a term proportional $\langle \partial J^{(0)} J^{(0)} \rangle \langle J^{(1)} J^{(1)} \rangle$ plus another term proportional to $\langle J^{(0)} J^{(0)} \rangle \langle \partial J^{(1)} J^{(1)} \rangle$. The weight under overall scaling of each of these expressions is $5 + 2\Delta_0$ where Δ_0 is the as yet unknown scaling dimension of the current operator $J^{(0)}$. On the other hand the weight under scaling of the divergence of the three pt function is $6 + \Delta_0$. Equating these weights we find that $\Delta_0 = 2$. We conclude that, just like their spin s counterparts, the scaling dimension of the scalar current $J^{(0)}$ is not renormalized as a function of λ.[13] This non-renormalization result may be argued for more intuitively as follows: the LHS of (4.3.13) is of dimension 5. The RHS of the same expression is also of dimension 5 if and only $J^{(0)}$ has dimension 2.

4.3.7 Vector models are dual to higher spin gauge theories

The non-renormalization of single trace currents in $U(N)$ Chern-Simons-fermion theory at leading order in $1/N$ indicates that its holographic dual has a free spectrum of purely higher spin gauge fields, of spin $s = 0, 1, 2, 3, \cdots$, and their bound states (which are dual to multi-trace operators). The divergences of the single trace currents are mixed with double and possibly triple trace operators, and the mixing is suppressed by $N^{-\frac{1}{2}}$. By comparing two and three point functions of the currents, we see that $N^{-\frac{1}{2}}$ should be identified with the bulk coupling constant.

Classically, the bulk nonlinear equations of motion must therefore preserve the higher spin gauge symmetries as well, for otherwise they would introduce extra longitudinal degrees of freedom for the higher spin fields even in the zero coupling limit. This means that the dual

[13] We thank O. Aharony for discussions on this topic.

bulk theory is a *pure* higher spin *gauge* theory. On the other hand, the higher spin symmetry in the Chern-Simons-fermion theory is broken by $1/N$ corrections, and we expect the HS gauge symmetry to be broken in the bulk by boundary conditions through loop effects,[14] and the HS gauge fields become massive through the mixing of its longitudinal mode with bound states.

Our argument of non-renormalization of singlet bilinear currents extends to general vector models coupled to Chern-Simons gauge fields, such as the various supersymmetric extensions of the Chern-Simons-fermion theory discussed so far. We therefore anticipate all such theories to be dual to some higher spin gauge theory in the bulk. The bulk theory is generally not parity invariant.

4.3.8 Appendix: Unitary Representations of the $d = 3$ Conformal Group

Unitary representations of the conformal group are labeled by the spin s and a scaling dimension Δ of their primary states. When $s \geq 1$ these labels are subject to the inequality $\Delta \geq s + 1$. In this case representations that saturate the inequality are short; the null states in this representation fall into a (long) representation with $\Delta = s + 2$ and spin $=s - 1$.

In the special case $s = \frac{1}{2}$ it turns out that $\Delta \geq 1$. The representation with $\Delta = 1$ is short and its null states fall into a (long) representation with $\Delta = 2$ and spin $=\frac{1}{2}$. The later is the representation of a free fermionic field, and the character (4.3.1) $F_F(x, \mu)$ of this field is given by

$$F_F(x, \mu) = \frac{x(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})}{(1 - \mu x)(1 - \mu^{-1} x)} \ . \tag{4.3.25}$$

Finally, when $s = 0$ we have $\Delta = 0$ or $\Delta \geq \frac{1}{2}$. The representation with $\Delta = 0$ has a single state. The representation with $\Delta = \frac{1}{2}$ has null states in a representation with $\Delta = \frac{5}{2}$ and $s = 0$. The states of a free scalar field fall into this representation; the character $F_S(x, \mu)$ of this representation is given by

$$F_S(x, \mu) = x^{\frac{1}{2}} \frac{(1 + x)}{(1 - \mu x)(1 - \mu^{-1} x)} \ . \tag{4.3.26}$$

We now present character formulae for all unitary representations of the 3d conformal algebra. Let us define

$$\chi_s(\mu) = \sum_{j=-s}^{s} \mu^j \tag{4.3.27}$$

$\chi_s(\mu)$ is the $SU(2)$ character at spin s (here s is a positive integer or half integer). Let us also define

$$G_{\Delta,s} = \frac{x^{\Delta} \chi_s(\mu)}{(1 - x)(1 - \mu x)(1 - \mu^{-1} x)} \ . \tag{4.3.28}$$

$G_{\Delta,s}$ is the partition function over states that are obtained by acting on an $SU(2)$ primary of spin s with an arbitrary number of derivatives, and so yields the character of any long

[14]This has been shown to occur, for instance, in Vasiliev's A-type theory in AdS_4 with the $\Delta = 2$ boundary condition, dual to the critical $O(N)$ model [68].

representation of the conformal algebra (i.e. any representation with $\Delta > s+1$ for $s \geq 1$ or $\Delta > 1$ for $s = \frac{1}{2}$ or $\Delta > \frac{1}{2}$ for $s = 0$).

