Phases in String Theory

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Sandip P Trivedi, at the Tata Institute of Fundamental Research, Mumbai.

(Prithvi Narayan P)

In my capacity as the supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Sandip P Trivedi)

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Collaborators

This thesis is based on work done in collaboration with several people.

The work presented in Chapter 2 is based on work done with S. P. Trivedi, and parts of it have appeared in print in JHEP 1007, 089 (2010). The work presented in Chapter 3 is based on work done with N.Iizuka, N. Kundu and S. P. Trivedi, and parts of it have appeared in print in JHEP 1201, 094 (2012). The work presented in Chapter 4 is based on work done with S. Minwalla, T. Sharma, V. Umesh and X. Yin, and parts of it have appeared in print in JHEP 1202, 022 (2012).

To

My Family

Synopsis

Introduction

By now there is considerable evidence that string theory is a consistent theory of quantum gravity. Although the theory is unique, it has diverse set of ground states. This complicated set of low energy vacua is usually referred to as landscape. The symmetries of each of these vacua may be different, and the low energy excitations about each of them forms a different effective field theory. This gives rise to the rich structure of string theory with each vacua describing a particular "phase" of the theory. The situation is reminiscent of condensed matter physics where the microscopic theory is unique, but the physics at large length scales show enormous number of emergent phases.

Exploring the string landscape has been a active area [1]. In the first part of this thesis, we study the stability of some nonsupersymmetric AdS_4 vacua in the landscape . These vacua arise in massive IIA supergravity compactifications [2] and the supersymmetric solutions were studied in considerable depth by [3]. In [4], we construct perturbatively stable non supersymmetric vacua in this setup. We find that, for a class of them, large number of nonperturbative decay channels are ruled out. This suggests that stable nonsupersymmetric AdS vacua might exist.

With the advent of AdS/CFT correspondence [5], it is now possible to map gravitational theories in some corners of landscape to quantum field theories. As AdS/CFT also happens to be a weak-strong duality, these vacua may prove useful in the study of strong coupling physics of quantum field theories which is otherwise hard to study using conventional techniques. It is interesting to consider these vacua and ask what one can learn about the dual strongly coupled CFTs from holography. In the second part of thesis we study a class of dilatonic black branes [6][7]. In [8] we compute the two point functions of a fermionic operator in the field theory dual to these gravity systems using holographic techniques. For this system with reasonable thermodynamics, we find violations of fermi liquid behaviour.

It is also interesting to ask what class of quantum field theories admit a gravity dual and if so, what are the properties of the gravity systems we expect? In the context of three-dimensional cherns simons theories, there are few examples of known gravity duals with simple matter content other than ABJM theory [9]. In the third and concluding part of the thesis based on [10], we consider some supersymmetric field theories with simple matter content and study certain protected quantities which gives hints about possible gravity duals. Although we have not identified gravity duals, some promising field theories are identified which could possibly have a supergravity dual.

On the Stability of Non-supersymmetric vacua

String theory landscape includes a large class of non supersymmetric vacua. Supersymmetric (susy) vacua have been studied extensively and are generally expected to be stable to both perturbative and nonperturbative decays [11]. Non supersymmetric vacua have not been as well understood as their susy counterparts. In this part of the thesis, we construct a class of non supersymmetric AdS vacua which are perturbatively stable and investigate their non perturbative stability.

A small decay rate of the AdS vacuum has dramatic consequences for the dual CFT living on its boundary. We can see this by considering AdS_{d+1} in Poincare coordinates:

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + \sum_{i=1}^{d-1} dx_i^2\right) + \frac{R^2}{r^2}dr^2. \quad (1)$$

with the decay rate per unit volume in the bulk being Γ. Taking the boundary metric to be flat, the decay rate per unit volume in the boundary theory is given by integrating the bulk decay rate in the radial direction as,

$$\Gamma_{boundary} = \int \sqrt{g} dr \Gamma \sim r_b^d \Gamma, \quad (2)$$

where r_b is the radial location of the boundary. The decay rate diverges as $r_b \to \infty$. We see that if the non-susy AdS vacua are unstable non-perturbatively, then non-supersymmetric CFT's which admit a gravity dual are unlikely to exist. If true, this is an important consequence since holography has emerged as a major tool with which to study strongly coupled conformal field theories. The perturbatively stable vacua that we find provides us with a setup in which we can explore this. Surprisingly, we find that large classes of decay channels are in fact ruled out.

The model

We consider a compactification of massive Type IIA theory on a Calabi Yau manifold (a blown up $\frac{T_6}{Z_3 \otimes Z_3}$) with fluxes and orientifold plane. This model was studied in [3]. First consider the orbifold limit $\frac{T_6}{Z_3 \otimes Z_3}$ which has 9 fixed points. Let each T^2 in the $T^6 = T^2 \otimes T^2 \otimes T^2$ be parametrized by complex coordinate $z_i, i = 1 \cdots 3$. Further consider a orientifold-6 plane filling the non-compact directions and wrapping a 3-cycle which is the locus of fixed points of the $\sigma : z \to -\bar{z}$ reflection symmetry. The resulting compactification now has $\mathcal{N} = 1$ supersymmetry. The three T^2 moduli, the dilaton-axion, and the nine blow up modes, all survive the orientifolding and form the bosonic components of chiral superfields. Turning on a blow up mode replaces the corresponding fixed point by a P^2 of non vanishing size.

Fluxes and Moduli

Now we incorporate the effects of fluxes. We will seek a solution which is close to the orbifold limit and incorporate the effects of blow up modes perturbatively. Let a basis of complex (1,1) forms which are odd under the reflection σ be ω_i, ω_A. Here ω_i , $i = 1, 2, 3$ are the poincare duals of cycles in the orbifold limit and ω_A , $A = 1, \cdots 9$ are the poincare duals to blow up cycles. We will use $a, b..$ to denote both i and A type of indices. The explicit form of ω_i is given by

$$\omega_i = (\kappa\sqrt{3})^{\frac{1}{3}} i dz^i \wedge d\bar{z}^i \tag{3}$$

where κ is the triple intersection number defined as $\kappa = \int_{T^6/Z_3^2} \omega_1 \wedge \omega_2 \wedge \omega_3$.

Let $\tilde{\omega}_i, \tilde{\omega}_A$, be the (2,2) forms which are hodge dual (in the internal manifold) to ω_i, ω_A respectively. Then the four form flux F_4 can be expanded in this basis as,

$$F_4 = \sum_i e_i \tilde{\omega}_i + \sum_A e_A \tilde{\omega}_A \tag{4}$$

.

If β_0 is the real part of the holomorphic three form $\Omega = \sqrt{2}3^{\frac{1}{4}} i dz_1 \wedge dz_2 \wedge dz_3$, then the three form flux H_3 can be expanded as [1]

$$H_3 = -p\beta_0 \tag{5}$$

Now we give a description of moduli. If J is the Kahler form, then the complexified Kahler two-form $J_c = B_2 + iJ$ is expanded as,

$$J_c = \sum_i t_i \omega_i + \sum_A t_A \omega_A \tag{6}$$

Here $t_a = b_a + iv_a$, where b_a refers to axions coming from B_2 and v_a are related to the sizes of various $(1, 1)$ cycles. Axion ξ which arises from C_3 pairs up with the 4 dimensional dilaton D to form a dilaton axion moduli.

The Kahler potential in the moduli space is [2].

$$K = -\log(8\kappa v_1 v_2 v_3 + \frac{4}{3}\beta \sum_{A=1}^{9} v_A^3) + 4D. \tag{7}$$

[1] Tadpole condition for the C_7 potential gives $m_0 p = -2\sqrt{2}\pi\sqrt{\alpha'}$, where m_0 is the Romans parameter.

[2] κ the triple intersection number and β takes definite values given in [3], but we will not need these explicit values

The resulting superpotential is

$$W = -p\xi - \sqrt{2}ipe^{-D} + e_i t_i + e_A t_A - m_0 \left(\kappa t_1 t_2 t_3 + \frac{\beta}{6} \sum_A t_A^3 \right) \tag{8}$$

From the above data we can determine the effective potential V in the 4 dimensional Einstein frame in terms of fluxes and moduli. The expectation value of the moduli in the resulting compactification is then given by the minima of V. In addition, supersymmetry imposes the condition

$$\text{sign}(m_0 e_i) < 0, \text{sign}(m_0 e_A) < 0, \tag{9}$$

Since the potential turns out to be invariant under $e_i \to -e_i$ for any i, this gives an easy way to construct a non supersymmetric vacuum from a given supersymmetric vacuum : change the sign of some or all of e_i so that the above condition is violated.

To keep the blow-up modes small and the Calabi-Yau moduli stabilized close to the orbifold point, we take the blow up fluxes e_A to satisfy the condition,

$$\frac{|e_A|}{|e_i|} \ll 1, \tag{10}$$

It turns out we can find a solution order by order in small parameter $\delta = \sqrt{|\frac{e_A^3}{e_1 e_2 e_3}|}$. The sizes of the blow up cycles v_A will turn out to be parametrically smaller than the sizes of cycles in the orbifold limit, i.e $v_A \ll v_i$.

The leading order solution shows that in some range of fluxes, when for all i, $e_i \sim e \gg 1$ (in string units) the compactification is Non Freund type : typical size of internal space l is small compared to R_{AdS}. In fact $\frac{l}{R_{AdS}} \sim e^{\frac{-1}{2}} \to 0$. This simplifies the analysis when it comes to checking for possible tachyons. Also the string coupling $e^{\phi} \sim e^{\frac{-3}{4}} \to 0$ so that the theory is weakly coupled in this regime.

We did a careful analysis of the perturbative stability and found two new solutions which break supersymmetry and have no unacceptable tachyons (i.e mass below the BF bound). We call them Type 2 and Type 3. We will also find it convenient to keep track of the susy solution which we call Type 1. Following is the list of perturbatively stable non susy vacua:

- Type 2: $\text{sign}(m_0 e_i) = \text{sign}(m_0 e_A) = +1$. Susy is broken. All modes are non-tachyonic.
- Type 3:$\text{sign}(m_0 e_i) = +1$, $\text{sign}(m_0 e_A) = -1$ Susy is broken, All modes are non-tachyonic.

Non Perturbative decays

To investigate the non-perturbative decay of an unstable vacuum, we follow the classic discussion of [12]. The decay of the unstable vacuum, called the false vacuum, into another state, the true vacuum is mediated by the nucleation of a bubble of true vacuum inside the false vacuum. In the semi-classical approximation one seeks a solution (called the bounce) to the Euclidean action which can interpolate between the false and true vacua. We work in the thin wall approximation which takes the thickness of the bubble wall (domain wall) to be much smaller than all the other length scales in the problem. These include the radius of the bubble and the radii of curvature of the inside and outside spacetimes.

In the thin wall approximation one can show that a necessary condition for bounce to exist is

$$\left(\frac{\epsilon}{3S_1} - \frac{S_1}{4}\right) > \sqrt{\frac{|V_+|}{3}} \tag{11}$$

where V_+, V_- are the vacuum energies of false and true vacuum respectively and ϵ is their difference. S_1 is the tension of the domain wall. If this condition is violated then the decay will not occur.

We present a description of D4 brane mediated decays here. It turns out that decays mediated by more general branes do not change the conclusions. The $D4$ brane wraps a two-cycle in the internal space and extends along two of the spatial directions of AdS_4. This causes the four-form flux F_4, along the 4-cycle dual to the two-cycle wrapped by the $D4$, to jump. This change in F_4 causes a change in the cosmological constant, and hence changes the vacuum energy.

Non-Susy to Susy decays

Consider a decay from a non susy vacuum to a susy vacuum. Since we will see that these decays are disallowed in the orbifold limit, effects of blow up fluxes can be neglected. The domain wall tension scales like

$$S_1 \sim \frac{|\delta e|}{|e|}(\frac{1}{|e|})^{9/4}, \tag{12}$$

where δe is the change in the four form flux across the wall ($e_i \sim e \gg 1$). That the tension S_1 is proportional to δe is easy to understand because the number of $D4$ branes is $\propto \delta e$. The thin wall approximation can be shown to hold if

$$\frac{|\delta e|}{|e|} \gg \frac{(\delta |e|)^2}{|e|^2}. \tag{13}$$

We can ensure this to be the case by choosing e to be of opposite signs on either side of the domain wall with almost the same magnitude, i.e $\delta e \sim 2|e|$ and $\delta e \ll |e|$. Note that this results in vacua which are close by in the moduli space but far apart in the flux space. As was mentioned before, the vacuum energy is independent of sign of e_i and the change in vacuum energy ϵ scales like

$$\epsilon \sim \frac{\delta |e|}{|e|}(\frac{1}{|e|})^{9/2}. \tag{14}$$

Hence the condition eq(11) is parametrically violated and this decay cannot occur.

Non-Susy to Non-Susy decays

Consider a decay from a non susy vacuum to another non susy vacuum. The tension of the interpolating domain wall satisfies a lower bound in terms of the jump in the superpotential caused by the domain wall as,

$$T \geq T_L \equiv 2e^{K/2}|\Delta W|, \tag{15}$$

where $\Delta W = \delta e_a v_a$ is the difference in superpotential across the domain wall. This bound is saturated for certain special cycles. The strategy we use is to rule out decays mediated by the domain walls for which lower bound T_L itself violates condition eq(11). In the orbifold limit, it turns out the lower bound T_L saturates the inequality eq(11) and hence inclusion of blow up fluxes is crucial for these results. Following is the summary of results.

- Type 2 to Type 2 decays are at most marginal. This means that upto the order we have worked, the inequality eq(11) is saturated.
- Type 3 to Type 3 decays are allowed.
- Type 2 to Type 3 decays are disallowed.

Holographic Fermi and Non-Fermi Liquids with Transitions in Dilaton Gravity

The Gauge/Gravity correspondence [5] provides us with a new tool to study strongly coupled field theories. It is worth exploring whether insights of relevance to condensed matter physics can be gained using this tool. One set of questions which have proved difficult to analyze using conventional techniques is the behaviour of fermions in strongly coupled systems in the presence of a chemical potential. This question is particularly interesting in view of considerable evidence now for non-Fermi liquid behaviour in condensed matter systems (see [13] for references). It is generally believed that strong coupling is required to explain these phenomenon.

Holographically, Non Fermi Liquids were initially realized in Extremal Reisnner Nordstrom black branes [14] [15]. While these branes are simple and explicit, they suffer from an important unphysical feature, namely, their entropy is nonvanishing at vanishing temperature and scales as a positive power of the chemical potential. Hence these systems may not be a good model for condensed matter systems. Later, gravity systems with vanishing entropy at extremality were explored in [7]. They found that these systems behaved like canonical fermi liquids. This raises the questions whether holographic systems with vanishing entropy at extremality always leads to fermi liquid behaviour. We address this question in this part of the thesis.

We find that a range of interesting behaviours arise by coupling fermions to a strongly coupled sector with a gravitational dual of Einstein Maxwell Dilaton system. This includes both Fermi liquid and non-Fermi liquid behaviour, transitions between them, and transitions from a non-Fermi liquid state to one where there are no well-defined quasi-particles. Moreover, this can happen when the strongly coupled sector has reasonable thermodynamic behaviour, consistent in particular with the third law of thermodynamics, since the gravity background has vanishing entropy at extremality.

The Einstein Maxwell Dilaton system

The system we consider consists of gravity , a $U(1)$ gauge field and a scalar ϕ which we call the dilaton, in four dimensions with the action

$$S = \int d^4x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - f(\phi)F^2 - V(\phi)\right] \tag{16}$$

We will be particularly interested in solutions where the dilaton has a run-away type of behaviour near the horizon of an extremal black brane. Such run-away behaviour was shown to result in vanishing entropy of the extremal brane [7]. This system was studied in [6], and we will be interested in the near horizon region as this is sufficient to determine the low-temperature or low frequency (compared to the chemical potential) response of the system. In the near horizon region, we take $f(\phi) = e^{2\alpha\phi}$ and $V(\phi) = V_0 e^{2\delta\phi}$. We can find a solution near horizon of the form [3]

$$ds^2 = \frac{dr^2}{C_a^2 r^{2\gamma}} - C_a^2 r^{2\gamma}dt^2 + r^{2\beta}(dx^2 + dy^2) \tag{17}$$

$$F = \frac{Q_e}{f(\phi)r^{2\beta}}dt\wedge dr \tag{18}$$

where β, γ are functions of α, β given as

$$\beta = \frac{(\alpha+\delta)^2}{4+(\alpha+\delta)^2} \qquad \gamma = 1 - \frac{2\delta(\alpha+\delta)}{4+(\alpha+\delta)^2}$$

C_a^2 and Q_e are determined in terms of α, δ.

We can also construct slightly non extremal black brane solutions (with $T \ll \mu$). From scaling arguments or from the explicit solution, we can show that the entropy density behaves as

$$S \sim T^{\frac{2\beta}{2\gamma-1}} \tag{19}$$

Note that this has the desired behaviour for entropy, i.e entropy vanishes at zero temperature. The parameters α, δ are chosen to be in a range where the solution and the thermodynamics is sensible. Although the geometry eq(17) has pathologies at $r \to 0$, a small temperature (parametrically small in large N) can be shown to control this.

[3] We showed numerically that such a near horizon can arise as the near-horizon limit of an asymptotically AdS_4 geometry (perturbed by a non-normalizable mode). To show this, we took the potential to be $V(\phi) = 2V_0\cosh(2\delta\phi)$.

We also computed the DC conductivity for this system holographically, as $\omega \to 0$, and for small temperature $\frac{T}{\mu} \ll 1$ it is,

$$Re(\sigma) \sim T^{\frac{2(4+\alpha^2-\delta^2)}{4+(\alpha-3\delta)(\alpha+\delta)}} \tag{20}$$

Fermionic Two point function

Following [14], consider a free fermion ψ in the bulk with mass m and charge q. Define[4]

$$\psi = (-g\ g^{rr})^{-\frac{1}{4}} e^{-i\omega t + ik_1 x} \begin{pmatrix} y_+(r) \\ 0 \\ 0 \\ -i\ z_-(r) \end{pmatrix} \tag{21}$$

where we have used translational in t, x_1, x_2 and rotational invariance in x_1, x_2 plane. The second and third components of the spinor have been chosen to vanish because it can be shown that it does not couple to other components. Then the equation of motion for the fermion is

$$\sqrt{\frac{g_{ii}}{g_{rr}}}\,(\partial_r - m\sqrt{g_{rr}})\,y_+ = -[k_1 - \sqrt{\frac{g_{ii}}{-g_{tt}}}(\omega + qA_t)]z_- \tag{22}$$

$$\sqrt{\frac{g_{ii}}{g_{rr}}}(\partial_r + m\sqrt{g_{rr}})z_- = -[k_1 + \sqrt{\frac{g_{ii}}{-g_{tt}}}(\omega + qA_t)]y_+ \tag{23}$$

We are interested in finding the two point function of the fermionic operator dual to ψ holographically using the AdS/CFT prescription. We impose ingoing boundary condition at the horizon and evolve the above equation to extract the normalizable and non normalizable part of ψ in the boundary (to leading order in ω). Then by the standard AdS/CFT prescription, the greens function is given by their ratio. This will show signatures of Fermi/Non fermi Liquid behaviour. Before we proceed to discuss the result we obtain, we give the Greens function (near the fermi momentum and for small frequencies) for a fermi liquid for reference.

$$G(\omega, k) = \frac{Z}{\omega - v_F(k - k_F) + i\Gamma} \tag{24}$$

where Z, v_F, k_F are constants. The decay width for Fermi Liquid is $\Gamma \sim \omega^2$.

It turns out that $\beta + \gamma$ is the parameter which governs the behaviour of the greens function. If $\beta + \gamma < 1$, it turns out there is no pole in the greens function and hence no fermi surface. Below we give the results obtained for other regimes of the parameter $\beta + \gamma$.

[4] for a metric of the form $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{11}(dx_1^2 + dx_2^2)$

$\beta + \gamma > 1$

In this case it turns out that the WKB approximation works well. Using this approximation, the greens function at momenta close to fermi momentum and small ω can be found to be of the form eq(24) with a decay width

$$\Gamma \sim exp\left[-2c_1\left(\frac{k_F^{2\gamma-1}}{\omega^{\beta+\gamma-1}}\right)^{\frac{1}{\gamma-\beta}}\right] \tag{25}$$

for some constant c_1. We see that the small frequency excitations have a linear dispersion relation, with a width which is exponentially suppressed at small frequency and therefore very narrow. Hence there is a very sharply defined quasi-particle and this behaviour is fermi liquid like. Inclusion of some other interactions could broaden the decay width to the canonical ω^2 behaviour.

$\beta + \gamma = 1$

Here we can solve the equation of motion exactly in the near horizon region. Greens function turns out to be of the form eq(24) but now with a decay width

$$\Gamma \sim e^{i\phi}\omega^{2\eta} \tag{26}$$

where $\eta = \frac{|k_1|}{2\gamma-1}$ and ϕ is a phase which depends on γ, β, k_1. Note that k_1 is close to fermi momentum k_F. This result is very similar to that obtained in extremal Reisnner Nordstrom case [16]. For $\eta > 1$ there is a well-defined quasi-particle with a linear dispersion and a width going like $\omega^{2\eta}$ which is quite narrow. Hence the behaviour is again fermi liquid like. For $1/2 < \eta < 1$, the width is broader than the Fermi liquid case and is hence a Non fermi Liquid. For $\eta < 1/2$ the behaviour is more novel. The last term in the denominator going like $\omega^{2\eta}$ dominates both the real and imaginary parts of the ω dependence. As a result there is no well-defined quasi-particle, since the residue vanishes at the pole. Finally for $\eta = 1/2$, the Green's function takes the form,

$$G_{R22} = \frac{Z}{v_F(|k| - |k_F|) + d_1\omega\log\omega + d_2\omega} \tag{27}$$

where d_1 is real and d_2 complex. This is called a marginal fermi liquid.

Supersymmetric states in Large N chern-simons theories

Chern Simons theories with arbitrary matter content have a sequence of fixed points parameterized by cherns simons number k. This is because the coefficient of Chern-simons term k is forced to be an integer by gauge invariance. Hence conformal field theories (CFT) are fairly easy to construct in these set ups. For $SU(N)$ or $U(N)$ theories at large N, this sequence of fixed points becomes a line of fixed points parametrized by the thooft coupling $\lambda = \frac{N}{k}$. They are interesting from the viewpoint of AdS/CFT correspondence, because line of fixed points parameterized by a coupling constant can interpolate between a field theory description at weak coupling and a dual gravity description at strong coupling. It is interesting to ask which class of chern-simons theories admit duals with a gravity approximation at strong coupling. Apart from ABJM theory, examples of gravity duals of theories with simple matter content have not yet been found.

In this part of the thesis, we compute the supersymmetric spectrum of a large class of large N Chern Simons matter theories. This is protected from renormalization under continuous deformation of λ and thus can be used to learn about strong coupling behaviour of the theories. We consider various Superconformal Chern-Simons-Matter theories with $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetry and compute (or present conjectures for) the BPS spectrum.

Unitary representation of $\mathcal{N} = 2, 3$ superconformal algebras

We describe here the notations required for presenting the results. In $2+1$ D, the bosonic part of $\mathcal{N} = 2$ superconformal algebra is $SO(3,2) \times SO(2)_R$ ($SO(3,2) \times SO(3)_R$ for $\mathcal{N} = 3$). Representations are labeled by the quantum numbers of the primary state (Δ, j, h). where Δ is the scaling dimension, j is spin and h is R-charge (or R symmetry highest weight). h can be any positive or negative real number for $\mathcal{N} = 2$, but is a positive half integer for $\mathcal{N} = 3$.

For $j \neq 0$, representations are unitary if

$$\Delta \geq |h| + j + 1$$

For $j = 0$ unitary representations occur when

$$\Delta \geq |h| + 1 \qquad \text{or} \qquad \Delta = |h|, \quad |h| \geq \frac{1}{2} \tag{28}$$

Representations saturating these bounds are called short while other unitary representations are called long.

One way to capture information about the state content of the conformal field theory is through Witten Index which is defined to be

$$\mathcal{I}_{+} = \mathrm{Tr}(-1)^F x^{H+J} e^{-\beta(H-J-R)} \tag{29}$$

The Witten index vanishes on all long representations of the supersymmetry algebra but is nonzero on short representations. The only way that the Witten index of a CFT can change under continuous variations of parameters like λ, is for the R-charge to be renormalized as a function of that parameter.

Theories with vanishing superpotential

R-charge as a function of λ

The first class of theories studied were $\mathcal{N} = 2$ $U(N)$ Chern Simons theories at level k, coupled to g adjoint chiral multiplets with vanishing superpotential. This class of theories was studied, and demonstrated to be superconformal (for all N, k and g) in [17]. They showed that the R-charge h of a chiral field is renormalized as a function of λ (to leading order in λ) as

$$h(\lambda) = \frac{1}{2} - (g+1)\lambda^2 \tag{30}$$

where $\lambda = \frac{N}{k}$. As the R-charge of a supersymmetric operator plays a key role in determining its scaling dimension (via the BPS formula), the exact characterization of the spectrum of supersymmetric states in this theory at large λ requires knowledge of function $h(\lambda)$ at large λ. This can be obtained by an application of the powerful recent results of Jafferis [18] to this problem. In [18] Jafferis used localization methods to derive a formula (in terms of an integral over r variables, where r is the rank of the gauge group) for the partition function on S^3 of the CFT in question, as a function of h_i the R-charges of all the chiral fields in the theory. He then demonstrated that the modulus of this partition function is extremized by the values of h_i at the conformal fixed point, assuming the absence of accidental global symmetries. In the large N limit of interest to this paper, Jafferis' matrix integral is dominated by a saddle point. Using a combination of analytic and numerical techniques, we extremized the action with respect to h, and thereby evaluate $h(\lambda)$. It turns out that $h(\lambda)$ is a monotonically decreasing function for all g. At large g (but all values of λ), we found analytical results which showed that $h(\lambda)$ tends to a constant value at $\lambda = \infty$ as

$$\frac{1}{2} - \frac{4}{\pi^2 g} + \mathcal{O}\left(\frac{1}{g^2}\right). \tag{31}$$

We numerically saw that this formula appears to work at the 10-15 percent level even down to $g = 2$. More specifically, at $g = 3$, our numerics indicates $h(\infty) \approx 0.35$, and at $g = 2$, $h(\infty) \approx 0.27$. these do not compare badly with 0.365 and 0.3 as predicted by eq(31).

Interestingly, however, at $g = 1$, $h(\infty) = 0$ i.e the R-charge of the chiral multiplet decreases without bound in this special case as was anticipated in [19].

Spectrum of single trace operators

Given the function $h(\lambda)$, one can evaluate the superconformal index $\mathcal{I}_+$ [20] of these theories as a function of $h(\lambda)$. As was already noted in [20], this index demonstrates that the spectrum of supersymmetric single trace operators grows exponentially with energy for $g \geq 3$. In the absence of a superpotential one can compute a more refined Witten index (adding in a chemical potential that couples to the global symmetry generators) which shows an exponentially growing density of states for the supersymmetric spectrum (in the theory without a superpotential) even for $g = 2$. This immediately suggests that the simplest possible dual description for all theories with $g \geq 3$ (and the theory without a superpotential at $g = 2$) is a string theory, with an exponential growth in supersymmetric string oscillator states.

However, the index indicates a sub exponential growth of supersymmetric states for all theories with $g = 1$ and theories with a nontrivial superpotential when $g = 2$. This leaves open the possibility of a simpler dual (one with a field theory's worth of degrees of freedom) in these cases. We now mention the results of $g = 1$ theories without a superpotential. Making the assumption that the spectrum of supersymmetric states in this theory is isomorphic to the cohomology of the classical action of the susy operator, we have computed the full spectrum of single trace supersymmetric operators in this theory. While the states do grow in number with energy in a roughly Kaluza-Klein fashion, they have arbitrarily high spins, ruling out a possible dual supergravity dual description.

The spectrum of supersymmetric state of g=1 theory includes the states in the chiral ring, $\text{Tr}\phi^n$ for all $n \geq 2$. The scaling dimensions of these operators is $nh(\lambda)$. Unitarity requires that the scaling dimension Δ of all the scalar operators in a 3D CFT must satisfy $\Delta \geq \frac{1}{2}$ and operators which saturates this bound are necessarily free. Since the R-charge in this theory decreases monotonically to zero, the scaling dimensions of these operators also decreases with λ and hit the unitarity bound one by one for larger and larger values of λ. One of the attractive scenario in this case is as follows. Let λ_n^f and be that λ where operator $Tr(\phi^n)$ becomes free . i.e

$$h(\lambda_n^f) = \frac{1}{2n} \tag{32}$$

As λ is increased past λ_2^f, λ_3^f and so on, $Tr(\phi^2), Tr(\phi^3)$.. becomes free and decouples from the theory one by one.

Theories with a superpotential

The second class of theories studied were $\mathcal{N} = 2$ $U(N)$ Chern Simons theories at level k, coupled to g adjoint chiral multiplets with appropriate superpotential. For theories that reduce to free systems as $\lambda \to 0$, the superconformal index is independent of the details of the superpotential, other than the fact that the index cannot now be weighted with respect to a chemical potential for any global symmetry under which the superpotential is charged, and depends only on the R-charge of matter fields which may be renormalized. So in particular, the index demonstrates the presence of an exponentially growing spectrum of supersymmetric states for $g \geq 3$, exactly as above. For this reason we focused our study on $g \leq 2$. We first start with $g = 1$ with various superpotentials.

$\mathrm{Tr}\,\Phi^4$ at $g = 1$

The superpotential deformation $\mathrm{Tr}\,\Phi^4$ is marginal at $\lambda = 0$, but is relevant at finite λ. It has been argued in [17] that the beta function for this superpotential term vanishes when its coefficient is of order λ (at small λ) leading to a weakly coupled CFT with a $\mathrm{Tr}\,\Phi^4$ su-

Cohomology states ($\mathcal{N}=2$ quantum numbers)	Multiplicity	$\mathcal{N}=2$ Primary	$\mathcal{N}=3$ Primary	Allowed h
$(h,0,h,h)$	1	$(h,0,h,h)$	$(h,0,h,h)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+\frac{1}{2},\frac{1}{2},h,h)$	1	$(h,0,h-1,h)$		$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{1}{2},h+1,h)$	1	$(h+1,0,h,h)$	$(h+1,0,h,h)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+2,1,h+1,h)$	1	$(h+\frac{3}{2},\frac{1}{2},h,h)$		$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+1,1,h,0)$	1	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$h\in\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{3}{2},h,0)$	1	$(h+1,1,h-1,0)$		$h\in\mathbb{Z}^+$

Table 1: Supersymmetric spectrum for 2 chiral adjoints at $\mathcal{N}=3$ fixed point with $SU(2)$ flavor symmetry.The notation is (Δ,j,h,y) where y is the $SU(2)$ flavour charge.

perpotential turned on. The presence of the superpotential forces the R-charge of the field ϕ to be fixed at $h=\frac{1}{2}$ at all values of λ in this new fixed line. While the superconformal index of this theory is blind to the presence of the superpotential, the spectrum of single trace supersymmetric operators is not. Again assuming that supersymmetric spectrum is accurately captured by the classical supercharge cohomology at all λ, we computed the supersymmetric spectrum. As in the case of theories without a superpotential, our conjectured supersymmetric spectrum grows with energy in a manner expected of Kaluza-Klein compactification, but continues to include states of arbitrarily high spin.

$\mathcal{N}=3$ theory at $g=2$

Let us now turn to $g=2$ theories with a superpotential. First consider superpotentials of the form $\mathrm{Tr}\,[\Phi_1,\Phi_2]^2$. This superpotential is marginal at $\lambda=0$, but is relevant at finite λ (regarded as a deformation about the theory with no superpotential). It was argued in [17] that the RG flow seeded by this operator ends with the coefficient of this superpotential stabilized at that finite value (of order λ) that enhances the supersymmetry of the theory to $\mathcal{N}=3$. This $\mathcal{N}=3$ theory enjoys invariance under an enhanced $SU(2)$ R symmetry group, and also enjoys invariance under an $SU(2)$ flavour symmetry group. We have computed the spectrum of supersymmetric states in this theory; our results are presented in Table 1. Interestingly, it turns out that the spins of supersymmetric states in this theory grow roughly in the manner one would expect of a Kaluza-Klein compactification of a supergravity theory on $AdS_4\times S^3$. In particular the spins of supersymmetric states in this theory never exceed two. We have not, however, managed to identify a specific supergravity compactification that could give rise to this spectrum.

Superconformal $\mathcal{N}=2$ deformations of the $\mathcal{N}=3$ theory

There exists a manifold of exactly marginal $\mathcal{N}=2$ deformations of the $\mathcal{N}=3$ theory described above and this can be characterized rather precisely in the neighbourhood of the $\mathcal{N}=3$ fixed point using the recent results of [21]. We have computed the spectrum of supersymmetric states in these deformed theories. Qualitatively, the results for these deformed theories are similar to those of the $\mathcal{N} = 3$ theory above. Although the spins of all supersymmetric states are less than or equal to two, we have not been able to identify a supergravity dual.

Discussions

In this thesis we study various aspects of the landscape of vacua in string theory. We explore different phases in the landscape both from the gravity side and from the field theory side by using the AdS/CFT correspondence.

From gravity side, in [4] we explore the stability of nonsupersymmetric vacua in the string landscape. Building on existing literature, we construct a large class of perturbatively stable vacua and then look for tunneling instabilities into nearby vacua in landscape. A large number of decay channels are ruled out by our analysis, suggesting that some non susy vacua could be stable after all. It would be instructive to construct more examples of such vacua with all moduli stabilized and investigate their non perturbative stability. This program might ultimately provide hints to constructing stable desitter vacua which is of relevance to string phenomenology.

Further continuing the exploration of the landscape, in [8] we construct gravity systems with reasonable thermodynamics and illustrate how non fermi liquid can arise in such systems using holography. We use two point functions of fermions to ascertain the nature of the excitations. Recently, the authors of [22],[23] have shown that entanglement entropy can be used as a probe of fermi surface. It would be interesting to explore the strong coupling phenomenon in field theories by studying their gravity duals by using these probes.

On the field theory side, in [10] we study supersymmetric cherns simons theories with simple matter content with the intent to check whether they admit gravity duals. We deduce the protected matter content which can prove useful in identifying the dual gravity system. Recently, localization techniques have been used to compute exact quantities like partition function in certain supersymmetric theories. This might provide hints for possible gravity dual and also provide evidence for duality among quantum field theories.

As string theory progresses, we expect to have a more complete understanding of string landscape.

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Publications

Papers relevant to the thesis work:

- P. Narayan and S. P. Trivedi,"On The Stability Of Non-Supersymmetric AdS Vacua," JHEP **1007**, 089 (2010) [arXiv:1002.4498 [hep-th]].
- S. Minwalla, P. Narayan, T. Sharma, V. Umesh, X. Yin,"Supersymmetric States in Large N Chern-Simons-Matter Theories,", JHEP **1202**, 022 (2012) [arXiv:1104.0680 [hep-th]].
- N.Iizuka, N. Kundu, P. Narayan, S. P. Trivedi, "Holographic Fermi and Non-Fermi Liquids with Transitions in Dilaton Gravity", JHEP **1201**, 094 (2012) [arxiv:1105.1162 [hep-th]].

Other Papers:

- N.Iizuka, S.Kachru, N.Kundu, P. Narayan, N. Sircar and S. P. Trivedi, "Bianchi Attractors: A Classification of Extremal Black Brane Geometries", arxiv:1201.4861 [hep-th].

Contents

Chapter 1

Introduction

String theory is believed to be a consistent theory of quantum gravity. Although there are different perturbative formulations of the theory [1], various dualities relate them to each other at the nonperturbative level and the full theory is expected to be unique. String theory contains various massless fields like the metric, scalar fields and other tensor fields which can take several different consistent background values giving rise to a multitude of solutions. Therefore, despite being a unique theory, string theory has many ground states. This complicated set of low energy vacua is usually referred to as the landscape. The symmetries of each of these vacua will in general be different, and the low energy excitations about each of them forms a different effective field theory. This gives rise to the rich structure of string theory with each vacua describing a particular "phase" of the theory. In fact authors of [2] estimate that the number of (perturbatively stable supersymmetric) vacua could exceed 10^{120}. The situation is reminiscent of condensed matter physics where the microscopic theory is unique, but the physics at large length scales shows an enormous number of emergent phases.

Exploring the landscape has been a active area of research in string theory [3][4]. One obvious motivation for this study comes from phenomenology. An important question in string phenomenology and string cosmology is finding vacua which are semi realistic. Exploring the landscape, one would hope to find vacua which share some of the essential features of the physical universe. Another possible motivation for a study of the landscape is to understand field theories better by employing the AdS/CFT correspondence. In fact one of the topics in this thesis deals with this subject as we will explain later. Also, understanding the web of dualities which connect different corners of the landscape is still an active research area. As these dualities provides a definition of what string theory is at nonperturbative level, a study of the landscape might prove useful in understanding the structure of string theory better.

In this thesis we study various aspects of the string landscape. We focus on the part of the landscape which have a effective four dimensional description. We also concentrate on theories with a negative cosmological constant. We now give a brief description of the three projects and describe how it helps us in understanding different aspects of the landscape.

Stability of Non-supersymmetric vacua

Critical string theories have 10 spacetime dimensions and compactifying 6 of the space dimensions results in a 4 dimensional effective theory. A typical vacua, at low energies has many massless scalar fields which are called moduli. Stabilizing these moduli by turning on masses for them (by say turning on background values for fluxes) is generally referred to as moduli stabilization. A lot of work has been done in recent years in stabilizing the moduli resulting in perturbatively stable supersymmetric vacua [3][4]. From phenomenology point of view, it would be most interesting to find de Sitter vacua (which necessarily breaks supersymmetry) with all moduli stabilized. But being time dependent they are harder to construct. Recently perturbatively stable non supersymmetric AdS vacua have been studied [5][6]. In the first part of this thesis we construct perturbatively stable nonsupersymmetric AdS_4 vacua in the landscape and also analyze their nonperturbative stability. This part of the thesis is based on the work done in [7].

In [8], the authors construct supersymmetric AdS_4 vacua with all moduli stabilized [1]. By changing some of their flux configurations, in [7] we construct a large class of non supersymmetric vacua with all the moduli can be stabilized.

