Black holes in Yang Mills theories

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by

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Shiraz Minwalla, at the Tata Institute of Fundamental Research, Mumbai.

Jyotirmoy Bhattacharya.

In my capacity as supervisor of the candidates thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Shiraz Minwalla.

Date:

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Synopsis

S1 Introduction

Nature provides us with an enormously wide variety of diverse phenomena. A theoretical physicist endeavors to understand these phenomena through a set of mathematically consistent theories. In several moments of triumph of theoretical physics, one is led to a connection between apparently disconnected phenomenon. This throws deeper insights into such phenomenon and enables us to understand them through a single theoretical framework. Often it is found that the mathematical power of such a framework leads us to unifications connecting distinct areas of physics. One important example of such a framework is quantum field theory, which has had enormous theoretical and experimental success. Another very promising example, with an elegant mathematical structure, is string theory.

String theory which started as a prominent attempt to quantize gravity, over the years, has grown into a much richer framework with several path breaking discoveries. As it stands today, it has a mathematically consistent structure through which we can not only study Plank-scale physics, but also attempt to address other problems of sufficient complexity and interest. One extremely important implication of string theory is the AdS/CFT correspondence [1], which constitutes an essential mathematical tool with wide applicability. This conjectured correspondence relates a theory of gravity in AdS space to a conformal gauge theory living on the boundary of the AdS space. This duality may be exploited to improve our understanding of both, gauge field theories (especially in the strongly coupled regime where there are very limited tools to study them directly) and gravity. In this thesis, we shall present several realizations of such studies uncovering new and interesting facts about gravity and quantum field theory, in a particular long wavelength limit.

In the long-wavelength limit, gauge theories admit a description in terms of a few effective degrees of freedom which constitutes a hydrodynamic description. The gravity analogs of these effective hydrodynamic degrees of freedom can be obtained through the AdS/CFT correspondence. This map between gravity and hydrodynamics has attracted much attention recently. This is because, it not only throws light on transport properties of certain exotic phases of matter (like the quark gluon plasma) but also has the potential to address some of the yet less understood phenomenon in hydrodynamics (like turbulence). We have worked out the precise details of the fluid-gravity correspondence for the case when there is a conserved global charge in the boundary gauge theory [2]. The bulk system in this case is a deformation of the charged black hole in AdS space which is a solution of Maxwell-Einstein system with a negative cosmological constant. In our analysis we have uncovered a novel transport phenomenon which occurs very generically in systems where the global symmetry is anomalous. To our knowledge, this kind of transport was never before considered in any treatment of hydrodynamics.

Continuing our endeavor to understand hydrodynamics with the help of gravity, we ventured into superfluids [3, 4]. The superfluid phase, is one in which an operator charged under a global symmetry gains expectation value in the symmetry broken phase. The expectation value of this charged operator can be thought of as the order parameter of the phase transition. A bulk gravity solution that is dual to such a phase is a charged black hole with a scalar hair. We perturbatively constructed such analytical hairy black hole solutions of the Einstein-Maxwell-Scalar systems and studied them away from equilibrium in derivative expansion. Subsequently, we derived the boundary hydrodynamics, yielding the transport properties in superfluids. Even in this case we were lead to discover a new transport phenomenon particular to superfluids which (as far as we know) was absent in the superfluid literature till now.

As we emphasized earlier, we may also use the fluid-gravity correspondence to learn about gravity solutions which may not be easily analyzed using Einstein equations. Gravity in five or more dimensions is very rich as it can potentially have black hole solutions with extremely exotic horizon topologies (e.g. black rings). Beside being of high importance to string theory these solutions are interesting from a purely gravity perspective. Using a Scherk-Schwarz compactification in the boundary directions it was possible to study localized plasma configurations which solved the Navier-Stokes equations. Then using the AdS/CFT correspondence these configurations could be mapped to horizon topologies of black objects in the bulk. Employing some numerical and perturbative analysis we were able to prove the existence of new black objects (in five or more dimensions) with non-trivial horizon topologies and predict some of their properties [5, 6].