Characters $\chi(x,\mu)$ of short representations of the conformal algebra are obtained by evaluating the character of a hypothetical long representation of that algebra and then subtracting out the character of its null states. It follows that, for $s \geq 1$

$$\chi_{s+1,s}(x,\mu) = G_{s+1,s}(x,\mu) - G_{s+2,s-1}(x,\mu) \ . \tag{4.3.29}$$

For $s = \frac{1}{2}$

$$\chi_{1,\frac{1}{2}}(x,\mu) = G_{1,\frac{1}{2}}(x,\mu) - G_{2,\frac{1}{2}}(x,\mu) = F_F(x,\mu) \tag{4.3.30}$$

while for $s = 0$

$$\chi_{\frac{1}{2},0}(x,\mu) = G_{\frac{1}{2},0}(x,\mu) - G_{\frac{5}{2},0}(x,\mu) = F_S(x,\mu) \ . \tag{4.3.31}$$

4.3.9 Appendix: Primary Operators in Free and Interacting Fermion Theories

In this appendix we present the details of some slightly tedious computations involving free and interacting fermions.

The generating function of conserved currents for free fermions

As we have explained in section 4.3 above, the single-trace primary operators of the free fermion theory satisfy the condition that their scaling dimension, ϵ and spin s satisfy the relation $\epsilon = s+1$, with the exception of $\bar{\psi}\psi$, which has scaling dimension 2 and spin 0. Here, we will determine explicit expressions for the corresponding primary operators.

How can we produce an operator of spin s and dimension $s+1$ in a theory of free fermions in $d = 3$? We are interested in operators built out of fermion bilinears (with color indices contracted). As ψ and $\bar{\psi}$ each have unit scaling dimension, the operator of interest must contain exactly $s-1$ derivatives. As $s-1$ derivatives can give rise to at most $s-1$ free traceless vector indices the remaining index (to make our operator spin s) must come from a γ matrix. [15] Consequently if we define the generating function F such that

$$\mathcal{O}(x;\epsilon) = \bar{\psi} F(\vec{\gamma}, \overrightarrow{\partial_\mu}, \overleftarrow{\partial_\mu}, \vec{\epsilon})\psi = \sum J^{(s)}_{\mu_1\mu_2\ldots\mu_s} \epsilon^{\mu_1} \ldots \epsilon^{\mu_s} \tag{4.3.32}$$

then F must take the form

$$F = \vec{\gamma}.\vec{\epsilon} f(\overrightarrow{\partial_\mu}, \overleftarrow{\partial_\mu}, \vec{\epsilon}) \ . \tag{4.3.33}$$

Following [66] we denote the arguments of f as vectors u and v, so that $\vec{u} = \overleftarrow{\partial}$ and $\vec{v} = \overrightarrow{\partial}$. The fermion equation of motion gives

$$\vec{u}.\vec{\gamma} = \vec{v}.\gamma = u^2 = v^2 = 0 \ . \tag{4.3.34}$$

[15] As $\gamma_{(\mu}\gamma_{\nu)} = \eta_{\mu\nu}$ we cannot have more than one of the current indices come from a γ matrix. Also, the equations of motion tell us that a γ matrix contracted with a derivative vanishes in its action on a fermion. The contraction of a derivative with a γ matrix, sandwiched by other γ matrices, can also be reduced to a form with a single γ matrix and derivatives using the equations of motion, (e.g. $\gamma_{(\mu}\gamma_\rho\partial^\rho\gamma_{\nu)} = -\gamma_{(\mu}\partial^\rho\gamma_{\nu)}\gamma_\rho + \gamma_{(\mu}\partial^\rho\eta_{\nu)\rho}$ where the first term on the RHS vanishes by the equations of motion and the last term has a single γ matrix).