We then go on to study the nonperturbative stability of these vacua. Note that the string landscape poses a problem for phenomenology : Even if we find a realistic vacuum, there is no principle which tells us why this particular vacuum among the host of other vacua in the landscape is more preferred. Thus it is not clear how to obtain testable predictions from the theory. Looking for vacua which are non perturbatively stable might alleviate the problem by narrowing the string landscape. Another motivation to look into the nonperturbative stability is the following. A small decay rate of the AdS vacua has dramatic consequences for the dual CFT living on its boundary. This is so because decay rate of the boundary theory is got by integrating the decay rate of the bulk theory over the bulk volume which diverges. In fact the divergence arises from the near boundary region or equivalently in the dual CFT language it arises by summing over instantons of very small size [9]. Hence even a small decay rate of the AdS_4 vacua might make a dual CFT unlikely to exist. Thus it is important to study the nonperturbative stability, especially in the light of recent excitement about applications of holography which we will mention later.

We examine non-perturbative decays of these non supersymmetric vacua to other supersymmetric and non-supersymmetric AdS_4 vacua mediated by instantons in the thin wall approximation. We find that for a class of these vacua a large number of decay channels are ruled out since the tension of the interpolating domain wall is too big compared to the energy difference in AdS units. We show that this can also be understood in terms of a "pairing

[1]The vacua also had weak string coupling and large AdS radius which is essential in controlling the approximations made.

symmetry" in the landscape which relate these vacua with supersymmetric ones. Then the stability of the nonsupersymmetric vacua follow from the stability of supersymmetric ones.

Non Fermi Liquids from Dilaton gravity

The gauge gravity duality [10] is one of the most fascinating ideas to have emerged from the study of string theory in recent years. This is a duality between some quantum field theories and a theory of gravity in one higher dimension. With the advent of this duality, it is now possible to map gravitational theories in some corners of the landscape to quantum field theories. Hence the multitude of vacua in the landscape translates to different emergent phases in the dual field theory. Therefore a study of the landscape might yield valuable insights for the dual field theory. Note that since AdS/CFT is a weak/strong duality, the strong coupling regime in field theory, which is otherwise hard to study using conventional techniques maps to gravity description which is weakly coupled and hence controllable.

With its large number of phases, condensed matter physics is a natural place to use these tools. Low energy effective theory of even a simple system like fermions in a fermi sea coupled to gauge field has been shown to be outside perturbative regime [11]. This is particularly relevant in view of considerable evidence now for non-Fermi liquid behaviour in condensed matter systems (see [12] for references). It is generally believed that strong coupling is required to explain these phenomenon. Studies on trying to use the tools of AdS/CFT to explore strong coupling in condensed matter physics have been underway for some time now [13][14]. The focus on the bulk side is on extremal branes, as these are universal objects in the string vacua and studying them reveal the universal features of the dual strongly coupled field theory. It is easy to turn on a chemical potential (or finite density) for the field theory by incorporating a gauge field in the bulk.

Initial studies focused on Extremal Reissner Nordstrom (eRN) black branes [13][14]. While these systems were found to have Non Fermi Liquid behaviour, they suffer from an important unphysical feature, namely, their entropy is nonvanishing at vanishing temperature and scales as a positive power of the chemical potential. Hence these systems may not be a good model for condensed matter systems. Later, gravity systems with vanishing entropy at extremality were explored in [15]. But these systems were found to have canonical fermi liquid behaviour [16]. This leads to an interesting question whether holographic systems with vanishing entropy at extremality always leads to fermi liquid behaviour.

In second part of thesis we study the two-point function for fermionic operators in a class of strongly coupled systems using the gauge-gravity correspondence. This part of the thesis is based on the work done in [17]. The gravity description includes a gauge field and a dilaton which determines the gauge coupling and the potential energy [18]. Extremal black brane solutions in this system have vanishing entropy. By analyzing a charged fermion in these extremal black brane backgrounds we calculate the two-point function of the corresponding boundary fermionic operator. We find that in some region of parameter space it is of Fermi liquid type. Outside this region no well-defined quasi-particles exist, with the excitations acquiring a non-vanishing width at zero frequency. At the transition, we find that the two-point function can exhibit non-Fermi liquid behaviour.

Supersymmetric states in Large N chern-simons theories

As explained earlier, the string landscape, through AdS/CFT correspondence, leads to a landscape of conformal field theories (CFTs). A study of the string landscape can be hence be also carried out by scanning the set of CFTs and asking which among them admits a gravity description at strong coupling.

For gauge theories with the Yang Mills kinetic term, a careful choice of matter content and couplings can be made to make the theory a conformal theory. This makes it difficult to construct a landscape of CFTs. The situation is very different in three dimensions where gauge fields can have a chern simons kinetic term. The coefficient of chern simons term (k) is forced by gauge invariance to be an integer and hence can not run under renormalization group flow. For any choice of matter content there is a sequence of fixed points parametrized by k. For $SU(N)$ or $U(N)$ theories at large N, this sequence of fixed points coalesce into a line of fixed points parametrized by the continuous thooft coupling $\lambda = \frac{N}{k}$.

Thus, in three dimensions it is fairly easy to construct landscape of three dimensional CFTs and moreover they have a tunable coupling λ. By AdS/CFT correspondence, line of fixed points parameterized by a coupling constant can interpolate between a field theory description at weak coupling and a dual gravity description at strong coupling. It would be interesting to scan through the landscape of these CFTs and ask which class of them admit duals with a gravity approximation at strong coupling.

In last and concluding part of this thesis we study the the supersymmetric spectrum of a large class of large N chern simons theories with simple matter contents. This part of the thesis is based on the work done in [19]. This is protected from renormalization under continuous deformation of λ and thus can be used to learn about strong coupling behaviour of the theories. In particular we study the $\mathcal{N} = 2$ superconformal $U(N)$ Chern-Simons-matter theories with adjoint chiral matter fields, with and without superpotential. We compute the superconformal indices and present conjectures on the full supersymmetric spectrum of the theories in the large N limit with up to two adjoint matter fields. Our results suggest that some of these theories may have supergravity duals at strong coupling, while some others may be dual to higher spin theories of gravity at strong coupling. For the $\mathcal{N} = 2$ theory with no superpotential, we study the renormalization of R-charge at finite 't Hooft coupling using "$\mathcal{Z}$-minimization".

This thesis is organized as follows. Exploring the landscape from the gravity side, in chapter 2 we study both perturbative and nonperturbative stability of a class of non supersymmetric vacua. We find some vacua which are surprisingly robust against many kinds of decays. Further continuing the exploration of the landscape from the gravity side, in chapter 3, we study Einstein dilaton gravity. By computing a fermionic two point function, we find non fermi liquid behaviour for the dual field theory. In chapter 4 we consider the landscape of Chern simons CFTs in a hope to find the dual gravity systems and identify some promising theories.

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Chapter 2

Stability Of Non-Supersymmetric AdS Vacua

2.1 Introduction

String Theory has a rich and complicated landscape of vacua. Non-supersymmetric anti-deSitter (AdS) vacua are an interesting class amongst these. Lacking supersymmetry, they are not as well understood as supersymmetric spacetimes. But being time independent, they should be easier to understand than deSitter spacetimes.

In this chapter we construct a class of non-supersymmetric AdS vacua which are perturbatively stable and investigate their non-perturbative stability.

Our construction is based on massive IIA theory compactified on a particular Calabi-Yau manifold. After suitably orientifolding and adding fluxes one obtains vacua where all moduli are stabilized and supersymmetry is broken with a negative cosmological constant. The Calabi-Yau manifold is obtained by blowing-up a $T^6/(Z_3 \times Z_3)$ orbifold. This model was studied in considerable depth by [1]. Here we carefully include the effects of the blow-up modes and related fluxes and analyze the stability of the resulting non-susy vacua in detail.

We find two classes of perturbatively stable non-susy vacua. These are called Type 2) and Type 3) vacua in our terminology (Type 1 vacua are supersymmetric). Both classes contain an infinite number of vacua.

Next we turn to a study of the non-perturbative stability of these vacua. At first sight one might expect that a non-supersymmetric AdS vacua in the landscape would always have some non-zero rate to decay to other vacua with lower vacuum energy. Such a decay rate, if it is small enough, would not have very drastic consequences for an observer in AdS space.

However, from the point of view of a dual CFT the small decay rate in the bulk leads to a divergence and has dramatic repercussions. For, consider AdS_{d+1} in Poincare coordinates:

$$ds^2 = (\frac{r}{R})^2(-dt^2 + \sum_{i=1,\cdots,d-1}(dx^i)^2) + (\frac{R}{r})^2 dr^2. \tag{2.1}$$

Let the decay rate per unit volume in the bulk be Γ. The corresponding decay rate per unit volume in the boundary is obtained by integrating the bulk decay rate in the radial direction. Taking the boundary metric to be flat, the decay rate per unit volume in the boundary theory is,

$$\Gamma_{boundary} = \int \sqrt{g} dr \Gamma \sim r_{boundary}^{d+1} \Gamma, \tag{2.2}$$

where $r_{boundary}$ is the radial location of the boundary. As $r_{boundary}$ is taken to infinity we see that this diverges.

Thus an arbitrarily small decay rate in the bulk leads to an infinitely fast decay in the boundary. The putative CFT dual meets with an instantaneous end and cannot exist. This consequence of a bulk decay was noted in [2] where the decay rate in a non-susy AdS spacetime, obtained by taking an orbifold of $AdS_5 \times S^5$, was discussed in some detail.

We see then that our expectation that the non-susy AdS vacua are unstable non-perturbatively suggests that non-supersymmetric CFT's which admit a gravity dual are unlikely to exist. If true, this is an important consequence since holography has emerged as a major tool with which to study strongly coupled conformal field theories.

The two large families of perturbatively stable non-susy vacua mentioned above provide us with a laboratory in which we can investigate this issue of non-perturbative stability. Our analysis reveals that in a large number of cases the decays to vacua with lower energy are in fact ruled out.

The essential reason for this is the geometry of AdS space. The dynamics of a decay is governed by competition between the volume gain in bulk energy and the surface cost due to the tension of the interpolating domain wall. For a system in flat space, the volume grows more rapidly than the surface area, so eventually the volume gain always wins out and an instanton always exists which allows a transition to the more stable phase. However in AdS space the volume and area grow at the same rate. This means in AdS space a decay can only happen if the tension of the interpolating domain wall is small enough compared to the difference in energies between the two vacua. Explicit calculations show that in several cases the tension turns out to be much too large thereby forbidding the decay.

In fact it turns out that working within our approximations, Type 2) vacua are stable and can at best decay marginally. Type 3) vacua, in contrast, do indeed decay to some other Type 3) and Type 2) vacua. We find no decays of the non-susy vacua to susy ones are allowed. In the marginal cases, one will have to go beyond our approximations to determine the stability.

Among our approximations one of the more significant ones is the thin wall approximation, as described in the classic paper by Coleman and Deluccia [3]. The domain walls involved cause a jump in flux and thus carry D-brane charges. At first sight one would expect them to be just D-branes and therefore well described in the thin wall approximation. However the change in flux also causes the moduli to vary and these typically have a mass of order the AdS scale. As a result the domain walls are no longer thin. To work within the thin wall approximation then we must restrict ourselves to cases where the moduli variation contributes little, compared to the D-brane contribution, to the tension of the domain wall. This allows the domain wall to be approximated as being just a D-brane, which is indeed thin. Another limitation comes from not having explored the full set of non-susy vacua. In particular we have only investigated choices of flux for which the Calabi-Yau manifold is a slightly blown-up version of the $T^6/(Z_3 \times Z_3)$ orbifold. General non-susy vacua, where the moduli could be stabilized far away from the orbifold point, have not been analyzed. There could be other non-susy vacua which we have not constructed which are allowed end points of decays.

Despite these limitations in our analysis, we find it significant that a large number of possible decays are in fact ruled out. This indicates that perhaps stable non-susy AdS vacua and associated CFT's might exist after all.

The stability of Type 2) vacua can also be understood in terms of a "pairing symmetry" in the landscape. By reversing the sign of the four-form flux in these vacua one obtains susy vacua with the same vacuum energy. The stability of the Type 2) vacua then follows from the stability of their partner susy vacua. While we have established the argument which relates the stability of the Type 2) vacua to that of their susy partners only within our approximations it could have a greater range of validity.

The basic strategy we employ in studying the non-perturbative stability is as follows. The tension of the interpolating domain wall satisfies a lower bound in terms of the jump in the superpotential caused by the domain wall. By comparing this lower bound against an upper bound which must be met for the decay to be allowed in AdS space, several decays can be ruled out.

Such a bound on the tension of domain walls is familiar in supersymmetric settings. It might seem puzzling at first that it arises in our study of non-susy vacua. The essential reason is that even in the non-supersymmetric case the compactification can be well approximated to be a Calabi-Yau space. The ratio of the size of the internal space to the AdS radius goes to zero for large flux in these compactifications making them of non Freund-Rubin type. This means that the main effect of the fluxes in these cases is to stabilize the moduli while the shifts of the Kaluza Klein modes is small. The tension of the domain wall, which is fixed to good approximation by the geometry of the Calabi-Yau space, is then determined by supersymmetric data and can be bounded by the jump in the superpotential. We expect that a similar strategy should be useful in other flux compactifications as well, where the internal compactification can continue to be well approximated as a supersymmetric one. It might also be useful in going beyond the thin wall approximation.

We end this introduction by discussing some related literature. "Skew whiffing", the idea of reversing the sign of fluxes to obtain non-susy vacua from susy ones, which we have used for Type 2) cases can be found in [4] and also in the context of black holes, [5], [6]. For other recent constructions of non-susy AdS_4 vacua see [7], [8]. Early papers on IIA compactifications include [10] which developed the 4D framework, and [11], [12], which discuss Moduli stabilization. A recent discussion on non-perturbative decays, especially decays of Minkowski nearly susy vacua to AdS space and related topics, is in [15].

This chapter is organized as follows. In section §2.2 we review the Model discussed in [1] and construct the non-susy vacua. In §2.3 we briefly review the discussion of vacuum decay in [3] with particular emphasis on AdS spacetimes. In §2.4 we turn to non-perturbative decays and analyze decays mediated by $D4$-branes. More general decays are briefly discussed in §2.5.

2.2 The Model

We will consider a simple compactification of massive Type IIA theory [13] on a slightly blown up $T^6/(Z_3 \times Z_3)$ orbifold [20]. By turning on flux one can get non-susy AdS vacua with all moduli being stabilized. This model was discussed extensively in [1]. We will follow their notation and discuss the essential features of this compactification.

We will show in this section that for appropriate choices of flux the vacua do not have any tachyons lying below the BF bound and thus are perturbatively stable. This will set the stage to consider their possible non-perturbative decays later.

We begin by describing the $T^6/(Z_3 \times Z_3)$ orbifold. This is an orbifold limit of a Calabi-Yau three-fold. Let $z_i = x_i + iy_i, i = 1 \cdots 3$ be three complex coordinates on the T^6, satisfying the periodicity conditions,

$$z_i \simeq z_i + 1, \quad z_i \simeq z_i + \alpha, \alpha = e^{\frac{\pi i}{3}}. \tag{2.3}$$

The two Z_3's, by which we do the identification are,

$$(z_1, z_2, z_3) \rightarrow (\alpha^2 z_1, \alpha^2 z_2, \alpha^2 z_3). \tag{2.4}$$

$$(z_1, z_2, z_3) \rightarrow (\alpha^2 z_1 + \frac{1+\alpha}{3}, \alpha^4 z_2 + \frac{1+\alpha}{3}, z_3 + \frac{1+\alpha}{3}). \tag{2.5}$$

This leaves 9 fixed points.

The resulting compactification has no complex structure moduli. The six -torus is a product manifold, $T^6 = T^2 \times T^2 \times T^2$. The resulting orbifold has three Kahler moduli corresponding to the sizes of the three T^2's. Each T^2 also gives rise to a zero mode for B_2 giving rise to three axions; together these give rise to three complex moduli. The dilaton and an axion which arises from C_3, give rise to one more complex modulus. Finally there are nine complex moduli which arise from metric and B_2 moduli associated with the 9 blow up modes.

We will need to consider a further Z_2 orientifold of this orbifold. This is obtained by modding out by $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$, where Ω_P is world sheet orientation reversal, F_L is left-moving fermion number, and σ is reflection,

$$\sigma : z_i \rightarrow -\bar{z}_i, i = 1, 2, 3. \tag{2.6}$$

There is a single O_6 plane which fills the non-compact directions and wraps a 3-cycle which is the locus of fixed points of the σ reflection symmetry. The resulting compactification now has $\mathcal{N}=1$ supersymmetry. The three T^2 moduli, the dilaton-axion, and the nine blow up modes, all survive the orientifolding and form the bosonic components of chiral superfields.

We now turn to incorporating the effects of flux. To begin, we discuss the effects of flux in the orbifold limit. Subsequently we will include the blow-up modes and related fluxes as well.

2.2.1 Fluxes, Superpotential and Potential

A basis of two-forms on the three T^2's is given by,

$$\omega_i = (\kappa\sqrt{3})^{1/3} i dz^i \wedge d\bar{z}^i, i = 1\cdots 3. \tag{2.7}$$

κ will be defined later in terms of the triple intersection number in eq.(2.26). Let the hodge duals of ω_i be $\tilde{\omega}_i$. The holomorphic three-form is,

$$\Omega = 3^{1/4} i dz_1 \wedge dz_2 \wedge dz_3. \tag{2.8}$$

Its real and imaginary parts are α_0, β_0. Note that under the Z_2 orientifold symmetry α_0, β_0 are respectively even and odd. The three- form H_3 has odd intrinsic particy under $\mathcal{O}$. Therefore a three-form flux can be expanded as,

$$H_3 = -p\beta_0, \tag{2.9}$$

with p constant. Similarly the form form flux F_4 can be expanded as

$$F_4 = e_i \tilde{\omega}_i. \tag{2.10}$$

After accounting for the presence of Cherns- Simons terms the tadpole condition for the C_7 potential is given by,

$$m_0 p = -2\sqrt{2}\kappa_{10}^2 \mu_6. \tag{2.11}$$

Here m_0 is the Romans parameter[1]. The metric takes the form,

$$ds^2 = \sum_{i=1}^{3} \gamma_i dz^i d\bar{z}^i. \tag{2.12}$$

It is convenient to work with the moduli,

$$v_i = \frac{1}{2}\frac{1}{(\kappa\sqrt{3})^{1/3}}\gamma_i. \tag{2.13}$$

[1]In our conventions, $2\kappa_{10}^2 = (2\pi)^7(\alpha')^4$, $\mu_6 = (2\pi)^{-6}(\alpha')^{-7/2}$

below. For now we suppress the dependence on the B_2, C_3 axions which will be considered later. The resulting potential in the 4 dimensional Einstein frame effective theory is then,

$$V = \frac{p^2}{4}\frac{e^{2\phi}}{(vol)^2} + \frac{1}{2}(\sum_{i=1}^{3} e_i^2 v_i^2)\frac{e^{4\phi}}{(vol^3)} + \frac{m_0^2}{2}\frac{e^{4\phi}}{vol} - \sqrt{2}|m_0 p|\frac{e^{3\phi}}{vol^{3/2}}. \quad (2.14)$$

where vol, which is related to the volume of the compactification[2], is defined to be

$$vol = \int_{T^6/(Z_3)^2} \sqrt{g_6} = \kappa v_1 v_2 v_3. \quad (2.15)$$

The four terms on the rhs of the potential above arise from the $|H_3|^2$, $|F_4|^4$, m_0^2 and the tension of the O_6 planes respectively.

The important point for the present analysis is that the potential is an even function of the fluxes e_i. Thus the minimum value of the potential and the location of the minimum in moduli space will only depend on the absolute values of e_i and not on their signs. We emphasize this because the conditions for supersymmetry do care about signs. These conditions take the form,

$$\text{sign}(m_0 e_i) < 0, \text{sign}(m_0 p) < 0. \quad (2.16)$$

The second condition is automatically met once the tadpole condition eq.(2.11) is satisfied. It follows from eq.(2.16) that all the e_i's must have the same sign to preserve susy.

This gives an easy way to construct non-supersymmetric minima. Starting with the supersymmetric case, we can change the sign of some or all of the e_i's (while keeping m_0, p fixed). The tadpole condition eq.(2.11) will continue to be met but the susy conditions will not be. However since the potential is an even function of the e_i fluxes, the susy minimum in moduli space will continue to be a minimum even for these non-susy choice of fluxes.

Now that we have understood the basic idea behind the construction of the non-supersymmetric vacua we turn to exploring them in more detail below.

2.2.2 The Superpotential

To begin, we discuss the general case of massive IIA on a CY_3 orientifold, and then specialize to the compactification at hand. Let $\omega_a, a = 1, \cdots h^{1,1}$ be a basis of $(1,1)$ forms in the CY_3. Let $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$ be the Z_2 orientifold symmetry, with σ being an antiholomorphic involution of the Calabi-Yau manifold. The space of $(1,1)$ forms splits into $H^{1,1} = H_-^{1,1} + H_+^{1,1}$ forms which are odd and even under σ, with dimension $h_-^{1,1}, h_+^{1,1}$ respectively. Only the moduli coming from H_- survive the orientifold projection [3]. Let $J_c = B_2 + iJ$ be the

[2]After the additional Z_2 orientifolding, the volume of the internal space becomes $\frac{vol}{2}$. For a more complete discussion of our conventions see subsection §2.2.2.

[3]This is easy to see from their partner B_2 moduli which are odd under $\mathcal{O}$

complexified Kahler two-form. Then we can expand J_c in terms of a basis of odd two-forms,

$$J_c = \sum_{a=1}^{h_-^{1,1}} t_a \omega^a, \tag{2.17}$$

Here,

$$t_a = b_a + iv_a. \tag{2.18}$$

are the complexified Kahler moduli. A single axion arises from C_3 in the compactification of interest to us,

$$C_3 = \xi \alpha_0, \tag{2.19}$$

where α_0 is defined as the real part of Ω. Susy pairs this with the $4-$ dimensional dilaton defined by,

$$e^{2D} = \frac{e^{2\phi}}{vol}. \tag{2.20}$$

The resulting Kahler potential for these moduli is

$$K = 4D - \log(\frac{4}{3}\kappa_{abc} v_a v_b v_c). \tag{2.21}$$

where the κ_{abc} are the triple intersection numbers,

$$\kappa_{abc} = \int \omega_a \wedge \omega_b \wedge \omega_c. \tag{2.22}$$

Let $\tilde{w}_a$ be a basis for $H_+^{2,2}$. These are are dual to the $(1,1)$ forms ω_a which are a basis of $H_-^{1,1}$ eq.(2.17). Then since F_4 is even under $\mathcal{O}$ we can expand it in this basis,

$$F_4 = e_a \tilde{w}_a. \tag{2.23}$$

We also turn on e_0 units of F_4 flux along the non-compact directions and m_a units of F_2 flux turned on along compact two -cycles. Then the full superpotential

$$W = e_0 - p\xi - \sqrt{2} i p e^{-D} + e_a t_a + \frac{1}{2}\kappa_{abc} m_a t_b t_c - \frac{m_0}{6}\kappa_{abc} t_a t_b t_c. \tag{2.24}$$

more We now restrict restrict ourself to the orbifold limit, working with only the untwisted Kahler moduli and the the F_4 fluxes eq.(2.10). There are three complexified Kahler moduli from the untwisted sector,

$$t_i = b_i + iv_i, i = 1, \cdots 3. \tag{2.25}$$

The triple intersection number on $T^6/(Z_3)^2$ is given by

$$\kappa_{123} = \int_{T^6/(Z_3)^2} \omega_1 \wedge \omega_2 \wedge \omega_3 = \kappa. \tag{2.26}$$

Now with the superpotential, eq.(2.24) and the Kahler potential eq.(2.21), one can get the potential. Minimizing this gives

$$v_i = \frac{1}{|e_i|}\sqrt{\frac{5}{3}|\frac{e_1e_2e_3}{\kappa m_0}|}, \tag{2.27}$$

$$e^D = |p|\sqrt{\frac{27}{160}|\frac{\kappa m_0}{e_1e_2e_3}|}, \tag{2.28}$$

and,

$$e^{\phi} = \frac{3}{4}|p|\left(\frac{5}{12}\frac{\kappa}{|m_0e_1e_2e_3|}\right)^{\frac{1}{4}}. \tag{2.29}$$

where $v_i = \frac{v}{|e_i|}$. The potential at the minimum takes the value

$$V_0 = -\sqrt{\frac{4}{15}}\left(\frac{27}{160}\right)^2 \frac{p^4\kappa^{\frac{3}{2}}|m_0|^{\frac{5}{2}}}{|e_1e_2e_3|^{\frac{3}{2}}}. \tag{2.30}$$

Keeping the terms in action which are quadratic in the axions gives a Mass matrix which can be diagonalized. Two distinct cases arise for the eigenvalues of the mass matrix:

• When sign$(m_0e_1e_2e_3) = -1$ it turns out that all eigenvalues are positive. This includes the susy case where sign$(m_0e_i) = -1$ for each value of i. But it also includes non-susy cases where the condition eq.(2.16), is not met and the condition sign$(m_0e_1e_2e_3) = -1$ still holds.

• When sign$(m_0e_1e_2e_3) = +1$ and susy is necessarily broken, there is one negative eigenvalue and thus one tachyon. Its mass is given by, $M^2 = -\frac{4}{15}|m_0|e^{4D}v$. The BF bound is $M^2_{BF} = -\frac{3}{4}V_{min}$. Then,

$$\frac{M^2}{M^2_{BF}} = \frac{-(\frac{4}{15})|m_0|e^{4D}v}{-(\frac{3}{10})|m_0|e^{4D}v} = \frac{8}{9}. \tag{2.31}$$

Thus we see that the mass lies above the BF bound and hence the resulting vacua are stable with respect to these axionic directions.

Some General Comments

Some important features of the vacua which arise from eq.(2.27)- eq.(2.30) are worth emphasizing at this stage. For the purpose of scalings, in this section we work in string frame with string scale set equal to one. We will be interested in vacua where the four-form flux $|e_i| \sim e \gg 1$. Note that the tadpole condition eq.(2.11) imposes no constraint on the four form flux e_i, which is allowed to get arbitrarily large. From eq.(2.29) we see that the dilaton $e^{\phi} \sim e^{-3/4} \to 0$, so that for large flux one has a weakly coupled theory.

For $e \gg 1$ we find parametrically that $\gamma_i \sim e^{1/2}$, so that the size of the internal space scales like $l \sim e^{1/4}$. In contrast the potential scales like $e^{3/2}$ and $M^{(4)}_{\rm pl} \sim e^{3/2}$ so that the radius of AdS space goes like $R_{AdS} \sim e^{3/4}$. Thus we find that both l, R_{AdS} become parametrically large as $e \gg 1$. As a result higher derivative corrections in the α' expansion will be suppressed. This still leaves corrections which involve higher powers of the field strengths without additional derivatives. Terms involving higher powers of F_4 were shown in [1] to be suppressed by an additional power of $g_s^2 \sim e^{-3/2}$ making them subdominant.

Note that the ratio $l/R_{AdS} \sim \frac{1}{\sqrt{e}} \to 0$, so that the compactification is not of Freund-Rubin type. The non Freund-Rubin nature of the compactification actually simplifies the analysis when it come to checking for possible tachyons. The KK modes have positive $(mass)^2$ in the absence of flux, including the effects of flux cannot make them tachyonic because of the parametric separation of scales. Thus it is sufficient to look for possible tachyons among the moduli, which are massless in the absence of flux.

Using the scalings, one can show that the F_4^2 terms scale like $O(1)$ in string units. In susy breaking vacua the flux sets the scale of susy breaking and one might therefore worry that our starting point, which is a Calabi-Yau orientifold with supersymmetry, is itself inconsistent. However this is not true. The gravitational backreaction of the flux is parametrically suppressed for large e, as we argued in the previous paragraph since $l/R_{AdS} \to 0$. From a ten dimensional point of view this follows from the fact that the gravitational back reaction is suppressed by g_s^2 which is small. Corrections to the Calabi-Yau metric can be systematically calculated in an expansion in $1/e$. Such corrections arise in the supersymmetric case as well and including them in the susy case alters the internal metric so that it is no longer Calabi-Yau but instead is a half-flat metric with $SU(3)$ structure. For some discussion of this see, [21], [22].

2.2.3 The Blow-up Modes

So far we have ignored the blow-up modes. We now include them and check if there are any unacceptable tachyons which arise from the blow-up moduli of their axionic partners. In ths work, we seek vacua where the Calabi-Yau manifold is close to its orbifold limit. There are 9 blow-up modes, turning on a blow-up mode replaces the corresponding singularities with a P^2 of non-vanishing size. We will introduce additional F_4 flux threading each of these P^2's.

The complexified Kahler two-form is now given by[4].

$$J_c = \sum_a t_a \omega_a = \sum_{i=1}^{3} t_i \omega_i + \sum_{A=1}^{9} t_A \omega_A, \tag{2.32}$$

with $t_A = b_A + iv_A$. ω_A are elements of $H_-^{1,1}$ dual to the blow-up two cycles, and v_A, b_A are the blow-up moduli and corresponding axions. The Kahler potential is modified to [5].

$$K = 4D - \log(8\kappa v_1 v_2 v_3 + \frac{4}{3}\beta \sum_{A=1}^{9} v_A^3). \tag{2.33}$$

Let us note that to stay within the Kahler cone, $v_A < 0$, [1]. This agrees with the intuition

[4] In our notation the index A which denotes the basis elements of $H_-^{1,1}$ takes values $i = 1, 2, 3, a = 1, \cdots 9$.

[5] κ, β take values, $\kappa = 81, \beta = 9$, according to [1], but we will not need these explicit values below.

that as v_A increases the total volume, which is the argument of the logarithm in the Kahler potential, decreases.

The four- form flux is,

$$F_4 = e_i\tilde{\omega}_i + e_A\tilde{\omega}_A. \tag{2.34}$$

The superpotential with four form fluxes is

$$W = -p\xi - i\sqrt{2}pe^{-D} + e_i t_i + e_A t_A - m_0(\kappa t_1 t_2 t_3 + \frac{\beta}{6}\sum_A t_A^3). \tag{2.35}$$

To keep the blow-up modes small and the Calabi-Yau close to the orbifold point, we take the extra flux e_A to satisfy the condition,

$$\frac{|e_A|}{|e|} \ll 1, \tag{2.36}$$

where $e \sim e_i$ denotes a generic flux along the $T^2 \times T^2$ four-cycles. We will see below that the resulting expectation value has $\frac{v_A}{v_i} \ll 1$ so that the blow up moduli have a comparatively small value and the Calabi-Yau moduli will then be stabilized close to the orbifold point.

For discussion below it is convenient to introduce the variable,

$$\delta = \sqrt{|\frac{e_A^3}{e_1 e_2 e_3}|}. \tag{2.37}$$

To calculate the value of the blow-up modes it is enough to expand the potential and keep only the first two terms in an expansion in δ. Also we will see that keeping terms upto order $(\frac{v_A}{v_i})^3$ will suffice.

We saw above that the leading order potential gives an acceptable extremum as far as the untwisted moduli are concerned. The first corrections will have a small effect on the untwisted moduli masses and they will continue to be safely above the BF bound. Also, the shifts in the values at the extremum for the Kahler moduli and the dilaton due to the first order correction can be ignored at this order. In contrast, for the blow-up modes and their axions the first corrections provide the dominant potential. In the analysis below we set the untwisted Kahler moduli and dilaton to their minimum values, eq.(2.27), eq.(2.29), and examine the effects of the first order corrections on the blow-up modes and their axionic partners.

First let us set the axions to be all zero. This gives,

$$\begin{aligned} V &= V_0 + \sqrt{\frac{15\kappa}{4\beta}}|V_0| \sum_A \sqrt{\frac{|e_A|^3}{|e_1 e_2 e_3|}} \left\{ \frac{1}{2x_A} - \frac{3}{10}\text{sign}(e_A)\sum_i \text{sign}(e_i) x_A \right. \\ &\left. + \frac{3x_A^3}{200}(1 + 2\sum_{<ij>} \text{sign}(e_i e_j) \right\}, \end{aligned} \tag{2.38}$$

where

$$x_A = -\sqrt{\frac{\beta|m_0|}{|e_A|}}v_A. \tag{2.39}$$

We note that an explicit minus sign has been introduced in the definition of x_A since the allowed values of $v_A < 0$. Also in the last summation on the rhs of eq.(2.38) the indices $i, j = 1, \cdots 3$ must take different values, and each distinct pair $< ij >$ appears once.

Let us now introduce the axion dependence in the potential which has so far been suppressed. From the superpotential, eq.(2.35) it follows that if the sign of all the axions is reversed, keeping the Kahler moduli and the dilaton the same, then $W \to -\bar{W}$ and therefore the potential is invariant. This means that the first term in a power series in the axions must be quadratic and therefore the extremum we find by setting them to zero is also an extremum once their dependence is included. We will examine whether this extremum is free of tachyonic modes lying below the BF bound in the following discussion. As was mentioned above, the leading term in the potential already provides an acceptable extremum as far as the untwisted axions are concerned.

The quadratic terms for the blow-up axions in the potential are,

$$\begin{aligned} V_{b_A} &= \frac{1}{40}\sqrt{\frac{15\kappa}{4\beta}}|V_0|\sum_A\sqrt{\frac{|e_A|^3}{|e_1e_2e_3|}}\frac{b_A^2}{(|e_A|/\beta|m_0|)}\left\{-\frac{20\text{sign}(m_0e_A)}{x_A}\right. \\ &\quad \left. + x_A(29 + 9\sum_i \text{sign}(m_0e_i))\right\}. \end{aligned} \tag{2.40}$$

For a solution to preserve supersymmetry eq.(2.16) must hold. In addition, since the fluxes e_A are also now turned on we have the conditions,

$$\text{sign}(m_0e_A) < 0. \tag{2.41}$$

We are now ready to discuss the different cases which arise when the signs of various fluxes are varied. The relative sign between m_0, p is fixed by the tadpole condition eq.(2.11). The different cases arise as we change the relative signs between m_0 and the fluxes e_A. We list here only the cases which have no unacceptable tachyons.

• Case 1): $\text{sign}(e_im_0) = \text{sign}(e_Am_0) = -1$. This case preserves susy. The potential has a minimum when the blow up modes take the value,

$$x_A = \sqrt{\frac{10}{3}}. \tag{2.42}$$

All axions are non-tachyonic including the b_A axions.

• Case 2): $\text{sign}(m_0e_i) = \text{sign}(m_0e_A) = +1$. Susy is broken. The extremum of the potential is at the same value, eq.(2.42). In this case there are no tachyons from the blow-up modes including the x_A directions and the blow-up axions.

• Case 3):sign$(m_0 e_i) = +1$, sign$(m_0 e_A) = -1$ Susy is broken, now the extremum of the potential lies at,

$$x_A = \sqrt{\frac{10}{21}}. \tag{2.43}$$

Again there are no tachyons from the blow-up modes and blow up axions.

Before proceeding let us note that there are 9 blow up modes. From eq.(2.38), eq.(2.40) we see that the potential for the blow-up modes and their axionic partners decouple to leading order in $(\frac{e_A}{e_i})^{3/2}$ from each other. Thus with sign$(m_0 e_i) = 1$ there are actually 2^9 cases with sign$(e_A m_0)$ for $A = 1, \cdots 9$ being ± 1. Depending on the sign the discussion of Case 2) or Case 3) applies for each blow-up mode and its axionic partner independently of the others.

We close this section by noting that the ground state energy for case 2), where $(m_0 e_i) > 0, (m_0 e_A) > 0$, is given by,

$$V = V_0 + V_0 \sqrt{\frac{2\kappa}{\beta}} \sum_A \frac{|e_A|^{3/2}}{\sqrt{|e_1 e_2 e_3|}}, \tag{2.44}$$

and for case 3), where $(m_0 e_A) < 0, (m_0 e_i) > 0$ by,

$$V = V_0 - V_0 \sqrt{\frac{50\kappa}{7\beta}} \sum_A \frac{(|e_A|)^{3/2}}{\sqrt{|e_1 e_2 e_3|}}. \tag{2.45}$$

These results for the ground state energy are correct [6] to order δ. We have neglected the shift in the untwisted Kahler moduli due to the blow-up fluxes, this contributes at order δ^2, since the first corrections in the energy about the minimum are second order in the values of the moduli shifts.

2.2.4 More General Fluxes

In this subsection we consider what happens when F_2 and F_6 flux are also activated. The fluxes are specified, in terms of the basis of two-forms ω_a by,

$$e_0 = \int F_6, \quad F_2 = -m_a \omega_a. \tag{2.46}$$

After they are turned on, the full superpotential is therefore,

$$W = e_0 + e_a t_a + \frac{1}{2}\kappa_{abc} m_a t_b t_c - \frac{m_0}{6}\kappa_{abc} t_a t_b t_c - p\xi - \sqrt{2} i p e^{-D}. \tag{2.47}$$

[6] By case 2) and 3) we mean here cases where all $e_A > 0$ or $e_A < 0$ respectively. In the mixed case where some e_A are positive and negative the terms within the sum in eq.(2.44), eq.(2.45) have to be changed appropriately.

The third term on rhs, which is quadratic in t_a contains the effects of the m_a flux. By shifting t_a by

$$t_a \to t_a - \frac{m_a}{m_0}, \tag{2.48}$$

and ξ by

$$\xi \to \xi - \frac{e_0}{p} - \frac{e_a m_a}{p} - \frac{1}{3}\kappa_{abc}\frac{m_a m_b m_c}{m_0^2 p}, \tag{2.49}$$

one can can reexpress the superpotential in terms of the shifted variables as,

$$W = \hat{e}_a t_a - \frac{m_0}{6}\kappa_{abc} t_a t_b t_c - p\xi - \sqrt{2} i p e^{-D} \tag{2.50}$$

with

$$\hat{e}_a = e_a + \frac{\kappa_{abc}}{2}\frac{m_b m_c}{m_0}. \tag{2.51}$$

Notice that m_a and e_0 have both disappeared in this superpotential. The shift, eq.(2.48), eq.(2.49), changes the real part of the chiral superfields and thus does not change the Kahler potential which is expressed in terms of the imaginary part of the chiral superfields, eq.(2.21), eq.(2.33). Thus we see that the theory can be mapped into the one we had studied earlier, without any m_a and e_0 flux. Hence the results of the previous sections hold with $e_a \to \hat{e_a}$.