S2 Hydrodynamics of charged fluids and Superfluids from gravity

It has recently been demonstrated that a class of long distance, regular, locally asymptotically AdS_{d+1} solutions to Einstein's equations with a negative cosmological constant is in one to one correspondence with solutions to the charge free Navier Stokes equations in d dimensions [8, 9, 10, 11, 12, 13, 14, 16] [1].

The connection between the equations of gravity and fluid dynamics, described above, was demonstrated essentially by use of the method of collective coordinates. The authors of [8, 10, 11, 12, 13, 16] noted that there exists a d parameter set of exact, asymptotically AdS_{d+1} black brane solutions of the gravity equations parameterized by temperature and velocity. They then used the 'Goldstone' philosophy to promote temperatures and velocities to fields. The Navier Stokes equations turn out to be the effective 'chiral Lagrangian equations' of

[1] There exists a large literature in deriving linearize hydrodynamics from AdS/CFT. See([17] - [50]). There have been some recent work on hydrodynamics with higher derivative corrections [52, 53].

the temperature and velocity collective fields.

This initially surprising connection between gravity in $d+1$ dimensions and fluid dynamics in d dimensions is beautifully explained by the AdS/CFT correspondence. Recall that a particular large N and strong coupling limit of that correspondence relates the dynamics of a classical gravitational theory (a two derivative theory of gravity interacting with other fields) on AdS$_{d+1}$ space to the dynamics of a strongly coupled conformal field theory in d flat dimensions. Now the dynamics of a conformal field theory, at length scales long compared to an effective mean free path (more accurately an equilibration length scale) is expected to be well described by the Navier Stokes equations. Consequently, the connection between long wavelength solutions of gravity and the equations of fluid dynamics - directly derived in [8] - is a natural prediction of the AdS/CFT correspondence. Using the AdS/CFT correspondence, the stress tensor as a function of velocities and temperatures obtained above from gravity was interpreted as the fluid stress tensor of the dual boundary field theory in its deconfined phase.

We generalize the above correspondence to the case where the boundary fluid has a conserved global $U(1)$ symmetry [2]. In such a system we have an additional degree of freedom which may be taken to be the chemical potential. In this case in addition to the stress tensor, We also have a charged current which is conserved. Thus we have two constitutive relations expressing the stress tensor and the charge current in terms of the velocities, temperature and the chemical potential (see §S2.1). The bulk description of this system comprises of a Einstein-Maxwell system. The charged deconfined phase in the boundary corresponds to a charged black hole in the bulk as we explain in more detail in §S2.2. For the five dimensional gravity system to be a consistent truncation of type-IIB supergravity a Chern-Simons terms was required to be present. This term manifested itself as an anomaly of the conserved $U(1)$ current and is found to have interesting hydrodynamic consequences.

We further probed this fluid-gravity connection including the situation when the global $U(1)$ is spontaneously broken [3, 4]. The degrees of freedom in this case are further enhanced to include the phase of the charged scalar operator which is the Goldstone boson for the spontaneously broken continuous symmetry (this degree of freedom is included in a slightly indirect way - see §S2.1). As explained in the introduction this situation corresponds to the phenomenon of superfluidity from the boundary point of view. The bulk dual of this phase are hairy black holes as explained in more detain in §S2.2.

S2.1 Hydrodynamic description

A quantum field theory in the long wavelength limit is assumed to admit a hydrodynamic descriptions for its near equilibrium dynamics. This description is generically based on a few classical fields (like local fluid velocities, temperature etc.) and bypasses all the complexities of interactions between more microscopic degrees of freedom. For this reason hydrodynamics is a good description at long wavelengths even for strongly interacting quantum field theories. The intrinsic quantum nature of theory is generically lost in such a macroscopic description but in certain special and interesting cases it may manifest even at the macroscopic level as we will see below.