It is convenient to change variables to $\vec{y} \equiv \vec{u} - \vec{v}$ and $\vec{z} \equiv \vec{u} + \vec{v}$. Then we have:

$$\vec{z} \cdot \vec{\gamma} = \vec{y} \cdot \vec{\gamma} = \vec{y} \cdot \vec{z} = 0, \quad -\vec{y}^2 = \vec{z}^2 = 2\vec{u} \cdot \vec{v} \neq 0 \ . \tag{4.3.35}$$

Terms in f will be of the form:

$$f = A + B\vec{\epsilon} \cdot \vec{y} + C\vec{\epsilon} \cdot \vec{z} + D\vec{u} \cdot \vec{v}\vec{\epsilon} \cdot \vec{\epsilon} + \dots \tag{4.3.36}$$

where each coefficient is a number.

If we define $w \equiv (\vec{u} \cdot \vec{v})\vec{\epsilon} \cdot \vec{\epsilon}$, $z \equiv \vec{\epsilon} \cdot \vec{z}$, and $y = \vec{\epsilon} \cdot \vec{y}$, then f can be thought of as a function of three variables $f(z, y, w)$.

The condition that each current is conserved can be expressed as:

$$(\overleftarrow{\partial_\mu} + \overrightarrow{\partial_\mu})\frac{\partial}{\partial \epsilon_\mu} F = 0 \tag{4.3.37}$$

which translates into the following condition on f:

$$z\partial_w f + \partial_z f = 0 \ . \tag{4.3.38}$$

The condition that each current is traceless can be expressed as:

$$\frac{\partial}{\partial \epsilon^\mu}\frac{\partial}{\partial \epsilon_\mu} F = 0 \tag{4.3.39}$$

which translates into:

$$\left(5\partial_w + 2w\partial_w^2 - \partial_y^2 + \partial_z^2 + 2z\partial_z\partial_w + 2y\partial_y\partial_w\right) f = 0 \ . \tag{4.3.40}$$

The solution to (4.3.38) is $f(w, y, z) = g(y, w - \frac{z^2}{2})$. If we define $t = w - \frac{z^2}{2}$, the equation for $g(y, t)$ is

$$\left(4\partial_t + 2t\partial_t^2 - \partial_y^2 + 2y\partial_y\partial_t\right) g(y, t) = 0 \ . \tag{4.3.41}$$

The general solution satisfying $g = 1$ at $t = y = 0$ is:

$$g = e^{2ky}\frac{\sinh 2k\sqrt{2t + y^2}}{2k\sqrt{2t + y^2}} \tag{4.3.42}$$

where k is any constant, which we take to be $1/2$.

The final form for f is thus

$$f(\vec{u}, \vec{v}, \vec{\epsilon}) = \frac{\exp\left(\vec{u} \cdot \vec{\epsilon} - \vec{v} \cdot \vec{\epsilon}\right) \sinh\sqrt{2\vec{u} \cdot \vec{v}\vec{\epsilon} \cdot \vec{\epsilon} - 4\vec{u} \cdot \vec{\epsilon}\vec{v} \cdot \vec{\epsilon}}}{\sqrt{2\vec{u} \cdot \vec{v}\vec{\epsilon} \cdot \vec{\epsilon} - 4\vec{u} \cdot \vec{\epsilon}\vec{v} \cdot \vec{\epsilon}}} \ . \tag{4.3.43}$$

Expanding the above expression in a power series around $\vec{\epsilon}$, we obtain:

$$\begin{aligned} f \ = \ & 1 + \epsilon(u - v) + \frac{1}{6}\epsilon^2\left(3u^2 - 10uv + 3v^2 + 2w\right) \\ & + \epsilon^3\left(\frac{u^3}{6} - \frac{7u^2v}{6} + \frac{7uv^2}{6} + \frac{uw}{3} - \frac{v^3}{6} - \frac{vw}{3}\right) \\ & + \frac{1}{120}\epsilon^4\left(10(u - v)^2(2w - 4uv) + (4uv - 2w)^2 + 5(u - v)^4\right) \end{aligned}$$

(above, $w = \vec{u} \cdot \vec{v}$, $u = \vec{u} \cdot \vec{\epsilon}$, $v = \vec{v} \cdot \vec{\epsilon}$.) which yields the currents reported in (4.3.9).

Two-point functions of primary operators in the free theory

In this subsection we explicitly compute the two-point function of conserved currents

$$\langle \mathcal{O}(\vec{x};\vec{\epsilon}_1)\mathcal{O}(0;\vec{\epsilon}_2)\rangle \tag{4.3.44}$$

determined in the previous subsection, and demonstrate that the two point functions of currents of different spin are orthonormal. We take the two-point function of the basic fermionic fields to be given by

$$\langle \psi(x)\bar{\psi}(0)\rangle = C_{\psi\psi}\frac{\gamma^\mu x_\mu}{x^3}. \tag{4.3.45}$$

Simple examples

To set up notation and get intuition we first work out some simple examples.