One final comment is worth making regarding this case. The compactification has gauge symmetries under which the flux and moduli transform, these can be thought of as the analogue of the $\tau \to \tau + 1$ subgroup of $SL(2, Z)$ which arises on a torus [1]. Configurations related by these symmetries are not distinct but should be identified. One set of such symmetries involve the shift in the b_a axions by integer units,

$$t_a \;\to\; t_a - u_a, \tag{2.52}$$

$$m_a \;\to\; m_a - m_0 u_a, \tag{2.53}$$

$$e_a \;\to\; e_a + \kappa_{abc} m_b u_c - \frac{m_0}{2}\kappa_{abc} u_b u_c, \tag{2.54}$$

$$e_0 \;\to\; e_0 + \frac{1}{2}\kappa_{abc} m_a u_b u_c - \frac{m_0}{6}\kappa_{abc} u_a u_b u_c + e_a u_a, \tag{2.55}$$

for integer u_a. The other involves the shift in ξ axion,

$$\xi \;\to\; \xi - u, \tag{2.56}$$

$$e_0 \;\to\; e_0 - pu. \tag{2.57}$$

Using these any m_a which is an integer multiple of m_0 and e_0 which is integer multiple of p can be set to zero. Since the m_a and e_0 fluxes satisfy quantization conditions, this only leaves a few physically distinct cases where these fluxes are non-vanishing.

2.3 Vacuum Decay in the Thin wall Approximation

We review the classic discussion of the non-perturbative decay of an unstable vacuum in [3]. Consider an unstable vacuum, called the false vacuum, which can decay to another state, the true vacuum. The decay is mediated by the nucleation of a bubble of true vacuum inside the false vacuum. This nucleation is a quantum tunneling process, and gives rise to a probability for decay per unit volume per unit time of the form

$$\Gamma/V = Ae^{-B/\hbar}. \tag{2.58}$$

In the semi-classical approximation one seeks a solution to the Euclidean action which can interpolate between the false and true vacua. Given such a solution, which is called the bounce, the coefficient in the exponent above is given by,

$$B = S_E - S_{False}, \tag{2.59}$$

where S_E is the euclidean action of the bounce and S_{False} is the action of the false vacuum.

We will work in the thin wall approximation in this paper. In this approximation the bounce solution or the bubble has three parts. The inside where the solution is well approximated by the true vacuum, the outside which is the false vacuum, and the bubble wall which interpolates between the two. In the thin wall approximation, the thickness of this wall is much smaller than all the other length scales in the problem. These include the radius of the bubble and the radii of curvature of the inside and outside spacetimes. Once these conditions are met, the tension of the bubble wall can be calculated by taking it to be a flat wall in flat space-time, neglecting both the curvature of spacetime and the curvature of the bubble wall. This simplifies the analysis considerably.

The Euclidean metric of the bounce solution can be taken to be S^3 symmetric and of form,

$$ds^2 = d\xi^2 + \rho(\xi)^2(d\Omega)^2, \tag{2.60}$$

where $(d\Omega)^2$ is the volume element on a unit S^3. We will be interested in decays where the inside and outside spacetime are both AdS. We denote the vacuum energy of the false vacuum, which is outside the bubble, and the true vacuum, which is inside, as V_+ and V_- respectively. Both are negative. The bubble wall lies at $\rho = \bar{\rho}$.

The bounce action gets contributions from the three parts of the solution, the inside, the wall and the outside,

$$B = B_{inside} + B_{Wall} + B_{outside}. \tag{2.61}$$

Since the outside region is essentially identical to the false vacuum, $B_{outside} = 0$. For a wall with tension S_1

$$B_{wall} = 2\pi^2\bar{\rho}^3 S_1. \tag{2.62}$$

Finally the inside region contributes,

$$B_{inside} \quad = \quad 12\pi^2 \left[\frac{\left(1 - \frac{1}{3}\bar{\rho}^2 V_-\right)^{\frac{3}{2}} - 1}{V_-} - \frac{\left(1 - \frac{1}{3}\bar{\rho}^2 V_+\right)^{\frac{3}{2}} - 1}{V_+} \right]. \tag{2.63}$$

The value of $\bar{\rho}$ can be calculated by extremizing B, i.e. requiring,

$$\frac{dB}{d\bar{\rho}} = 0. \tag{2.64}$$

Notice from eq.(2.63) that with $V_\pm < 0$, for large $\bar{\rho}$

$$B_{inside} = -\frac{4}{\sqrt{3}}\pi^2 \bar{\rho}^3 (\sqrt{|V_-|} - \sqrt{|V_+|}), \tag{2.65}$$

so that the bulk gain in energy grows like $\bar{\rho}^3$. This is a consequence of the fact that the volume and the area both grow in the same fashion in AdS space. The wall contribution which goes like the area also grows like $\bar{\rho}^3$ with a positive coefficient, eq.(2.62). A sufficient condition for an extremum value of B to exists is that the net coefficient of the $\bar{\rho}^3$ dependence at large $\bar{\rho}$ is negative. For $|V_-| > |V_+|$ this yields the condition,

$$S_1 < \sqrt{\frac{4}{3}}[\sqrt{|V_-|} - \sqrt{|V_+|}]. \tag{2.66}$$

A little more analysis shows that this is also a necessary condition. The above condition can also be expressed as

$$\left(\frac{\epsilon}{3S_1} - \frac{S_1}{4}\right) > \sqrt{\frac{|V_+|}{3}}, \tag{2.67}$$

2.4 $D4$-Brane Mediated Decays

In this section we consider non-perturbative decays mediated by a domain wall that carries only $D4$ brane charge. More general brane configurations carrying other charges as well will be discussed later. We then consider a $D4$ brane which wraps a two-cycle that is a combination of the three T^2's. The $D4$ brane wraps a two-cycle in the internal space and extends along two of the 3 spatially non-compact directions of AdS_4 thereby giving rise to a domain wall which separates the true vacuum from the false one. This causes the four-form flux F_4, along the 4-cycle dual to the two-cycle wrapped by the $D4$, to jump. And this change in F_4 cause a change in the cosmological constant.

We will work within the thin wall approximation below. For this approximation to hold the domain wall must have a thickness which is much smaller than all the other relevant length scales, namely, the radius of the S^3, and the AdS radii of the true and false vacua. At first sight it would seem that this condition is obviously met since a $D4$-brane is much thinner than all these distance scales, when supergravity is valid. However there is an important caveat, which was also mentioned in the introduction. In the situations at hand a change in flux also causes the vacuum expectation value of the moduli to change. As a result the moduli also begin to vary across the domain wall. Now the moduli can be shown to result in a wall with thickness of order R_{AdS}, which is not thin.

To stay within the thin wall approximation we will only consider decays where the change in the moduli from one vacuum to the other is sufficiently small. The moduli contribution to the tension will then be much smaller than the $D4$ brane contribution and can therefore be neglected. The domain wall can then be well approximated by a $D4$-brane which is indeed thin. The precise conditions ensuring that the moduli contribution is small will be worked out for various cases as we proceed.

2.4.1 Non-Susy to Susy Decay

We begin by considering the decay of a non-susy vacuum to a susy one. We will see below that for all these decays the tension of the domain wall is larger than the energy difference between the two vacua, resulting in the decays being forbidden in the thin wall approximation. This mismatch is parametric in the flux, therefore in this subsection we do not need to keep track of precise numerical factors.

To begin, we work in the orbifold limit, neglecting the blow-up modes and the related fluxes, e_A. This leaves three two-cycles, namely the three T^2's, and three fluxes, $e_i, i = 1, 2, 3$. The essential argument will become clear if we take all the three fluxes e_i to be of the same order, $e_i \sim e$. For the supergravity description to be valid $|e| \gg 1$.

Now consider a single $D4$ brane which wraps the first T^2. Its tension arises from the Nambu-Gotto action,

$$\text{Action} = \mu_4 e^{-\phi} \int \sqrt{-g} d^3\xi_i dx^1 dy^1 \sim \gamma_1 \mu_4 e^{-\phi} \int \sqrt{-g} d^3\xi_i. \tag{2.68}$$

where $\xi_i, i = 1, \cdots 3$ are the 3 directions in AdS space along which it extends, and we have used eq.(2.12) and done the integral over the T^2.

We will work in 4 dim Einstein frame below. This is related to the string frame by

$$g_{\mu\nu} = \frac{e^{2\phi}}{vol} g^E_{\mu\nu}. \tag{2.69}$$

Accounting for this, gives the Einstein frame tension for a single $D4$ wrapping the first T^2 to be

$$S_1 \sim \gamma_1 \mu_4 e^{-\phi} (\frac{e^{2\phi}}{vol})^{3/2}, \tag{2.70}$$

where γ_1 is the size of the T^2.

The domain wall of interest to us is obtained by wrapping all three T^2's in general, since all three fluxes must reverse sign. Its tension is of order

$$S_1 \sim \gamma\mu_4 e^{-\phi}(\frac{e^{2\phi}}{vol})^{3/2}|\delta e| \sim \frac{|\delta e|}{|e|}(\frac{1}{|e|})^{9/4}, \tag{2.71}$$

where $\gamma \sim \gamma_i$ is the size of the 3 T^2's and $\delta e \sim \delta e_i$ is the change in flux. Now let us take into account the conditions imposed by the thin wall approximation. The moduli make a contribution to the tension that can be shown to be

$$T_{mod} \sim M(\Delta\Phi)^2. \tag{2.72}$$

Here $M \sim 1/R_{AdS} \sim 1/|e|^{9/4}$, is the mass of canonically normalized moduli field, and

$$\Delta\Phi \sim \frac{\delta v_i}{v_i} \sim \frac{|\delta e_i|}{e_i}. \tag{2.73}$$

is the total change in the vacuum expectation value of the canonically normalized field across the domain wall. Substituting in eq. (2.72) yields

$$T_{mod} \sim \frac{(\delta|e|)^2}{|e|^2}(\frac{1}{|e|})^{9/4}. \tag{2.74}$$

For the thin wall approximation to hold, S_1 must dominate over T_{mod}. This gives

$$\frac{|\delta e|}{|e|} \gg \frac{(\delta|e|)^2}{|e|^2}. \tag{2.75}$$

This condition can be met if the susy vacuum has fluxes which are opposite in sign but approximately the same in magnitude as the non-susy vacuum we start with. That is,

$$e_i^{susy} \sim e^{susy} \sim -e. \tag{2.76}$$

Then the non-susy vacuum we start with and the susy vacuum it could decay to, lie in approximately the same region of moduli space, but are very far apart in flux space. As a result

$$|\delta e| \sim 2|e|, \tag{2.77}$$

and eq.(2.75) becomes,

$$1 \gg \frac{(\delta|e|)^2}{|e|^2}. \tag{2.78}$$

From eq.(2.76), $|e^{susy}| \sim |e|$ and therefore $\delta|e| = |e| - |e^{susy}|$ is small and this condition is indeed met.

Having ensured that the decay process lies within the thin wall approximation let us now see why it is not allowed. The important point is that since the absolute value of the flux in the non-susy and susy vacua are close, their energy difference is also small. From eq.(2.30),

$$\epsilon \sim \frac{\delta|e|}{|e|}(\frac{1}{|e|})^{9/2}. \tag{2.79}$$

For the decay to proceed, a necessary condition which follows from eq.(2.67) is that $\frac{\epsilon}{3S_1} > \frac{S_1}{4}$. From, eq.(2.71) and eq.(2.79), this condition becomes,

$$\frac{\delta|e|}{|e|} \geq (\frac{|\delta e|}{|e|})^2 \sim O(1), \tag{2.80}$$

where the last relation follows from eq.(2.77). We see now that the condition in eq.(2.80) is incompatible with eq.(2.78).

Thus we see that the decay of a non-susy vacuum to a susy vacuum is not allowed in the thin wall approximation.

So far we have neglected the blow-up modes and also neglected the related blow-up fluxes. After including these one can have perturbatively stable non-susy vacua of Type 2) or Type 3) as discussed earlier. The obstruction we found above disallowing a non-susy to susy decay was parametric in the e_i fluxes for large $|e_i|$. Including the blow-up fluxes cannot overcome this parametric obstruction as long as the blow-up fluxes are small and meet the condition, eq.(2.36). Therefore we conclude that non-susy vacua of Type 2) and 3), which arise when the flux meets the condition eq.(2.36), cannot decay to susy vacua in the thin wall approximation.

2.4.2 Decays From Non-Susy to Other Non-Susy Vacua

We now turn to examining whether a non-susy vacuum can decay to other non-susy vacua. We will need to calculate the tension of a $D4$ brane wrapping a two-cycle in the internal space. The $D4$-brane causes a jump in the flux to occur and therefore a jump in the superpotential. It is well known that the tension of the resulting domain wall satisfies a lower bound

$$T \geq T_L, \tag{2.81}$$

where T_L is given by

$$T_L = 2e^{K/2}|\Delta W|, \tag{2.82}$$

with

$$\Delta W = \delta e_a v_a, \tag{2.83}$$

being the change in the superpotential caused by the jump in the flux.

Our basic strategy will be to compare T_L with an upper bound in terms of the energy difference between the two vacua. This will allow us to rule out various decays.

Let us note here that the formula, eq.(2.82) is true more generally as well, when the D brane carries other charges too. It arises because the tension of the domain wall is only determined by the geometry of the Calabi-Yau space, in our approximations. In fact, in the absence of fluxes, the Calabi-Yau manifold preserves supersymmetry and the lower bound for the tension, in terms of the jump in the superpotential, is really a BPS bound. Branes which saturate the bound preserve supersymmetry, in the absence of flux.

In the $D4$-brane case the lower bound follows from the fact that the Kahler form on the Calabi-Yau is a calibration [7]. For the sake of clarity let us pause to quickly review how this comes about.

Consider a two- cycle in the Calabi-Yau manifold. Let $\sigma, \bar{\sigma}$ be holomorphic and anti-holomorphic coordinates on the world volume. And let the Pull back of the Kahler form of the Calabi-Yau onto the world volume be,

$$P[J] = K_{\sigma\bar{\sigma}} d\sigma d\bar{\sigma}. \tag{2.84}$$

Also let $P[g]$ be the induced metric on the world volume (we are supressing indices here). The area element is then given by,

$$\sqrt{det(P[g])} d\sigma \wedge d\bar{\sigma}. \tag{2.85}$$

Now since the Kahler form of the Calabi-Yau is a calibration we know that for any two-cycle,

$$|K_{\sigma\bar{\sigma}}| \leq \sqrt{det(P[g])}. \tag{2.86}$$

The equality is met only when the cycle is either holomorphic, or antiholomorphic. In the holomorphic case $z^i(\sigma)$ where z^i are coordinates of the Calabi-Yau manifold; in the antiholomorphic case, $z^i(\bar{\sigma})$. In these cases the $D4$ brane wrapping the two-cycle is supersymmetric

The tension of the resulting domain wall is given by

$$T = \mu_4 e^{-\phi} \int d^2\sigma \sqrt{det(P[g])}. \tag{2.87}$$

Using eq.(2.86) we get a lower bound on the tension,

$$T \geq T_L \equiv \mu_4 e^{-\phi} |\int P[J]|. \tag{2.88}$$

Now if γ_a is a basis of two-cycles and ω_a a basis of dual-two forms, and if the two-cycle wrapped by the $D4$ brane is $\gamma = \delta n_a \gamma_a$ then we have

$$\int P[J] = \delta n_a v_a, \tag{2.89}$$

where the Kahler moduli v_a are defined in eq.(2.17). This leads to the lower bound

$$T_L = \mu_4 e^{-\phi} |\delta n_a v_a|. \tag{2.90}$$

[7] For an early reference see [24]. For a pedagogical discussion see, [25].

We now go to 4 dim Einstein frame using eq.(2.69). In addition we can relate the winding numbers δn_a to the jump in the four form flux δe_a by

$$\delta e_a = 3^{2/3}\sqrt{2}\kappa^{1/3}\mu_4 \delta n_a, \tag{2.91}$$

Using these, we get eq.(2.82), with eq.(2.83).

Let us now turn to evaluating the upper bound on the tension. We saw in §2.3 that for the decay to be allowed it must meet the condition, eq.(2.66). Working to leading order in ϵ (the diffference between the vacuum energies) we get,

$$T \leq T_U \equiv \frac{\epsilon}{\sqrt{3|V_0|}}. \tag{2.92}$$

The justification for working to leading order in ϵ comes from the thin wall approximation, as we will see below.

In the discussion below we will ask if the lower bound T_L is bigger than the upper bound T_U. If this is true the decay will not be allowed. In cases where $T_L < T_U$ there will be interpolating $D4$ branes.e.g. wrapping susy cycles which saturate the lower bound, which will lead to allowed decays.

Before proceeding let us make one more comment. When we calculated T_L above we assumed that the moduli are fixed and calculated the tension of the $D4$ brane in this fixed moduli background. Actually the change in flux caused by the $D4$ brane also causes the moduli to change. But as long as the fractional change in expectation value of the moduli is small, i.e.,

$$|\frac{\delta v_a}{v_a}| \ll 1, \tag{2.93}$$

the resulting effect on the $D4$ brane tension can be neglected. In the thin wall approximation the variation of the moduli must make a smaller contribution to the domain wall tension than the $D4$-brane makes, this requirement gives rise to the condition, eq.(2.93), as we will see below.

2.4.3 Decays in the Orbifold limit

To begin let us set the flux e_A along the blow-up 4-cycles to be zero. Only the e_i fluxes are then activated and we only consider $D4$ branes which wrap the T^2 two-cycles and cause these fluxes to jump.

We first consider the limitations imposed by the thin wall approximation. To save clutter we set $m_0 > 0$ below. Then $e_i > 0$ in these vacua. To begin let us consider a case where all the fluxes are comparable, $e_1 \sim e_2 \sim e_3 \sim e$, and where the change in flux caused by the domain wall is also comparable, $\delta e_i \sim \delta e$. A $D4$ brane which changes the flux by amount δe contributes a tension,

$$T_{brane} \sim |\frac{\delta e}{e}|\frac{1}{|e|^{9/4}}. \tag{2.94}$$

The moduli contribute a tension which is now,

$$T_{moduli} \sim |\frac{\delta e}{e}|^2 \frac{1}{|e|^{9/4}}. \tag{2.95}$$

For $T_{brane} \gg T_{moduli}$ we get,

$$|\frac{\delta e}{e}| \ll 1. \tag{2.96}$$

From eq.(2.27) we see that the moduli change in response to the flux by

$$\frac{\delta v_i}{v_i} \sim \frac{\delta e}{e}. \tag{2.97}$$

Thus eq.(2.93) follows from the condition, eq.(2.96). From eq.(2.30) the potential at the minimum changes by

$$|\frac{\epsilon}{V_0}| \sim |\frac{\delta e}{e}| \ll 1. \tag{2.98}$$

This justifies working to leading order in ϵ in eq.(2.92).

If the three e_i fluxes and/or their changes are not comparable, a similar argument goes through with the factor $|\frac{\delta e}{e}|$ being replaced by that for the flux with the largest fractional change, i.e. the largest values of $|\frac{\delta e_i}{e_i}|$. Once again, both eq.(2.93), and eq.(2.92) follow.

We now calculate both T_U and T_L for such decays. The energy difference ϵ can be calculated in terms of the change in fluxes δe_i from eq.(2.30). This gives,

$$T_U = \frac{3}{2}(\sum_i \frac{\delta e_i}{e_i})\sqrt{\frac{|V_0|}{3}}. \tag{2.99}$$

From eq(2.15), eq(2.27) and eq(2.30), we get

$$\frac{\sqrt{|V_0|}}{e_i} = v_i\sqrt{\frac{2}{3}}\frac{e^{2D}}{\sqrt{vol}}. \tag{2.100}$$

Eq.(2.99) then becomes,

$$T_U = \frac{e^{2D}}{\sqrt{2vol}}\sum_i \delta e_i v_i. \tag{2.101}$$

Next we calculate T_L. Since $\Delta W = \delta e_i v_i$, and Kahler potential K is as given in eq(2.21), using eq(2.27), we get

$$T_L = 2e^{\frac{K}{2}}|\Delta W| = \frac{e^{2D}}{\sqrt{2vol}}\sum_i \delta e_i v_i. \tag{2.102}$$

Comparing eq(2.101) and eq(2.102), we see that $T_U = T_L$.

This means the decay can at best be marginally allowed. The marginal cases arises when the $D4$ brane wraps a supersymmetric cycle. Since supersymmetry is broken one expects that corrections to the approximation we are working in will result in the marginal case becoming either allowed or disallowed [8]. We will incorporate some of these corrections in the following discussion and also comment on which cases remain marginal after including some of these corrections further below.

In the discussion above we have set the e_A fluxes to vanish. If they are turned on but are small so that δ defined in eq.(2.37) is small, then for $D4$ branes which only wrap the T^2 two-cycles the calculations above still give the leading answers in δ for T_U, T_L. In the discussion below we will now turn to including $D4$ branes which can cause a change in the e_A fluxes.

Explicit Example of a Disallowed Decay

The advantage of working in the orbifold limit is that one can explicitly calculate the size of the T^2 two-cycles and associated tension of branes. This allows us to give a simple example of a situation where the brane tension is too big, because the cycle is not holomorphic resulting in the decay being disallowed.

Consider a $D4$ which wraps the first two T^2's. Let $\sigma, \bar{\sigma}$ be the holomorphic, and anti-holomorphic coordinates on the world volume, and let the mapping from the world volume to the first T^2 be linear and holomorphic, $z^1(\sigma) = \sigma$, and to second T^2 be linear and anti-holomorphic, $z^2(\bar{\sigma}) = \bar{\sigma}$. The resulting cycle is clearly not holomorphic, and the resulting wrapping numbers for the two T^2's are $+1$ and -1 respectively. The tension of the resulting $D4$ brane is

$$T = \frac{e^{2D}}{\sqrt{2vol}}\delta|e_1|(v_1 + v_2). \tag{2.103}$$

From the discussion above we have,

$$|T_U| = T_L = \frac{e^{2D}}{\sqrt{2vol}}\delta|e_1||(v_1 - v_2)|. \tag{2.104}$$

Thus we see that $T > T_U$ and the decay is not allowed.

2.4.4 General Decays With Blow-up Fluxes

Let us first examine the conditions imposed by the thin wall approximation on the allowed change in the blow-up fluxes. From the Kahler potential eq.(2.33) it is easy to see that a change in canonically normalized blow-up modes is,

$$\Delta\phi_{bu} \sim \sqrt{\delta}\frac{\delta v_A}{v_A}, \tag{2.105}$$

where δv_A is the change in the blow-up moduli. It then follows that the blow up modes also have a mass,

$$M_{bu} \sim \sqrt{|V_0|} \sim R^{-1}_{Ads}, \tag{2.106}$$

[8]The marginal case corresponds to a no-force condition on the $D4$ brane and a flat direction in the AdS_4 theory. Such a flat direction should get lifted without susy.

and their contribution to the tension is

$$T_{bu} \sim M_{bu}\Delta(\phi_{bu})^2 \sim \sqrt{|V_0|}\delta(\frac{\delta e_A}{e_A})^2, \tag{2.107}$$

where we have used the fact that the vacuum expectation value of $v_A \sim \sqrt{|e_A|}$. The $D4$ brane wrapping the dual two-cycle which causes this jump in flux has a tension,

$$T_{brane} \propto |\delta e_A v_A|. \tag{2.108}$$

Inserting the correct proportionality factors and converting to Einstein frame as in the previous subsection now gives,

$$T_{brane} \sim \sqrt{|V_0|}\delta|\frac{\delta e_A}{e_A}|. \tag{2.109}$$

Thus comparing eq.(2.107), eq.(2.109), gives the condition,

$$|\frac{\delta e_A}{e_A}| \ll 1, \tag{2.110}$$

which must be met for the thin wall approximation to hold.

We now turn to various different cases. In the following discussion we set $m_0 > 0$ for simplicity.

2.4.5 Type 2) to Type 2) Decays

The vacuum energy is given in eq.(2.44). In this case, $e_A, e_i > 0$. The change in blow up fluxes δe_A contributes to the difference in energy density ϵ and thus to T_u,

$$\delta T_U = -\sqrt{\frac{3}{2\beta}}\sqrt{|V_0|}\sqrt{\frac{e_A^3}{e_1 e_2 e_3}}\sum_A \frac{\delta e_A}{e_A}. \tag{2.111}$$

Using eq(2.30),eq(2.15), eq(2.27) and the fact that for Type 2 vacuum $v_A = -\sqrt{\frac{10 e_A}{3\beta|m_0|}}$, we get

$$T_U = \frac{1}{\sqrt{2}}\frac{e^{2D}}{\sqrt{vol}}(\sum_i \delta e_i v_i + \sum_A \delta e_A v_A). \tag{2.112}$$

In obtaining this formula we had also added the contribution due to the change in the e_i flux which was obtained in eq.(2.101) above.

Since $\Delta W = \delta e_i v_i + \delta e_A v_A$, and the Kahler potential eq(2.33), to leading order in δ, we get

$$T_L = 2e^{\frac{K}{2}}|\Delta W| = \frac{e^{2D}}{\sqrt{2vol}}|\sum_i \delta e_i v_i + \sum_A \delta e_A v_A|. \tag{2.113}$$

For the decay to occur the rhs of eq.(2.112) must be positive, thus we see that again $T_U = T_L$. Therefore the decays can again be at most marginal.

In the calculation above the effects due to the jump in the non-blow up fluxes were calculated as in the previous subsection and thus are correct only to leading order in δ. Thus we are assuming that

$$|\delta e_i v_i|\delta \ll |\delta e_A v_A|. \tag{2.114}$$

Using the relation that $|\frac{v^A}{v^i}| \sim |\frac{e_i}{e_A}|\delta$ this gives,

$$|\frac{\delta e_i}{e_i}| \ll |\frac{\delta e_A}{e_A}|. \tag{2.115}$$

In fact the Type 2) vacua are stable with at best marginal decays upto a high order of approximation. This is due to their being related (after a change in the sign of all fluxes) with supersymmetric vacua, as will be explained in section §2.4.8.

2.4.6 Type 3) to Type 3) Decays

Here we consider the analogous decays for Type 3) vacua. In this case $e_A < 0$, $e_i > 0$. The ground state energy is given in eq. (2.45). Including a contribution due to the change in the e_i flux gives,

$$T_U = \frac{1}{\sqrt{2}}\frac{e^{2D}}{\sqrt{vol}}(\sum_i \delta e_i v_i + 5\sum_A \delta e_A v_A). \tag{2.116}$$

Note that the contribution proportional to δe_A on the rhs comes with a coefficient 5. T_L continues to be given by eq(2.113). Therefore now there can be situations where $T_U > T_L$.

As an example consider the case where δe_i vanishes, and one of the $\delta e_A \neq 0$. For the energy difference to be positive, $\epsilon > 0$, which means $\delta e_A < 0$, since $v_A < 0$. As we will argue below, in this case there is a susy cycle which saturates the lower bound $T = T_L$. Thus $T < T_U$ and the decay will proceed. eq.(2.115) is met.

The argument establishing that there is a susy cycle with $\delta e_A \neq 0, \delta e_i = 0$ is as follows. Blowing up the orbifold slightly gives rise to a P^2 at every fixed point. There is a $P^1 \subset P^2$. It is easy to see that this P^1 is a holomorphic cycle and is non-trivial in homology [9]. Its size, a, is proportional to v_A the blow-up modulus. Now being holomorphic a must be proportional to the resulting jump in the superpotential. This can only happen if the δe_i coefficients vanish for this cycle, since $v_i \gg |v_A|$ [10].

We now estimate the decay rate which results in this case from changing δe_A. The bounce action can be shown to be of order

$$B \sim \frac{\epsilon M_{Pl}^4}{V_0^2} \tag{2.117}$$

[9] In the coordinates used in [26], eq.(3.1), this cycle is given by setting $z_2 = w = 0$, so it is clearly holomorphic. To include the point at infinity, $z_1 \to \infty$ a second patch is needed. The Kahler form integrates to a non-zero value on this cycle so it is clearly non-trivial in homology.

[10] Ideally we should have calculated the intersection numbers of this cycle with the P^2 divisor and the other four-cycles from first principles and shown that these are of the required form. We will not attempt this here.

where we have reinstated the dependence on the four dimensional Planck scale M_{Pl} on dimensional grounds. The rate of decay goes like $\Gamma \sim e^{-B}$, so the fastest decays are those with the smallest jumps in flux. Working out the resulting discharge of a particular vacuum due to all the competing decays is a fascinating question that we leave for the future [11].

2.4.7 Type 3) to Type 2) Decays

Here we discuss the decays of Type 3) to Type 2) vacua. The former have $e_A < 0$ while the latter have $e_A > 0$. The expectation value of the v_A moduli depend actually on the absolute value of e_A. So to meet the thin wall approximation we can now adjust $|e_A|$ so that the variation in v_A is small and therefore its contribution to the domain wall tension can be neglected. To illustrate this we infact adjust $|e_A|$ so that this variation vanishes.

Using $v_A = -\sqrt{\frac{10}{3}}\sqrt{\frac{|e_A|}{\beta|m_0|}}$ for Type 2) and $v_A = -\sqrt{\frac{10}{21}}\sqrt{\frac{|e_A|}{\beta|m_0|}}$ for Type 3, we learn that for v_A to be the same,

$$e_{A \text{ type } 3} = -7e_{A \text{ type } 2}. \tag{2.118}$$

Using this, we can calculate the difference in energy

$$\epsilon = V_{\text{type } 3} - V_{\text{type } 2} = \frac{3}{2}\sum \frac{\delta e_i}{e_i}|V_0| + c|V_0|\sqrt{\frac{\kappa}{\beta e_1 e_2 e_3}}\sum_A |e_{A \text{ type } 2}|^{\frac{3}{2}}. \tag{2.119}$$

where $c = \sqrt{2} + 7\sqrt{50}$. Note that for the decay to be possible $\epsilon > 0$. Using eq(2.100) T_U can be calculated to be

$$T_U = \left(\delta e_i v_i + 24\sum_A e_{A \text{ type } 2}\ |v_A|\right)\frac{e^{2D}}{\sqrt{2vol}}. \tag{2.120}$$

Note that for $\epsilon > 0$ the term in the brackets in the above equation is greater than zero.

T_L can be calculated to be,

$$T_L = |\delta e_i v_i + \sum_A \delta e_A v_A|\frac{e^{2D}}{\sqrt{2vol}}. \tag{2.121}$$

Now $v_A < 0$ and from eq(2.118) we know that $\delta e_A = -8e_{A \text{ type } 2}$, therefore

$$T_L = |\delta e_i v_i + 8\sum_A e_{A \text{ type } 2}|v_A|)|\frac{e^{2D}}{\sqrt{2vol}}. \tag{2.122}$$

It is now clear that as long as $\delta e_i v_i > 0$, $T_U > T_L$. A decay will be allowed if $T < T_U$. Like in the Type 3) -Type 3) case, concrete examples can be given where this is true. For example in the case where $\delta e_i = 0$ the $D4$ brane tension saturates the lower bound with $T = T_L$, since it is a holomorphic cycle, leading to an allowed decay.

[11] For this we also need to take into account the fact that inside the bubble is a negatively curved FRW universe which ends in a big crunch.

We can also ask about the possibility of Type 2) vacua decaying to Type 3). Running the above argument again the coefficient 24 in the second term on the rhs of eq.(2.120) and 8 in the second term of eq.(2.122) both reverse sign making both these terms negative. Since $\epsilon > 0$ for the decay to happen, we find that $T_U < T_L$. This shows that such decays are disallowed.

We have adjusted the fluxes so that the v_A moduli have the same value in the two vacua, thereby ensuring that the moduli contribution to the domain wall tension is small. Our conclusions will remain unchanged if the fluxes took different values, allowing for a variation in v_A, as long as one stays in the thin wall approximation.

2.4.8 Supersymmetric Partners in the Landscape and Marginality

We have seen above that Type 2) vacua are stable and can at most decay marginally, within our approximations. We will now see that this stability is quite general and can be understood by relating these vacua to supersymmetric ones.

We had seen in eq.(2.14), that when only e_i fluxes are excited the potential energy is invariant under a change in sign of the four-form fluxes, $e_i \to -e_i$, as long as the axions, b_a all vanish. In fact this is more generally true and follows directly from the IIA supergravity action where the F_4 dependence arises in the term,

$$S_{IIA} = -\frac{1}{2}\int d^{10}x\sqrt{-g}|\tilde{F}_4|^2 + \cdots \tag{2.123}$$

with,

$$\tilde{F}_4 = F_4 - F_2 \wedge B_2 - \frac{m_0}{2} B_2 \wedge B_2, \tag{2.124}$$

As long as B_2 vanishes[12], taking

$$F_4 \to -F_4, \tag{2.125}$$

gives the same action.

In contrast the conditions for supersymmetry *do* care about the sign of the fluxes, as we have discussed extensively above. Now in the Type 2) vacua all the four- form flux has a sign opposite to that required by supersymmetry. This means that starting with a vacuum of Type 2) we can construct a susy vacuum with the same energy by reversing all the F_4 fluxes. This susy vacuum will also have the same expectation values for the Kahler moduli and the dilaton.

Now consider a possible decay of a Type 2) vacuum to a susy vacuum of this type. By reversing the sign of all the fluxes we can relate this to the decay of a susy vacuum to a non-susy Type 2) vacuum. The vacuum energies of the initial and final vacua in the first decay and its partner decay are the same. The domain wall in the second case carries charges opposite to the first one. If the first decay is mediated by a $D4$ brane wrapping some cycle, the partner is mediated by the anti D4 brane wrapping the same cycle. Thus

[12] More correctly we mean the axions which arise from B_2 should vanish.

the two domain walls must also have the same tension. It then follows that the first decay of the non-susy vacuum can be allowed iff the partner susy vacuum can decay. But on general grounds one expects the susy vacua to be stable. We therefore conclude that the Type 2) non-susy vacuum we started with also cannot decay.

It is clear that a similar argument would also work if instead of considering the decay of the Type 2) vacuum to a susy vacuum we considered its decay to another Type 2) or a Type 3) vacuum. In both of these cases the axions are not turned on. By reversing all the four- form fluxes we can relate this to the decay of the susy vacuum to a susy vacuum in the first case, or the decay of a susy vacuum to the partner of a Type 3) vacuum in the second case. Both should not occur, given the stability of susy vacua. The partner for the Type 3) case is a vacuum with $e_i < 0, e_A > 0$. These belong to cases where there are tachyons below the BF bound, but this does not invalidate the argument above.

How general is this argument which ensures the stability of the Type 2) vacua by relating it to partner susy vacua? Our discussion above is based on the thin wall approximation in supergravity. And holds if the true and false vacuum have vanishing values for the axions. In the thin wall approximation only the $D4$ brane contribution to the domain wall tension is important, and this is clearly the same in the non-susy vacuum decay and its partner. Going beyond, one can argue that the domain wall tension continues to be equal in the two cases if the moduli contribution is included in the tension, as long as the axions are not activated in the domain wall. This follows from the fact that the potential energy and Kinetic energy terms all respect the flux reversal symmetry in the absence of axions. Since the axions vanish in both the true and false vacuum there is no reason as such for them to get activated, but for thick enough walls where the moduli under go big excursions this could happen anyways as a way of reducing the tension. If so, eventually for a thick enough wall the argument would break down.

Even for decays which are well described by the thin wall approximation subleading correction are important in the marginal case. We had found above that decays of Type 2) to Type 2) vacua are marginal if the $D4$ brane wraps a susy cycle. This result is easy to understand in light of the above discussion, since the partner susy decay would be now mediated by a BPS domain wall. However in the non-susy Type 2) decay case, one expects that the marginal nature is only approximate and eventually corrections lead to the decay being either allowed or disallowed. The corrections responsible for this might arise as corrections to the thin wall approximation itself, as we have mentioned above, or they might require going beyond the sugra approximation and including α' and g_s corrections. We leave an exploration of such questions for the future.

Finally the argument above applies only for decays of the Type 2) vacua to others where the axions are not turned on. All the stable vacua we have explored in this paper are of this type, but there could be other vacua where the b_A axions have non-zero expectation values. The argument above says nothing about the possible decays of Type 2) vacua in such cases and this would have to be examined on a case by case basis.

2.4.9 More on Supersymmetric Domain Walls

Let us end this section with some more comments on susy domain walls. We had mentioned in the discussion around eq.(2.86) that $D4$ branes which wrap holomorphic or antiholomorphic cycles preserve susy. More accurately if we take Type IIA on the Calabi-Yau manifold without flux the $D4$ brane wrapping such a cycle will preserve half the supersymmetries, i.e., $\mathcal{N} = 1$. If we now turn on flux to preserve $\mathcal{N} = 1$ susy then only one of the two cases, either the holomorphic or antiholomorphic cycle, preserves the surviving $\mathcal{N} = 1$ susy [27]. That only one of the two cases could preserve susy at best is easy to understand from the requirement of force balance. The antiholomorphic case can be thought of as the anti D4 wrapping the same cycle. If the attractive gravitational force cancells the RR repulsion for the brane it will not cancell for the anti-brane and vice-versa.

It is easy to see that a susy brane leads to a marginal decay. In this case the tension is given by T_L and the energy difference, $\epsilon = -3e^K \Delta|W|^2$. It is then easy to see that the condition for marginality,

$$\frac{\epsilon}{3T} - \frac{T}{4} = \sqrt{\frac{|U|}{3}}, \tag{2.126}$$

is met, where U is the cosmological constant. The tension is given by T_L in the probe approximation. Going beyond would require including changes in the moduli which arise because the brane causes the flux to jump. One expects the susy branes to continue to be marginal even then. Susy domain walls where moduli fields vary have been discussed in [28], [29], [30], [31], where it was shown that the walls are indeed marginal [13]. In this analysis the fluxes (which are parameters in the superpotential) were held fixed. One could try to include the changes of flux in the analysis of these authors as well, but we leave this for the future.

Finally, in practice given the charges carried by the domain wall it is not always easy to decide whether a corresponding supersymmetric cycle exists. As a special case we can consider the orbifold theory and linear branes, for which the z^i coordinates are linear functions of $\sigma, \bar{\sigma}$. Even in this simple case, the existence of a supersymmetric cycle translates into a fairly intricate number theoretic constraint on the wrapping numbers of the $D4$ brane, as discussed in [27]. Things simplify if the integers δn_i are large, $|\delta n_i| \gg 1$. Now, upto fractional corrections, which are of order $1/\sqrt{|n|}$, we can approximate, $\delta n_i \simeq \pm m_i^2$, to be a perfect square. The only obstruction to having a susy brane then arises due to the signs of the δn_i. If the δn_i's all have the same sign then a susy cycle exists, else it does not exist.

2.5 More General Decays

In this section we consider domain walls which carry more general charges.