The equation of motion for fluid dynamics in the relativistic case are merely the conservation principles corresponding to the symmetries of the system (like the conservation of the stress tensor corresponding to space-time translational invariance, or the conservation of a charge current corresponding to some globally conserved charge). Once these equations are written in terms of the hydrodynamic fields the conservation equations provide us with a consistent set of equations which we can solve (with a given set boundary conditions) to obtain various allowed fluid configuration. This is realized by expressing the stress tensor and the charged current, for example, in terms of the hydrodynamic field variables. Such relations are called constitutive relations and contain information about the microscopic theory at large wavelengths where hydrodynamics is well defined.

These constitutive relations are the key ingredient in the hydrodynamic description of a system and therefore will remain the key focus of discussions. Since hydrodynamics is a long wavelength phenomenon therefore derivative expansion provides a natural method to write these constitutive relations. The form of the such constitutive relations in a derivative expansion is fixed by Lorentz symmetry and two other very powerful principles. The first one is a statement of local form of the second law of thermodynamics (which implies the divergence of the entropy current is positive semi-definite) and the second one is a statement of time translation invariance known as the Onsager's principle. We shall elucidate these abstract discussions below with two examples - one with a ordinary fluid with globally conserved $U(1)$ charge and the other in which such a global $U(1)$ is spontaneously broken giving rise to an additional long-wavelength mode by the Goldstone theorem (i.e. superfluids). In the case of ordinary charged fluids we shall consider the effects of a parity violating anomaly while in the case of superfluids we shall confine our discussions to the parity even sector.

Charged fluids

In this subsection we construct the most general equations of Lorentz invariant charged fluid dynamics consistent with the second law of thermodynamics upto first order in derivative expansion. The long-wavelength degrees of freedom of a locally equilibrated system with a single global $U(1)$ charge can be taken to be the velocity field $u_\mu(x)$ (normalized so that $u^\mu u_\mu = -1$), the temperature field $T(x)$ and a chemical potential field $\mu(x)$. As mentioned previously, both the energy momentum tensor and the charged current can be expressed in terms of these five fields and their gradients through the constitutive relations. The equations of motion of charged fluid dynamics are the conservation of the stress tensor and charge current

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu \\ \nabla_\mu J^\mu &= -\frac{c}{8}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \end{aligned} \tag{1}$$

which provides the five equations for the five hydrodynamic fields. In these equations we have allowed for the possibility that the current in question has a $U(1)$ anomaly. We call the coefficient c the anomaly coefficient. We have also allowed the current to be coupled to an external source with field strength $F_{\mu\nu}$. At this point it would also be convenient to define the background electric and magnetic fields by the relations

$$E_\mu = F_{\mu\nu}u^\nu; \quad B_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}u^\nu F^{\lambda\sigma}. \tag{2}$$

To completely determine the equations of motion it remains to determine the dependence of $T^{\mu\nu}$ and J^μ on the fields $u^\mu(x)$, $T(x)$, $\mu(x)$ and their derivatives.

By considering a stationary fluid for which $u^\mu = (1, 0, 0, 0)$ and using boost invariance one can argue that the stress tensor and charge current take the form

$$\begin{aligned} T^{\mu\nu} &= (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} + T^{\mu\nu}_{diss} \\ J^\mu &= qu^\mu + J^\mu_{diss} \end{aligned} \tag{3}$$

where $\eta_{\mu\nu} = \text{diag}(-+++)$ is the Minkowski-metric. $T^{\mu\nu}_{diss}$ and J^μ_{diss} are the contributions to the stress tensor and charge current that involve derivatives of μ, T and u^μ. The equations that express $T^{\mu\nu}_{diss}$ and J^μ_{diss} in terms of fluid dynamical fields and their derivatives are termed constitutive relations. In the long wavelength fluid dynamical limit it is sensible to expand the constitutive relations in powers of derivatives of the fluid dynamical fields u^μ, T and μ. We refer to such an expansion as a derivative expansion and the terms which are linear in gradients as first order terms.