As a first example, consider the two-point function of two scalar currents:

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle\ . \tag{4.3.46}$$

Using Wick's theorem we rewrite this as:

$$\mathrm{Tr}\langle -\psi(0)\bar{\psi}(x)\rangle\langle\psi(x)\bar{\psi}(0)\rangle \tag{4.3.47}$$

where the trace is over gamma matrix indices. We then compute it to be:

$$\begin{aligned} C_{\psi\psi}^2\mathrm{Tr}\frac{\gamma^\mu x_\mu}{x^3}\frac{\gamma^\nu x_\nu}{x^3} &= C_{\psi\psi}^2\frac{x_\nu x_\mu}{x^6}\mathrm{Tr}\gamma^\mu\gamma^\nu \\ = C_{\psi\psi}^2\frac{x_\nu x_\mu}{x^6}2\eta^{\mu\nu} &= C_{\psi\psi}^2\frac{2}{x^4}. \end{aligned}$$

To evaluate more complicated two-point functions, we make use of the identity

$$\mathrm{Tr}\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma \ \ = \ \ 2\left(\eta^{\rho\nu}\eta^{\sigma\mu}+\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\rho\sigma}\eta^{\mu\nu}\right). \tag{4.3.48}$$

Note that (4.3.48) is symmetric under interchange of ρ and σ.

We next consider the two-point function of two spin-1 currents:

$$\langle \bar{\psi}(x)\gamma^\nu\psi(x)\bar{\psi}(y)\gamma^\mu\psi(y)\rangle\ . \tag{4.3.49}$$

We have:

$$\begin{aligned} \langle \bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(y)\gamma^\nu\psi(y)\rangle &= \mathrm{Tr}\langle -\psi(y)\bar{\psi}(x)\rangle\gamma^\mu\langle\psi(x)\bar{\psi}(y)\rangle\gamma^\nu \\ &= C_{\psi\psi}^2\frac{x_\rho}{x^3}\frac{x_\sigma}{x^3}\mathrm{Tr}\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu \\ &= C_{\psi\psi}^2\frac{x_\rho}{x^3}\frac{x_\sigma}{x^3}2\left(\eta^{\rho\nu}\eta^{\sigma\mu}+\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\rho\sigma}\eta^{\mu\nu}\right) \\ &= 2C_{\psi\psi}^2\left(\frac{2x^\mu x^\nu}{x^6}-\frac{\eta^{\mu\nu}}{x^4}\right). \end{aligned}$$

As above, we will often set $y=0$ in the last line.

Results for all spins

Let ε be a *null* polarization vector. The two point function of the generating operator $\mathcal{O}(x;\varepsilon) = \sum J^{(s)}_{\mu_1\cdots\mu_s}\varepsilon^{\mu_1}\cdots\varepsilon^{\mu_s}$ is evaluated in the theory of N free complex fermions to be

$$\langle \mathcal{O}(x;\varepsilon)\mathcal{O}(0;\varepsilon)\rangle = \frac{N}{32\pi^2 x^2}\left\{\left[1-\left(\frac{4\varepsilon\cdot x}{x^2}\right)^2\right]^{-\frac{1}{2}} - 1\right\}. \tag{4.3.50}$$

Expanding this in ε, we have

$$\langle J^{(s)}(x;\varepsilon)J^{(s)}(0;\varepsilon)\rangle = \frac{N}{32\pi^{\frac{5}{2}}}\frac{2^{4s}\Gamma(s+\frac{1}{2})}{s!}\frac{(\varepsilon\cdot x)^{2s}}{(x^2)^{2s+1}}, \tag{4.3.51}$$

where $J^{(s)}(x;\varepsilon)$ is the spin-s part of $\mathcal{O}(x;\varepsilon)$. The spin 0 case is special, where we have $\langle J^{(0)}(x)J^{(0)}(0)\rangle = \frac{N}{8\pi^2}|x|^{-4}$.