[13]This is true only when the superpotential does not vanish in between the two vacua, otherwise the wall tension is too big. In our case, starting with the probe approximation and including corrections, the superpotential will not vanish. However for larger changes the resulting analysis might be more involved.

The general vacuum with all fluxes turned on was discussed in §2.4. The ground state energy for different vacua can be calculated by replacing e_a in formulae obtained in the case with $m_a = 0$, with eq.(2.51).

Our discussion of domain walls will follow that in §2.4 above. Given a domain wall with some charges, the change in the superpotential provides a lower bound on its tension. Below we will then calculate this lower bound, T_L and compare with an upper bound T_U defined in eq.(2.92).

We calculate T_L by keeping the moduli which appear in the superpotential to be fixed. We will come back to justifying this probe approximation below when we also discuss the validity of the thin wall approximation. For the superpotential, eq.(2.47), the change due to a domain wall carrying charges, $(\delta e_0, \delta e_a, \delta m_a)$ is,

$$\Delta W = \delta e_0 + \delta e_a t_a + \kappa_{abc} \delta m_a t_b t_c. \tag{2.127}$$

It is useful to express this in terms of the real and imaginary parts of $t_a = b_a + iv^a$, and in terms of $\hat{e}_a$,

$$\delta \hat{e}_a = \delta e_a + m_0 \kappa_{abc} \delta m_b b_c. \tag{2.128}$$

This gives,

$$\Delta W = \delta e_0 + \delta e_a b_a + \frac{\kappa_{abc}}{2} \delta m_a (b_b b_c - v_b v_c) + i \delta \hat{e}_a v_a. \tag{2.129}$$

From here using eq.(2.82) we get,

$$T_L = 2e^{K/2} |\Delta W| = \frac{e^{2D}}{\sqrt{2vol}} \sqrt{(\delta \hat{e}_a v_a)^2 + ([\delta e_0 + \delta e_a b_a + \frac{\kappa_{abc}}{2} \delta m_a b_b b_c] - \frac{\kappa_{abc}}{2} \delta m_a v_b v_c)^2}. \tag{2.130}$$

The axion fields b_a which appear above have a vacuum expectation value,

$$b_a = \frac{m_a}{m_0}. \tag{2.131}$$

It is worth pausing to discuss the physics behind this expression. The 6-brane component of the domain wall gives rise to an induced 4 brane component because the axions b_a are now-non-zero. This is responsible for the shift in δe_a, eq.(2.128). The second term within the square root arises from a $D2$ brane and a $D6$ brane component. The $D2$ brane component included a contribution due to induced $D2$ brane charge which arises from the $D4$ brane and $D6$ brane components in the presence of the axions. These together with e_0 account for the term within the square brackets.

Let us now compare this with T_U. For a Type 2) - Type 2) decay this is given by replacing $(\delta e_i, \delta e_a)$ in eq.(2.112) by their hatted counterparts giving,

$$T_U = \frac{e^{2D}}{\sqrt{2vol}} |\delta \hat{e}_a v_a|. \tag{2.132}$$

Thus we see that for this case now $T_L \geq T_U$ and the decay is at best marginal. The marginal case arises when the second term within the square root in eq.(2.130) vanishes. For this to happen the sum total of the $D2$ brane charge and $D6$ brane must vanish. In addition the tension must equal the lower bound, this would require the brane configuration to be supersymmetric.

For Type 3) -Type 3) we obtain T_U by replacing the δe_a by $\delta \hat{e}_a$ in eq.(2.116). This gives,

$$T_U = \frac{e^{2D}}{\sqrt{2vol}} |\delta \hat{e}_i v_i + 5\delta \hat{e}_A v_A|. \tag{2.133}$$

Now we see that T_U can be greater than T_L. For example this can happen if the sum of the D2 and D6 brane charges vanish and the δe_i and δe_A fluxes have opposite sign. In such cases if the tension is equal to T_L or close to it the decay will occur.

Similarly one finds that decays of Type 3) to Type 2) can indeed occur. And also one finds that decays of Type 2) to Type 3) cannot occur because $T_L > T_U$. We skip some of the details here.

Let us end with three comments. First, in this section we have not discussed the constraints imposed by the thin wall approximation. This requires that the contribution the moduli make to the domain wall tension is smaller than the D brane contribution. It is straightforward to evaluate the moduli contribution and impose this constraint but the results are unaffected.

Second, one of the conclusions that follows from our analysis above is that Type 2) vacua continue to be stable even after decays involving the most general kind of brane are considered. We had argued in §2.4.8 that there was a pairing symmetry which related the Type 2) vacua to susy vacua and this explained their stability. However this symmetry required that the b_A axions are not turned on. Now, with the most general kind of interpolating brane, m_a will in general undergo a change so that the b_a fields will become non-zero even if they vanish to begin with.

It turns out that while there is no exact symmetry which relates Type 2) vacua to susy ones in general, one can show that there is an approximate symmetry of this type to leading order in the change in fluxes. Finally, one can also consider $D8$ and $NS5$ brane mediated decays. These can be shown to lie outside the thin wall approximation.

2.6 Discussion

• We have constructed two explicit classes of non-susy AdS vacua, denoted as Type 2) and Type 3). Both are perturbatively stable. We have found that several possible decays of these vacua to other susy and non-susy vacua with lower energy are disallowed since the tension of the interpolating domain wall is much too big. The underlying reason for this is the geometric nature of AdS space where volume and surface area grow at the same rate for a large bubble.

The Type 3) vacua do have allowed decays to some other Type 3) and Type 2) vacua.

The Type 2) vacua were found to be stable in our approximations, although some decays are only marginally disallowed. It is important to go beyond our approximations to decide what happens in these marginal cases. By changing the sign of all the four-form fluxes the Type 2) vacua are turned into susy vacua with the same energy. We argued, within our approximations, that the stability of the susy vacua then ensures the stability of their Type 2) partners. This protection mechanism might well be more robust and perhaps extends even beyond leading order, but one expects it to eventually fail, tipping the marginal decays one way or another. We leave an analysis of this for the future.

• Our analysis was carried out by considering a specific model of IIA theory on the blown-up $T^6/(Z_3 \times Z_3)$, after including the effects of flux and a further orientifolding. However some of our conclusions are more general and apply to IIA on any Calabi-Yau manifold with fluxes. E.g., Type 2) vacua, obtained by flipping the sign of all the four-form fluxes exist as extremum of the potential in general, since in the absence of axions coming from the B_2 field, IIA sugra will continue to have the symmetry, eq.(2.125). However their perturbative stability is not guaranteed in general, since some of the axions could lie below the BF bound in these vacua. If the vacua are perturbatively stable they will also be stable with respect to non-perturbative decays, within the approximations used here.

• A small bulk rate of decay leads to a diverging decay rate in the boundary as we had discussed in the introduction. What is the dual description of this in the boundary CFT? In the bulk the divergence arises after integrating over all radial locations of the instanton, due to the diverging bulk volume. It is tempting to speculate that in the boundary there is a corresponding one-parameter family of instantons, parametrised by their size. And summing over the different sizes then gives rise to this divergence, which arises in the CFT due to instantons of very small size.

It might seem that the the divergence mentioned above can be controlled by introducing a cut-off at a large and finite radial location in the bulk. Conformal invariance would not be exact now but would be an approximate symmetry in the deep IR. However a more detailed analysis is needed, depending on the kind of instability one is dealing with, before one can be sure. It could be that the detailed nature of the boundary conditions at the cut-off play a significant role even in the IR [14]. We leave a detailed understanding of this divergence in the boundary theory and related issues about controlling it also for the future.

[14]This is more of a worry, in our minds, for the kind of decays discussed in this paper nucleated by a D brane rather than decays in non-susy orbifolds [2] nucleated by an instanton analogous to the one responsible for the decay of the KK vacuum [32]. In the D-brane case the RR repulsion dominates over gravitational attraction and results in a runaway $-\phi^6$ potential arising in its world volume action. This could potentially cause an instability in the quantum theory whose cure depends delicately on the correct boundary conditions.

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Chapter 3

Holographic Non-Fermi Liquids in Dilaton Gravity

3.1 Introduction

The Gauge/Gravity correspondence [1], [2], [3] provides us with a new tool to study strongly coupled field theories. It is worth exploring whether insights of relevance to condensed matter physics can be gained using this tool. One set of questions which have proved difficult to analyze using conventional techniques is the behaviour of fermions in strongly coupled systems in the presence of a chemical potential. There has been considerable activity exploring this issue on the the gravitational side recently and some interesting lessons have been learnt from such studies. This question is particularly interesting in view of considerable evidence now for non-Fermi liquid behaviour in condensed matter systems, e.g., in High T_c materials and in heavy fermion systems close to quantum phase transitions.

Extremal black branes, which are at non-zero chemical potential and typically at zero temperature, are of particular interest in the gravity description in exploring this question. Some of the early studies have focussed on analyzing the behavior of fermionic fields in extremal Reissner Nordstrom (eRN) Black brane backgrounds [4], [5], [6], [7], [8], [9]. While these black branes have the virtue of being simple and explicit they suffer from an important unphysical feature, namely, their entropy does not vanish despite their vanishing temperature. Instead, the entropy of these extremal solutions scales with the appropriate power of the chemical potential and increases as the chemical potential increases. It is widely believed that this big violation of the third law of thermodynamics is an artifact of the large N limit, and in the absence of supersymmetry or say the infinite number of symmetries in 1+1 dim. CFT's, this degeneracy should be lifted once finite N corrections are included [1]. In this context it is also relevant to note that quite often in string constructions extremal RN black branes have been found to be unstable, for example due to the presence of light charged scalars, [11].

[1]For some discussion of related issues in extremal black holes see [10].

In this chapter we will consider $3+1$ gravity systems with a holographic field theory dual which is $2+1$ dimensional. The reservations discussed above for extreme RN black branes make it worth looking for other gravity systems where the extremal black branes are different so that their entropy in particular vanishes at extremality. Such a class of systems was explored in [12], [13], [14]. The key new ingredient was to include a dilaton which allows the gauge coupling of the Maxwell field to vary. It was found that as a result the black branes have zero entropy at extremality [2]. The dilatonic systems were further generalized by [15], see also, [16], with both the gauge coupling and the potential energy now depending on the dilaton. Extremal black branes were often found to posses zero entropy in such systems as well.

These dilatonic systems, in particular their extremal black brane solution, are therefore a promising starting point for exploring questions related to the behaviour of fermionic fields. The behaviour of a bulk fermion in an extremal black brane solution of the type studied in [13], [14], was analyzed in [17], [18]. It was found that the two-point function of the corresponding fermionic operator in the boundary theory was qualitatively quite different from the non-Fermi liquid behaviour found in the eRN case and much more akin to a Fermi-liquid. The two-point function showed that there is a sharp Fermi surface in the system with well-defined quasi-particles excitations which have a linear, i.e., relativistic, dispersion relation at small frequency, and a width which has an essential singularity at vanishing frequency and which is therefore very narrow at small frequency [3].

Here we analyze the behaviour of a charged bulk fermion for the more general class of extremal dilaton systems studied in [15] and use it to calculate the two-point function of the corresponding fermionic operator in the boundary. We find that there is a wide range of behaviours that the fermion two-point function exhibits. One parameter in particular determines this behaviour, it is denoted by $\beta+\gamma$ below (for a definition of β, γ in terms of the parameters appearing in the Lagrangian eq.(3.4), see eq.(3.14), (3.15)). For $\beta+\gamma>1$ one gets Fermi-liquid behaviour. At $\beta+\gamma=1$ there is a transition. For $\beta+\gamma<1$ there are no well-defined quasi-particle excitations since they acquire a big width which is non- vanishing as $\omega \to 0$. The behaviour at the transition, when $\beta+\gamma=1$, is also quite interesting. The geometries which correspond to this case include both extreme RN type solutions and other backgrounds where the entropy vanishes. These additional backgrounds, we find, also give rise to non-Fermi liquid behaviour of a type very similar to that seen in the extreme RN case first.

From the field theory point of view, the systems we analyze can be thought of as essen-

[2] More precisely to ensure that higher derivative corrections are small one should introduce a small temperature. One then finds that the entropy density vanishes as a positive power of the temperature.

[3] For a Fermi liquid the width is $O(\omega^2)$, which is much broader. It could easily be that additional interactions, e.g., of 4-Fermi type, which are suppressed in the large N limit, when incorporated can broaden out this width to the ω^2 behaviour of Fermi liquid theory. Keeping this in mind we will refer to such behaviour as being of Fermi-liquid type below. It is also worth mentioning that there are many additional gapless excitations in the system which contribute to the specific heat and the conductivity. For this reason such a phase is described as a fractionalised Fermi-Liquid (FL*) phase rather than a Fermi Liquid phase in [19], [20], [21].

tially free fermions coupled to a fermionic operator of a strongly interacting sector [17]. The near-horizon gravity solution provides a dual description of the strongly coupled sector. By varying the parameters β, γ we explore different kinds of strongly coupled sectors and the resulting change in the behaviour of the fermionic two-point function. Our central result is that the class of strongly coupled sectors which are described by our gravity backgrounds can give rise to the different types of behaviour mentioned above and to transitions among these kinds of behaviours. For example, at the risk of belaboring this point, our results show that Non-Fermi liquid behaviour of the type found first in [4] - [9] is more common and can occur without the large entropy of the eRN case. It also shows that transitions can occur across which well-defined quasi-particle excitations acquire a big width and cease to exist.

On general grounds we expect to be able to model only strongly coupled systems in the large N limit using a classical gravity description. This is a central limitation of our analysis. As a result the fermions we are studying are only a small subsector of a much bigger system with many degrees of freedom. It turns out that while the fermionic two-point function undergoes dramatic changes as the parameter $\beta + \gamma$ is varied, as was mentioned above, the geometry and other background fields change smoothly, signalling that most of the degrees of freedom of the large N "heat bath" in fact do not change their behaviour in a significant way. As a result, one finds that the thermodynamics and transport properties like electrical conductivity also do not change significantly; in particular the qualitatively big changes in the fermionic two-point function do not correspond to phase transitions. Once one goes beyond the large N limit one expects that the significant changes in the behaviour of the fermions, should they continue to occur, would also be accompanied by significant changes in thermodynamics and transport. A preliminary indication of this is provided by $1/N$ corrections to the electrical conductivity which is sensitive to the change in the Fermion two-point function and therefore to a change in its properties, [8], [9]. The results of this paper showing that behaviours other than of Fermi liquid type can arise in a fairly robust way may be taken as preliminary evidence that such behaviour is fairly generic in strongly coupled field theories and could occur beyond the large N limit as well.

Before proceeding it is worth commenting on some of the related literature. For a general discussion about phase transitions where the Fermi surface disappears and non-Fermi liquid behaviour can arise see [22]. A system with fermions living on probe branes with examples of Non-Fermi liquid phases and transitions due to the excitations getting gapped was found in [23]. Some discussion of the Holographic description of a Fermi liquid can be found in [24]. Progress towards constructing an holographic description of the strange metal phase can be found in [25]. Recent progress in understanding the Holographic non-Fermi liquid phases often found in gravity systems in terms of fractionalised Fermi liquids and related ideas is contained in [19], [20], [21].

This chapter is organized as follows. We begin by reviewing the dilaton system of interest and discuss the near-horizon geometry of extremal and near-extremal black branes in this system, along with some aspects of their thermodynamics and transport in §3.2. A scenario in which such near horizon geometry can arise in a asymptotically AdS_4 solution is presented in §3.3. The fermionic two-point function, for various ranges of parameters, is discussed in §3.4. §3.5 contains a summary of main results and conclusions.

3.2 The Dilaton Gravity System

The system we consider consists of gravity, a $U(1)$ gauge field, and a scalar, ϕ, which we call the dilaton, with action,

$$S = \int d^4x\sqrt{-g}\left\{R - 2(\nabla\phi)^2 - f(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)\right\}. \tag{3.1}$$

Note that for simplicity we have taken the kinetic energy term of the dilaton to be canonical. This restriction can be easily relaxed although we will not do so here. The gauge coupling $g^2 \equiv (f(\phi))^{-1}$ and the potential $V(\phi)$ are both a function of the dilaton.

We will be particularly interested in solutions where the dilaton has a run-away type of behaviour near the horizon of an extremal black brane. Such run-away behaviour can result in the entropy of the extremal brane vanishing [13]. Also, we will be mainly concerned with the low-temperature or low frequency (compared to the chemical potential) response of the system. On general grounds one expects that this response will be determined by the near-horizon geometry. Thus for our purposes we will mainly be interested in the behaviour of $f(\phi)$ and $V(\phi)$ when the dilaton has evolved sufficiently far along the run-away direction. We will take this behaviour to be of exponential type,

$$f(\phi) = e^{2\alpha\phi}, \tag{3.2}$$

$$V(\phi) = V_0 e^{2\delta\phi}. \tag{3.3}$$

The parameters α, δ thus characterize the run-away behaviour which occurs for $\phi = \pm\infty$. These parameters will repeatedly enter the discussion below. Substituting in eq.(3.1) then gives the action,

$$S = \int d^4x\sqrt{-g}\left\{R - 2(\nabla\phi)^2 - e^{2\alpha\phi}F_{\mu\nu}F^{\mu\nu} - V_0 e^{2\delta\phi}\right\} \tag{3.4}$$

It is worth noting before we proceed that the full dependence of $f(\phi), V(\phi)$ away from the run-away region can be very different from these exponential forms. In fact, to obtain a solution which is asymptotically AdS_4 space, the potential $V(\phi)$ will need to have an extremum at a negative value of the cosmological constant and the dilaton will have to asymptote to this extremum far away from the horizon as we will later see. However, these features of $f(\phi), V(\phi)$, and the corresponding features of the geometry, will not be very significant for determining the low-energy behaviour which will arise essentially from the near-horizon region. In field theory terminology these features correspond to UV data which is irrelevant for IR physics. The action eq.(3.4) therefore determines only the IR physics of the field theory. In the analysis below we will also take V_0 appearing in eq.(3.3) to satisfy the condition [4] ,

$$V_0 < 0. \tag{3.5}$$

In this chapter we will be interested in electrically charged black branes. Using the expected symmetries of the solution (translations and rotation in the x, y directions and time independence), the metric can be chosen to be of the form,

$$ds^2 = -a(r)^2 dt^2 + \frac{dr^2}{a(r)^2} + b(r)^2(dx^2 + dy^2) \tag{3.6}$$

The horizon of the extremal black brane will be taken to lie at $r = 0$. The gauge field equation of motion gives,

$$F = \frac{Q_e}{f(\phi)b^2} dt \wedge dr. \tag{3.7}$$

The remaining equations of motion can be conveniently expressed in terms of an effective potential as [26]

$$V_{\text{eff}} = \frac{1}{b^2}\left(e^{-2\alpha\phi}Q_e^2\right) + \frac{b^2 V_0}{2} e^{2\delta\phi}, \tag{3.8}$$

and are given by,

$$(a^2b^2)'' = -\, 2V_0 e^{2\delta\phi} b^2 \tag{3.9}$$

$$\frac{b''}{b} = -\, \phi'^2 \tag{3.10}$$

$$(a^2b^2\phi')' = \frac{1}{2}\partial_\phi V_{\text{eff}} \tag{3.11}$$

$$a^2b'^2 + \frac{1}{2}a^{2\prime}b^{2\prime} = a^2b^2\phi'^2 - V_{\text{eff}}. \tag{3.12}$$

3.2.1 The Solutions

In this subsection we will construct the near-horizon geometry for a class of extremal black brane solutions to these equations. Consider an ansatz [5]

$$a = C_a r^\gamma \qquad b = r^\beta \qquad \phi = k \log r \tag{3.13}$$

Note that a multiplicative constant in b can be set to unity by rescaling x, y, and an additive constant in ϕ, while subdominant at small r can be absorbed into V_0 and Q. With this

[4] Since a negative cosmological constant is easier to obtain in string/M theory this choice for the sign of V_0 might be also easier to obtain in a string/M construction.

[5] Actually the only assumption in this ansatz is that b has a power law dependence on r. Given this fact, eq. (3.10) implies that $\phi \propto \log r$, and eq.(3.9) implies (except for some special cases) that $a(r)$ is also a power law.

ansatz, the equations eq(3.9) to eq(3.12) can be solved to give,

$$\beta = \frac{(\alpha+\delta)^2}{4+(\alpha+\delta)^2} \qquad \gamma = 1 - \frac{2\delta(\alpha+\delta)}{4+(\alpha+\delta)^2} \qquad k = -\frac{2(\alpha+\delta)}{4+(\alpha+\delta)^2} \tag{3.14}$$

$$C_a^2 = -V_0 \frac{\left(4+(\alpha+\delta)^2\right)^2}{2\left(2+\alpha(\alpha+\delta)\right)\left(4+(3\alpha-\delta)(\alpha+\delta)\right)} \qquad Q_e^2 = -V_0 \frac{2-\delta(\alpha+\delta)}{2\left(2+\alpha(\alpha+\delta)\right)} \tag{3.15}$$

The following three conditions must be satisfied for this solution to be valid :

$$Q_e^2 > 0 \Rightarrow \frac{2-\delta(\alpha+\delta)}{2+\alpha(\alpha+\delta)} > 0 \tag{3.16}$$

$$C_a^2 > 0 \ \Rightarrow \left(2+\alpha(\alpha+\delta)\right)\left(4+(3\alpha-\delta)(\alpha+\delta)\right) > 0 \tag{3.17}$$

$$\gamma > 0 \ \Rightarrow 1 - \frac{2\delta(\alpha+\delta)}{4+(\alpha+\delta)^2} > 0 \tag{3.18}$$

The last condition arises from the requirement that g_{tt} vanish at the horizon, which we have taken to lie at $r = 0$. The above conditions can be reexpressed as,,

$$2-\delta(\alpha+\delta) \ > \ 0 \tag{3.19}$$

$$2+\alpha(\alpha+\delta) \ > \ 0 \tag{3.20}$$

$$4+(3\alpha-\delta)(\alpha+\delta) \ > \ 0. \tag{3.21}$$

Note that from eq.(3.14), (3.15) and eq. (3.19),

$$\gamma - \beta = \frac{4-2\delta(\alpha+\delta)}{4+(\alpha+\delta)^2} > 0. \tag{3.22}$$

The parameter $\beta+\gamma$ will play an important role in the subsequent discussion. From eq.(3.14), (3.15) it takes the value,

$$\beta+\gamma = 1 + \frac{(\alpha+\delta)(\alpha-\delta)}{4+(\alpha+\delta)^2}. \tag{3.23}$$

Note that $\beta+\gamma = 1$ when $\alpha = \pm\delta$. The case $\alpha+\delta = 0$ has $\gamma = 1, \beta = 0$ and therefore corresponds to an $AdS_2 \times R^2$ geometry which is also the near horizon geometry in the extreme RN case. The case $\alpha = \delta$ has $\beta > 0$ and therefore corresponds to an extremal brane with vanishing horizon area.

3.2.2 More on the Solutions

Here we comment on some properties of the solutions in more detail.

The solution in eq.(3.14), (3.15) has only one parameter V_0, in particular the charge too gets fixed in it in terms of this parameter. In the full solution, including the asymptotic region near the boundary, the charge or the chemical potential would of course be an additional parameter, however this parameter does not appear in the near-horizon solution.

The solution above eq.(3.14), (3.15) is actually an exact solution to the equations of motion, but in a situation where the asymptotic boundary conditions are different, say AdS_4, it will only be approximately valid at small values of r. And the chemical potential will enter in the determination for when the near-horizon geometry stops being a good approximation [6]

If V_0 and μ are the only two scales in the geometry one expects that the near horizon geometry is a good approximation for

$$r \ll \frac{\mu}{\sqrt{|V_0|}}. \tag{3.24}$$

Note that r is dimensionless, μ and $\sqrt{|V_0|}$ have units of Mass [7], thus this formula is consistent with dimensional analysis [8].

Let us make a few more comments. When $\alpha + \delta$ does not vanish, $\beta > 0$, and therefore the area of the horizon and thus the entropy vanishes. Second, the solution has a smooth limit when $\delta \to 0$, and reduces to the black brane found in [13], [27] in this limit. Third, the solution is somewhat analogous to Lifhsitz type solutions [28], however, in general the metric in the solution does not have any scaling symmetry. Exceptions arise when $\gamma = 1$, which requires either $\alpha + \delta = 0$, the eRN case mentioned above, or $\delta = 0$, the case studied in [13]. Finally, after a suitable coordinate transformation it is easy to see that the solution we have obtained above, eq.(3.14), (3.15) agrees with the solution discussed in [15] in §8, eq.(8.1a) - (8.1d), with the non-extremality parameter m set to zero [9].

We will examine the thermodynamics of the near extremal solution next and also comment on electrical conductivity.

3.2.3 Thermodynamics of the Slightly Non-Extremal Black Brane

Next we turn to constructing slightly non-extremal black brane solutions (these would have temperature $T \ll |\mu|$, where μ is the chemical potential). We can construct a one parameter deformation of the extremal solutions, where

$$a^2 = C_a^2 r^{2\gamma} \left(1 - \left(\frac{r_h}{r}\right)^{2\beta+2\gamma-1}\right) \tag{3.25}$$

[6] More generally, there could be additional scales, e.g., if a relevant operator is turned on in the boundary CFT, besides the chemical potential, to obtain the full geometry. In such a situation our comments apply if these additional scales are also of order the chemical potential. In §3.3 we will in fact construct examples of such solutions where the relevant operator is dual to the dilaton. For typical values of parameters considered there the additional scale which corresponds to the coupling constant of this operator in the Lagrangian is of order μ.

[7] There is a hidden overall Newton constant G_N in the action (3.1).

[8] Another way to obtain (3.24) in the full solution viewpoint is to note that we need $r - r_h \ll r_h \sim \mu$ where r_h is horizon. By restoring the length scale of the system $\sim 1/\sqrt{|V_0|}$ and by coordinate transformation so that we set horizon to be $r = 0$, we obtain (3.24).

[9] For comparison purposes the parameters (α, δ) defined here should be related to (γ, δ) in [15] as follows: $(\alpha, \delta) \to (\gamma, -\delta)$.

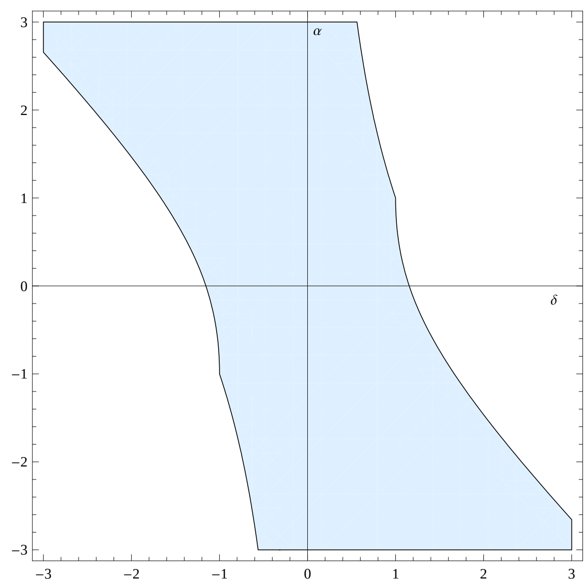

Figure 3.1: Region Allowed by the Constraints

and b^2, ϕ take the form in eq.(3.13), with C_a, γ, β, k as given in eq.(3.14), eq.(3.15). The parameter r_h characterizes the deformation and corresponds to the location of the horizon. It is easy to see that the deformed solutions have a first order zero at the horizon and thus are non-extremal. For $r_h \ll 1$ these solutions are close to extremal.

It is simple to see that the temperature of the non-extremal black brane goes like,

$$T \sim r_h^{2\gamma-1} \tag{3.26}$$

and the entropy density scales like,

$$s \sim r_h^{2\beta} \sim T^{\frac{2\beta}{2\gamma-1}} \tag{3.27}$$

A physically acceptable extremal black brane, which corresponds to the ground state of a conventional field theory on the boundary, should have a positive specific heat when it is heated up. This leads to the additional condition for an acceptable solution,

$$2\gamma - 1 > 0 \tag{3.28}$$

When expressed in terms of α, δ this becomes,

$$4 + (\alpha - 3\delta)(\alpha + \delta) > 0 \tag{3.29}$$

This condition must be added to the three discused earlier, eq.(3.19) - (3.21). In Figure 3.1 we show the region in the (α, δ) plane which meets all these four conditions.

3.2.4 Taming the Singularity

In the discussion above we used classical Einstein gravity and worked in the two-derivative approximation. These approximations can break down at sufficiently small values of the radial coordinate r. For example, a curvature singularity could arise or the dilaton can diverge signalling such a breakdown. In this subsection, we argue that turning on a temperature which is very small in the large N limit can often help control this breakdown. Our arguments are only suggestive at the moment, a definitive discussion would require an embedding of these dilaton systems in string theory which has not been done as yet.

The parameter

$$L = \frac{1}{\sqrt{|V_0|}} \tag{3.30}$$

is an important length that characterizes the system. We will assume that L and the chemical potential μ are the only two scales in the system in the two derivative approximation. For example, if the geometry is asymptotically AdS_4 the radius of AdS_4 would be of order L. From eq.(3.14), (3.15) we see that L is also the only scale in the near-horizon solution. A measure of the number of degrees of freedom in the system is given by

$$N^2 = \frac{L^2}{l_{Pl}^2}. \tag{3.31}$$

In the near-horizon geometry of the extremal solution the Ricci scalar, $\mathcal{R} \sim r^{2(\gamma-1)}/L^2$. The higher invariants $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are also of the same order, i.e. $R_{\mu\nu}R^{\mu\nu} \sim (r^{2(\gamma-1)}/L^2)^2$ etc. We see that these invariants diverge at $r = 0$ for $\gamma < 1$ [10]. At finite temperature the divergence is cutoff at the horizon located at $r = r_h$. We get

$$\mathcal{R}l_{pl}^2 \sim \frac{r_h^{2(\gamma-1)}}{N^2} \tag{3.32}$$

Expressing this in terms of the temperature

$$T \sim \frac{r_h^{2\gamma-1}}{L} \tag{3.33}$$

leads to

$$\mathcal{R}l_{pl}^2 \sim \frac{(TL)^{\frac{2\gamma-2}{2\gamma-1}}}{N^2} \tag{3.34}$$

We see that for $N^2 \gg 1$ the curvature can be made much smaller than the Planck scale by taking

$$\left(\frac{1}{N^2}\right)^{\frac{2\gamma-1}{2(1-\gamma)}} \ll TL \ll 1 \tag{3.35}$$

Thus the curvature can be made much smaller than the Planck scale, while keeping the temperature much smaller than $1/L$. In the large N limit, where $N \to \infty$, keeping L fixed,

[10] For $\gamma \geq 1$ tidal forces could still blow up, as happens in the Lifshitz solutions obtained when $\gamma = 1, \beta \neq 0$.

this condition is in fact met for any non-zero temperature.

However this analysis might be incomplete, since the 4 dimensional Planck scale is a derived quantity in string theory and the criterion for breakdown of classical two-derivative gravity involves the curvature in units of the string scale, which is related to l_{Pl} via the values of moduli, and also involves the string coupling. It could be that requiring the curvature to be much smaller than the string scale imposes a stronger condition than eq.(3.35), or that a stronger condition arises by requiring that quantum effects remain small [11] .

For example, it could be that the dilaton enters in the relation between the string and Planck scale [12], since the dilaton also varies with r this could change the condition for the validity of the two-derivative approximation. Similarly, the dilaton might also enter in the string coupling and the requirement that quantum corrections are small could impose significant restrictions. In fact this is likely to be the case. The gauge coupling in the action eq.(3.4) goes like $g^2 = e^{-2\alpha\phi}$. One would expect the theory to be weakly coupled only when $g^2 \ll 1$. The dilaton in the near horizon region is given by,

$$\phi = k \log(r/r_c) \tag{3.36}$$

where we have introduced a radial cut-off r_c on the RHS. This leads to the condition,

$$e^{-2\alpha\phi} = \left(\frac{r_h}{r_c}\right)^{-2\alpha k} \ll 1 \tag{3.37}$$

When $\alpha k > 0$ this does not allow the temperature to become very small. The parameter r_c depends on the chemical potential, which determines how far out in r the geometry begins to depart from the near-horizon solution, and also can depend on the asymptotic value of ϕ. It could be that this condition can be met only if T is large compared to μ, this would require a temperature which is much too big, the resulting finite temperature black brane would not be described by the near-horizon metric we have found. Or it could be that if one starts with the asymptotic value of the dilaton being small enough this condition can be met while remaining within the scope of the solution we have found.

Clearly, this question about the validity of our solution, in the presence of a small temperature, will need to be revisited in a more complete string construction.

3.2.5 Conductivity

For completeness let us also comment on the electrical conductivity of this system. Our discussion follows [13] §3 and [14] §3 closely, we omit some details. Before proceeding let us note that the compution of the electrical conductivity in AdS/CFT is discussed in several other papers as well, e.g., [29], [30], [31], [32]. The essential idea of our calculation, [30], is

[11] Another reason for thinking that there is more to this analysis is the condition eq.(3.35) involves L which does not directly have an interpretation in the boundary theory.

[12] The scalar we are calling the dilaton may not literally be the dilaton field of string theory whose expectation value determines the string coupling.

to cast the equation governing a perturbation of say the A_x component of the gauge field in the black brane background in the form of a Schroedinger problem,

$$-\frac{d^2\psi}{dz^2} + V(z)\psi = \omega^2\psi \tag{3.38}$$

where ω is the frequency. Starting at the boundary with an ingoing pulse one can calculate the reflection amplitude R from the potential $V(z)$. The conductivity is then given by

$$\sigma = \frac{1-R}{1+R}. \tag{3.39}$$

The dependence of the conductivity on ω, T, for small values of these parameters can be obtained, upto overall coefficients, by analyzing the behaviour of $V(z)$ in the near-horizon region. Thus our lack of knowledge of the full solution in the problem at hand will not be a limitation in extracting this information, although conceptually it is useful to assume that there is a screen eventually located in an asymptotically AdS region.

We computed the schrodinger potential $V(z)$, which turns out to be

$$V(z) = \frac{c}{z^2} \tag{3.40}$$

where the coefficient c, which is important for this calculation, takes the value,

$$c = 2\frac{(4+\alpha^2-\delta^2)(4+(\alpha-2\delta)(\alpha+\delta))}{(4+(\alpha-3\delta)(\alpha+\delta))^2} \tag{3.41}$$

Defining

$$\nu = \sqrt{c+\frac{1}{4}}, \tag{3.42}$$

it then follows from the analysis of §3 in [13] for example that the optical conductivity, for $\omega \ll \mu$ at zero temperature, is given by,

$$Re(\sigma) \sim \omega^{2\nu-1} \sim \omega^{\frac{2(4+\alpha^2-\delta^2)}{4+(\alpha-3\delta)(\alpha+\delta)}} \tag{3.43}$$

Similarly, the DC conductivity, for $\omega \to 0$, for small temperature, $T/\mu \ll 1$, goes like,

$$Re(\sigma) \sim T^{2\nu-1} \sim T^{\frac{2(4+\alpha^2-\delta^2)}{4+(\alpha-3\delta)(\alpha+\delta)}} \tag{3.44}$$

as follows from the analysis in [14] §3 for example. There is in addition a delta function at zero frequency in $Re(\sigma)$ which we have omitted above.

Note that using eq.(3.14), eq.(3.15), we can express the exponent

$$2\nu - 1 = \frac{2\gamma}{(2\gamma-1)} \tag{3.45}$$

Since $2\gamma - 1 > 0$ from eq. (3.28), the RHS is always positive thus the exponent in both the optical conductivity and DC conductivity are positive. This means the optical conductivity increases with increasing frequency and the DC conductivity increases with increasing temperature, with the system behaving in effect as one with a "soft gap" [13]. As (α, δ) are varied $2\gamma - 1$ can become arbitrarily small (while remaining positive) and thus the exponent eq.(3.45) can become very large so that the increase with frequency or temperature is very gradual.

The result for the optical conductivity agrees with §8 of [15]. The DC conductivity does not agree. The answer above corresponds to the DC conductivity as defined by the two-point current -current correlation function using the Kubo formula. The definition in [15], §5, for the DC conductivity is different and related to the drag force on a massive charge.

3.3 Extremal Branes: from near-horizon to boundary of AdS

In §3.2.1 we investigated a system of dilaton gravity described by the action eq.(3.1). Since we were interested in the behaviour when the dilaton had evolved sufficiently far along a run-away direction we took $f(\phi)$ and $V(\phi)$ to be of the form, eq.(3.2), eq.(3.3). The resulting solution was then of the form eq. (3.13). In this section we will show that such a solution can indeed arise as the near-horizon limit starting from an asymptotic AdS_4 geometry perturbed by a varying dilaton. We do this for a particular form of the potential[14],

$$V(\phi) = 2V_0 \cosh(2\delta\phi) \tag{3.46}$$

with $V_0 < 0$.

This potential has the property that along the run-away direction where, $\phi \to \infty$, $V(\phi) \to V_0 e^{2\delta\phi}$ and therefore agrees with eq.(3.3). As a result the solution eq.(3.13) continues to be a good approximate solution for this potential as well. In addition, the potential has a maximum at $\phi = 0$, with $V(\phi = 0) = 2V_0 < 0$ which can support a AdS_4 solution. Working in a coordinate system of the form eq.(3.6), we will construct a numerical solution which asymptotes between this AdS_4 solution and a near-horizon geometry given by eq.(3.13).

Note that near $\phi = 0$ the potential eq.(3.46) is tachyonic with $m^2 < 0$. For the masss to be above the BF bound, δ must meet the condition

$$\delta^2 < \frac{3}{8}. \tag{3.47}$$

Also since $m^2 < 0$, both normalizable and nonnormalizable modes of dilaton are falling near the boundary $r \to \infty$. This corresponds to the fact that with $m^2 < 0$ the dilaton

[13] In a system with a conventional gap the conductivity would be exponentially sensitive to the temperature, going like, $Re(\sigma) \sim e^{\frac{-\Delta}{T}}$, instead of having the power-law behaviour we find.

[14] We expect similar results for other potentials with the same qualitative behaviour.

corresponds to a relevant operator in the CFT dual to the asymptotic AdS_4 space-time. In the solution we obtain numerically, in general, the dilaton will go like a linear combination of normalizable and nonnormalizable modes. Accordingly, in the dual field theory the Lagrangian will be deformed by turning on the relevant operator dual to the dilaton.

In the subsequent discussion it will be convenient to choose units such that $|V_0| = 1$.

3.3.1 Identifying The Perturbation

It is actually convenient to start in the near-horizon region and then integrate outwards, towards the boundary, to construct the full solution.