Now the possibility of redefinitions of the hydrodynamic fields (u^μ, T and μ) by first (or higher) order quantities introduces arbitrariness in the quantities $T^{\mu\nu}_{diss}$ and J^μ_{diss}. We require 5 conditions to fix this arbitrariness. This is realized by imposing 5 frame choice conditions on $T^{\mu\nu}_{diss}$ and J^μ_{diss}. Although there are several choice of frames that are adopted in the for the purpose of discussion in this synopsis we shall adhere to the so called transverse (Landau) frame where the following conditions hold

$$u_\mu T^{\mu\nu}_{diss} = 0; \quad u_\mu J^\mu_{diss} = 0. \tag{4}$$

We shall specify all our results in this transverse frame.

The entropy current at first order has the canonical form

$$J^\mu_S = su^\mu - \frac{1}{T}u_\mu T^{\mu\nu}_{diss} - \frac{\mu}{T}J^\mu_{diss}\,. \tag{5}$$

s being the thermodynamic entropy density of our fluid. Note that the second term is zero in the transverse frame that we have chosen.

Let us now focus on the case when the anomaly is absent i.e. $c = 0$. In parity even sector, this canonical form in (5) is in fact unique if we require the divergence of this entropy current is positive semi-definite [4]. We show this by first considering the most general entropy current that is allowed by symmetry. We then demand that the divergence of J^μ_S be positive semi-definite for any solution of the equations of superfluid hydrodynamics *on an arbitrary background spacetime* and that the Onsager relations are satisfied.[2] These restrictions forces

[2] Recall that the second law of thermodynamics must apply in any conceivable consistent situation. In particular it must apply when the system is formulated on an arbitrary background spacetime provided the system is free of diffeomorphism anomalies. This condition is true of all experimental superfluids as well as all superfluids obtained via the AdS/CFT correspondence.

the entropy current to take the canonical form.

We would like to emphasize that the requirement that the entropy current be of positive divergence *in an arbitrary background spacetime* provides powerful constraints on the form of the entropy current and through it, on the form of the possible dissipative corrections to the hydrodynamic constitutive relations *even in flat space*. For instance, the divergence of the entropy current could contain a term proportional to

$$\nabla_\mu J^\mu_S \propto v_1 R_{\mu\nu} u^\mu u^\nu + \dots \tag{6}$$

where u^μ is the fluid velocity, $R_{\mu\nu}$ is the Ricci tensor, and v_1 is some arbitrary coefficient function. The divergence of the entropy current may also contain many other terms independent of curvatures. However, for any given fluid flow these other terms can be held fixed while $R_{\mu\nu}u^\mu u^\nu$ is made arbitrarily negative by tuning the curvature tensor.[3] It follows that the divergence of the entropy current is positive for an arbitrary fluid flow on an arbitrary spacetime only if $v_1 = 0$. Thus, we find a constraint on the entropy current for fluid motion in a flat space background, even though we needed to move to a curved spacetime in order to obtain this constraint.

The divergence of the entropy current in (5) is given by

$$\partial_\mu J^\mu_S = -\frac{1}{T}\partial_\mu u_\nu T^{\mu\nu}_{diss} + \left(\frac{E_\mu}{T} - P_{\mu\nu}\partial^\nu \frac{\mu}{T}\right) J^\mu_{diss}. \tag{7}$$

the requirement of positivity of (7) yields

$$T^{\mu\nu}_{diss} = -\eta\; \sigma^{\mu\nu} - \frac{\eta'}{3}\; P^{\mu\nu}\Theta; \quad J^\mu_{diss} = \sigma\left(\frac{E^\mu}{T} - P^{\mu\nu}\partial_\nu \frac{\mu}{T}\right). \tag{8}$$

where

$$\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\frac{\Theta}{3}\right); \quad \Theta = \partial_\mu u^\mu; \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu. \tag{9}$$

Note that the coefficient of bulk viscosity η' is zero for conformal fluids (which is true in particular for the gravitational fluid).