We have defined the set of currents $j^{(s)}$ with a different normalization convention, namely normalizing the norm of the corresponding state in radial quantization. The relative normalization between $J^{(s)}$ and $j^{(s)}$ can be determined as follows. If we define $j^{(s)}(x;\varepsilon) = j^{(s)}_{\mu_1\cdots\mu_s}(x)\varepsilon^{\mu_1}\cdots\varepsilon^{\mu_s}$, then

$$\langle j^{(s)}(x;\varepsilon)j^{(s)}(0;\varepsilon)\rangle = 2^s\frac{(\varepsilon\cdot x)^{2s}}{(x^2)^{2s+1+\delta_s}}, \tag{4.3.52}$$

for $s>0$. In the spin 0 case, we have $\langle j^{(0)}(x)j^{(0)}(0)\rangle = |x|^{-4}$. From this we deduce

$$J^{(s)}_{\underline{\mu}}(x) = a_s j^{(s)}_{\underline{\mu}}(x), \qquad a_s = \left[\frac{N}{32\pi^{\frac{5}{2}}}\frac{2^{3s}\Gamma(s+\frac{1}{2})}{s!}\right]^{\frac{1}{2}}, \tag{4.3.53}$$

for $s>0$. In the spin 0 case, $a_0 = \frac{\sqrt{2N}}{4\pi}$. In the interacting Chern-Simons-fermion theory, a_s receives quantum corrections.

Explicit computation of the divergence of $J^{(3)}$

In carrying out all our manipulations below, we use the fermion equation of motion

$$D_\mu\gamma^\mu\psi = D_\mu\bar{\psi}\gamma^\mu = 0. \tag{4.3.54}$$

Some useful identities are:

$$\begin{aligned}
&\gamma^\mu D_\mu D_\nu\psi = \gamma^\mu(D_\nu D_\mu - iF_{\mu\nu})\psi = -i\gamma^\mu F_{\mu\nu}\psi,\\
&D_\mu D_\nu\bar{\psi}\gamma^\mu = (D_\nu D_\mu + iF_{\mu\nu})\bar{\psi}\gamma^\mu = i\bar{\psi}\gamma^\mu F_{\mu\nu},\\
&\gamma^\mu D_\mu\gamma^\nu D_\nu\psi = 0,\\
&(\eta^{\mu\nu}+\gamma^{\mu\nu})D_\mu D_\nu\psi = 0,\\
&D_\mu D^\mu\psi = \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\psi,\\
&D_\mu D_\nu\bar{\psi}\gamma^\nu\gamma^\mu = 0,\\
&D_\mu D^\mu\bar{\psi} = \frac{i}{2}\bar{\psi}\gamma^{\mu\nu}F_{\mu\nu},
\end{aligned} \tag{4.3.55}$$

Note our convention is such that $[\overrightarrow{D}_\mu, \overrightarrow{D}_\nu]\psi = -iF_{\mu\nu}\psi$. We now use the equation of motion for $F_{\mu\nu} = F^a_{\mu\nu}T^a$, namely

$$\epsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{4\pi}{k}J^\mu, \quad \text{or} \quad F^a_{\mu\nu} = \frac{2\pi}{k}\epsilon_{\mu\nu\rho}J^{a\,\rho}, \tag{4.3.56}$$

where $J_\mu = J^a_\mu T^a$, $J^a_\mu = \bar{\psi}\gamma_\mu T^a\psi$. It is also useful to have

$$\begin{aligned} J^a_\rho T^a\psi &= (\bar{\psi}\gamma_\rho T^a\psi)\,T^a\psi \\ &= -\frac{1}{4}\left(\gamma_\rho\psi(\bar{\psi}\psi) + \gamma^\mu\gamma^\rho\psi(\bar{\psi}\gamma_\mu\psi)\right) \\ &= -\frac{1}{4}\left(\gamma_\rho\psi J^{(0)} + \gamma^\mu\gamma_\rho\psi J^{(1)}_\mu\right), \end{aligned} \tag{4.3.57}$$

and

$$\bar{\psi}J^a_\rho T^a = -\frac{1}{4}\left(J^{(0)}\bar{\psi}\gamma_\rho + J^{(1)}_\mu\bar{\psi}\gamma_\rho\gamma^\mu\right). \tag{4.3.58}$$

To derive these relations, we used $(T^a)^i_j(T^a)^l_m = \frac{1}{2}\delta^i_m\delta^l_j$ and the 3d Fierz identity

$$\chi\bar{\lambda} = -\frac{1}{2}\bar{\lambda}\chi - \frac{1}{2}\bar{\lambda}\gamma_\mu\chi\gamma^\mu.$$

We can now proceed to explicitly compute $\partial^\mu J^{(3)}_{\mu\nu_1\nu_2}$. First, consider the current $\hat{J}^{(3)}$ which is not traceless,

$$\hat{J}^{(3)}_{\mu_1\mu_2\mu_3} = \frac{1}{6}\bar{\psi}\gamma_{\mu_1}\left(3\overleftarrow{D_{\mu_2}}\overleftarrow{D_{\mu_3}} - 10\overleftarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 3\overrightarrow{D_{\mu_2}}\overrightarrow{D_{\mu_3}} + 2(\overleftarrow{D_\sigma}\overrightarrow{D^\sigma})\eta_{\mu_2\mu_3}\right)\psi.$$