To start, we first identify a perturbation in the near-horizon region which grows as one goes towards the UV (larger values of r). For this purpose, we will approximate the potential as $V = -e^{2\delta\phi}$ and ignore the correction going like $e^{-2\delta\phi}$ to it, this will lead to a condition on the parameters (α, δ) which we will specify shortly.

Including a perturbation in the metric gives,

$$a(r) = C_a r^\gamma (1 + \ d_1 \ r^\nu) \quad ; \quad b(r) = r^\beta (1 + \ d_2 \ r^\nu); \quad \phi(r) = k \log r + \ d_3 \ r^\nu \tag{3.48}$$

Solving the equations of motion eq.3.9 to eq.3.12 to leading order in r, determine ν as

$$\nu_1 = -\frac{3}{2} + \frac{4 + 2\delta(\alpha+\delta)}{4 + (\alpha+\delta)^2} + \frac{\sqrt{(4 + (3\alpha - \delta)(\alpha+\delta))\left[36 - (\alpha+\delta)(17\delta - 19\alpha + 8\alpha^2\delta + 8\alpha\delta^2)\right]}}{2(4 + (\alpha+\delta)^2)^2} \tag{3.49}$$

We can also determine d_2, d_3 in terms of d_1 which is left undetermined and hence is a free parameter that characterises the resulting solution.

In our analysis above to determine the perturbation, we approximated the potential $V = -2\cosh(2\delta\phi) \simeq -e^{2\delta\phi}$, while keeping the leading corrections due to the perturbation in eq.(3.48)). This is justified, for small r if,

$$\nu < -4\delta k. \tag{3.50}$$

which has a overlap with the acceptable regions of α, δ mentioned in previous subsection. In the numerical analysis we will choose values for (α, δ) which lie in this region, and which also meet the condition eq.(3.47).

3.3.2 Numerical integration

Staring with the perturbed solution in the near-horizon region the equations can be now be numerically integrated to obtain the solution for larger values of r. For this purpose the full potential eq.(3.46) is used.

Figure(3.2) and Figure(3.3) show the resulting solution for $\alpha = 1$, $\delta = 0.6$, these values satisfy the conditions, eq.(3.47), eq.(3.50). The strength of the perturbation was chosen to be $d_1 = 0.01$.

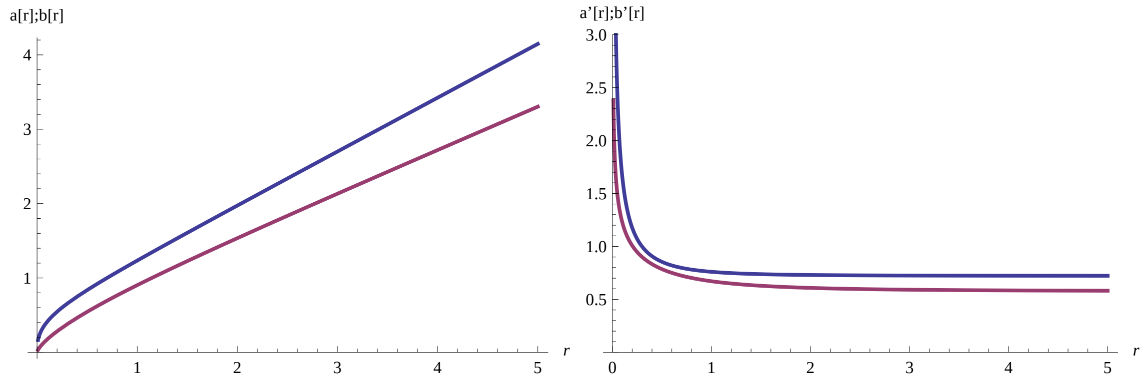

Figure 3.2: Numerical solution interpolating between the near horizon solution and AdS_4 for $\alpha = 1$, $\delta = 0.6$ and $d_1 = 0.01$. The second plot shows that $a'(r)$ and $b'(r)$ approach 1. Red lines denote a, Blue lines denote b.

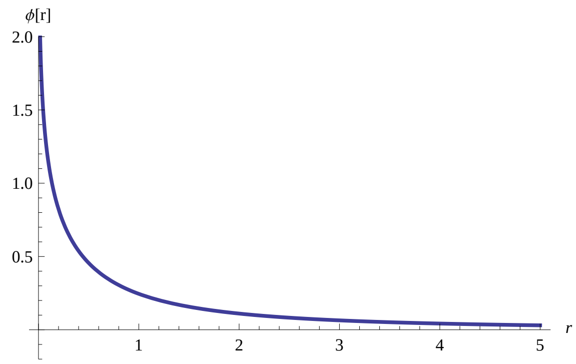

Figure 3.3: Numerical solution for ϕ, for $\alpha = 1$, $\delta = 0.6$.

Figure(3.2) and Figure(3.3) clearly show that $a(r) \propto r$ and $b(r) \propto r$ for large r, so the solution is asymptotically AdS_4. The dilaton approaches 0, the extrema of the potential $\cosh(2\delta\phi)$. Thus the solution interpolates between AdS_4 and solution discussed in §3.2.1 in the near-horizon region. Qualitatively similar results are obtained if the parameters α, δ and d_1 are varied within a range [15].

One final comment about parameters. For a given α and δ there should be a two-parameter family of solutions corresponding to the chemical potential μ and the coupling constant of the relevant operator dual to the dilaton in the boundary theory. Our solution above has one parameter d_1 which is the strength of the perturbation in the IR. Another parameter, which can be thought of as changing the overall energy scale in the boundary theory, corresponds to a coordinate rescaling in the bulk, $(r, x^\mu) \to (\lambda r, x^\mu/\lambda)$. Under this coordinate transformation the charge Q, eq.(3.7), transforms as $Q_e \to \lambda^2 Q_e$.

3.4 Fermionic Two Point Function

We will consider a free fermion in the bulk with mass m and charge q. Its action is

$$S_{fermion} = \int d^{3+1}x\sqrt{-g}i[\bar{\psi}\Gamma^M D_M\psi - m\bar{\psi}\psi] \tag{3.51}$$

We will mostly follow the spinor and related Dirac matrix notation of [5] and specifically comment on any differences below. In our notation then,

$$\bar{\psi} = \psi^\dagger\Gamma^{\bar{t}}, D_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} - iqA_M \tag{3.52}$$

where ω_{abM} is the spin connection and A_M is the vector potential. The gamma matrices,

$$\Gamma^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \tag{3.53}$$

with

$$\gamma^0 = i\sigma_2, \gamma^1 = \sigma_1, \gamma^2 = \sigma_3. \tag{3.54}$$

where $\sigma_i, i = 1, \cdots 3$ denote the Pauli matrices.

The spinor ψ has four components and we define $\psi_\pm$ to be the upper and lower two components respectively,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{3.55}$$

Using the translational symmetries we can take,

$$\psi_\pm = (-gg^{rr})^{-\frac{1}{4}}e^{-i\omega t + ik_i x^i}\phi_\pm \tag{3.56}$$

[15] We work in the region where eq.(3.50) and eq.(3.47) are met.

Asymptotically, near the AdS_4 boundary, the spinor components take the form,

$$\phi_+ = Ar^m + Br^{-m-1}, \phi_- = Cr^{m-1} + Dr^{-m} \tag{3.57}$$

where A, B, C, D are 2 dim. column vectors. If we define $D = SA$ then the two-point function on the boundary for the dual operator is

$$G_R = -iS\gamma^0 \tag{3.58}$$

We can simplify the calculations by using the rotational symmetry in x, y plane to choose $k = k_1$. Then writing

$$\phi_\pm = \begin{pmatrix} y_\pm \\ -iz'_\pm \end{pmatrix} \tag{3.59}$$

The equation of motion for ψ then breaks up into block 2×2 form coupling only (y_+, z'_-) and (y_-, z'_+) together respectively. The two-point function G_R has two non-vanishing components, G_{R11} and G_{R22}, these are related by, $G_{R11}(\omega, k_1) = G_{R22}(\omega, -k_1)$ and are not independent. In what follows we will set (y_-, z'_+) to vanish, this is sufficient to extract G_{R22} and then also G_{R11}. The equations for (y_+, z'_-) take the form,

$$\sqrt{\frac{g_{ii}}{g_{rr}}} \left(\partial_r - m\sqrt{g_{rr}}\right) y_+ = -(k_1 - u)z'_- \tag{3.60}$$

$$\sqrt{\frac{g_{ii}}{g_{rr}}} (\partial_r + m\sqrt{g_{rr}})z'_- = -(k_1 + u)y_+ \tag{3.61}$$

with

$$u = \sqrt{\frac{g_{ii}}{-g_{tt}}}(\omega + qA_t) \tag{3.62}$$

Asymptotically, towards the boundary , it is easy to see that the solution take the form,

$$\begin{pmatrix} y_+ \\ z'_- \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -\frac{(\omega+\mu q+k_1)}{(2m-1)r} \end{pmatrix} r^m + C_2 \begin{pmatrix} \frac{-(\omega+\mu q-k_1)}{(2m+1)r} \\ 1 \end{pmatrix} r^{-m} \tag{3.63}$$

where μ is the asymptotic value of the gauge potential A_t. Comparing with eq.(3.57) and eq.(3.58) we find that

$$G_{R22} = -\frac{C_2}{C_1}. \tag{3.64}$$

In the near horizon region, eq.(3.14), (3.15) the fermion equations of motion, eq.(3.60), (3.61) take the form,

$$r^{\beta+\gamma} \left(\partial_r - \frac{m}{r^\gamma}\right) y_+ = -\left(k_1 - r^{\beta-\gamma}(\omega + qA_t)\right) z'_- \tag{3.65}$$

$$r^{\beta+\gamma} \left(\partial_r + \frac{m}{r^\gamma}\right) z'_- = -\left(k_1 + r^{\beta-\gamma}(\omega + qA_t)\right) y_+ \tag{3.66}$$

where C_a dependence has been removed by some rescalings.

We will be interested in the retarded Green's function in the bulk, this is obtained by imposing in-going boundary conditions at the horizon. Very close to the horizon where ω term dominates, eq.(3.65), (3.66) become,

$$r^{\beta+\gamma}\,\partial_r \begin{pmatrix} y_+ \\ z'_- \end{pmatrix} = \omega r^{\beta-\gamma}\, i\sigma_2 \begin{pmatrix} y_+ \\ z'_- \end{pmatrix} \tag{3.67}$$

The ingoing solution is obtained by taking

$$\begin{pmatrix} y_+ \\ z'_- \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\omega z} \tag{3.68}$$

where

$$z = \frac{1}{(1-2\gamma)r^{2\gamma-1}} \tag{3.69}$$

Note that the time dependence has been taken to be of form $e^{-i\omega t}$, eq.(3.56), and since $z \to -\infty$, at the horizon, where $r \to 0$, $e^{-i\omega(t+z)}$ is well behaved at the future horizon where $t \to \infty$.

We are interested in the small frequency behaviour of the boundary two-point function. At a Fermi surface, where $k_1 = k_F$, the boundary two-point function has a singularity, for $\omega \to 0$. We will be interested in asking whether such a surface can arise in this system and what is the nature of small frequency excitations near this surface. It will be convenient to divide our analysis into three parts, depending on the value the parameter $\beta + \gamma$ takes.

When $\beta + \gamma > 1$ we will see that the boundary fermionic two-point function is of Fermi liquid type. More correctly, as was discussed in the introduction the small frequency excitations have a linear dispersion relation, with a width which is narrower than ω^2. When $\beta + \gamma < 1$ we will find that the low-frequency excitations acquire a width which is non-vanishing even in the $\omega \to 0$ limit, and thus is very broad. The transition region, $\beta + \gamma = 1$ consists of two lines. One of them corresponds to extremal RN type geometries, which are well known to give Non-Fermi liquid behaviour [4], [5], [6]. The other corresponds to geometries which have vanishing entropy, here we find that the behaviour can be of both Fermi or non-Fermi liquid type with width $\Gamma \sim \omega^p$. The power $p > 0$ and can be bigger, equal to, or less than two, so that one can get both Fermi-liquid and non-Fermi liquid behaviour.

We now turn to discussing these three cases in turn. In Figure 3.4 we plot the regions where β, γ take different values, in the (α, δ) plane.

Before proceeding let us comment on the significance of the parameter $\beta + \gamma$. We denote [16]

$$\psi = \begin{pmatrix} y_+ \\ z'_- \end{pmatrix}. \tag{3.70}$$

[16] To save clutter, henceforth will refer to this two component spinor itself as ψ.

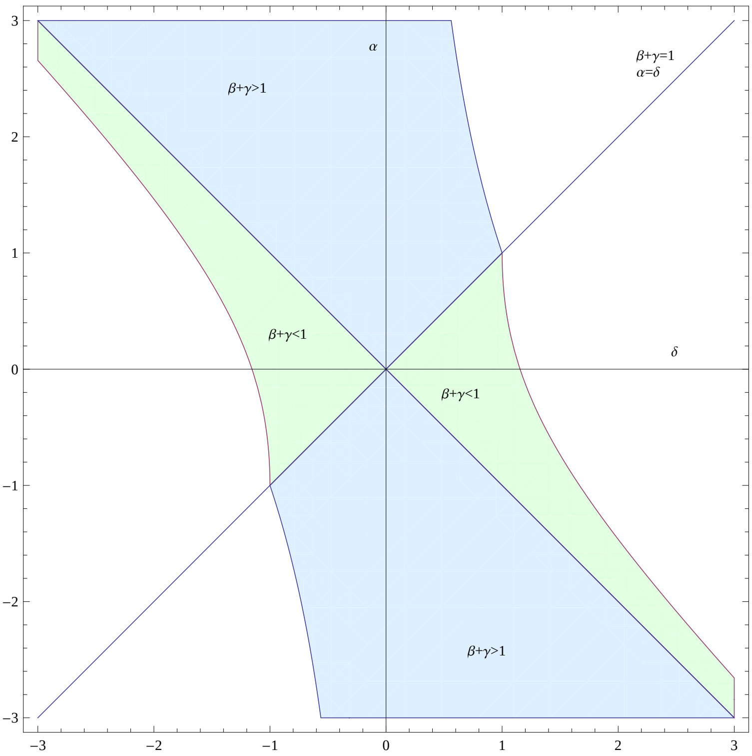

Figure 3.4: Region with $\beta+\gamma>1$ in Blue; $\beta+\gamma<1$ in Green.

There is a convenient way to recast eq.(3.65) and (3.66) as a second order equation of Schrodinger form, with a spin dependent potential:

$$\begin{aligned} r^{\gamma+\beta}\partial_r(r^{\gamma+\beta}\partial_r\psi) &= [m^2r^{2\beta}+k_1^2-(\omega+qA_t)^2r^{2(\beta-\gamma)}]\psi+\beta m r^{2\beta+\gamma-1}\sigma_3\psi \\ &\quad +i[(\beta-\gamma)r^{2\beta-1}(\omega+qA_t)+qr^{2\beta}\partial_rA_t]\sigma_2\psi\,. \end{aligned} \tag{3.71}$$

The Schrodinger variable is

$$\zeta=\int\frac{dr}{r^{\gamma+\beta}}=\frac{r^{1-\beta-\gamma}}{(1-\beta-\gamma)}. \tag{3.72}$$

The distance as measured in this variable to the horizon ($r=0$) has a power-law divergence for $\beta+\gamma>1$, a logarithmic divergence for $\beta+\gamma=1$ and no divergence for $\beta+\gamma<1$. This difference ultimately results in the three qualitatively different types of behaviour we will see below.

Eq.(3.71) is convenient for our analysis for the following reason. We are interested in the dominant ω dependence at small frequency. As we will see, this dependence arises from a region close to the horizon where the m, A_t dependent terms on the RHS of eq.(3.71) can be neglected. This only leaves the k_1 and ω dependent terms in eq.(3.71). Of these the k_1 dependent term is particularly important, since it is repulsive and hinders the particle from falling into the horizon. Physically this is because the shrinking size of the x, y directions gives rise to a cost in the k_1 dependent energy that increases closer to the horizon. In eq.(3.71) this is seen clearly since the k_1 dependent term does not give rise to a spin dependent potential (which would be proportional to the σ matrices) and only multiplies the identity matrix in spin space, with a sign which corresponds to its providing positive potential energy.

3.4.1 $\beta + \gamma > 1$

In this case we will see that the WKB approximation can be used to calculate the boundary Green's function. The width of the boundary correlator is related to tunneling through a classically disallowed region in the bulk and will be exponentially suppressed at small frequency.

In the WKB approximation radial derivatives on (y_+, z'_-) are more important than those on the metric or gauge potential, From eq.(3.65) and eq.(3.66) it then follows that for example y_+ satisfies the equation [17],

$$r^{\beta+\gamma}\partial_r(r^{\beta+\gamma}\partial_r y_+) - m^2 r^{2\beta} y_+ - \left(k_1^2 - r^{2(\beta-\gamma)}(\omega + qA_t)^2\right) y_+ = 0. \tag{3.73}$$

We will be mainly interested in the small frequency behaviour, which is essentially determined by the near horizon region, where

$$r \ll 1. \tag{3.74}$$

In this region we see that the mass dependent terms are subdominant compared to the ω and k_1 dependent terms. Also, from eq.(3.7) we see that in this region the gauge potential is given by,

$$A_t \sim r^{1-2\alpha k-2\beta}. \tag{3.75}$$

From eq.(3.14), (3.15) we see that for $\beta + \gamma > 1, (\alpha + \delta)(\alpha - \delta) > 0$. It will turn out that the small frequency behavior can in fact be extracted from the region where

$$r^{1-(2\beta+2\alpha k)} \ll \omega \tag{3.76}$$

so A_t can also be neglected compared to the ω dependence [18] in eq.(3.73). Note that $1 - (2\alpha k + 2\beta) > 0$ under eq. (3.29), therefore eq. (3.74) and (3.76) are compatible.

The equation eq.(3.73) can be cast in the form of a Schroedinger equation for a zero energy eigenstate,

$$-\frac{d^2 y_+}{d\zeta^2} + V y_+ = 0 \tag{3.77}$$

[17] This equation has additional terms which are small in the WKB approximation.
[18] We take $q \sim O(1)$.

where ζ is defined in eq.(3.72), and V, the potential, is

$$V = k_1^2 - \frac{\omega^2}{r^{2(\gamma-\beta)}} \tag{3.78}$$

There are three regions of interest in the calculation. The region very close to the horizon, which we will call R_1 is where $r \to 0$. This region is classically allowed, as follows from eq(3.22) which imply that $\gamma - \beta > 0$, therefore $V < 0$.

The second region which we call R_2 is where

$$1 \gg r^{\gamma-\beta} \gg \frac{\omega}{k_1} \tag{3.79}$$

so that the k_1 term dominates over the ω dependent term in V. Note we still have to meet eq.(3.76); at the end of this section we will return to this point and show that eq.(3.76) and eq.(3.79) are indeed compatible. Note that the R_2 region is classically disallowed and in this region the frequency dependence is unimportant. Finally the third region R_3 is close to the boundary where $r \to \infty$.

In region R_1, as mentioned earlier, the wavefunction behaviour is as in eq(3.68) The regions R_1 and R_2 are separated by a turning point at

$$r_{tp} = \left(\frac{\omega}{k_1}\right)^{\frac{1}{\gamma-\beta}} \tag{3.80}$$

In region R_2 there are two independent solutions to eq.(3.77) which in the WKB approximation go like

$$f_{\pm} = e^{\mp\left(\frac{k_1}{(\gamma+\beta-1)r^{(\gamma+\beta-1)}}\right)}. \tag{3.81}$$

Matching to solution in region R_1 using standard turning point formulae, see, e.g, [33], gives,

$$y_+ = \frac{A}{\sqrt{\hat{k}}}[f_+ + \frac{i}{2}e^{-2I}f_-] \tag{3.82}$$

where $\hat{k} = k_1 r^{\beta-\gamma}$ and,

$$I = c_1 \left(\frac{k_1^{2\gamma-1}}{\omega^{\beta+\gamma-1}}\right)^{\frac{1}{\gamma-\beta}}, \tag{3.83}$$

$$c_1 = \int_1^{\infty} \frac{dx}{x^{2\gamma}} \sqrt{x^{2(\gamma-\beta)} - 1}\,, \tag{3.84}$$

and A is an overall coefficient. Notice that e^{-2I} is exponentially suppressed when $\omega \to 0$. Thus the f_- term has a very small coefficient and is subdominant at small ω.

From eq.(3.65), (3.66) it is easy to find the solution for z'_- in this region. The result can be stated as follows. There are two linearly independent solutions in region R_2,

$$\psi_\pm = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} f_\pm. \tag{3.85}$$

The solution which agrees with eq.(3.82) is

$$\psi = \frac{A}{\sqrt{\hat{k}}}[\psi_+ + \frac{i}{2}e^{-2I}\psi_-] \tag{3.86}$$

Note that in this solution the frequency dependence is summarized in the coefficient e^{-2I}, the solutions, $\psi_\pm$ are independent of frequency.

Now $\psi_\pm$ can further be extended from region R_2 to region R_3 which lies close to the boundary. Let the coefficients C_1, C_2, eq.(3.63), which arise from $\psi_\pm$ be denoted by $C_{1\pm}, C_{2\pm}$ respectively. Then the boundary two-point function is given by

$$G_{R22} = -\frac{C_{2+} + \frac{i}{2}e^{-2I}C_{2-}}{C_{1+} + \frac{i}{2}e^{-2I}C_{1-}} \tag{3.87}$$

A Fermi surface arises when the coefficient C_{1+} vanishes for $\omega \to 0$. In general this imposes one real condition on the momentum k_1 and for a suitably chosen $k_1 = k_F$ and if necessary by adjusting the geometry [19] for the spacetime which interpolates between the near horizon region and the AdS boundary etc it should be possible to meet this condition. By rotational invariance this will then be true for all $|\vec{k}| = k_F$. Expanding C_1 near $k = k_F$ then gives,

$$G_{R22} = \frac{c_3}{\omega - v_F(|\vec{k}| - k_F) + ic_2 e^{-2I}} \tag{3.88}$$

v_F and c_2 arise from the Taylor series expansion of C_{1+} and the leading behaviour of C_{1-}, we have neglected the term proportional to C_{2-} in the numerator, and the leading C_{2+} dependence feeds into the numerator c_3.

We see that the small frequency excitations have a linear dispersion relation, with a width given by,

$$\Gamma \sim e^{-2I} = exp\left[-2c_1\left(\frac{k_F^{2\gamma-1}}{\omega^{(\beta+\gamma-1)}}\right)^{\frac{1}{\gamma-\beta}}\right]. \tag{3.89}$$

This width is exponentially suppressed at small frequency and therefore very narrow.

Let us end by checking the validity of our approximations. Our use of the WKB approximation in regions R_1, R_2 imposes restrictions. This approximation requires that radial derivatives acting on $\psi = \begin{pmatrix} y_+ \\ z'_- \end{pmatrix}$ are more important than derivatives of the metric. In

[19] One can also vary the dilaton dependence of the gauge coupling and potential once the dilaton is not in the run-away region.

region R_2 this gives the condition

$$r^{\gamma+\beta-1} \ll 1 \tag{3.90}$$

(we have set $k_1 \sim O(1)$). Note that eq.(3.90) can be met in the near horizon geometry where $r \ll 1$ only if $\beta + \gamma > 1$. The fact that the WKB approximation breaks when $\beta + \gamma \leq 1$ is also suggested by the decay width eq.(3.89) which is no longer suppressed at small ω. In addition, we have assumed that eq.(3.76) is correct so that the gauge potential dependent terms can be dropped. In region R_2 this has to be compatible with the condition eq.(3.79). One can show from eq.(3.14), eq.(3.15) that in the region where $\beta + \gamma > 1$, $\gamma - \beta < 1 - (2\beta + 2\alpha k)$, it therefore follows that for small ω these two conditions are compatible.

In Region R_1 far from the turning point validity of WKB approximation requires,

$$r \ll \omega^{\frac{1}{2\gamma-1}} \tag{3.91}$$

In addition eq. (3.76) needs to be met. These are clearly compatible, in fact eq.(3.91) is more restrictive.

3.4.2 Some General Comments for the Cases $\beta + \gamma \leq 1$

We will now turn to analyzing what happens when $\beta + \gamma \leq 1$.

A few general comments are worth making before we go into details. From eq.(3.14), eq.(3.15) we see that $\beta + \gamma = 1$ corresponds to the lines, $\alpha = -\delta$ and $\alpha = +\delta$. The first case, $\alpha = -\delta$ corresponds to an $AdS_2 \times R^2$ metric which is the near-horizon geometry of the extreme Reissner Nordstrom Black Brane. This has been analyzed extensively in [4] - [9], and we will not elaborate on this case further. The second case, $\alpha = \delta$, necessarily has $\beta \neq 0$ (for $\alpha \neq 0$), it is not $AdS_2 \times R^2$ and has vanishing area.

In the extremal RN case while studying the fermion equation of motion eq.(3.65), eq.(3.66) at small frequency the dependence on m, k_1 and charge through A_t dependent terms are all important. In contrast for the $\alpha = \delta$ case and for all cases where $\beta + \gamma < 1$ both the A_t and m dependent terms in eq.(3.65), (3.66) can be neglected, in the near-horizon region relevant for determining the small frequency behaviour. This results in considerable simplification of the analysis.

It turns out that extracting the ω dependence requires us to solve the equations from the horizon upto a radial location where

$$1 \gg r \gg \omega^{\frac{1}{2\gamma-1}}. \tag{3.92}$$

Beyond that the ω dependence turns out to be subdominant and can be neglected [20]. Now from eq.(3.65 3.66) we see that the A_t dependence is unimportant if $|A_t| \ll \omega$. From

[20] One expects on general grounds that the gravitational redshift is monotonic as one goes from the black brane horizon to the boundary making the ω dependence increasingly negligible.

eq.(3.76) this leads to the condition,

$$r \ll \omega^{\frac{1}{1-2\alpha k-2\beta}} \tag{3.93}$$

which is compatible with eq.(3.92) if

$$\omega^{\frac{1}{2\gamma-1}} \ll \omega^{\frac{1}{1-2\alpha k-2\beta}}. \tag{3.94}$$

This last condition is true for $\omega \ll 1$ because eq.(3.14), eq.(3.15) imply that $2\gamma - 1 < 1 - 2\alpha k - 2\beta$ for $\delta \neq -\alpha$.

The m dependent terms can be neglected if they are small compared to the effect of ∂_r. Now ψ will be vary at least as rapidly as a power of r, from eq.(3.65), (3.66) the condition for neglecting the m dependent term then becomes

$$\frac{m}{r^\gamma} \ll \frac{1}{r}. \tag{3.95}$$

For $m \sim O(1)$, $r \ll 1$ this gives, $\gamma < 1$, which is true since $\beta + \gamma \leq 1$ and $\beta > 0$ when $\alpha \neq -\delta$.

Henceforth we will study the cases $\beta+\gamma < 1$ and the branch $\alpha = \delta$ for the case $\beta+\gamma = 1$ and therefore we can set the A_t, m dependent terms to be zero in [21] eq.(3.65), eq.(3.66).

Eq.(3.65), (3.66) then become

$$r^{\beta+\gamma}\partial_r\psi = (-k_1\sigma_1 + ir^{\beta-\gamma}\omega\sigma_2)\psi \tag{3.96}$$

The behaviour of the solution can be understood qualitatively as follows. Very close to the horizon, the ω dependent term on the RHS will dominate over the k_1 dependent one since $\beta - \gamma < 0$ as eq. (3.22). Thus ψ will be of the form given in eq.(3.68). The effects of the frequency will become subdominant to k_1 dependent ones when the ω dependent term on the RHS of eq.(3.96) becomes less important compared to the k_1 dependent term giving the condition,

$$r \gg \left(\frac{\omega}{k_1}\right)^{\frac{1}{\gamma-\beta}}. \tag{3.97}$$

Now another way to estimate when the effects of frequency become small is when the the phase in eq.(3.68) becomes small. Using eq.(3.69) this gives

$$|\omega z| \sim |\frac{\omega}{r^{2\gamma-1}}| \ll 1 \tag{3.98}$$

which implies,

$$r \gg \omega^{\frac{1}{2\gamma-1}} \tag{3.99}$$

Now it is easy to see that for $\beta + \gamma < 1$, $\omega^{\frac{1}{2\gamma-1}} < \omega^{\frac{1}{\gamma-\beta}}$ for $\omega \ll 1$. Thus as r is

[21] To ensure clarity let us reiterate that below when we refer to $\beta + \gamma = 1$ we only mean the case where $\alpha = \delta$, for which the metric is not $AdS_2 \times R^2$.

increased from the horizon eq.(3.99) will be met before eq.(3.97) is met [22]. For the case when $\beta + \gamma < 1$ then in the region where

$$\omega^{\frac{1}{2\gamma-1}} \ll r \ll \left(\frac{\omega}{k_1}\right)^{\frac{1}{\gamma-\beta}} \tag{3.100}$$

the solution can be obtained by simply expanding the exponential in eq.(3.68) and gives,

$$\psi = \begin{pmatrix} 1 \\ -i \end{pmatrix} + O(\omega z). \tag{3.101}$$

Going to large values of r the k_1, m and A_t dependence will become important, but in this region the ω dependence can be neglected. We will return to studying the consequences of our analysis above for the $\beta + \gamma < 1$ case in §3.4.4.

In contrast, for the $\beta + \gamma = 1$ case the two exponents in eq.(3.97) and (3.99) are the same, since $\beta = 1 - \gamma$. Thus, a careful analysis is requires which we will turn to now.

3.4.3 More on the $\beta + \gamma = 1$ case

Here we will be interested in solving the fermion equation in the background, eq.(3.14), eq.(3.15), with $\alpha = \delta$. Our starting point is eq.(3.96). It is convenient to define,

$$\chi_\pm = y_+ \pm z'_-. \tag{3.102}$$

And work with the variable,

$$\tilde{z} = \frac{\omega}{(2\gamma-1)r^{2\gamma-1}}, \tag{3.103}$$

which goes to infinity at the horizon. For now we specialize to the case when $k_1 > 0$ and define

$$\eta = \frac{k_1}{2\gamma - 1}. \tag{3.104}$$

Then eq.(3.96) can be solved with ingoing boundary conditions to give

$$\chi_- = \sqrt{\tilde{z}} H^{(1)}_{\frac{1}{2}+\eta}(\tilde{z}), \tag{3.105}$$

$$\chi_+ = -\tilde{z}^{-\eta}\partial_{\tilde{z}}(\tilde{z}^\eta \chi_-). \tag{3.106}$$

In the region where $\tilde{z} \ll 1$, i.e., $\omega \ll r^{2\gamma-1}$, the solution for $\psi = \begin{pmatrix} y_+ \\ z'_- \end{pmatrix}$, in terms of the radial variable r and upto an overall ω dependent normalization, turns out to be

$$\psi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} r^{k_1} + d e^{i\phi}\omega^{2\eta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} r^{-k_1}. \tag{3.107}$$

[22] We are assuming k_1 is $O(1)$ here.

Here d and the phase $e^{i\phi}$ depend on γ, β, k_1, but are independent of ω. In the case of $\eta < \frac{1}{2}$, ϕ is generically non zero and for $\eta > \frac{1}{2}$, it turns out that $e^{i\phi} = i$.

There are subleading corrections on the RHS to the real part of ψ which are suppressed by a power of ω, these are not kept in eq.(3.107) since their effect on the Green's function is comparable to ω dependent contributions generated when evolving the solution further out towards the boundary.

It is also worth noting that the result eq.(3.107) agrees with what one would get by continuing the WKB results of §3.4.1 to the case $\beta + \gamma = 1$. More precisely, from eq.(3.72) we see that for $\beta + \gamma = 1$, $\zeta = \log r$. As a result for $r \gg r_{tp}$ the exponential factor in the two solutions go like $r^{\pm k_1}$ which agrees with eq.(3.107). The WKB suppression factor which was exponentially suppressed in ω has now turned into a power-law suppression in eq.(3.107). This is because the fermion wave function can now penetrate the barrier more easily and thus can have a bigger mixing with the modes in the vicinity of the horizon. This crossover from the exponential suppression to a power-law has also been discussed in [17].

It is now easy to follow the discussion in §3.4.1 to calculate the two-point function on the boundary in this case. A Fermi surface will arise for $k_1 = k_F$ if the growing solution is purely normalisable in AdS_4. The Green's function one gets by expanding around this value of momentum is [23]

$$G_{R22} = \frac{Z}{\omega - v_F(|k| - |k_F|) + iDe^{i\phi}\omega^{2\eta}} \tag{3.108}$$

where Z, D are constants. For $\eta > 1/2$, the phase $e^{i\phi} = 1$, whereas for $\eta < 1/2$ the phase is in general complex.

This result is very similar to what was obtained in the eRN case in [4],[5],[6],[7]. The result also agrees with the general considerations in [24] [22].

For $\eta > 1/2$ there is a well-defined quasi-particle with a linear dispersion and a width which goes like $\omega^{2\eta}$. For $1/2 < \eta < 1$, the width is broader than the Fermi liquid case. For $\eta < 1/2$ the behaviour is more novel. The last term in the denominator going like $\omega^{2\eta}$ dominates both the real and imaginary parts of the ω dependence. As a result there is no well-defined quasi-particle, since the residue vanishes at the pole. Finally for $\eta = 1/2$, the Green's function actually needs to be modified and can be shown to take the form,

$$G_{R22} = \frac{Z}{v_F(|k| - |k_F|) + d_1\omega\log\omega + d_2\omega} \tag{3.109}$$

where d_1 is real and d_2 complex.

Unlike the eRN case, when $\alpha = \delta$ and $\gamma \neq 1$, the geometry has no scaling symmetry. Despite this fact eq.(3.96) has a scaling symmetry for all values of γ, when $\beta + \gamma = 1$, under

[23] For $k < 0$ the same result goes through with now $\eta = \frac{|k_1|}{2\gamma - 1}$

[24] In fact eq.(3.108), for $\eta < 1/2$, is of the scaling form proposed in [22] and satisfies the inequalities in eq.(9) and (18) of [22].

which $r \to \lambda r, \omega \to \lambda^{2\gamma-1}\omega$ with k_1 being invariant [25]. This scaling symmetry results in the the complex part characterized by the exponent η being of power law type in the frequency. One difference is that in our case the mass and charge of the bulk fermion do not enter in η explicitly but only through k_F, which does depend on these parameters.

Before closing this subsection it is worth commenting that while on general grounds we expect a value of k_F to exist for which the bulk solution is purely normalisable, leading to a singularity in G_{R22}, we have not investigated this feature in detail. We leave such an analysis, along with the related calculation of Z, v_F, D which appear in eq.(3.108), for the future.

3.4.4 Case $\beta + \gamma < 1$

In this case the essential features of the solution can be deduced by setting the k_1 dependent terms in eq.(3.96) to zero. This can be seen to be self-consistently true. In fact the essential point was already made in §3.4.2. Setting the k_1 dependent term to vanish in eq.(3.96) gives the solution eq.(3.68). When eq.(3.98) is met the solution reduces to (3.101). This happens before the k_1 term becomes important because

$$\omega^{\frac{1}{2\gamma-1}} \ll \omega^{\frac{1}{\gamma-\beta}} \tag{3.110}$$

when $\beta+\gamma < 1$ as discussed in §3.4.2 around eq.(3.100). A more careful analysis shows that the region eq.(3.100) where the solutions reduces to the form, eq.(3.101) should be thought of as being obtained by keeping $r/\omega^{1/(2\gamma-1)}$ fixed and large while taking $\omega \to 0$.

Let us now examine the consequences of eq.(3.101). Note in particular that at leading order y_+ and z'_- have a relative phase which is imaginary.

In the following discussion it will be useful to define two basis vectors,

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.111}$$

and express the leading order answer as

$$\begin{pmatrix} y_+ \\ z'_- \end{pmatrix} = \psi_+ - i\psi- \tag{3.112}$$

Starting from the region eq.(3.100) and going further towards the boundary the k_1, m, A_t dependent terms we have been neglecting will come into play and the form of the solution will deviate from eq.(3.112). Asymptotically, towards the boundary, the solution obtained from both $\psi_\pm$ will take the form given in eq.(3.63). Let $C_{1\pm}, C_{2\pm}$ be the values for the coefficients $C_{1,2}$ which appear in eq.(3.63) when we start with $\psi_\pm$ and evolve the solution towards the boundary respectively. Then, the net value of C_1 we get starting from eq.(3.112)

[25] In fact a suitable change of variables can map the eq.(3.96) for all values of γ, and $\gamma + \beta = 1$, to the case $\gamma = 1, \beta = 0$.

is

$$C_1 = C_{1+} - iC_{1-} \tag{3.113}$$

Now notice that the equations eq.(3.65), eq.(3.66) are both real, therefore $C_{1\pm}$ will be both real as well.

As in the discussion for the $\beta + \gamma > 1$ case a Fermi surface arises at $k_1 = k_F$ when C_1 vanishes at this momentum as $\omega \to 0$. However, since C_{1+} and C_{1-} are real this actually imposes two conditions

$$C_{1+} = 0\,,\, C_{1-} = 0 \tag{3.114}$$

which must both be met by adjusting only one real variable - the momentum k_1. Generically, this will be impossible to do.

Our conclusions will be discussed more throughly in the following subsection. We will see that starting from $\beta + \gamma = 1$ as we go into the region where $\beta + \gamma < 1$, there is no locus in momentum space about which there are quasi-particle excitations with a width that vanishes as ω vanishes. However, for $\beta + \gamma < 1$, but close to unity, there is a surface about which the excitations have a frequency independent width (at small ω) which is much smaller than the chemical potential and which vanishes as $\beta + \gamma \to 1$.

3.4.5 The Transition from $\beta + \gamma = 1$ to $\beta + \gamma < 1$

It is useful to discuss the transition from $\beta + \gamma = 1$ to $\beta + \gamma < 1$ in more detail.

Let us start with the case $\beta + \gamma = 1$ and first consider the case when the exponent η in eq.(3.108) satisfies the condition $2\eta > 1$. In this case, at small ω,

$$G_{R22} = \frac{Z}{\omega - v_F(|k| - k_F) + id_1\omega^{2\eta}}, \tag{3.115}$$

and as was mentioned above there are well-defined quasi-particle excitations about the Fermi surface.