This analysis was generalized in [68] to include the effects of the anomaly. The chief difference from the $c = 0$ case is the fact that the divergence of the canonical form of the entropy current has an additional piece

$$\partial_\mu J^\mu_s = -\frac{1}{T}\partial_\mu u_\nu T^{\mu\nu}_{diss} + \left(\frac{E_\mu}{T} - P_{\mu\nu}\partial^\nu \left[\frac{\mu}{T}\right]\right) J^\mu_{diss} - \frac{c\mu}{T} E_\mu B^\mu. \tag{10}$$

The sign of this additional term may be easily manipulated by choosing a particular configuration of electric and magnetic fields. Thus in the presence of the background electromagnetic field and a non-zero c we are forced to modify the canonical form of the entropy current so as to ensure the positivity of its divergence. This modification in turn forces

[3]Note that curvature tensors do not contribute to the fluid equations at first order, so it is consistent to hold fluid flows fixed while taking curvatures to be very large.

a modification in the constitutive relations. The authors of [68] found that most general modification to the canonical form of the entropy current (and hence the most general entropy current including the parity odd sector) [4] consistent with the positivity condition was given by

$$J^\mu_S = su^\mu - \frac{1}{T} u_\mu T^{\mu\nu}_{diss} - \frac{\mu}{T} J^\mu_{diss} + \alpha\ l^\mu + \alpha_B\ B^\mu\,. \tag{11}$$

where $l^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda u_\sigma$. Note that the second term is zero in the transverse frame in which we work. Although $T^{\mu\nu}_{diss}$ does not require any further modification the charge current has to be modified in the following way

$$J^\mu_{diss} = \sigma\left(\frac{E^\mu}{T} - P^{\mu\nu}\partial_\nu \frac{\mu}{T}\right) + D\ l^\mu + D_B\ B^\mu. \tag{12}$$

The restrictions of positivity further do not allow for any new free parameters in the constitutive relations or in the entropy current and the parameters α, α_B, D and D_B are completely determined to be [5]

$$\begin{aligned} D &= c\frac{\mu^3}{3T} \\ D_B &= c\frac{\mu^2}{2T} \\ \alpha &= c\left(\mu^2 - \frac{2}{3}\frac{q}{\rho+P}\mu^3\right) \\ \alpha_B &= c\left(\mu - \frac{1}{2}\frac{q}{\rho+P}\mu^2\right) \end{aligned} \tag{13}$$

If we set the background electromagnetic fields to zero then the only remaining physical effect is a new term in the charge current proportional to l^μ whose coefficient is completely determined by the anomaly coefficient, the chemical potential and the temperature. Note that before this analysis was performed in [68], this new term was discovered in a gravity calculation in [2, 66] as we shall describe in §S2.2.

superfluids

By definition, a superfluid is a fluid phase of a system with a spontaneously broken global symmetry. When discussing superfluids this forces us to consider the gradient of the Goldstone boson as an extra hydrodynamical degrees of freedom in addition to the standard variables u^μ, T and μ. More precisely, if we denote the Goldstone Boson by ψ (ψ is the phase of the condensate of the charged scalar operator) and we also wish turn on a background gauge field A_μ then

$$\xi_\mu = -\partial_\mu\psi + A_\mu \tag{14}$$

represents the covariant derivative of the Goldstone Boson and is an extra hydrodynamic degree of freedom. According to the Landau-Tisza two fluid model the superfluid should

[4] Note that the fact that in the absence of anomalies, the canonical form of the entropy current was the most general entropy current consistent with the second law of thermodynamics was shown in [4] which appeared after [68].