Using the identities above we find that (before subtracting the trace) the divergence is given by:

$$\begin{aligned} 6\partial^\mu\hat{J}^{(3)}_{\mu\nu_1\nu_2} &= -i\,\bar{\psi}\gamma^\mu\left(16\overleftarrow{D}_{\nu_1}F_{\nu_2\mu} + 16F_{\mu\nu_1}\overrightarrow{D}_{\nu_2} + 2\eta_{\nu_1\nu_2}\left(\overleftarrow{D}_\lambda F_{\mu\lambda} + F_{\lambda\mu}\overrightarrow{D}_\lambda\right)\right)\psi \\ &\quad -i\,\bar{\psi}\gamma^{\nu_1}\left(16(\overleftarrow{D}^\mu F_{\nu_2\mu} + F_{\mu\nu_2}\overrightarrow{D}^\mu)\right)\psi \\ &\quad +\bar{\psi}\gamma^{\nu_1}\left(6(\overleftarrow{D}^2\overleftarrow{D}_{\nu_2} + \overrightarrow{D}_{\nu_2}\overrightarrow{D}^2) - 10(\overleftarrow{D}^2\overrightarrow{D}_{\nu_2} + \overleftarrow{D}_{\nu_2}\overrightarrow{D}^2)\right)\psi\,. \end{aligned} \tag{4.3.59}$$

We now substitute for D^2 and further simplify:

$$\begin{aligned} 6\partial^\mu\hat{J}^{(3)}_{\mu\nu_1\nu_2} &= -i\bar{\psi}\gamma^\mu\left(32\overleftarrow{D}_{\nu_1}F_{\nu_2\mu} + 32F_{\mu\nu_1}\overrightarrow{D}_{\nu_2} + 2\eta_{\nu_1\nu_2}\left(\overleftarrow{D}_\lambda F_{\mu\lambda} + F_{\lambda\mu}\overrightarrow{D}_\lambda\right)\right)\psi \\ &\quad - i\bar{\psi}\gamma^{\nu_1}\left(16(\overleftarrow{D}^\mu F_{\nu_2\mu} + F_{\mu\nu_2}\overrightarrow{D}^\mu)\right)\psi - i\bar{\psi}\left(2\overleftarrow{D}_{\nu_1}\tilde{F}_{\nu_2} + 2\tilde{F}_{\nu_1}\overrightarrow{D}_{\nu_2} - 6(D_{\nu_1}\tilde{F}_{\nu_2})\right)\psi, \end{aligned} \tag{4.3.60}$$

where $\tilde{F}_\mu = \epsilon_{\mu\nu\rho}F^{\nu\rho}$.

Now substituting in F and using Fierz identity, we have

$$
\begin{aligned}
&-32i\bar{\psi}\gamma^{\mu}\left(\overleftarrow{D}_{\nu_1}F_{\nu_2\mu}+F_{\mu\nu_1}\overrightarrow{D}_{\nu_2}\right)\psi=\frac{32\pi}{k}\Big[\partial_{\nu_1}(\bar{\psi}\psi)\bar{\psi}\gamma_{\nu_2}\psi-\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi\Big],\\
&-2i\eta_{\nu_1\nu_2}\bar{\psi}\gamma^{\mu}\left(\overleftarrow{D}_{\lambda}F_{\mu\lambda}+F_{\lambda\mu}\overrightarrow{D}_{\lambda}\right)\psi=-\frac{2\pi}{k}\eta_{\nu_1\nu_2}\partial_{\mu}(\bar{\psi}\psi)\bar{\psi}\gamma^{\mu}\psi,\\
&-i\bar{\psi}\gamma^{\nu_1}\left(16(\overleftarrow{D}^{\mu}F_{\nu_2\mu}+F_{\mu\nu_2}\overrightarrow{D}^{\mu})\right)\psi=\frac{8\pi}{k}\Big[-\eta_{\nu_1\nu_2}\partial_{\mu}(\bar{\psi}\psi)\bar{\psi}\gamma^{\mu}\psi+2\partial_{\nu_1}(\bar{\psi}\psi)\bar{\psi}\gamma_{\nu_2}\psi\\
&\qquad-\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi+\epsilon_{\nu_2\lambda\mu}(\bar{\psi}\overleftrightarrow{D}^{\mu}\gamma_{\nu_1}\psi)(\bar{\psi}\gamma^{\lambda}\psi)\Big],\\
&-i\bar{\psi}\left(2\overleftarrow{D}_{\nu_1}\tilde{F}_{\nu_2}+2\tilde{F}_{\nu_1}\overrightarrow{D}_{\nu_2}\right)\psi=-\frac{2\pi}{k}\left[\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi+(\bar{\psi}\gamma_{\nu_2}\psi)\partial_{\nu_1}(\bar{\psi}\psi)+\epsilon_{\nu_2\lambda\mu}(\bar{\psi}\overleftrightarrow{D}_{\nu_1}\gamma^{\mu}\psi)(\bar{\psi}\gamma^{\lambda}\psi)\right],\\
&-6i\bar{\psi}\left(D_{\nu_1}\tilde{F}_{\nu_2}\right)\psi=\frac{6\pi}{k}\left[(\bar{\psi}\gamma_{\nu_2}\psi)\partial_{\nu_1}(\bar{\psi}\psi)+\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi-\epsilon_{\nu_2\lambda\mu}(\bar{\psi}\overleftrightarrow{D}_{\nu_1}\gamma^{\mu}\psi)(\bar{\psi}\gamma^{\lambda}\psi)\right].
\end{aligned}
\tag{4.3.61}
$$