Suppose we now lower the value of $\beta + \gamma$ so that $\beta + \gamma = 1 - \epsilon, \epsilon \ll 1$. The bulk fermion solution with momentum k_F will not be purely normalisable any more and our arguments in the previous subsection show that the Green's function takes the form,

$$G_{R22} = \frac{Z}{\omega - v_F(|k| - k_F) + \Delta_1 + i\Delta_2 + id_2\omega^{2\eta}} \tag{3.116}$$

where Δ_1, Δ_2, are ω independent and vanish when $\epsilon \to 0$. We see that Δ_1 can be absorbed by a shift in [26] k_F, $k_F \to k_F - \frac{\Delta_1}{v_F}$. About this new Fermi momentum we get,

$$G_{R22} = \frac{Z}{\omega - v_F(|k| - k_F) + i\Delta_2 + id_2\omega^{2\eta}} \tag{3.117}$$

so that the excitations have a width Δ_2, which does not vanish as $|k| \to k_F, \omega \to 0$, and is

[26] Alternatively, e.g., in the canonical ensemble, we can absorb it into a shift in the chemical potential μ.

therefore very broad. In summary, the well-defined quasi-particle which existed at $\beta+\gamma=1$ has therefore disappeared at $\beta+\gamma<1$ [27].

Next, let us turn to the case when the exponent η in eq.(3.108) satisfies the condition $2\eta<1$. In this case there is no sharply defined quasi-particle even when $\beta+\gamma=1$. We define,

$$k_T \equiv (|k|-k_F). \tag{3.118}$$

Taking $|k_T|$ fixed and small compared to k_F, and regarding G_{R22} as a function of ω there is a pole [9] at $\omega=\omega_*+i\Gamma$ with

$$\omega_* \sim \Gamma \sim |k_T|^{\frac{1}{2\eta}}. \tag{3.119}$$

The pole has vanishing residue,

$$Z_{res} \to 0, \tag{3.120}$$

and is also broad,

$$\frac{\Gamma}{w_*} \to 1. \tag{3.121}$$

as $k_T \to 0$.

Let us now lower $\beta+\gamma$ in this case to the value $\beta+\gamma=1-\epsilon$. This results[28] in a Green's function,

$$G_{R22} = \frac{Z}{-v_F(|k|-k_F)+\Delta_1+i\Delta_2+De^{i\phi}\omega^{2\eta}}, \tag{3.122}$$

where $\Delta_{1,2}$ vanish as $\epsilon\to 0$. Shifting k_F this can be written as

$$G_{R22} = \frac{Z}{-v_F(|k|-k_F)+|\Delta|De^{i\phi}e^{-i\pi\eta}+De^{i\phi}\omega^{2\eta}} \tag{3.123}$$

where $|\Delta|$ is determined by Δ_1, Δ_2 and the shift in k_F. For $|k|\to k_F$ the pole in ω lies at

$$\omega_* = -i|\Delta| \tag{3.124}$$

This gives rise to a width which does not vanish when $(k-k_F)\to 0$.

In summary we see that when $\beta+\gamma<1$ the excitations become very broad and acquire a width which is non-vanishing at zero frequency. There is still a locus in momentum space, at $|k|=k_F$, which we can call the Fermi surface, with the energy of the excitations, defined as the real part of ω, extending down to zero energy as the momentum approaches k_F. However, a more precise definition of the Fermi surface can be taken to be the locus where Green's function with $\omega=0$ has a pole in momentum, and across which it changes sign. With this definition, there is no Fermi surface for $\beta+\gamma<1$, since the excitations have a non-zero width even at zero energy, as mentioned above.

[27] The width Δ_2, while it does not vanish when $|k|\to k_F$, is small compared to the chemical potential for $\epsilon \ll 1$.

[28] The term linear in ω in the denominator is dropped compared to $\omega^{2\eta}$ at small ω.

3.5 Conclusions

The gravity system studied in this chapter has a scalar, the dilaton, and two couplings α and δ which appear in the action given in eq.(3.2), (3.3) and which determine how the dilaton enters in the gauge coupling and the potential respectively. Instead of α, δ it is sometimes more convenient to use the parameters, β, γ, which appear in the metric eq.(3.13) and are given in terms of α, δ, in eq.(3.14), (3.15).

Extremal black branes in this system were studied in [15]. Here we have studied a charged fermion in the extremal black brane background and calculated the two-point function for the corresponding fermionic operator in the dual strongly coupled field theory. The black brane background has rotational symmetry in the two spatial directions and the two-point function inherits this symmetry, At small frequency, which is the focus of our investigation, the essential features of this two -point function can be deduced from the near-horizon geometry of the extremal black hole.

3.5.1 Results

Our results depend on the parameters β, γ, in particular on the combination [29] $\beta + \gamma$.

• When $\beta + \gamma > 1$ we find that close to the Fermi-surface there are well-defined quasi-particles, with a linear (i.e. relativistic) dispersion relation and a width which is exponentially suppressed in ω. The precise form of the Green's function is given in eq.(3.88).

• This behaviour undergoes a transition when $\beta + \gamma < 1$. In this case there are no well-defined quasi-particle excitations. Instead the low-energy excitations become very broad with a width which does not vanish at small frequency. See the concluding paragraph of §3.4.5 for more discussion on the Fermi surface.

• The transition region $\beta + \gamma = 1$ is also very interesting. In terms of the parameters α, δ which appear in the Lagrangian for the system, this corresponds to two lines, $\alpha = \pm\delta$. The $\alpha = -\delta$ line corresponds to an extremal RN geometry. The fermionic two-point function in this case is well studied and known to exhibit interesting non-Fermi liquid behaviour. Here we focus on the other case, the $\alpha = \delta$ line, for which the extremal geometry has vanishing entropy and the near-horizon geometry has no scaling symmetry. Despite this difference we find that bulk fermion equation acquires a scaling symmetry analogous to that in the eRN case and the two-point function again exhibits non-Fermi liquid behaviour. The precise form of the Green's function is given in eq.(3.108) and depends on the parameter η. When $2\eta < 1$, there are no well-defined quasi-particles excitations close to the Fermi surface. When $2\eta > 1$, there are well-defined quasi-particle, with a width which vanishes as $\omega \to 0$, although this width can be much broader than in Fermi-liquid theory. When $2\eta = 1$, one

[29] The distance to the horizon for the variable ζ, eq.(3.72), in terms of which the fermion equation of motion becomes of Schrodinger form is governed by $\beta + \gamma$. It is infinite for $\beta + \gamma > 1$, logarithmically infinite for $\beta + \gamma = 1$, and finite for $\beta + \gamma < 1$.

gets a marginal Fermi liquid [30]. The quasiparticles, when they exist for the $\beta+\gamma=1$ case, get very broad when $\beta+\gamma$ becomes less than unity.

• The transition between these behaviours occurs in a smooth way. More precisely, the Green's function evolves in a smooth manner as the parameter $\beta+\gamma$ is varied. The underlying reason for this is that the background geometry itself evolves smoothly.

3.5.2 Discussion

It is worth trying to phrase our results in terms of the semi-holographic description which was proposed in [17]. The near-horizon region of the geometry corresponds to a strongly coupled field theory sector in this description, which is coupled to bulk fermionic excitations localized away from the near-horizon region. The bulk fermionic excitations by themselves are weakly coupled and form a sea which is essentially responsible for the Fermi surface in the boundary theory. The coupling between the two sectors allows the bulk fermions to decay and gives rise to their width. When $\beta+\gamma>1$ this decay width is small, since it arises due to tunneling through a WKB barrier. This results in a width which is highly suppressed with an essential singularity as $\omega\to 0$. As $\beta+\gamma\to 1$ the barrier is lowered and the bulk fermions can decay more easily into degrees of freedom in the strongly coupled sector, resulting in a decay width which is only power law suppressed. Finally, when $\beta+\gamma<1$ the decay process is sufficiently enhanced and leads to a width which is non-vanishing even as $\omega\to 0$, leaving no sharply defined quasi-particles in the excitation spectrum.

In fact in our analysis we did not use the information about the full geometry but only the geometry in the near-horizon region. The full geometry depends on many more details of the model including the dependence, even away from the run-away region where $\phi\to\pm\infty$, of the gauge coupling function and the potential on the dilaton. It is therefore less universal than the near-horizon geometry which is in fact often an attractor. This is very much in the spirit of the semi-holographic description, in effect we only relied on the gravity dual for the strongly coupled sector, and did not use much information about the gravity solution away from the horizon since in the end that would have given rise to a weakly coupled bulk fermion whose dynamics can be understood in field theoretic terms anyways [31].

The basic lesson then from this work is that a range of interesting behaviours can arise by coupling fermions to a strongly coupled sector with a gravitational dual of the kind considered here. This includes both Fermi liquid and non-Fermi liquid behaviour, transitions between them, and transitions from a non-Fermi liquid state to one where there are no well-defined quasi-particles since the excitations have become very broad and essentially disappeared. Moreover, this can happen when the strongly coupled sector has reasonable

[30] One difference with the eRN case is that the mass and charge of the bulk fermion enters in η only through their dependence on the Fermi-momentum, k_F.

[31] Some examples of gravity solutions which interpolate between AdS_4 and the solution eq.(3.13) in the near-horizon region are discussed in §3.3. These are obtained with reasonable potentials and gauge coupling functions. It is worth studying these examples further to calculate the value of k_F (for $\beta+\gamma\geq 1$) and the the value of the residue and v_F in eq.(3.108), (3.117), (3.123), in them.

thermodynamics behaviour consistent in particular with the third law of thermodynamics, since the gravity background has vanishing entropy at extremality.

An important feature about our system is that the dramatic changes in the behaviour of the fermionic Green's function which we have found are *not* accompanied by any phase transition or significant changes in the thermodynamics or transport properties. The entropy densty or specific heat, for example, scale as given by eq.(3.27) and smoothly changes as $\beta + \gamma$ is lowered from a value greater than unity to less than unity. Similarly, the DC or optical conductivity also changes smoothly, eq.(3.44), eq.(3.43), eq.(3.45). In fact the background geometry itself changes smoothly, as was mentioned above, this is the root cause for the smooth behaviour in transport and conductivity. On general grounds the gravity system should correspond to a strongly coupled field theory in the large N limit. In this limit there are many extra degrees of freedom besides the fermionic ones we have focussed on. And these extra degrees of freedom do not undergo any significant change in their properties even though the fermionic ones we have focussed on do, resulting in the smooth changes in thermodynamics and transport.

The large N limit is the price we pay for the having a tractable gravity description. At finite N one would expect that the transitions seen in the behaviour of the fermion correlator will also manifest itself in phase transitions or big qualitative changes in thermodynamics and transport. Preliminary evidence for this is the fact that the conductivity in our set up already has $1/N$ corrections which see the changes in the nature of the fermion two-point function. This was investigated in [8], [9], where it was found that for a Green's function of the type in eq.(3.108) there would be corrections to conductivity of the form $\sigma \sim \frac{1}{N}T^{-2\eta}$. Since we have found non-Fermi liquid behaviour to arise from in a wide variety of gravitational backgrounds it is reasonable to hope that it will persist for some finite N strongly coupled theories as well.

There are several directions for future work. It will be interesting to generalize these investigations to higher dimensions [32].

Going beyond effective field theory, it is important to try and embed the class of gravity systems studied here in string/M theory. This would put constraints on allowed values of α, β and also the charges and masses for the Fermion fields which determine k_F and the exponent η in eq. (3.108). Allowed ranges of these parameters would then determine which kinds of non-Fermi liquid theories are theoretically speaking allowed and when transitions of various kinds are allowed. Embedding in string/M theory is also important for deciding whether our approximation of classical two -derivative gravity is a controlled one, as was discussed in §3.2.4. For some progress towards providing such embeddings see [35], [36] [33]. It will also be useful to ask whether this analysis can be extended beyond the case where

[32]See for example [34].

[33]For example, our IR effective action (3.1) with parameter $\alpha = \sqrt{3}, \delta = -1/\sqrt{3}, V_0 = -12\sqrt{3}$ can be obtained from M-theory on Sasaki-Einstein space from eq.(4.3) of [35], by setting $\chi = 0$ and $h \equiv \pm 1 \mp e^{4\phi/\sqrt{3}}$, in the regime where $|e^{4\phi/\sqrt{3}}| \ll 1$. However, the near horizon behaviour of the numerically obtained solution in [35] is different from our solution (3.14), (3.15). It is worth studying this point further.

the gravity theory is analyzed in the two-derivative approximation for example in Vasiliev theory [37].

Another direction would be to couple charged matter and study superconducting instabilities [38] along the lines of [11]. Or to allow for a bulk Fermi sea in the near-horizon region and incorporate the changes this leads to [25], [39], [40].

Investigating transitions of the kind we have found in the presence of a magnetic field would also be an interesting extension. In this context it would be natural to also include an axion in the bulk theory, [14].

Finally, only a very small class of possible attractor geometries have been studied here [34]. There is clearly a vast zoo waiting to be explored and the behaviour of fermions in these additional backgrounds might hold even more surprises.

[34]Some references pertaining to the attractor mechanism are, [26], [41], [42], [43], [44], and more recently [45].

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Chapter 4

Supersymmetric States in Large N Chern-Simons-Matter Theories

4.1 Introduction

The coupling constant of a four dimensional gauge theory coupled to matter generically runs under the renormalization group. While it is sometimes possible to choose the matter content and couplings of the theory so that the gauge β function vanishes, such choices are very special. In three dimensions, on the other hand, gauge fields are naturally self coupled by a Chern-Simons type action. As the coefficient of the Chern-Simons term in the action is forced by gauge invariance to be integrally quantized, the low energy gauge coupling (inverse of coefficient of the Chern-Simons term) cannot be continuously renormalized and so does not run under the renormalization group. All these statements are for every choice of matter content and couplings of the theory. As a consequence CFTs are much easier to construct starting with Chern-Simons coupled gauge fields in $d = 3$ than with Yang Mills coupled gauge fields in $d = 4$ [1, 2, 3, 4, 5].

Precisely because the coefficient, k, of the Chern Simons term is an integer, the Chern-Simons coupling cannot be varied continuously. The set of Chern-Simons CFTs obtained, by varying a given Lagrangian over the allowed values of k, yields a sequence rather than a fixed line of CFTs. Consider, however an $SU(N)$ Chern Simons theory at level k. Such a theory admits a natural 't Hooft limit in which we take $N \to \infty$, $k \to \infty$ with $\lambda = \frac{N}{k}$ held fixed. As explained by 't Hooft, λ is the true loop counting parameter or coupling constant in this limit. Several physical quantities - like the spectrum of operators with finite scaling dimension- are smooth functions of λ. Now a unit change in k changes λ by $-\frac{\lambda^2}{N}$, a quantity that is infinitesimal in the large N limit. As a consequence, even though k and N are both integers, λ is an effectively continuous parameter in the large N limit. Effectively, the discretum of Chern-Simon-matter CFTs at finite N merges into an effective fixed line of Chern-Simon-matter theories at large N, parameterized by the effectively continuous variable λ.

Lines of fixed points of large N CFTs map to families of theories of quantum gravity, under the AdS/CFT correspondence [6]. CFTs at weak or finite 't Hooft coupling λ are generically expected to map to relatively complicated higher spin theories of gravity [7, 8, 9] or string theories on AdS spaces of string scale radii. In many examples of explicit realization of AdS/CFT, the bulk description simplifies in some manner at strong λ. It is then natural to ask whether the large class of fixed lines of Chern-Simons-matter theories admit simple dual descriptions at large λ [5]. The first explicit realization of the gravity dual of a large N Chern-Simons-matter theory, as a critical string theory, was achieved by ABJM [10]. At infinitely strong coupling the ABJM Chern-Simons-matter theory develops a supergravity dual description, which is a considerable simplification over the highly curved stringy dual description at finite coupling. A direct field theoretic hint for the nature of the dual of ABJM theory [10] at strong coupling comes from the observation that the set of single trace supersymmetric states in ABJM theory have spins ≤ 2 (and in fact match the spectrum of supergravitons of IIA theory on $AdS_4 \times CP^3$).

While many examples of gravity duals of supersymmetric Chern-Simons-matter theories have been proposed following the work of ABJM (see for instance [11, 12, 13, 14, 15, 16]), essentially all such proposals involve quiver type matter content in the field theory. The gravity duals of seemingly simpler Chern-Simons-matter fixed points, both with and without supersymmetry, remain unknown (and may well be most interesting in the non supersymmetric context). On the other hand, it is of significant interest to find the CFT duals of gravity theories in AdS_4 with as few four-dimensional bulk fields (apart from gravity itself) as possible, and one may hope that the Chern-Simons theories with simple matter content are good candidates.

In order to maintain a degree of technical control, however, here we study only supersymmetric theories with at least four supercharges. We will consider large N $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Chern Simons theories with a single $U(N)$ gauge group and g adjoint chiral multiplets (for all integer g). Such theories have been studied perturbatively in [5, 17] (see also [18]). We will study theories both with and without superpotentials. We address and largely answer the following question: what is the spectrum of supersymmetric operators as a function of the 't Hooft coupling λ? In the rest of this introduction we elaborate on our motivation for asking this question.

The results we find for the supersymmetric spectrum of several fixed lines is quite simple, and has several intriguing features. As we describe in more detail below, an important difference between some theories with $\mathcal{N} = 2$ supersymmetry and theories with $\mathcal{N} = 3$ and higher supersymmetry is that the R-charge of the chiral multiplets of the $\mathcal{N} = 2$ theories (and hence the charges and dimensions of supersymmetric operators) may sometimes be continuously renormalized as a function of λ [5] (and sometimes not [17]). Luckily, Jafferis [19] has presented a proposal that effectively allows the computation of the R-charge as a function of λ in many of these theories. We assume the correctness of Jafferis' proposal and proceed to use a combination of analytic and numerical techniques to present a complete qualitative picture of this R-charge as a function of λ and the number of chiral multiplets g.

While we hope that our results will eventually inspire conjectures for relatively simple large λ descriptions of some of the theories we study, in no case that we have studied have our results proven familiar enough to already suggest a concrete conjecture for the dual description of large λ dynamics. Of the supersymmetric spectra we encounter here, the one that appears most familiar is the spectrum of the $\mathcal{N} = 3$ theory with two chiral multiplets. As we will describe later, this spectrum includes only states of spins ≤ 2, and so might plausibly agree with the spectrum of some supergravity compactification: however a detailed study of the spectrum as a function of global charges reveals some unexpected features that has prevented us (as yet) from finding a supergravity compactification with precisely this spectrum. The spectrum of supersymmetric states in $\mathcal{N} = 2$ deformations of the $\mathcal{N} = 3$ theory also has similar features. We hope to return to an investigation into the possible meanings of these spectra in the future.

Upon completion of this work, we received [20] which overlaps with section 4 of this chapter.

4.2 $\mathcal{N} = 2, 3$ superconformal algebras and their unitary representations

In order to lay out the background (and notation) for our analysis of supersymmetric states, in this section we present a brief review of the structure of the $\mathcal{N} = 2, 3$ superconformal algebras, their unitary representations and their Witten indices[1]. We also explicitly decompose every representation of the relevant superconformal algebras into irreducible representations of the conformal algebra. The paper [21] is useful background material for this section.

4.2.1 The superconformal algebras and their Witten indices

In this section we briefly review the representation theory of the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ superconformal algebras. The bosonic subalgebras of these Lie super algebras is given by $SO(3,2) \times SO(2)$ (for $\mathcal{N} = 2$) and $SO(3,2) \times SO(3)$ (for $\mathcal{N} = 3$). Primary states of these algebras are labeled by (Δ, j, h) where Δ is the scaling dimension of the primaries, j is its spin and h is its R-charge (or R symmetry highest weight). h can be any positive or negative real number for $\mathcal{N} = 2$, but is a positive half integer for $\mathcal{N} = 3$.

The labels of unitary representations of the superconformal algebra obey the inequalities forced by unitarity. When $j \neq 0$ the condition

$$\Delta \geq |h| + j + 1$$

[1] This section was worked out in collaboration with Jyotirmoy Bhattacharya.

is necessary and sufficient for unitarity. When $j = 0$ unitary representations occur when

$$\Delta = |h|, \quad (|h| \geq \frac{1}{2})$$

or when

$$\Delta \geq |h| + 1$$

The representations saturating the above bounds are all short.

The Witten index $\text{Tr}(-1)^F x^{\Delta+j}$ vanishes on all long representations of the supersymmetry algebra but is nonzero on short representations. This index captures information about the state content of a conformal field theory. The only way that the Witten index of a CFT can change under continuous variations of parameters (like the parameter λ in our theory), is for the R-charge to be renormalized as a function of that parameter. Note that the R-charge is fixed to be a half integer at $\mathcal{N} = 3$, but can in principle be continuously renormalized at $\mathcal{N} = 2$.

In the case of $\mathcal{N} = 2$ theories, we have glossed over a detail. At the purely algebraic level, in this case, there are really two independent Witten indices; $\mathcal{I}_+$ and $\mathcal{I}_-$. These are defined as

$$\mathcal{I}_+ = \text{Tr}(-1)^F x^{H+J} e^{-\beta(H-J-R)}$$

and

$$\mathcal{I}_- = \text{Tr}(-1)^F x^{H+J} e^{-\beta(H-J+R)}$$

respectively. We used the notation H for the dilatation operator, J the third component of the spin, and R the R symmetry generator. The indices above are distinct, even though they both evaluate to quantities independent of β. The first index receives contributions only from states with $\Delta = j + h$; all such states are annihilated by and lie in the cohomology of the supercharge with charges $(\frac{1}{2}, -\frac{1}{2}, 1)$. The second index receives contributions only from states with $\Delta = j - h$; all such states are annihilated by and lie in the cohomology of the supercharge with charges $(\frac{1}{2}, \frac{1}{2}, 1)$. The existence of two algebraically independent Witten Indices is less useful than it might, at first seem, in the study of quantum field theories, as the two indices are closely linked by the requirement of CPT invariance.

4.2.2 State content of all unitary representations of the $\mathcal{N} = 2$ superconformal algebra

In the rest of this section we will list the conformal representation content of all unitary representations of the superconformal algebra, and use our listing to compute the index of all short representations of this algebra.

To start with, we present a group theoretic listing of the state content of an antisymmetrized product of supersymmetries. This is given in Table (4.1).

The conformal primary content of a long representation of the superconformal algebra is given by the Clebsh Gordon product of the state content of the product of susy generators

Operator	States
I	$(0,0,0)$
Q	$(\frac{1}{2},\frac{1}{2},1)$, $(\frac{1}{2},\frac{1}{2},-1)$
Q^2	$(1,0,2)$, $(1,0,-2)$, $(1,0,0)$, $(1,1,0)$
Q^3	$(\frac{3}{2},\frac{1}{2},1)$, $(\frac{3}{2},\frac{1}{2},-1)$
Q^4	$(2,0,0)$

Table 4.1: A decomposition of the antisymmetrized products of supersymmetries into irreducible representations of the maximal compact bosonic subgroup of the relevant superalgebras. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation. The same labeling convention is used in all tables in this section.

Primary	Conformal content	Index
(Δ,j,h)	(Δ,j,h), $(\Delta+\frac{1}{2},j+\frac{1}{2},h+1)$, $(\Delta+\frac{1}{2},j-\frac{1}{2},h+1)$, $(\Delta+\frac{1}{2},j+\frac{1}{2},h-1)$, $(\Delta+\frac{1}{2},j-\frac{1}{2},h-1)$, $(\Delta+1,j,h+2)$, $(\Delta+1,j,h-2)$, $(\Delta+1,j+1,h)$, $2(\Delta+1,j,h)$, $(\Delta+1,j-1,h)$, $(\Delta+\frac{3}{2},j+\frac{1}{2},h+1)$, $(\Delta+\frac{3}{2},j-\frac{1}{2},h+1)$, $(\Delta+\frac{3}{2},j+\frac{1}{2},h-1)$, $(\Delta+\frac{3}{2},j-\frac{1}{2},h-1)$, $(\Delta+2,j,h)$	0
$(\Delta,0,h)$	$(\Delta,0,h)$, $(\Delta+\frac{1}{2},\frac{1}{2},h+1)$, $(\Delta+\frac{1}{2},\frac{1}{2},h-1)$, $(\Delta+1,0,h+2)$, $(\Delta+1,0,h-2)$, $(\Delta+1,1,h)$, $2(\Delta+1,0,h)$, $(\Delta+\frac{3}{2},\frac{1}{2},h+1)$, $(\Delta+\frac{3}{2},\frac{1}{2},h-1)$, $(\Delta+2,0,h)$	0

Table 4.2: Conformal primary content of long representations. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

above with that of the primary. We list the conformal primary content of an arbitrary long representation in Table (4.2).

Note that a long representation of the susy algebra decomposes into 16 long representations of the conformal algebra when $j \neq 0$; when $j=0$ we must delete the representations with negative values for the $SO(3)$ highest weight ($j-\frac{1}{2}$ and $j-1$) from the generic j result leaving us with a total of 11 conformal representations. The Witten index of all long representations automatically vanishes.

Let us now turn to the short representations. To start with consider representations with $h \neq 0$, $j \neq 0$ and $\Delta = |h|+j+1$. These representations are short because they include a family of null states. These null states themselves transform in a short representation of the superconformal algebra, with quantum numbers $(|h|+j+\frac{3}{2}, j-\frac{1}{2}, h+1)$ (when h is positive) and $(|h|+j+\frac{3}{2}, j-\frac{1}{2}, h-1)$ (when h is negative). It is not too difficult to verify that the conformal primary content of such a short representation (represented by $\chi_S(j,h)$) and the Witten index of these representations is as given in Table (4.3) (we list the result

Primary	Conformal content	Indices ($h>0$; $h<0$)
$(j+h+1,j,h)$ ($j\neq 0$, $h\neq 0$)	$(j+h+1,j,h)$, $(j+h+\frac{3}{2},j+\frac{1}{2},h+1)$, $(j+h+\frac{3}{2},j+\frac{1}{2},h-1)$, $(j+h+\frac{3}{2},j+\frac{1}{2},h-1)$, $(j+h+2,j,h-2)$, $(j+h+2,j+1,h)$, $(j+h+2,j,h)$, $(j+h+\frac{5}{2},j+\frac{1}{2},h-1)$	$\mathcal{I}_-=0$, $\mathcal{I}_+=(-1)^{2j+1}\frac{x^{2j+h+2}}{1-x^2}$; $\mathcal{I}_+=0$, $\mathcal{I}_-=(-1)^{2j+1}\frac{x^{2j-h+2}}{1-x^2}$

Table 4.3: Conformal primary content and index of generic short representation. Representations are labeled by (Δ,j,h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

Primary	Conformal content	Index
$(j+1,j,0)$	$(j+1,j,0)$, $(j+\frac{3}{2},j+\frac{1}{2},1)$, $(j+\frac{3}{2},j+\frac{1}{2},-1)$, $(j+2,j+1,0)$	$\mathcal{I}_+=\mathcal{I}_-=(-1)^{2j+1}\frac{x^{2j+2}}{1-x^2}$

Table 4.4: Conformal primary content and index for $(j+1,j,0)$ representation. Representations are labeled by (Δ,j,h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

for positive h; the result for negative h follows from symmetry).[2] Note that all 8 conformal primaries that occur in this decomposition are long (recall we have assumed $h\neq 0$).

It is not difficult to verify that $\chi_L(h+j+1,j,h)=\chi_S(j,h)+\chi_S(j-\frac{1}{2},h+1)$. This expresses the fact that the state content of a long representation just above unitarity is equal to the sum of the state content of the short representation it descends to plus the state content of the short representation of null states.

Let us now turn to the special case of short representation $(j+1,j,0)$. The null states of this representation consist of a sum of two irreducible representations with quantum numbers $(j+\frac{3}{2},j-\frac{1}{2},1)$ and $(j-\frac{3}{2},j-\frac{1}{2},-1)$. It is not too difficult to convince oneself that the primary content of such a short representation is as given in Table (4.4). Note that all 4 conformal representations that appear in this split are short.[3] It may be verified[4] that $\chi_L(j+1,j,0)=\chi_S(j+1,j,0)+\chi_S(j+\frac{3}{2},j-\frac{1}{2},1)+\chi_S(j+\frac{3}{2},j-\frac{1}{2},-1)$.

Let us next turn to the special case of a short representation with $j=0$ and with

[2]The Witten index of these representations may be evaluated as follows. When $h>0$ there are no states with $\Delta=j-h$ and so $\mathcal{I}_-=0$. States with $\Delta=j+h$ occur only in the representation $(j+h+\frac{3}{2},j+\frac{1}{2},h+1)$ and we find $\mathcal{I}_+=(-1)^{2j+1}\frac{x^{2j+h+2}}{1-x^2}$ where we have used the spin statistics theorem to assert that the fermion number of a primary of angular momentum j is $(-1)^{2j}$. Similarly when $h<0$ we have $\mathcal{I}_+=0$ and $\mathcal{I}_-=(-1)^{2j+1}\frac{x^{2j-h+2}}{1-x^2}$.

[3]The Witten index of these representations may be evaluated as follows. States with $\Delta=j+h$ occur only in the representation $(j+\frac{3}{2},j+\frac{1}{2},1)$ while states with $\Delta=j-h$ occur only in the representation $(j+\frac{3}{2},j+\frac{1}{2},-1)$. The Witten indices of this representation are given by $\mathcal{I}_+=\mathcal{I}_-=(-1)^{2j+1}\frac{x^{2j+2}}{1-x^2}$.

[4]In order to perform this verification, it is important to recall that $\chi_L(j+1,j,0)$ is the sum of 16 *long* characters of the conformal group, 4 of which are at the unitarity threshold. Equivalently we may write this as the sum of $12+4=16$ long conformal characters and 4 short conformal characters (where we have used the fact that a long conformal character, at its unitarity bound, decomposes into the sum of a short and a long character). The 4 short characters in this decomposition simply yield $\chi_S(j+1,j,0)$ above, while the 16 long characters constitute $\chi_S(j+\frac{3}{2},j-\frac{1}{2},1)+\chi_S(j+\frac{3}{2},j-\frac{1}{2},1)$.

Primary	Conformal content	Index($h > 0$; $h < 0$)
$(h+1, 0, h)$	$(h+1, 0, h)$, $(h+\frac{3}{2}, \frac{1}{2}, h+1)$, $(h+\frac{3}{2}, \frac{1}{2}, h-1)$, $(h+2, 0, h-2)$, $(h+2, 1, h)$, $(h+2, 0, h)$, $(h+\frac{5}{2}, \frac{1}{2}, h-1)$	$\mathcal{I}_- = 0$, $\mathcal{I}_+ = -\frac{x^{h+2}}{1-x^2}$; $\mathcal{I}_+ = 0$, $\mathcal{I}_- = -\frac{x^{-h+2}}{1-x^2}$

Table 4.5: Conformal content and index of representation $(h+1, 0, h)$. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

Primary	Conformal content	Index
$(1, 0, 0)$	$(1, 0, 0)$ $(\frac{3}{2}, \frac{1}{2}, 1)$, $(\frac{3}{2}, \frac{1}{2}, -1)$ $(2, 0, 0)$, $(2, 1, 0)$	$\mathcal{I}_+ = \mathcal{I}_- = -\frac{x^2}{1-x^2}$

Table 4.6: Conformal content and index for representation $(1, 0, 0)$. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

quantum numbers $(h+1, 0, h)$. Such representations are often referred to as semi short, to distinguish them from the 'short' $j = 0$ representations we will deal with next. We will deal with the case $h > 0$ (the results for $h < 0$ can then be deduced from symmetry). The primary for the null states of this representation has quantum numbers $(h+2, 0, h+2)$. Note that the null states transform in an isolated short representation. The state content and Witten index of a semishort $j = 0$ representation are listed in Table (4.5). Note that $\chi_L(h+1, 0, h) = \chi_S(h+1, 0, h) + \chi_S(h+2, 0, h+2)$; this formula captures the split of a long representation into the short representation and null states.

The short representation with primary labels $(1, 0, 0)$ is a bit special; its null states have primaries with quantum numbers $(2, 0, 2)$ and $(2, 0, -2)$ (these are isolated short representations, see below). The state content and Witten index of this representation are given in Table (4.6). Of the 5 conformal primaries that appear in this split, only the representation with quantum numbers $(2, 1, 0)$ is short.[5] Using the results we present below, it is possible to verify the character decomposition rule:[6]

$$\chi_L(1, 0, 0) = \chi_S(1, 0, 0) + \chi_S(2, 0, 2) + \chi_S(2, 0, -2).$$

Now let us turn to the isolated short representations $(h, 0, h)$ for $|h| \geq 1$. The primaries for the null states of these representations have quantum numbers $(h+\frac{1}{2}, \frac{1}{2}, h+1)$. The null states transform in a (short) non-unitary representation, reflecting the fact that the isolated

[5] The conformal representation $(1, 0, 0)$ and $(\frac{3}{2}, \frac{1}{2}, \pm 1)$ are not short as spin 0 and spin $\frac{1}{2}$ are exceptions to the general rule.

[6] The character on the LHS is a sum of 10 long conformal representations or $11 = 5 + 3 + 3$ short conformal representations (the extra representation is the conformal shortening vector of $(2, 1, 0)$ and is given by $(3, 0, 0)$). States with $\Delta = j + h$ occur only in the representation $(\frac{3}{2}, \frac{1}{2}, 1)$ while states with $\Delta = j - h$ occur only in the representation $(\frac{3}{2}, \frac{1}{2}, -1)$.

Primary	Conformal content	Index($h>0$; $h<0$)
$(h,0,h)$	$(h,0,h)$, $(h+\frac{1}{2},\frac{1}{2},h-1)$, $(h+1,0,h-2)$	$\mathcal{I}_-=0$, $\mathcal{I}_+=\frac{x^h}{1-x^2}$; $\mathcal{I}_+=0$, $\mathcal{I}_-=\frac{x^{-h}}{1-x^2}$

Table 4.7: Conformal content and index for representation $(h,0,h)$.

Primary	Conformal content	Index
$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2})$, $(1,\frac{1}{2},-\frac{1}{2})$	$\mathcal{I}_-=0$, $\mathcal{I}_+=\frac{x^{\frac{1}{2}}}{1-x^2}$
$(\frac{1}{2},0,-\frac{1}{2})$	$(\frac{1}{2},0,-\frac{1}{2})$ $(1,\frac{1}{2},\frac{1}{2})$	$\mathcal{I}_+=0$, $\mathcal{I}_-=\frac{x^{\frac{1}{2}}}{1-x^2}$

Table 4.8: Conformal content and index for representations $(\frac{1}{2},0,\frac{1}{2})$ and $(\frac{1}{2},0,-\frac{1}{2})$

representations cannot be regarded as a limit of unitary long representations but can be regarded as the limit of non-unitary long reps. The conformal primary content and Witten indices for these representations are given in Table (4.7). Here we have written conformal content for h positive; the result for negative h is given by symmetry. Recall that $h\geq\frac{1}{2}$ for the representations we have just discussed. The lower bound of this inequality, $h=\frac{1}{2}$, is a special case. The conformal decomposition and index for the $h=\frac{1}{2}$ and $-\frac{1}{2}$ cases are given in Table (4.8).

4.2.3 Decomposition of all unitary representations of the $\mathcal{N}=3$ algebra into $\mathcal{N}=2$ representations

In this subsection we record the decomposition of all $\mathcal{N}=3$ representations into $\mathcal{N}=2$ representations. Representations of the $\mathcal{N}=3$ algebra are labeled as (Δ,j,h), where h is the highest weight under the Cartan of the $SO(3)$ R symmetry (normalized to be a half integer). A generic $\mathcal{N}=3$ long representation with $j\neq 0$ breaks as follows

$$(\Delta,j,h)_3=\bigoplus_{m=-h}^{h}\left[(\Delta,j,m)_2\oplus(\Delta+\frac{1}{2},j+\frac{1}{2},m)_2\oplus(\Delta+\frac{1}{2},j-\frac{1}{2},m)_2\oplus(\Delta+1,j,m)_2\right]. \quad (4.1)$$

where $\oplus$ denotes a direct sum and $\bigoplus_{m=-h}^{h}$ denotes the direct sum of representations.

Here the subscript denotes $\mathcal{N}$ of the algebra; i.e. $()_3$ denotes a representation of the $\mathcal{N}=3$ algebra, while $()_2$ denotes a representation of the $\mathcal{N}=2$ algebra.

The summation outside the brackets on the RHS of (4.1) reflects the fact that a primary that transforms in a given irreducible $SO(3)$ (R symmetry in $\mathcal{N}=3$ algebra) representation consists of several different $SO(2)$ primaries (with distinct R-charges). The four terms in the bracket on the RHS of (4.1) represent the states obtained by acting on the $\mathcal{N}=3$ primary with supercharges that belong to the $\mathcal{N}=3$ algebra, but are absent in the $\mathcal{N}=2$ algebra.

Spin j	$\mathcal{N}=3$ primary	$\mathcal{N}=2$ primaries
$j\neq 0$	$(j+h+1,j,h)$	$\bigoplus_{m=-h}^{h}\Big[(j+h+1,j,m)\oplus$ $(j+h+\frac{3}{2},j+\frac{1}{2},m)\Big]$
Generic short $j=0$	$(h+1,0,h)$	$\bigoplus_{m=-h}^{h}\Big[(h+1,0,m)\oplus(h+\frac{3}{2},\frac{1}{2},m)\Big]$ $\oplus\bigoplus_{m=-(h+2)}^{h+2}(h+2,0,m)$
Isolated short $j=0$	$(h,0,h)$	$\bigoplus_{m=-h}^{h}(h,0,m)$

Table 4.9: Decomposition of short $\mathcal{N}=3$ representations into $\mathcal{N}=2$ representations.

The decomposition (4.1) may be rewritten as follows:

$$(j+h+1+\epsilon,j,h)_3=\bigoplus_{m=-h}^{h}\Big[(j+h+1+\epsilon,j,m)_2\oplus(j+h+\frac{3}{2}+\epsilon,j+\frac{1}{2},m)_2\Big]$$
$$\oplus\bigoplus_{m=-(h+1)}^{h+1}\Big[(j+h+\frac{3}{2}+\epsilon,j-\frac{1}{2},m)_2\oplus(j+h+2+\epsilon,j,m)_2\Big]. \tag{4.2}$$

In this equation we have grouped together terms on the RHS for the following reason. Recall that the decomposition of a long representation - with $j\neq 0$ - at the unitarity bound, into short unitary representations of the superconformal algebra, is given both at $\mathcal{N}=3$ and at $\mathcal{N}=2$ by

$$(j+h+1+\epsilon,j,h)\stackrel{\epsilon\to 0}{\longrightarrow}(j+h+1,j,h)\oplus(j+h+\frac{3}{2},j-\frac{1}{2},h+1) \tag{4.3}$$

This formula should apply to (4.2). Comparing (4.3) and (4.2), it is plausible (and correct) that the generic short $\mathcal{N}=3$ representation decomposes into $\mathcal{N}=2$ representations according to the formula

$$(j+h+1,j,h)_3=\bigoplus_{m=-h}^{h}\Big[(j+h+1,j,m)_2\oplus(j+h+\frac{3}{2},j+\frac{1}{2},m)_2\Big] \tag{4.4}$$

where all representations that saturate the unitarity bound, on the RHS of (4.4), are short.