[5] upto integration constants which vanished in all holographic calculations using gravity.

be thought of as a two component fluid: a condensed component and a non condensed or normal component. The velocity field of the normal fluid is given by u_μ and the velocity of the condensed phase is proportional to ξ_μ. It is often convenient to define the component of ξ orthogonal to u,

$$\zeta^\mu = P^{\mu\nu}\xi_\nu \,. \tag{15}$$

We shall also find it useful to define the quantities

$$n^\mu = \frac{\zeta^\mu}{\zeta}; \quad \tilde{P}^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu - n^\mu n^\nu. \tag{16}$$

where ζ is the magnitude of the vector ζ^μ and $\tilde{P}^{\mu\nu}$ is a projector orthogonal to both u^μ and ξ^μ (or ζ^μ or n^μ).

The equations of motion of the superfluid are given by

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu \\ \partial_\mu J^\mu &= c E_\mu B^\mu \\ \partial_\mu \xi_\nu - \partial_\nu \xi_\mu &= F_{\mu\nu} \end{aligned} \tag{17}$$

together with the constitutive relations

$$\begin{aligned} T^{\mu\nu} &= (\rho + P) u^\mu u^\nu + P\eta^{\mu\nu} + f\xi^\mu\xi^\nu + T^{\mu\nu}_{diss} \\ J^\mu &= q u^\mu - f\xi^\mu + J^\mu_{diss} \\ u{\cdot}\xi &= \mu + \mu_{diss} \end{aligned} \tag{18}$$

Note that ξ^μ being the gradient of the phase is a microscopically defined quantity and therefore we shall not allow any field redefinition of it. Writing the equilibrium stress tensor and charge current in terms of ξ^μ as in (18) itself involves a partial choice of frame (in a frame where $\mu_d iss$ is not set to zero) [6]. Apart from this choice we also have to specify five more condition to fix the redefinition ambiguity of five other hydrodynamic fields. In order to fix that, just like in §S2.1, we shall adhere to the transverse frame condition specified in (4).

As was the case for the theory of charged fluids which we described in the previous section, superfluids also allow for a simple 'canonical' entropy current [3]

$$J^\mu_{S\,\text{canon}} = s u^\mu - \frac{\mu}{T} J^\mu_{diss} - \frac{u_\nu T^{\mu\nu}_{diss}}{T} \tag{19}$$

s where s is the thermodynamical entropy density of our fluid and is related to ρ and P through the Gibbs-Duhem relation

$$\rho + P = sT + \mu q \tag{20}$$

and

$$dP = s dT + q d\mu + \frac{1}{2} f d\xi^2 \tag{21}$$

[6]This frame has been referred to as a 'fluid frame' in [3]

where

$$\xi = \sqrt{-\xi^\mu \xi_\mu}\,. \tag{22}$$

It has been demonstrated in [3] that the entropy current (19) is invariant under field redefinitions. Following arguments very close to the that for ordinary charged fluids we went on to construct the most general entropy current [4]. Even in this case of parity even superfluids we found that demanding the divergence of J^μ_S be positive semi-definite for any solution of the equations of superfluid hydrodynamics on an arbitrary background spacetime almost completely fixes the entropy current to the canonical form (19). However, in this case, consistent with the above conditions we can add to the canonical entropy current a term of the form $\partial_\nu \left(c_0 \left(\xi^\mu u^\nu - \xi^\nu u^\mu\right)\right)$, c_0 being an arbitrary function of the scalar fields T, μ and ξ. Such a term is unphysical because being manifestly divergenceless it does not contribute to the divergence of the entropy current and hence does not play any role in determining the constitutive relations. Also from the bulk point of view this term is related to a trivial ambiguity in the pullback of the area form on the horizon, which gives the boundary entropy density (see [4, 9] for more details).