We now use the identity

$$
\epsilon_{\nu_2\lambda\mu}\bar{\psi}(-\overleftrightarrow{D}_{\nu_1}\gamma^{\mu}+\overleftrightarrow{D}^{\mu}\gamma_{\nu_1})\psi(\bar{\psi}\gamma^{\lambda}\psi)=-\eta_{\nu_1\nu_2}\partial^{\lambda}(\bar{\psi}\psi)(\bar{\psi}\gamma_{\lambda}\psi)+\partial_{\nu_1}(\bar{\psi}\psi)\bar{\psi}\gamma_{\nu_2}\psi \tag{4.3.62}
$$

to obtain the following total for the above sum:

$$
6\partial^{\mu}\hat{J}^{(3)}_{\mu\nu_1\nu_2}=\frac{2\pi}{k}\left[-9\eta_{\nu_1\nu_2}\partial_{\mu}(\bar{\psi}\psi)(\bar{\psi}\gamma^{\mu}\psi)+30\partial_{\nu_1}(\bar{\psi}\psi)\bar{\psi}\gamma_{\nu_2}\psi-18\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi\right]. \tag{4.3.63}
$$

Subtracting the trace we obtain:

$$
\partial^{\mu}J^{(3)}_{\mu\nu_1\nu_2}=\partial^{\mu}\hat{J}^{(3)}_{\mu\nu_1\nu_2}-\frac{\pi}{5k}\left[\eta_{\nu_1\nu_2}(\bar{\psi}\gamma^{\mu}\psi)\partial_{\mu}(\bar{\psi}\psi)+2\partial_{\nu_1}(\bar{\psi}\gamma_{\nu_2}\psi)\bar{\psi}\psi+2(\bar{\psi}\gamma_{\nu_2}\psi)\partial_{\nu_1}(\bar{\psi}\psi)\right]. \tag{4.3.64}
$$

The indices on the RHS are understood to be symmetrized.

4.4 Discussion

This chapter has been devoted to the study of $U(N)$ Chern-Simons theory coupled to a single massless fundamental fermion. We set out to to solve this theory in the 't Hooft large N limit and have been partly successful in this task. In particular, we were able to compute the partition function on $\mathbb{R}^2$ of the theory at finite temperature at all values of the 't Hooft coupling. In order to perform this computation we employed a lightcone gauge, and used a dimensional reduction regularization scheme. It would be useful to perform further checks of the consistency of our gauge and regularization scheme.

It has recently been realized that the partition functions of supersymmetric Chern-Simons theories on S^3 are often exactly computable via the techniques of supersymmetric localization [133, 134, 135, 136]. This quantity appears to play the role of a c function under the renormalization group flow and so is of clear physical interest. It would be interesting to see if the techniques employed here could allow the exact computation of the S^3 partition function of our non-supersymmetric theory in the large N limit, as a function of λ.

It should also be possible to generalize our discussion of the finite temperature behavior of our theory to include a chemical potential for fermion number. The finite λ behavior of such a system describes an interacting Fermi sea in three dimensions, and so may be of interest for various condensed matter problems.

In the course of our analysis of this theory we have encountered the fact that the higher spin currents obey anomalous conservation equations. In the classical theory we have computed the explicit form of these conservation equation at low values of the spin (see e.g. (4.3.13)). These nonlinear anomalous conservation equations contain a large amount of information; for example they encode the anomalous dimensions of the spin s currents in a $\frac{1}{N}$ expansion (see subsection 4.3.4). It would be interesting to determine the explicit form of these anomalous conservation equations, if possible, as a function of λ.