We may deduce the split of other short $\mathcal{N}=3$ representations into representations of the $\mathcal{N}=2$ algebra using identical reasoning. The result is given in Table (4.9).

4.3 The R-charge as a function of λ in the absence of a superpotential

The first class of theories we consider consists of $\mathcal{N}=2$ $U(N)$ Chern Simons theories at level k, coupled to g adjoint chiral multiplets with vanishing superpotential. This class of theories was studied, and demonstrated to be superconformal (for all N, k and g) in [5]. In particular, it is superconformal at every value of λ, in the large N limit.

In the free limit the R-charge of the chiral fields in this theory equals $\frac{1}{2}$. However, it was demonstrated in [5] that this R-charge is renormalized as a function of λ; indeed at first nontrivial order in perturbation theory [5] demonstrated that the R-charge of a chiral field is given by

$$h(\lambda) = \frac{1}{2} - (g+1)\lambda^2 \tag{4.5}$$

where $\lambda = \frac{N}{k}$. As the R-charge of a supersymmetric operator plays a key role in determining its scaling dimension (via the BPS formula), the exact characterization of the spectrum of supersymmetric states in this theory at large λ clearly requires control over the function $h(\lambda)$ at large λ. Such control cannot be achieved by perturbative techniques, but is relatively easily obtained by an application of the extremely powerful recent results of Jafferis [19] to this problem. In this section we will adopt a proposal by Jafferis [19] to perform this determination.

4.3.1 The large N saddle point equations

According to the prescription of [19], the superconformal R-charge of the theories we study is determined by extremizing the magnitude of their partition function on S^3 with respect to the trial R-charge,[7] h, assigned to a chiral multiplet. The partition function itself is determined by the method of supersymmetric localization to be given by the finite dimensional integral

$$\mathcal{Z}(h) = \int \prod_{i=1}^{N} du_i \, \exp\left\{ N^2 \left[\frac{i\pi}{\lambda}\frac{1}{N}\sum_i u_i^2 + \frac{1}{N^2}\sum_{i\neq j} \log\sinh\left(\pi u_{ij}\right) + \frac{g}{N^2}\sum_{i,j} \ell(1-h+iu_{ij}) \right] \right\} \tag{4.6}$$

where $\lambda = N/k$ is the 't Hooft coupling, u_i ($i = 1 \dots N$) are real numbers (and the integration range is from $-\infty$ to ∞), $u_{ij} \equiv u_i - u_j$, and the function $\ell(z)$ satisfies $\partial_z \ell(z) = -\pi z \cot(\pi z)$.

According to [19], once the partition function $\mathcal{Z}$ is obtained by performing the integral in (4.6), the R-charge of the chiral fields is determined (up to caveats we will revisit below) by solving the equation $\partial_h |\mathcal{Z}(h)|^2 = 0$. This gives the exact superconformal R-charge.

[7] More precisely, as shown in [19], a supersymmetric theory on S^3 can be defined with an arbitrary choice of R-charge h, and the partition function of this theory on S^3 is $\mathcal{Z}(h)$. The superconformal R-charge is such that $|\mathcal{Z}(h)|^2$ is minimized.

In the large N limit the integral in (4.6) may be determined by saddle point techniques. The saddle point equations, together with the equation $\partial_h|\mathcal{Z}(h)|^2 = 0$ (which determines h, given the saddle point) are given by

$$0 = \frac{iu_k}{\lambda} + \frac{1}{N}\sum_{j(\neq k)}^{N}\left\{\coth(\pi u_{kj}) - \frac{i}{2}g\Big[(1-h+iu_{kj})\cot\pi(1-h+iu_{kj}) \right. \tag{4.7}$$
$$\left. -(1-h-iu_{kj})\cot\pi(1-h-iu_{kj})\Big]\right\},$$

$$0 = \mathrm{Re}\left[\sum_{i,j=1}^{N}(1-h+iu_{ij})\cot\pi(1-h+iu_{ij})\right]. \tag{4.8}$$

4.3.2 Perturbative solution at small λ

While we have been unable to solve the equations (4.7) in general even at large N, it is not difficult to solve these equations either at small λ (at all g) or at large g (for all λ). In this subsection we describe the perturbative solution to these equations at small λ (for all g). In the next subsection we will outline the perturbative procedure that determines $h(\lambda)$ at all λ but large g.

It is apparent from a cursory inspection of (4.7) that the eigenvalues u_i must become small in magnitude (in fact must scale like $\sqrt{\lambda}$) at small λ. It follows that complicated functions of u simplify to their Taylor series expansion in a power expansion in λ. This is the basis of the perturbative technique described in this subsection.

More quantitatively, at small λ we expand u_i and h as

$$u_k = \sqrt{\lambda}\left(u_k^{(0)} + \lambda u_k^{(1)} + \cdots\right), \tag{4.9}$$

$$h = h^{(0)} + \lambda h^{(1)} + \lambda^2 h^{(2)} + \cdots, \tag{4.10}$$

and attempt to solve our equations order by order in λ. At leading nontrivial order, $\mathcal{O}(\lambda^0)$, equation (4.8) reduces to $h^{(0)} = \frac{1}{2}$. On the other hand, equation (4.7) at its leading nontrivial order, namely $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$, reduces to

$$iu_i^{(0)} = -\frac{1}{\pi}\frac{1}{N}\sum_{j(\neq i)}\frac{1}{u_i^{(0)} - u_j^{(0)}}. \tag{4.11}$$

Apart from an unusual factor of i, this is precisely the large N saddle point equations of the Wigner model. The extra factor of i may be dealt with by working with the rescaled variable

$$y_j = e^{-\frac{\pi i}{4}}u_j$$

in terms of which

$$y_i^{(0)} = \frac{1}{\pi}\frac{1}{N}\sum_{j(\neq i)}\frac{1}{y_i^{(0)} - y_j^{(0)}}. \tag{4.12}$$

The solution to this equation is well known in the large N limit. The eigenvalues $y_i^{(0)}$ cluster themselves into a "cut" along the interval $(-a, a)$ with

$$a = \sqrt{\frac{2}{\pi}} \, .$$

The density of eigenvalues, $\rho(y) = \sum_{i=1}^{N} \delta(y - y_i^i)$, in this interval is given by

$$\rho(y) = \frac{2}{\pi a^2}\sqrt{a^2 - y^2}. \tag{4.13}$$

Using the fact that $u \approx e^{\frac{\pi i}{4}}\sqrt{\lambda}y$, we see that, at leading order in λ, the saddle point is given by the eigenvalues u_i clustering along a straight line of length of order $\sqrt{\lambda}$, oriented at 45 degrees in the complex plane.

Note that the distribution of eigenvalues has $u \to -u$ symmetry and in particular the average value of eigenvalues is zero. The $u \to -u$ symmetry is an exact symmetry of the saddle point equations, and we assume that it is preserved in the solution (i.e. not spontaneously broken).

Let us now proceed beyond the leading order. (4.8) is automatically satisfied at $\mathcal{O}(\sqrt{\lambda})$ (this is because $\mathrm{Im}\left[\sum_{i\neq j}(u_i^{(0)} - u_j^{(0)})\right] = 0$). At order λ the same equation reduces to

$$h^{(1)} = -2\,\mathrm{Re}\left[\frac{1}{N^2}\sum_{i\neq j}(u_i^{(0)} - u_j^{(0)})^2\right] \tag{4.14}$$

Now recall that the phase of u_i is $e^{\frac{\pi i}{4}}$. As a consequence the real part vanishes and hence, from (4.14), $h^{(1)} = 0$.

In order to compute the correction to $h(\lambda)$ at $\mathcal{O}(\lambda^2)$ we need to find the first correction u_i^1 to the eigenvalue distribution. At order $\mathcal{O}(\lambda^0)$, by using (4.11) and $\sum_j u_j = 0$, (4.7) reduces to

$$\frac{1}{\pi N}\sum_{j(\neq k)}\frac{1}{(u_k^{(0)} - u_j^{(0)})^2}u_j^{(1)} = \frac{\pi}{6}(3g-2)u_k^{(0)}. \tag{4.15}$$

In order to solve this equation we once again move to the "real" variable y. That is we define $u_j^{(0)} = e^{\frac{\pi i}{4}}y_j$ as above. Let us also define $u_j^{(1)} = e^{\frac{\pi i}{4}}v_j(y_j)$. In the large N limit $u_i^{(0)}$ is effectively a continuous variable on the 45 degree cut on the complex plane, and

$$u^{(1)} = e^{\frac{\pi i}{4}}v(y)$$

for a continuous function $v(y)$ that we now determine. The equation for $v(y)$ is given by

$$\frac{1}{\pi}\mathcal{P}\int\frac{v(y)\rho(y)}{(y_1 - y)^2}dy = \frac{i\pi}{6}(3g-2)y_1. \tag{4.16}$$

Integrating both sides of this equation with respect to y_1 we find

$$\mathcal{P}\int \frac{v(y)\rho(y)}{(y_1-y)}dy = -\frac{i\pi^2}{12}(3g-2)y_1^2 + k_1, \tag{4.17}$$

where k_1 is the constant of integration.

We can solve this integral equation by a similar trick as before. Either by this method or by directly solving for the coefficients for a polynomial ansatz for $v(y)$, we get [8]

$$v(y) = -\frac{i\pi}{12}(3g-2)y. \tag{4.18}$$

With the first nontrivial correction to the eigenvalue distribution in hand, it is now a simple matter to compute the shift in the scaling dimension $h(\lambda)$ at leading order nontrivial order in λ. (4.8) is automatically obeyed at $\mathcal{O}(\lambda^{\frac{3}{2}})$. However at $\mathcal{O}(\lambda^2)$, the same equation yields

$$\mathrm{Re}\left[\sum_{k\neq j} -2\pi^3(u_i^{(0)}-u_j^{(0)})^4 + 3\pi\left(4(u_i^{(0)}-u_j^{(0)})(u_i^{(1)}-u_j^{(1)}) + h^{(2)}\right)\right] = 0 \tag{4.19}$$

In other words

$$\begin{aligned} 3h^{(2)} =& \mathrm{Re}\left[\left[\int dy_1 dy_2 \rho(y_1)\rho(y_2)\left\{-2\pi^2(y_1-y_2)^4 - 12i(y_1-y_2)(v(y_1)-v(y_2))\right\}\right]\right. \\ =& \mathrm{Re}\left[-2\pi^2\left(2\langle y^4\rangle + 6\langle y^2\rangle^2\right) - 24i\langle y\ v(y)\rangle\right] \end{aligned} \tag{4.20}$$

where we have defined

$$\langle O(y)\rangle \equiv \int O(y)\rho(y)dy.$$

Evaluating the integrals we find

$$h^{(2)} = -(1+g) \tag{4.21}$$

This exactly matches the explicit perturbative result of Gaiotto and Yin [5]. Similar agreement was also found with [26].

The method presented here is easily iterated to higher orders in λ. We have explicitly implemented this perturbative procedure to a few orders in perturbation theory. At $g=1$ we find

$$h = \frac{1}{2} - 2\lambda^2 + \frac{13\pi^2}{3}\lambda^4 - \left(\frac{207\pi^4}{10} - 32\pi^2\right)\lambda^6 + \left(\frac{339019\pi^6}{2520} - \frac{3355\pi^4}{9} + \frac{160\pi^2}{3}\right)\lambda^8 + \cdots \tag{4.22}$$

[8] This correction to the eigenvalue distribution tilts the eigenvalue cut - originally at 45 degrees in the complex plane - a little nearer to the real axis.

while for general g we have

$$\begin{aligned} h &= \frac{1}{2} - (1+g)\lambda^2 + \frac{1}{12}(1+g)\left[-24(-1+g) + \pi^2(3g^2+15g+8)\right]\lambda^4 \\ &+ \Big[-8 - 25\frac{\pi^2}{3} - \frac{61\pi^4}{60} + g(-\frac{4\pi^2}{3} - \frac{637\pi^4}{120}) + g^2(8 + \frac{64\pi^2}{3} - \frac{395\pi^4}{48}) \\ &+ g^3(\frac{52\pi^2}{3} - \frac{239\pi^4}{48}) + g^4(3\pi^2 - \frac{53\pi^4}{48}) - g^5\frac{\pi^4}{16}\Big]\lambda^6 + \cdots \end{aligned}$$

4.3.3 Perturbative solution at large g

As we have described in the previous subsection, in the limit of small 't Hooft coupling the solution to the saddle point equation, (4.7), is obtained by balancing the first term (large because of the inverse power of λ) against the second (large because of the singularity at small u), and treating the third term as a perturbation.

Let us now consider another limit; one in which the number of flavours g becomes large, without making any assumptions on λ. In this case the small u singularity of the second term in (4.7) balances against either the largeness of g (in the third term) or the smallness of λ (in the first term). The important point is that u is necessarily small for this balance to work, so perturbative techniques apply. The most interesting regime is one in which $\lambda = \mathcal{O}(1/g)$. The perturbative procedure proceeds exactly as in the previous subsection.

We get, at leading order

$$h(\lambda) = \frac{1}{2} - \frac{g\pi^2}{\left(\frac{g\pi^2}{2}\right)^2 + \left(\frac{\pi}{\lambda}\right)^2} + \text{higher order}. \tag{4.23}$$

4.3.4 Numerical study of R-charge for small g and large λ

In the previous subsections we used perturbative techniques to establish the following. The function $h(\lambda)$ starts out at the value $\frac{1}{2}$ at $\lambda = 0$. It always decreases at small λ; at leading order $h(\lambda) = \frac{1}{2} - (g+1)\lambda^2$. What happens at larger values of λ ? This question turned out to be easy to answer at large g. In this limit the decrease of $h(\lambda)$ as a function of λ stops at $\lambda \sim \frac{1}{g}$, after which $h(\lambda)$ settles down at its asymptotic value

$$h(\lambda) = \frac{1}{2} - \frac{4}{\pi^2 g} + \mathcal{O}(\frac{1}{g^2}). \tag{4.24}$$

Neither of the analyses we have performed, however, reliably predict the behaviour of $h(\lambda)$ at large λ when g is of order unity. An unjustified extrapolation of (4.24) suggests that $h(\lambda)$ always monotonically decreases, asymptoting to a constant value at $\lambda = \infty$. In order to check whether all this is really the case we have numerically solved a discretized version of the saddle point equations with Ne eigenvalues using Mathematica. [9]

[9] In the rest of this chapter Ne denotes the number of eigenvalues used for the purposes of discretized numerics. Ne is a composite symbol - (it is not equal to product of N with e).

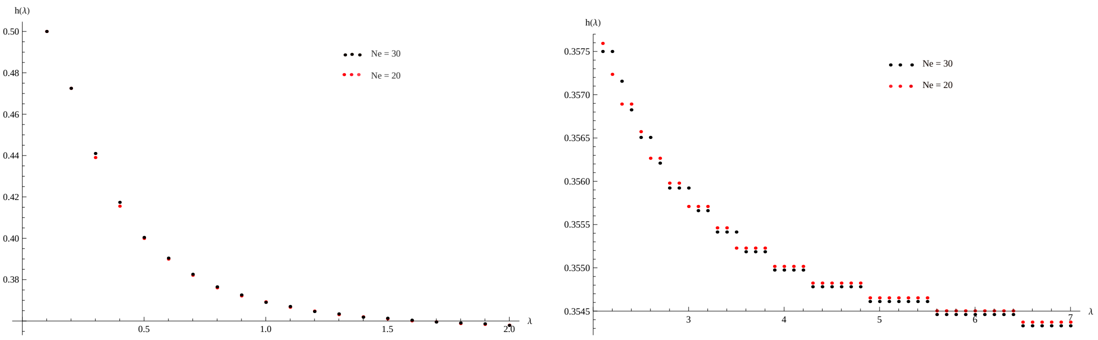

Figure 4.1: h vs. λ for $g = 3$, $Ne = 20$ and $Ne = 30$. In the figure on the left, λ varies from 0 to 2. In the figure on the right λ varies from 2 to 7. Note that $h(\lambda)$ scale is different in the two figures. R-charge saturates to around 0.354. While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.

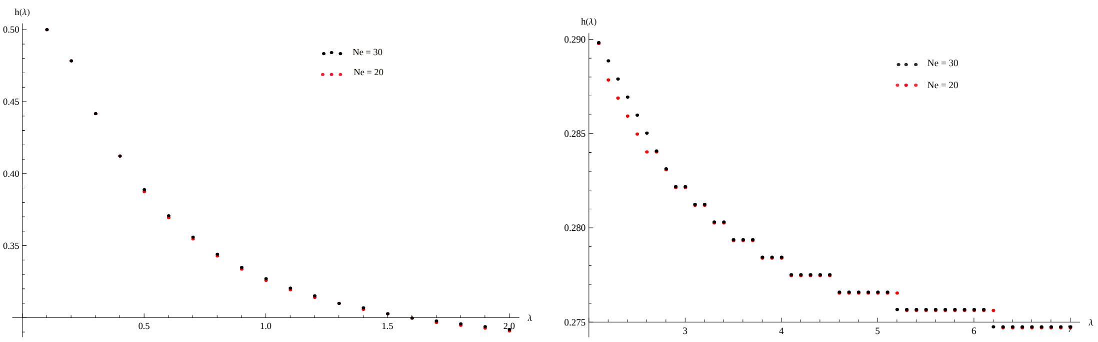

Figure 4.2: h vs. λ for $g = 2$, $Ne = 20$ and $Ne = 30$. In the figure on the left, λ varies from 0 to 2. In the figure on the right λ varies from 2 to 7. Note that $h(\lambda)$ scale is different in the two figures. R-charge saturates to around 0.274.While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.

At $g = 3$ the function is as given by the graph in Fig. 4.1. Note that $h(\lambda)$ asymptotes to almost a constant value by $\lambda = 2$, and varies only slightly in the λ range 2 to 7 (this constant is approximately 0.354).

At $g = 2$ we find the function $h(\lambda)$ given by Fig. 4.2. Note that $h(\lambda)$ asymptotes to almost a constant value by $\lambda = 2$, and varies only slightly in the λ range 2 to 7 (the asymptote value is approximately 0.274).[10]

As is apparent, in both these cases the function $h(\lambda)$ displays the qualitative behaviour predicted by the large g formula (4.23); $h(\lambda)$ monotonically decreases from $h = \frac{1}{2}$ at $\lambda = 0$ to a finite value of h (greater than $\frac{1}{4}$) at $\lambda = \infty$. Notice that at $g = 3$, $\frac{1}{2} - h(\infty) = \frac{1}{2} - .354 \approx 0.146$. This shift from $\frac{1}{2}$ agrees to 10 percent with the first order prediction of large g perturbation theory (4.24), $\frac{4}{3\pi^2} = 0.135$. At $g = 2$, $1 - h(\infty) = \frac{1}{2} - .274 = 0.226$ which

[10]For both $g = 2$ and $g = 3$, we have also computed $h(\infty)$ directly both at $Ne = 20$ and at $Ne = 30$ simply by setting $\lambda = \infty$ in equations (4.7) and (4.8). In order to do this in practice we used our $\lambda = 7$ results for the eigenvalues as a trial guess in Mathematica's equation solver routine. From this direct analysis we once again found $h(\infty) = 0.354$ for $g = 3$ and $h(\infty) = 0.273$ for $g = 2$.

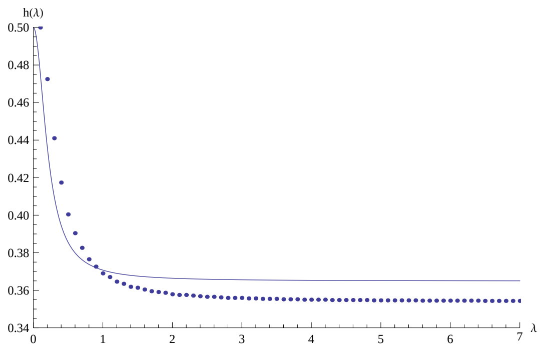

Figure 4.3: h vs. λ for $g = 3$ and $Ne = 30$. The blue line is large g perturbation theory prediction. R-charge saturates to around 0.354. While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.

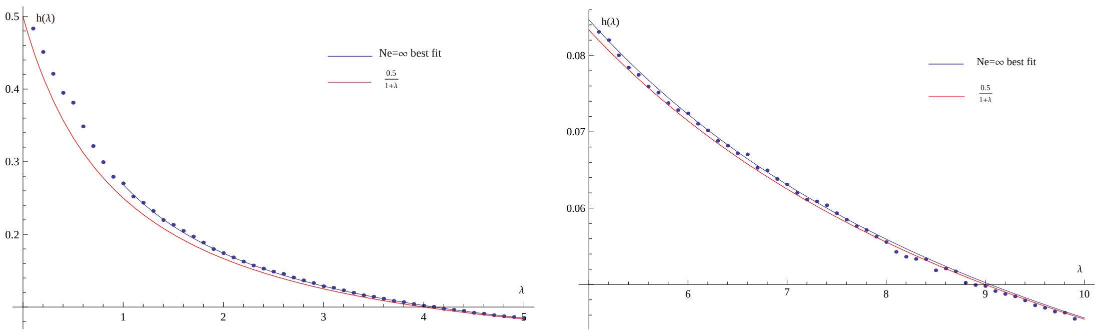

Figure 4.4: h vs. λ for $g = 1$. Blue line is best fit of the data points to the form $\frac{\alpha}{\beta+\lambda}$. The red curve is $\frac{1}{2(1+\lambda)}$.

agrees to 12 percent with the first order prediction of large g perturbation theory (4.24), $\frac{4}{2\pi^2} = 0.203$. It thus appears that, even quantitatively, the results of large g perturbation theory are not too far off the mark from the correct answer all the way down to $g = 2$. In order to see this more clearly, in Fig 4.3 we replot our results for $h(\lambda)$ versus λ at $g = 3$ and compare with the predictions of large g perturbation theory at first order in the perturbative expansion. Note the semi quantitative agreement between the curves.

On the other hand our numerical result for the function $h(\lambda)$ at $g = 1$ is presented in Fig. 4.4 for $\lambda \in (0, 10)$ [11] At every λ we have solved the saddle point equations at $Ne = 20, 30, \ldots 100$ and bestfitted our results to the form

$$h(\lambda) + \frac{b(\lambda)}{c^2(\lambda) + (Ne)^2}. \tag{4.25}$$

In the plot in Fig. 4.4 the data points represent the values of $h(\lambda)$ (obtained out of this

[11] It would of course be possible to obtain data at larger values of λ as well. However this process becomes computationally increasingly expensive, as $\frac{1}{Ne}$ errors appear to increase upon increasing λ. In order to generate reliable data at larger and larger λ requires solving the equations at larger and larger Ne.

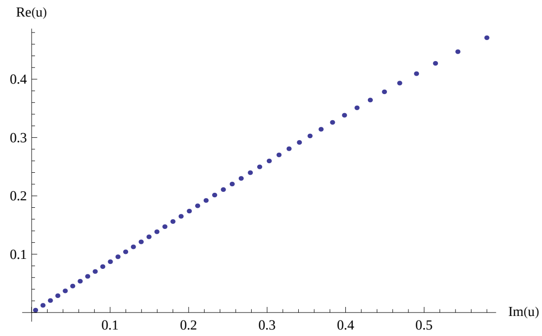

Figure 4.5: Eigenvalue distribution for Ne=90 at $\lambda = 11$.

best fit procedure) versus λ. We have also done a very crude estimate of the likely magnitude of the error in $h(\lambda)$; we estimate that this error is not larger than a few (conservatively, say, 5) percent.

In Fig. 4.4 we have also presented two curves. The blue (upper) curve represents the bestfit of $h(\lambda)$ versus λ to the form $h(\lambda) = \frac{\alpha}{\beta+\lambda}$. The bestfit values turn out to be $\alpha = 0.495$ and $\beta = 0.841$. The red (lower) curve in Fig. 4.4 is simply a graph of the function $f(\lambda) = \frac{1}{2(\lambda+1)}$. Notice that our data (the blue curve) always lies above the red curve, but appears to asymptote rather accurately to the later at large λ. As will be explained later, the red curve is a theoretical lower bound for $h(\lambda)$. Based on our data we conjectrure that $h(\lambda)$ asymptotes to from above to the curve $\frac{1}{2(\lambda+1)}$ at large λ. It would be interesting (and may be possible) to establish this fact by a direct analytic study of the saddle point equations at large λ; however we leave this exercise for future work.

As we have explained above, in order to obtain $h(\lambda)$ above we have to solve a saddle point eigenvalue equation. In Fig 4.5 we present a scatter plot of the saddle point value of the eigenvalues at $\lambda = 11$ and $Ne = 90$. Note that the eigenvalues appear to orient themselves along a curve that does not deviate too far from a straight line (in the complex plane). The magnitude of this 'cut' is approximately 0.7469 and its angle with the real axis in the complex plane is approximately 39.69 degrees.

To study the variation of eigenvalue distribution with λ we plot the length and angle of the eigenvalue distribution for $Ne = 50$ from λ=1 to 10 in Fig. 4.6. These plot show that the length of the eigenvalue distribution first increases with increasing λ (we know from small λ perturbation theory that this length scales like $\sqrt{\lambda}$ at extremely small λ) but then reaches a maximum at λ somewhere between 5 and 6, and then decreases again. On the other hand the angle made by the cut continues to decrease upon increasing λ (see the second graph in 4.6. It would be interesting to continue this analysis to larger λ, but we leave that for future work.

We now proceed to explain the siginificance of our results for $h(\lambda)$ at $g = 1$. The supersymmetric states in the $g = 1$ theory must of course include the states in the chiral ring $\mathrm{Tr}\,\phi^n$ for all n where ϕ is the scalar component of the chiral field. The scaling dimension

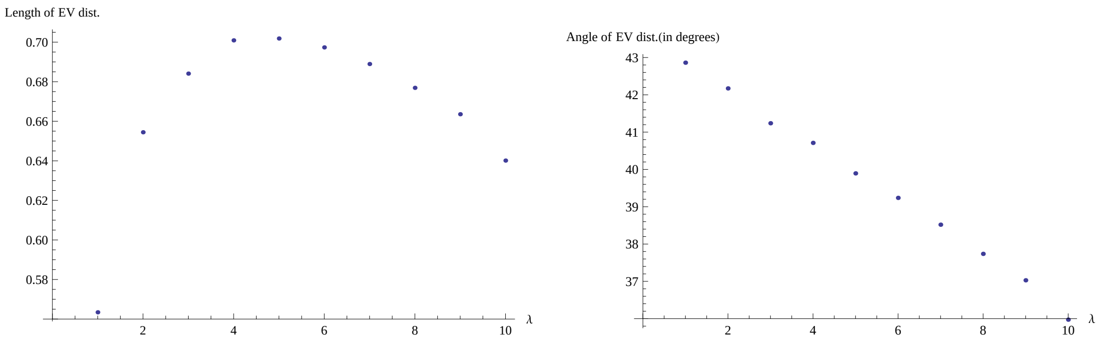

Figure 4.6: Variation of size and angle(with real axis) of eigenvalue distribution for Ne=50 from λ=1 to 10.

of these chiral ring operators is given by $nh(\lambda)$. Unitarity, however, requires that every scalar operator in any 3 d CFT has scaling dimension $\geq \frac{1}{2}$, and that an operator with dimension $\frac{1}{2}$ is necessarily free (i.e. decoupled from the rest of the theory). As we have seen, $h(\lambda)$ decreases monotonically to zero as λ is increased. Let λ_n^f denote the unique solution to the equation

$$h(\lambda_n^f) = \frac{1}{2n}.$$

For $\lambda > \lambda_n^f$, the scaling dimension of $\mathrm{Tr}\,\phi^n$ descends below its unitarity bound $\frac{1}{2}$. (For later use we will also find it useful to define

$$h(\lambda_n^m) = \frac{2}{n}.$$

λ_n^m is the value of the coupling at which a superpotential deformation by $\mathrm{Tr}\,\Phi^n$ becomes marginal. Note that $\lambda_n^m < \lambda_n^f$.) It follows from unitarity that our theory must either cease to exist [12] or must undergo a phase transition at a critical value, $\lambda = \lambda_c \leq \lambda_2^f$. While many possibilities are logically open, one attractive scenario (which is close to the scenario suggested in [18]) is the following. As λ is increased past λ_2^f then $\mathrm{Tr}\,\phi^2$ becomes free and decouples from the theory. As λ is further increased past λ_3^f then $\mathrm{Tr}\,\phi^3$ also becomes free and decouples. This process continues ad infinitum, leading to an infinite number of phase transitions.

The picture oulined above for the $g = 1$ theory, namely that each of the $\mathrm{Tr}\phi^{n+1}$ decouple at successively larger values of n can be subjected to a consistency check. It was demonstrated in [18], using brane constructions, that the deformation of the zero superpotential system by the operator $\mathrm{Tr}\phi^{n+1}$ breaks supersymmetry precisely at $\lambda = n$ (and in particular susy is not broken at smaller values of λ). However the deformation of a superpotential by a free field always breaks supersymmetry. Consistency with the scenario outlined above

[12] However Niarchos's study of this theory in [18] makes this possibility unlikely

therefore requires that $\lambda_n^f > n - 1$. In other words

$$h(\lambda) \geq \frac{1}{2(\lambda + 1)}.$$

Our data seems consistent with this bound, and as mentioned before, suggests (and we conjecture that) $h(\lambda)$ asymptotes to $\frac{1}{2(\lambda+1)}$ from above. It would be very interesting to establish this conjecture by analytic methods, but we leave that for future work.

If this picture outlined in the last two paragraphs is correct, then, in the limit $\lambda \to \infty$ we have an effective continuum of chiral primaries, with scaling dimension $\geq \frac{1}{2}$ (all primaries with lower dimension have decoupled). This suggests that the large λ behavior of this theory is intriguing, and possibly singular.[13]

4.4 Supersymmetric states of a theory with a single adjoint in the absence of a superpotential

In the last section, we presented the qualitative picture of the R charge $h(\lambda)$ for all λ, g. Given the function $h(\lambda)$, it is not difficult to evaluate the superconformal index [21] of these theories as a function of $h(\lambda)$. As was already noted in [21], this index demonstrates that the spectrum of supersymmetric single trace operators grows exponentially with energy for $g \geq 3$. In the absence of a superpotential, by computing a slightly more refined Witten index , the exponential growth of the operators can be shown to persist even for $g = 2$ theories as will be shown in §4.6. This immediately suggests that the simplest possible dual description for all theories with $g \geq 3$ (and the theory without a superpotential at $g = 2$) is a string theory, with an exponential growth in *supersymmetric* string oscillator states.

We now turn to a detailed investigation of the theory with $g = 1$ and no superpotential.

4.4.1 Superconformal index

The free theory

We begin by analyzing the index of the free theory of a single $\mathcal{N} = 2$ $U(N)$ adjoint chiral multiplet Φ (the $U(N)$ is gauged), in the large N limit. Let the generator of the global flavour symmetry of this theory - corresponding to the rephasing of Φ - be denoted by G. G is normalized so that the field ϕ has charge $\frac{1}{2}$ under G. In this case the refined Witten index

$$\mathcal{I}_+ = \text{Tr}\left[(-1)^F x^{H+J} e^{-\beta(H-J-R)} y^G\right] \tag{4.26}$$

is easily computed in the free theory. The letter index relevant to this computation is

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}} y^{\frac{1}{2}} - x^{\frac{3}{2}} y^{-\frac{1}{2}}}{1 - x^2}.$$

[13] Note, on the other hand, that when $g \geq 2$, $h(\lambda) > \frac{1}{4}$ for all λ. The large λ behaviour for these theories shows no hint of any singular behaviour.

Note that the two supersymmetric letters in the basic multiplet are ϕ and $\bar{\psi}_+$(from now on denoted as $\bar{\psi}$) and that the flavour charge of $\bar{\psi}$ is opposite to that of ϕ. It follows that the refined index is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{(1-x^{\frac{n}{2}}y^{\frac{n}{2}})(1+x^{\frac{3n}{2}}y^{-\frac{n}{2}})}. \tag{4.27}$$

It is possible to rewrite this index in terms of an index over single trace primaries as

$$\mathcal{I}_+ = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n, y^n)}{n(1-x^{2n})}\right], \tag{4.28}$$

where $\mathcal{I}_{st}$ is found to be [14]

$$\begin{aligned}\mathcal{I}_{st} = (xy)^{\frac{1}{2}} + xy + (xy)^{\frac{3}{2}} - x^{\frac{3}{2}}y^{-\frac{1}{2}} + \frac{1}{1-(xy)^3}\Big[(xy)^2 - x^2 + (xy)^{\frac{5}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}} + (xy)^3 - x^3 y \\ + (xy)^{\frac{7}{2}} - x^{\frac{7}{2}}y^{\frac{3}{2}} + (xy)^4 - x^4y^2 + (xy)^{\frac{9}{2}} - x^{\frac{9}{2}}y^{\frac{5}{2}}\Big] + \frac{1}{1-x^3y^{-1}}\left(x^{\frac{7}{2}}y^{-\frac{1}{2}} - x^{\frac{9}{2}}y^{-\frac{3}{2}}\right).\end{aligned} \tag{4.29}$$

Note that (4.29) receives contributions from (effectively) either two or three states at each energy level[15]. At every energy level we see the contribution of the chiral ring (in the form of $(xy)^E$ for every E). At every level we also, however, see the contribution of either one or two additional "particles".

The large N theory at finite 't Hooft coupling

Now consider the $\mathcal{N} = 2$ $U(N)$ CS theory at level k with one adjoint matter field and no superpotential, in the 't Hooft limit. As we have seen previously, the R-charge of this theory is renormalized as a function of λ. The renormalization of the R-charge is, really, more accurately a mixing of the R-charge and the flavour charge; the extent of this mixing varies with λ.

The effect of this mixing on the superconformal index may be dealt with very simply. If we perform the replacement

$$y \to y x^{2h(\lambda)-1}$$

on all the formulas for the (gauged) free theory in the previous subsubsection, then they apply to the interacting theory at arbitrary λ (and infinite N).

[14]Similar techniques were used to compute the Index of $\mathcal{N} = 4$ Yang Mills theory in [27], ABJM theory in [21], and the partition function of free gauge theories in, for instance, [28]

[15]Note that (4.29) reduces to (4.30) below upon setting $y = 1$.

4.4.2 Conjecture for the supersymmetric spectrum at all couplings

In this section we (conjecturally) enumerate all single trace primary operators that are annihilated by the supercharge Q that was preserved in the superconformal index. This will allow us to enumerate the full single trace supersymmetric operator spectrum of our theory. We perform our enumeration of operators in the classical (though nonlinear) theory; and assume without proof that the same result continues to hold in the full quantum theory. We initially ignore the effect of the renormalization of R-charge in this theory as a function of coupling; as in the previous section, it will prove extremely easy to insert this effect into our final answer right at the end.[16]

We now describe the procedure we will use for the enumeration in this subsection in more detail. Every supersymmetric operator of the sort we seek must be built out of ϕ, $\bar{\psi}_+$ and D_{++} where ϕ and $\bar{\psi}_+$ both have R-charge $\frac{1}{2}$ and the + subscript denotes $SO(3)$ charge (these are the only letters with $\Delta = h + j$). As we have explained above, the letters ϕ and $\bar{\psi}_+$ have flavour charge $\pm\frac{1}{2}$ respectively.

In the free theory any operator constructed out of these elements is annihilated by our special supercharge Q (with R-charge unity and $SO(3)$ charge $-\frac{1}{2}$). In the interacting theory, on the other hand, while it continues to be true that $[Q, \phi] = \{Q, \bar{\psi}\} = 0$ we now have

$$[Q, D_{++}O] \sim [[\phi, \bar{\psi}_+], O]$$

As illustrated by this equation, the derivative D carries the same x and y charges as the combination of letters $\phi\bar{\psi}_+$. Consequently a given term in the index counts an infinite number of distinct possible operator structures. For instance, at order $x^5 y$ the cohomology potentially has operators of the form $\mathrm{Tr}\,(\phi^4\bar{\psi}^2)$, $\mathrm{Tr}\,(\phi^3 D\bar{\psi})$ and $\mathrm{Tr}\,(\phi D^2\phi)$. In any given charge sector we refer to the *number of derivatives* in the operator as its *level*. As we see from the equation above, the operator Q preserves charge but maps states of level l to states of level $l-1$. At any given fixed value of the charge let the number of states in the free theory at level l be $n(l)$. Let the number of states in the kernel of the action of Q (those that are annihilated by Q) at level l be denoted by $c(l)$. It follows that the number of states in Q cohomology at level l is given by $c(l) - (n(l+1) - c(l+1))$ (because $n(l+1) - c(l+1)$ gives the number of exact states at level l).

The precise value of this cohomology, as defined above, contains some redundant information. This is because a total D_{++} derivative (outside the trace) of any member of cohomology obviously itself belongs to the cohomology. Such descendant elements of coho-

[16]The reason that the index is insufficient to completely characterize the supersymmetric spectrum is that it is blind to Bose-Fermi pairs of supersymmetric states, whose contribution to the index cancel. The reader may feel that such conspiratorial cancellations are unlikely, but that is far from the truth. In fact it is immediately clear on general grounds that our result for the index in the previous section must include some important cancellations. In order to see this recall that the R current and stress tensor appear in the supersymmetry multiplet with quantum numbers $(\Delta = 2, J = 1, R = 0, G = 0)$. It follows that the contribution of this multiplet to the index is $-x^4 y^0$: however such a term is absent in (4.29). It must be that the full susy spectrum of the theory includes a bosonic state whose contribution to the index cancels that of the stress tensor multiplet. Below we will see in some detail how this works.

mology are trivial (they do not map to new particles under the AdS/CFT map) and should be removed from the analysis. This is easy to do in a self consistent manner.

We have written a Mathematica routine that computes the numbers $c(l)$ and $n(l)$ and hence the cohomology at low orders. Our routine then proceeds to eliminate descendant. All the results we have obtained so far are consistent with the following conjectures for the structure of primary states in cohomology:

- The cohomology (obviously) contains only states with x and y quantum numbers of $\mathrm{Tr}\,(\phi^m \bar{\psi}^n)$ for positive m and n.