We have also argued that the spectrum of local gauge invariant operators in our theory, at dimensions of order unity, is not renormalized as a function of the 't Hooft coupling at leading order in large N, by combining conformal representation theory with the sparseness of the single trace spectrum in vector models.

An outstanding question about the theory studied in this chapter is "what is its bulk dual description?" We have argued above that this dual description is a higher spin theory. However we do not yet have a precise conjecture for the nature of this dual.

To end this discussion we note that the non-renormalization argument presented here does not apply in theories with adjoint or bifundamental matter fields as the single trace spectrum of these theories is not sparse.[16] On the other hand, as we have explained in the introduction, non supersymmetric effective large N Chern-Simons fixed lines are easily constructed with matter fields in adjoint or bifundamental representations. The scaling dimension of single trace operators in such theories is protected neither by conformal nor supersymmetric representation theory. In these theories all single trace operators baring the stress tensor and currents for global symmetries can, and presumably do get renormalized. It is at least conceivable that some theory with adjoint or bifundamental fermions admits a strong coupling limit in which all but a finite number of single trace operators are infinitely renormalized, and the dual description of the theory is Einstein gravity in AdS_4, coupled to a minimal number of additional fields. It would clearly be very interesting to identify any theory with this property.

As is clear from the discussion above, the computations presented in this chapter have merely scratched the surface of a large and potentially very interesting area of investigation. We hope to report on some of the topics discussed above in the future.

Note Added:

In this chapter we computed the free energy of fundamental fermions coupled to Chern Simons theory in the special case that the holonomy of the gauge field around the thermal circle is the identity matrix. The calculation of this chapter was later generalized to the computation of the free energy in a different holonomy background in [138] and in an arbi-

[16] In supersymmetric theories with such matter content, supersymmetric indices and representation theory give rise to a new source of non-renormalization theorems for single trace operators even in theories with bifundamental and adjoint matter, see e.g. the paper [137].

trary holonomy background in [139]. In particular it was pointed out in [138] that the free energy thus computed agrees with the free energy of a bosonic theory (in agreement with a conjectured duality between the fermionic and bosonic system) if the holonomy is given not by the identity matrix but instead by a different universal function. Once the computation of this chapter is modified to include this nontrivial holonomy, the free energy computed herein agrees perfectly with the free energy of the conjectured bosonic dual, providing a spectacular check of the conjectured 'bosonization' duality in 3 dimensions.

Chapter 5

Conclusion

In this thesis, we studied the behavior of strongly-interacting quantum field theories in three dimensions in the large-N limit using both the AdS_4/CFT_3 correspondence and traditional field theory techniques.

In Chapters 2 and 3, we calculated thermodynamic properties as well as quantities related to transport in a class of gravitational systems expected to be dual to strongly interacting conformal field theories, at finite temperature, chemical potential and in the presence of a magnetic field. We incorporated dilatonic and axionic couplings to generate models with very realistic thermodynamic properties – in particular, a vanishing entropy at zero temperature – not present in the traditional gravitational theories without a dilaton. While we do not have a weakly-coupled description of the dual field theories, (and strictly speaking, therefore cannot be sure that they exist) it is natural to expect that the results obtained, in combination with previous work on the subject, provide at least a qualitative picture of the possible dynamics of strongly interacting conformal field theories.

From the field theory side, in Chapter 4, we considered $U(N)$ Chern-Simons theory coupled to fundamental fermions in the large N limit. We were able to calculate the free energy of the theory on $R^2 \times S^1$ for all values of the 't Hooft coupling λ. We also studied the operator spectrum of the theory – the results suggest that the holographic dual is some sort of higher-spin gauge theory, even in the strongly interacting limit.

In Chapter 4, we also noted that many Chern-Simons theories exist that, although non-supersymmetric, are conformal and have a large N limit (e.g., the $U(N)\times U(N)$ Chern-Simons theory coupled to a massless fermion in the bifundamental representation mentioned earlier.) The methods we used to solve the Chern-Simons vector model in Chapter 4 would clearly fail for theories with adjoint or bifundamental matter, as would our argument for the protection of the scaling dimensions of the higher-spin current operators. This illustrates the power as well as the potential applicability of the gravitational techniques we used in the first part of this thesis – theories with matrix matter are extremely difficult to solve from the field theory point of view, yet it is precisely these theories that have a chance of posessing a holographic dual that is a traditional Einstein-Maxwell theory of gravity (or some variant thereof).

We are eagerly looking forward to taking part in future developments in this field.

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