- We conjecture that, at level zero, all such charges admit a unique (conformal primary) state in cohomology unless $m = 0$ and n is even. This state in cohomology may be thought of as the trace of the completely symmetric combination of $\bar{\psi}$ and ϕ. The exception ($m = 0$ and n even) is a consequence of the fact that $\mathrm{Tr}\,\bar{\psi}^{2k}$ vanishes because of the fermionic nature of $\bar{\psi}$.

- We conjecture that, at level one, there also always exists a unique (conformal primary) state in the cohomology subject to the following exceptions. There are no states at level one when $n = 0$ or when $n = 1$.[17] There are also no level one states when $m = 0$ or when $m = 1$ and n is even.[18]

- We conjecture that the cohomology has no (conformal primary) states at levels higher than one.

- That our conjecture is consistent with the index listed above as may be seen as follows. A pair of a level zero and level one state gives a vanishing contribution to the index. It follows that we have a contribution to the index only in those cases in which level zero and level one states are unpaired. According to our conjecture, unparied states occur at charges at which there exists a level one state but no level zero state. This occurs for states of the charges $\mathrm{Tr}\,\phi^m$, $\mathrm{Tr}\,(\phi^m\bar{\psi})$, $\mathrm{Tr}\,\bar{\psi}^{2m+1}$ and $\mathrm{Tr}\,(\bar{\psi}^{2m}\phi)$. This precisely accounts for the index computed above.

We summarize the conjecture described above in Table (4.10). This is found to agree with the index calculated for this theory as calculated in section 4.4.1. In that table we have also used the fact that every short representation of the $\mathcal{N} = 2$ superconformal algebra has a unique (conformal primary) state in Q cohomology to read off the full spectrum of the short (or supersymmetric) single trace primary operators of the theory implied by our conjecture for Q cohomology.

[17] This statement is obvious when $n = 0$ because then there do not exist any level one states with the right quantum numbers. Moreover when $n = 1$ the unique level one state with the right quantum numbers is a descendant of $\mathrm{Tr}\,\phi^{m-1}$.

[18] When $m = 0$ there are no level one states with the right quantum numbers. When $m = 1$ and n is even, the unique such state is a descendant of $\mathrm{Tr}\,\bar{\psi}^{n-1}$. Of course no such statement can be (or is) true when n is odd because $\mathrm{Tr}\,\bar{\psi}^{n-1}$ vanishes.

Cohomology state	#	$\mathcal{N}=2$ Primary	Allowed h and j
$(h,0,h,h)$	1	$(h,0,h,h)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(j+h,j,h,h-2j)$	1	$(j+h-\frac{1}{2},j-\frac{1}{2},h-1,h-2j)$	$j,h\in\frac{1}{2}\mathbb{Z}^+,\ h\geq j$ $h=j\in\mathbb{Z}$ not allowed.
$(j+h,j,h,h-2j+2)$	1	$(j+h-\frac{1}{2},j-\frac{1}{2},h-1,h-2j+2)$	$h,j\in\frac{1}{2}\mathbb{Z}^+,\ j\geq\frac{3}{2},\ h\geq j-\frac{1}{2}$
$(2h+1,h+1,h,-2h)$	1	$(2h+\frac{1}{2},h+\frac{1}{2},h-1,-2h)$	$h\in\frac{1}{2}\mathbb{Z}^+$

Table 4.10: $\mathcal{N}=2$ primary content of single adjoint matter theory with zero superpotential. The second column stands for the multiplicity of the cohomology states. The notation is (Δ,j,h,g) which respectively are scaling dimension, spin, R-charge and $U(1)$ flavor charge. The flavor charges are normalized to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. The results above apply when $h(\lambda)=\frac{1}{2}$. The results for the general case are obtained from the results of the table above by the replacement $\Delta\to\Delta-(1-2h(\lambda))g,\ h\to h-(1-2h(\lambda))g$ for every primary in the table.

While the states listed in Table (4.10) do grow in number with energy in a roughly Kaluza-Klein fashion, notice that the primaries listed include states of arbitrarily high spins, ruling out a possible dual supergravity dual description.

4.5 Supersymmetric states of theories with a single adjoint field with nonzero superpotential

4.5.1 Space of theories

Let us now turn to the study of superconformal theories with superpotential. First we start with theories with a single adjoint chiral multiplet. One may construct several superconformal theories by perturbing the theory with no superpotential. The simplest theory of this sort may be constructed by perturbing the superpotential of the theory of the previous subsection with a $\mathrm{Tr}\,\Phi^4$ term. While this perturbation is marginal at zero λ, it is relevant at every finite λ (this follows as $h(\lambda)<\frac{1}{2}$ for all finite λ). As arugued in [5], the RG flow seeded by this operator terminates at a new fixed point at which the coefficient c of $\mathrm{Tr}\,\Phi^4$ in the superpotential is of order unity, in units in which a uniform factor of $k=\frac{N}{\lambda}$ sits outside the whole action.[19] This line of CFTs also reduces to the free theory as $\lambda\to 0$. The $\mathrm{Tr}\,\Phi^4$ superpotential in this theory breaks the flavour symmetry of the zero superpotential theory down to Z_4. The requirement of the invariance of the superpotential under R symmetry transformations forces the R-charge of this system to equal $\frac{1}{2}$ at every value of the coupling constant λ.

There exist no other lines of CFTs with this matter content that reduce to the free CFT in the limit $\lambda\to 0$. However there plausibly exist many other lines of superconformal fixed

[19]The argument that the RG flow ends at a finite value of c is simple. When $c\gg 1$, the gauge interaction in the theory is negligibly weak compared to the $\mathrm{Tr}\,\Phi^4$ interaction and may be ignored. The model is then effectively the Wess Zumino theory whose β function towards the IR is known to be negative. Consequently the sign of the β function flips from positive for small c to negative at large c, and so must have a zero at c of order unity.

points that cannot be deformed to the free theory. To start with the superpotential term $\mathrm{Tr}\,\Phi^3$ is relevant at all values of λ. At small λ the RG flow seeded by this term presumably terminates at a large N and supersymmetric analogue of the Wilson Fisher fixed point. It thus seems plausible that the new fixed point exists at every value of λ. The R-charge of ϕ is fixed at $\frac{2}{3}$ at this fixed point.

More exotically, as we have explained towards end of §4.3, the fact that $h(\lambda)$ decreases without bound (as λ is increased) plausibly suggests the existence of fixed points seeded by $\mathrm{Tr}\,\Phi^n$ induced RG flows, for all values of n, at sufficiently large λ. The R-charge of the operator ϕ is fixed at $\frac{2}{n}$ along these fixed lines of theories.

4.5.2 Superconformal index of the theory with a $\mathrm{Tr}\,\Phi^4$ superpotential

Let us compute the Witten index

$$\mathrm{Tr}(-1)^F x^{H+J}$$

for the theory with the $\mathrm{Tr}\,\Phi^4$ superpotential. The letter index of this theory is the Witten index of the sum of the $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, 0, -\frac{1}{2})$ representations. We find the single letter contribution

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}}}{1+x}.$$

It follows that the Witten index of the theory - in the large N limit - is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1}{1-\mathcal{I}_L(x^n)} = \prod_{n=1}^{\infty} \frac{1+x^n}{1-x^{\frac{n}{2}}+x^n}.$$

The formula above gives the Witten index of the full theory (including all multi trace operators) in the large N limit. It is interesting to inquire about the single trace index for the same theory. Now the full index is obtained from the single trace index by Bose exponentiation. The single trace index actually receives contributions both from single trace conformal primaries and single trace conformal descendants. It is of most interest to isolate the index over all single trace primaries (as this gives the index over the particle spectrum of the dual theory). Let us define the index over single trace primaries as $\mathcal{I}_{st}$. It then follows that

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1+x^n}{1-x^{\frac{n}{2}}+x^n} = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n)}{n(1-x^{2n})}\right]$$

This equation completely determines $\mathcal{I}_{st}$. It is possible to show that

$$\begin{aligned} \mathcal{I}_{st} &= x^{\frac{1}{2}} + x + x^{\frac{7}{2}} - x^{\frac{9}{2}} + x^{\frac{13}{2}} - x^{\frac{15}{2}} + x^{\frac{21}{2}} - x^{\frac{23}{2}} + \ldots \\ &= x^{\frac{1}{2}} + x + \frac{x^{\frac{7}{2}}}{1+x+x^2}. \end{aligned} \tag{4.30}$$

Note that the spectrum of single traces is periodic at large enough energies, with one new boson and one new fermion being added at energy intervals of three.

Cohomology states	Multiplicity	Protected primaries
$(\frac{1}{2},0,\frac{1}{2})$	1	$(\frac{1}{2},0,\frac{1}{2})$
$(1,0,1)$	1	$(1,0,1)$
$(2k+\frac{5}{2},k+1,k+\frac{3}{2})$	1	$(2k+2,k+\frac{1}{2},k+\frac{1}{2})$
$(2k+3,k+1,k+2)$	1	$(2k+\frac{5}{2},k+\frac{1}{2},k+1)$
$(2k+\frac{5}{2},k+\frac{3}{2},k+1)$	1	$(2k+2,k+1,k)$
$(2k+3,k+\frac{3}{2},k+\frac{3}{2})$	1	$(2k+\frac{5}{2},k+1,k+\frac{1}{2})$

Table 4.11: Supersymmetric spectrum for $\mathcal{N}=2$ single adjoint with superpotential $\mathrm{Tr}\,\Phi^4$, $k \in 0, \mathbb{Z}^+$. The notation again is (Δ, j, h).

4.5.3 Conjecture for the supersymmetric spectrum of the theory with a $\mathrm{Tr}\,\Phi^4$ superpotential

In this subsection we present a conjecture for the partition function over single trace supersymmetric operators in the theory with a $\mathrm{Tr}\,\Phi^4$ term in the superpotential. Our method is the same as that employed in the previous section: we list the classical cohomology of the special supercharge Q (the supercharge that annihilates the superconformal index).

The only difference between the cohomology of Q in this section, and the cohomology of Q in the absence of a superpotential (computed in the previous section) is that Q no longer annihilates $\bar{\psi}$, but we instead have

$$Q\bar{\psi} \sim \phi^3$$

.

As in the previous section the cohomology of interest can be calculated separately for operators with distinct values of $\Delta + j$. As in the previous section, we have written a Mathematica code to compute the cohomology for all states upto $\Delta + j \leq \frac{23}{2}$.

Our conjecture has been summarized in Table (4.11) in terms of charges (Δ, j, h) of the cohomology states. This is found to agree with the index calculated for this theory in section 4.5.2. As states of the form $(j+h, j, h)$ (for $j > 0$) are descendants of the superconformal primary $(j+h-\frac{1}{2}, j-\frac{1}{2}, h-1)$, the corresponding superconformal primaries are also listed out in the table.

As in the case of theories without a superpotential, our conjectured supersymmetric spectrum grows with energy in a manner expected of Kaluza-Klein compactification, but continues to include states of arbitrarily high spin.

4.5.4 Conjecture for the supersymmetric spectrum of the theory with a $\mathrm{Tr}\,\Phi^3$ superpotential

In this subsection we present a conjecture for the classical cohomology of the particular supercharge Q for the theory with a $\mathrm{Tr}\,\Phi^3$ superpotential. A theory with such a superpotential is always strongly coupled, and so we cannot use a free calculation to compute its superconformal index. However, the computation of Q cohomology in such a theory is easily

Cohomology states	Multiplicity	Protected primaries
$(\frac{2}{3},0,\frac{2}{3})$	1	$(\frac{2}{3},0,\frac{2}{3})$
$(\frac{5k+2}{3},k,\frac{2}{3}(k+1))$	1	$(\frac{5k}{3}+\frac{1}{6},k-\frac{1}{2},\frac{2k-1}{3})$
$(\frac{5}{6}(2k+1),k+\frac{1}{2},\frac{2k+1}{3})$	1	$(\frac{5k+1}{3},k,\frac{2}{3}(k-1))$

Table 4.12: Supersymmetric spectrum for $\mathcal{N}=2$ single adjoint with superpotential Φ^3, $k \in \mathbb{Z}^+$. The notation is (Δ, j, h)

performed (under the same assumption of the previous subsection, i.e. that it is sufficient to use the classical supersymmetry transformation rules), using the same method as in the previous subsection.

The difference between the computation of this subsection and that of the previous one is as follows. In this case the action of Q on $\bar{\psi}$ is given by $Q\bar{\psi} \sim \phi^2$. Further the fact that we have a superconformal theory with a Φ^3 superpotential forces the following charge assignments: the R-charges of $\phi, \bar{\psi}$ are $\frac{2}{3}, \frac{1}{3}$ respectively, while the scaling dimensions of $\phi, \bar{\psi}$ as determined by the BPS relation $\Delta = j + h$, are $\frac{2}{3}, \frac{5}{6}$ respectively.

As in the previous subsection, the cohomology must be computed separately at every value of $\Delta + j$. The calculation has been done up to $\Delta + j = 12$ on Mathematica. Our conjecture has been summarized in Table (4.12) in terms of charges (Δ, j, h) of the cohomology states. This can be shown to agree with a index calculation done using localization techniques. As states of the form $(j+h, j, h)$ (for $j > 0$) are descendants of the superconformal primary $(j+h-\frac{1}{2}, j-\frac{1}{2}, h-1)$, the corresponding superconformal primaries are also listed out in the table. Note that the spectrum includes states of arbitrarily high spin.

4.6 Supersymmetric states in the theory with two adjoint fields and vanishing superpotential

The superconformal theory with two adjoints and no superpotential has a $U(2)$ flavour symmetry, realized as the rotation of the chiral multiplets Φ_1 and Φ_2 as a doublet. We denote the two $U(1)$ Cartan charges of this $U(2)$ by G_1 and G_2. Our conventions are that the field ϕ_1 has charges $(G_1, G_2) = (1, 0)$ while the charges of ϕ_2 are $(0, 1)$. We compute the refined Witten index defined by

$$\mathcal{I} = \text{Tr}\left[(-1)^F x^{H+J} y_1^{G_1} y_2^{G_2}\right]. \tag{4.31}$$

The letter index is given by

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}}(y_1+y_2) - x^{\frac{3}{2}}(y_1^{-1}+y_2^{-1})}{1-x^2}.$$

As usual the full multitrace index of the free theory is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1}{1-\mathcal{I}_L(x^n, y^n)}.$$

This index captures an exponential growth of density of supersymmetric states. In order to see that this must be the case, note that the number of chiral primaries at level L is of order 2^L. As there are no fermionic states with the quantum numbers of the chiral primaries, it is impossible for this contribution to be cancelled in the full index. It follows that the index, in this state, receives contributions from an exponentially growing number of states.

4.7 Supersymmetric states in the $\mathcal{N}=3$ theory with a adjoint hypermultiplet

Of all the possible superpotential deformations of the zero superpotential $\mathcal{N}=2$ theory with two adjoint chiral multiplets, the deformation $W = \alpha \mathrm{Tr}\,[\Phi_1, \Phi_2]^2$ has a special role. This deformation is relevant at nonzero λ. As was shown in [5] the RG flow seeded by this deformation has a fixed point at $\alpha = \frac{2\pi}{k}$; at this value of the coefficient, the theory is conformally invariant; further its supersymmetry is enhanced to $\mathcal{N}=3$ and the theory enjoys invariance under the full $\mathcal{N}=3$ superconformal algebra.

$\mathcal{N}=3$ superconformal symmetry forces the theory to have an $SU(2)$ R symmetry; ϕ_1 and ϕ_2^* transform as a doublet under this symmetry. ϕ_1 and ϕ_2 each have R-charge $\frac{1}{2}$ under the canonical $U(1)$ subgroup of this R symmetry group. Note of course that the value of this R-charge is protected by $SU(2)$ representation theory, and cannot be renormalized as a function of λ. In addition, the fact that the superpotential is proportional to $\mathrm{Tr}\,(\epsilon^{ij}\Phi_i\Phi_j)^2$ reveals that the superpotential deformation preserves an $SU(2)$ flavour subgroup of the $U(2)$ flavour isometry group of the theory without a superpotential.

In this section we will compute the supersymmetric spectrum of this theory at large N.

4.7.1 Superconformal index

As the $\mathcal{N}=3$ theory reduces to a free theory at $\lambda=0$, its superconformal index is easily computed. The superconformal index is defined by

$$\mathcal{I} = \mathrm{Tr}\left[(-1)^F x^{H+J} y^G\right] \tag{4.32}$$

where G is the $U(1)$ component of the $SU(2)$ flavor group (under which ϕ_1 has charge $\frac{1}{2}$ and ϕ_2 has charge $-\frac{1}{2}$). The relevant letter index is given by

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{3}{2}}y^{-\frac{1}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{\frac{3}{2}}y^{\frac{1}{2}}}{1-x^2} = \frac{x^{\frac{1}{2}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})}{1+x},$$

so that the index over the theory is

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1+x^n}{(1-x^{\frac{n}{2}}y^{\frac{n}{2}})(1-x^{\frac{n}{2}}y^{-\frac{n}{2}})}.$$

As in previous sections, it is possible to rewrite this index in terms of an index over single trace primaries as

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1+x^n}{(1-x^{\frac{n}{2}}y^{\frac{n}{2}})(1-x^{\frac{n}{2}}y^{-\frac{n}{2}})} = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n, y^n)}{n(1-x^{2n})}\right]. \tag{4.33}$$

We find

$$\begin{aligned} \mathcal{I}_{st} &= x^{\frac{1}{2}}(y+\frac{1}{y}) + x(1+y^2+\frac{1}{y^2}) + x^{\frac{3}{2}}(y^3+\frac{1}{y^3}) + x^2(y^4+\frac{1}{y^4}) \\ &+ \sum_{n=5}^{\infty} x^{\frac{n}{2}}\left(y^n + \frac{1}{y^n} - y^{n-4} - \frac{1}{y^{n-4}}\right). \end{aligned} \tag{4.34}$$

This simple result describes a spectrum with 4 states - two bosons and two fermions - at every energy higher than a minimum value. Note that (4.34) reduces to (4.42) upon setting y to unity.

4.7.2 Supersymmetric cohomology

We now proceed to compute the single trace supersymmetric cohomology of the $\mathcal{N}=3$ theory.[20] We are instructed to count traces built out of $D_+^n\phi_i$ and $D_+^n\bar{\psi}_i$ ($i=1,2$). We are only interested in Q cohomology, where the action of Q on the basic fields is given by

$$\begin{aligned} &Q\phi_i = 0, \\ &Q\bar{\psi}_i = [\phi_i, [\phi_1, \phi_2]], \\ &Q[D_{++}, \cdot\,] = \big[[\phi_1, \bar{\psi}_1] + [\phi_2, \bar{\psi}_2], \cdot\,\big]. \end{aligned} \tag{4.35}$$

As in previous subsections, we have explicitly enumerated this cohomology (with the help of Mathematica) at low quantum numbers, and used the results of this numerical experiment to suggest a relatively simple conjecture for the conformal primary content of this cohomology. As in previous subsections, each conformal primary member of cohomology implies the existence of a single short $\mathcal{N}=2$ superconformal representation. Unlike the situation in previous sections, however, this spectrum has to satisfy an additional consistency check, as $\mathcal{N}=2$ representations must group together into $\mathcal{N}=3$ representations. Our conjecture for Q cohomology passes this consistency check, and leads us to conjecture that the short supersymmetric operator content of the theory is given as in Table (4.13). This is found to

[20] As in the previous subsubsection, we know that there must exist states that contribute to the partition function but are invisible in the index simply from the observation that the contribution of the stress tensor multiplet -x^4y^0 - is not visible in (4.34).

Cohomology states ($\mathcal{N}=2$ quantum numbers)	Multiplicity	$\mathcal{N}=2$ Primary	$\mathcal{N}=3$ Primary	Allowed h
$(h,0,h,h)$	1	$(h,0,h,h)$	$(h,0,h,h)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+\frac{1}{2},\frac{1}{2},h,h)$	1	$(h,0,h-1,h)$		$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{1}{2},h+1,h)$	1	$(h+1,0,h,h)$	$(h+1,0,h,h)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+2,1,h+1,h)$	1	$(h+\frac{3}{2},\frac{1}{2},h,h)$		$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+1,1,h,0)$	1	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$h\in\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{3}{2},h,0)$	1	$(h+1,1,h-1,0)$		$h\in\mathbb{Z}^+$

Table 4.13: Supersymmetric spectrum for 2 chiral adjoints at $\mathcal{N}=\ni$ fixed point with $SU(2)$ flavor symmetry. The flavor charges for ϕ and $\bar{\psi}$ are normalized to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively

agree with the index calculated for this theory in section 4.7.1. (Our notation for quantum numbers of states as well as primaries is (Δ, j, h, g) where g is the $SU(2)$ flavour charge.)

A striking feature of this conjectured supersymmetric spectrum is that it includes no states with spin ≥ 2. This suggests that the $\mathcal{N}=3$ theory might admit a supergravity-like dual description at large λ.

As the full global symmetry group of our theory is $SU(2)\times SU(2)$ it is tempting to conjecture that the supergravity description in question is obtained by a compactification of a 7-dimensional supergravity on S^3. Indeed the states in the first 2 rows (of the second last column) of Table (4.13) have $SU(2)\times SU(2)$ quantum numbers that are strongly reminiscent of scalar and vector spherical harmonics on S^3. However the states in the last line of this table do not appear to fit well into this pattern; as the difference between the two $SU(2)$ quantum numbers of these states grows without bound; this never happens for S^3 spherical harmonics for states with a fixed (or bounded) value of spin. For this reason we are unsure whether our results for the supersymmetric spectrum of this theory are consistent with a possible dual description in terms of a higher dimensional supergravity theory. We leave further discussion of this question to future work.

4.8 Marginal $\mathcal{N}=2$ deformations of the $\mathcal{N}=3$ theory

It was demonstrated in [25] that the infinitesimal manifold of exactly marginal deformations of a given SCFT has a very simple characterization.[21] This space is simply given by modding out the space of marginal (but not necessarily exactly marginal) classical deformations by the complexified action of the global (non R) symmetry group $\mathcal{G}_C$ of the theory.

The space of marginal scalar deformations of the $\mathcal{N}=2$ theory , at the level of the superpotential it is given by operators $\mathrm{Tr}(\Phi^{a_1}\Phi^{a_2}\Phi^{a_3}\Phi^{a_4})$ with the indices $a_1\ldots a_4$ completely symmetrized. These operators transform in the 5 dimensional (spin 2) representation of the global symmetry group $SU(2)\sim SO(3)$. In colloquial terms they constitute a complex traceless symmetric 3×3 matrix M on which complexified $SO(3)$ transformations O act

[21] In the perturbative regime, a global characterization of the "conformal manifold" was given in [17].

Points on conformal manifold	Flavour symmetry	Charges
$\lambda_1 = 0 = \lambda_2$	$SU(2)$	Φ_1, Φ_2 form a doublet
$\lambda_1 = \lambda_2 \neq 0$	$U(1)$	Φ_1: 1, Φ_2: -1
$\lambda_1 = 0, \lambda_2 \neq 0$	$U(1)$	$\Phi_1 + i\Phi_2$: 1, $\Phi_1 - i\Phi_2$: -1
$\lambda_1 \neq 0, \lambda_2 = 0$	$U(1)$	$\Phi_1 + \Phi_2$: 1, $\Phi_1 - \Phi_2$: -1

Table 4.14: Symmetries on the conformal manifold

according to the law

$$M \to OMO^T$$

The classification of all 3×3 matrices M that are inequivalent under this transformation law is well studied. Generic complex symmetric matrices M can be diagonalized by the action of O; consequently generic exactly marginal deformations of the $\mathcal{N} = 3$ theory are in one to one correspondence with complex diagonal traceless 3×3 matrices and are labeled by two complex eigenvalues. In addition to the generic case, however, there exist two special classes of matrices M that cannot be diagonalized. Instead, one of them can be put in the form [29]

$$\begin{pmatrix} \lambda_1 + \frac{i}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \lambda_1 - \frac{i}{2} & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix} \tag{4.36}$$

This gives a new one parameter set of exactly marginal deformations of the $\mathcal{N} = 3$ theory. The second possible form is

$$\begin{pmatrix} 0 & \frac{1+i}{2} & 0 \\ \frac{1+i}{2} & 0 & \frac{1-i}{2} \\ 0 & \frac{1-i}{2} & 0 \end{pmatrix} \tag{4.37}$$

Let us first focus on the generic marginal deformations of the $\mathcal{N} = 3$ theory. The generic deformation can be put in the form

$$W = \lambda_1 \mathrm{Tr}\,(\Phi_1^2 + \Phi_2^2)^2 - \lambda_2 \mathrm{Tr}\,(\Phi_1^2 - \Phi_2^2)^2 - (\lambda_1 + \lambda_2)\mathrm{Tr}\,(\Phi_1\Phi_2 + \Phi_2\Phi_1)^2 \tag{4.38}$$

which is better written as

$$W = \lambda_1 \mathrm{Tr}\,\left[(\Phi_1 + \Phi_2)^2(\Phi_1 - \Phi_2)^2\right] - \lambda_2 \mathrm{Tr}\,\left[(\Phi_1 + i\Phi_2)^2(\Phi_1 - i\Phi_2)^2\right] \tag{4.39}$$

At a generic point on the conformal manifold the flavor symmetry is completely broken. For special values of $\lambda_{1,2}$, listed in Table (4.14), a $U(1)$ flavor symmetry is restored. The space of generic exactly marginal deformations of the $\mathcal{N} = 3$ theory is a two complex dimensional manifold.

Next, for the nongeneric deformation as parametrized by (4.36), the superpotential deformation is

$$W = \lambda_2 \left(\mathrm{Tr}\,(\Phi_2^4) - \lambda_1 \mathrm{Tr}\,(\Phi_1\Phi_2\Phi_1\Phi_2)\right) \tag{4.40}$$

There is no flavour symmetry in this case.

As for the nongeneric deformation as parametrized by (4.37), the superpotential deformation is

$$W = \lambda_2 \left((1+i)\mathrm{Tr}\,(\Phi_1^4 - \Phi_2^4) + 2(1-i)\mathrm{Tr}\,(\Phi_1^3\Phi_2 - \Phi_1\Phi_2^3) \right) \tag{4.41}$$

There is no flavour symmetry in this case also.

The $U(1)$ R-charge of chiral multiplets in these theories is fixed to $\frac{1}{2}$ (merely by the observation that each of these theories has a space of exactly marginal deformations, labeled by different quartic superpotential deformations).

4.8.1 Superconformal index of these theories

The Witten index of the deformed theories that preserve $U(1)$ flavor symmetry is simply identical to the index of the $\mathcal{N}=3$ theory determined in the previous section.

The Witten index of deformed theories that break the $U(1)$ symmetry is also equal to to that of the $\mathcal{N}=3$ theory, but with $y=1$ (as there is no flavor charge with respect to which states can be weighted in this case). We find the remarkably simple result

$$\mathcal{I}_{st} = 2x^{\frac{1}{2}} + 3x + 2x^{\frac{3}{2}} + 2x^2. \tag{4.42}$$

Thus the single trace index sees a total of only 9 conformal primaries!

4.8.2 Conjecture for the supersymmetric cohomology

For cohomology calculations we consider the general marginal superpotential as given in (4.38) along with the original $\mathcal{N}=3$ superpotential with a coefficient normalized to one. With this superpotential the action of the special supercharge on the basic letters is as follows

$$Q(\bar{\psi}_1) = -\phi_1\phi_2^2 - \phi_2^2\phi_1 + 2\phi_2\phi_1\phi_2 + (\lambda_1-\lambda_2)\phi_1^3 - 4(\lambda_1+\lambda_2)(\phi_2\phi_1\phi_2) \tag{4.43}$$
$$Q(\bar{\psi}_2) = -\phi_2\phi_1^2 - \phi_1^2\phi_2 + 2\phi_1\phi_2\phi_1 + (\lambda_1-\lambda_2)\phi_2^3 - 4(\lambda_1+\lambda_2)(\phi_1\phi_2\phi_1) \tag{4.44}$$
$$Q[D_{++},\cdot] = \big[[\phi_1,\bar{\psi}_1] + [\phi_2,\bar{\psi}_2],\cdot\big] \tag{4.45}$$

Although it is not obvious, it (experimentally) appears that the cohomology is largely independent of the complex ratio $\frac{\lambda_1}{\lambda_2}$ but instead depends only on whether the flavour symmetry of the theory is broken or restored. Using the methods described in earlier sections we have generated data that suggests that the cohomology of these theories takes the following form.

For the generic $\mathcal{N}=2$ deformations (4.39), the conformal primary states in the cohomology, and the corresponding $\mathcal{N}=2$ superconformal representations, are given, in the case that a $U(1)$ flavor symmetry is preserved $((\lambda_1=0,\lambda_2\neq 0)$, $(\lambda_1\neq 0,\lambda_2=0)$, $(\lambda_1=\lambda_2\neq 0))$ in Table (4.15).

Cohomology states ($\mathcal{N}=2$ quantum numbers)	Multiplicity	$\mathcal{N}=2$ Primary	Allowed h
$(h,0,h,h)$ $(h,0,h,-h)$ $(h,0,h,0)$	1 1 1	$(h,0,h,h)$ $(h,0,h,-h)$ $(h,0,h,0)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(\frac{3}{2},\frac{1}{2},1,0)$	1	$(1,0,0,0)$	
$(h+\frac{3}{2},\frac{1}{2},h+1,h)$ $(h+\frac{3}{2},\frac{1}{2},h+1,-h)$ $(h+\frac{3}{2},\frac{1}{2},h+1,0)$	1 1 1	$(h+1,0,h,h)$ $(h+1,0,h,-h)$ $(h+1,0,h,0)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+1,1,h,0)$	1	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$h\in\frac{1}{2}\mathbb{Z}^+$
$(h+\frac{1}{2},\frac{3}{2},h-1,0)$	1	$(h,1,h-2,0)$	$h\in 2\mathbb{Z}^+$

Table 4.15: Supersymmetric spectrum for 2 chiral adjoints at $\mathcal{N}=2$ fixed point with $U(1)$ flavor symmetry. Notation is (Δ,j,h,g) with g normalized to be $\frac{1}{2}$ for ϕ.

Cohomology state	Primary	Multiplicity	Allowed values
$(\frac{k}{2},0,\frac{k}{2})$	$(\frac{k}{2},0,\frac{k}{2})$	3 if $\frac{k}{2}$ is odd, 2 if else	$k\in\mathbb{Z}^+$
$(\frac{k+1}{2},\frac{1}{2},\frac{k}{2})$	$(\frac{k}{2},0,\frac{k}{2}-1)$	3 if $\frac{k}{2}$ is even, 1 if $\frac{k}{2}$ is odd, 2 if $\frac{k}{2}$ is half an odd integer	$k\in\mathbb{Z}^+$
$(2k+\frac{1}{2},\frac{3}{2},2k-1)$	$(2k,1,2k-2)$	1	$k\in\mathbb{Z}^+$

Table 4.16: Cohomology and primary content of theories with no flavor symmetry. The notation is (Δ,j,h), since there is no flavor symmetry.

On the other hand the conformal primary cohomology and the superconformal primary content of $\mathcal{N}=2$ cases when there is no flavour symmetry i.e. when $\lambda_1\neq\lambda_2$ and $\lambda_1,\lambda_2\neq 0$ is given in Table (4.16). Also for the first non generic $\mathcal{N}=2$ deformation (4.40) with $\lambda_1\neq 0$ and for the second nongeneric $\mathcal{N}=2$ deformation (4.41) the superconformal primary content is the same as in Table (4.16).

For the first non generic $\mathcal{N}=2$ deformation (4.40), if $\lambda_1=0$, the superconformal primary content is given in Table (4.17)

Note that in each case the supersymmetric spectrum has no states with spins greater than two, suggesting again the possibility of a dual supergravity description for these theories at strong coupling.

4.8.3 Theories with three or more chiral multiplets

In this case the letter partition function equals unity at a value of $x<1$. It follows that the Witten index undergoes a Hagedorn transition at finite 'temperature'. In other words the number of supersymmetric operators protected by susy grows exponentially with energy in these theories. Restated, our system has a stringy growth in its degrees of freedom; the effective string scale is the AdS scale (unity in our units). It is clearly impossible for such theories to have a gravitational description (in any dimension).

Cohomology states	Multiplicity	$\mathcal{N}=2$ Primary	Allowed k
$(\frac{1}{2},0,\frac{1}{2})$ $(1,\frac{1}{2},\frac{1}{2})$ $(\frac{3}{2},\frac{1}{2},1)$ $(2,\frac{1}{2},\frac{3}{2})$	2 1 1 3	$(\frac{1}{2},0,\frac{1}{2})$ $(\frac{1}{2},0,-\frac{1}{2})$ $(1,0,0)$ $(\frac{3}{2},0,\frac{1}{2})$	
$(k,0,k)$	3	$(k,0,k)$	$k\in\frac{Z^+}{2}$,$k\geq 1$
$(k+\frac{1}{2},\frac{1}{2},k)$	3 , if $k=$ even 4 , if $k\neq$ even	$(k,0,k-1)$	$k\in\frac{Z^+}{2}$, $k\geq 2$
$(2k+1,1,2k)$ $(k+\frac{3}{2},1,k+\frac{1}{2})$	2 1	$(2k+\frac{1}{2},\frac{1}{2},2k-1)$ $(k+1,\frac{1}{2},k-\frac{1}{2})$	$k\in Z^+$ $k\in Z^+$
$(2k+\frac{1}{2},\frac{3}{2},2k-1)$	1	$(2k,1,2k-2)$	$k\in Z^+$

Table 4.17: Supersymmetric spectrum for $\mathcal{N}=2$ nongeneric deformation with $\lambda_1=0$

Note that in these theories the index undergoes a phase transition at a finite value of the chemical potential. In the 'high temperature' (more accurately small x) phase the logarithm of the index is of the order N^2. It seems possible that this index captures the entropy of supersymmetric black holes in the as yet mysterious bulk dual of these theories.

4.9 Discussion

The "nicest" theories we studied are the $\mathcal{N}=3$ theory with one adjoint hypermultiplet and the $\mathcal{N}=2$ superpotential deformed theories with two adjoint chiral multiplets and with $U(1)$ or no flavor symmetry. We found that their supersymmetric spectrum consists of only operators of spin ≤ 2, suggesting a possible supergravity dual in the strong coupling limit. In the $\mathcal{N}=3$ case, while part of the supersymmetric spectrum looks like the Kaluza-Klein spectrum of 7-dimensional supergravity compactified on S^3, there is an additional tower of states in spectrum that do not seem to come from standard KK modes. In the $\mathcal{N}=2$ deformed theories, the spectrum contains states of arbitrarily high $U(1)$ charges, suggesting that they could come from KK modes of S^1-compactification of supergravity theories, but to identify their duals appears difficult due to some unusual features of the spectrum.

The $\mathcal{N}=2$ theories with one adjoint chiral multiplet are even more intriguing. With either $\mathrm{Tr}\,\Phi^4$ or $\mathrm{Tr}\,\Phi^3$ superpotential, there is a line of fixed points. At these fixed point theories, in the large N limit, the supersymmetric spectrum involves a single tower of operators/states of arbitrarily high spin as well as R-charge. This rules out the possibility of a supergravity dual, but leaves open the possibility that the duals of the strongly coupled SCFTs are higher spin theories of gravity in AdS_4.

The most mysterious case is the $\mathcal{N}=2$ theory with one adjoint chiral multiplet and no superpotential. The R-charge of this theory is renormalized and decreases monotonically with the 't Hooft coupling λ. At some point, when $\lambda=\lambda_2^f\approx 1.23$, the operator $\mathrm{Tr}\,\Phi^2$ becomes a free field and decouples from the theory. At this point, a new $U(1)$ global symmetry emerges and in principle the $\mathcal{Z}$-minimization prescription no longer determines the superconformal R-charge. If we assume that the naive $\mathcal{Z}$-minimization is still valid at large N for $\lambda > \lambda_2^f$, then we find that the renormalized R-charge approaches zero asymptotically at strong coupling. If this is true, apart from the decoupled free fields, the BPS spectrum involves a discretum of states starting at dimension $\Delta = 1/2$. While at general λ the BPS spectrum consists of towers of states of arbitrarily high spin and R-charge, the R-charge form a discretum at strong coupling, suggesting that a new noncompact dimension emerges in the higher spin gravity dual.

Finally, in the cases with more than two adjoint flavours, the number of supersymmetric states grow exponentially with the dimension. It suggests that their dual theories are string theories in AdS_4 with an exponentially growing tower of *supersymmetric* string oscillator excitations. The superconformal index of these theories as a functional of the chemical potential undergoes a phase transition. After this phase transition, these theories are likely to be dual to supersymmetric black holes in the yet to be determined dual string theories in AdS_4.

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Chapter 5

Conclusions

In this thesis we studied various aspects of the landscape of vacua in string theory. We concentrated on theories with four dimensional effective gravity theory description and a negative cosmological constant. We explored different phases in the landscape from the gravity side and also from the field theory side by using the AdS/CFT correspondence.

From gravity side, we explored the stability of nonsupersymmetric vacua in the string landscape. Building on existing literature, we constructed a large class of perturbatively stable vacua and then looked for tunneling instabilities into nearby vacua in landscape. A large number of decay channels were ruled out by our analysis, suggesting that some non supersymmetric vacua could be stable after all. It would be instructive to construct more examples of such vacua with all moduli stabilized and investigate their non perturbative stability. This program might ultimately provide hints to constructing stable desitter vacua which is of relevance to string phenomenology.

Further continuing the exploration of the landscape, we constructed gravity systems with reasonable thermodynamics and illustrated how non fermi liquid can arise in such systems using holography. We used two point functions of fermions to ascertain the nature of the excitations. Recently, entanglement entropy has emerged as another probe for the existence of fermi surfaces. It would be interesting to explore the strong coupling phenomenon in field theory by studying their gravity duals with all the probes that holography provides us and to build a consistent picture.

On the field theory side, we studied supersymmetric chern simons theories with simple matter content with the intent to check whether they admit gravity duals. We deduced the protected matter content which can prove useful in identifying the dual gravity system. Recently, localization techniques have been used to compute exact quantities like partition function in certain supersymmetric theories. This might provide hints for possible gravity dual and also provide evidence for duality among quantum field theories.

As string theory progresses, we expect to have a more complete understanding of the string landscape. We hope this leads to a better understanding of the nonperturbative nature of string theory and the structure of quantum field theories.