It was also shown in [3, 4] that the divergence of this entropy current in (19) is given by

$$\partial_\mu J^\mu_s = -\partial_\mu \left(\frac{u_\nu}{T}\right) T^{\mu\nu}_{diss} - \left(\partial_\mu \left(\frac{\mu}{T}\right) - \frac{E_\mu}{T}\right) J^\mu_{diss} + \frac{\mu_{diss}}{T} \partial_\mu \left(f \xi^\mu\right) \tag{23}$$

In the case of superfluid dynamics, the $SO(3,1)$ tangent space symmetry at any point is generically broken down to $SO(2)$ by the nonzero velocity fields u^μ and ξ^μ. Representations of $SO(2)$ are all one dimensional. We refer to fluid dynamical data that is invariant under $SO(2)$ as scalar data. All other fluid data has charge $\pm m$ under $SO(2)$, where m is an integer. There is always as much $+m$ as m data. We will find it useful to group together $+1$ and 1 charge data into a two column which we refer to as vector data; similarly we group $+2$ and 2 data together into tensor data. In all the sections below we shall use the terminology of scalar, vector and tensor of $SO(2)$ in the above sense.

Constraints from positivity of entropy production and Onsager relations

We will now explore the constraints on dissipative coefficients from the physical requirements of positivity of entropy production and the Onsager reciprocity relations. We will find these requirements cut down the 36 parameter set of possible dissipative coefficients (assuming parity invariance) to a 14 parameter set of coefficients that are further constrained by positivity requirements. For concreteness we present our analysis in the transverse frame.

Constraints from positivity of entropy production

The divergence of the 'canonical' entropy current, given by (23), involves only terms proportional to $\partial_\mu u_\nu T^{\mu\nu}_{diss}$, $\partial_\nu(\mu/T) J^\nu_{diss}$ and $\mu_{diss} \partial_\mu (q_s \xi^\mu/\xi)$. Let us examine these terms one by one. In the transverse frame

$$\partial_\mu u_\nu T^{\mu\nu}_{diss} = \sigma_{\mu\nu} T^{\mu\nu}_{diss} + \left(\frac{\Theta}{3}\right) \left(T_{diss}\right)^\theta_\theta$$

where $\sigma_{\mu\nu}$ and Θ are defined in (9).

Now the field $\sigma_{\mu\nu}$ has one scalar piece of data

$$S_w = n^\mu n^\nu \sigma_{\mu\nu}$$

one vector piece of data

$$[V_b]_\mu = \tilde{P}_\mu^\nu n^\alpha \sigma_{\nu\alpha}$$

and a tensor piece of data

$$\mathcal{T}_{\mu\nu} = \tilde{P}_\mu^\alpha \tilde{P}_\nu^\beta \sigma_{\alpha\beta}$$

The trace of $T^{\mu\nu}_{diss}$ couples to another scalar piece of data

$$S_{w'} = \partial_\mu u^\mu.$$

Similarly, in the transverse gauge

$$\partial_\nu(\mu/T) J^\nu_{diss} = P^\nu_\alpha \partial_\nu(\mu/T) J^\alpha_{diss},$$

where $P_{\mu\nu}$ is the projection operator (defined in (9)) that projects orthogonal to u_μ only. The quantity $P^\nu_\alpha \partial_\nu(\mu/T)$ has one scalar piece of data

$$S_b = (n^\mu \partial_\mu)\,(\mu/T)$$

and one vector piece of data

$$[V_a]_\mu = \tilde{P}^\sigma_\mu \partial_\sigma\,(\mu/T)$$

Finally

$$S_a = \frac{\partial_\mu(q_s \xi^\mu/\xi)}{T^3}$$

is itself a scalar piece of data.

In other words we conclude that the expression for the divergence of the entropy current, (23), depends explicitly (i.e. apart from the dependence of $T^{\mu\nu}_{diss}$ J^μ_{diss} and μ_{diss} on these terms) only on 4 scalar expressions, 2 vector expressions and one tensor expression. At first order in derivatives the number of on-shell inequivalent scalar vector and tensors are respectively 7, 5 and 2 in the parity even sector [3, 4]. Let us choose these 4 vectors scalars S_a, S_b, S_w and $S_{w'}$, supplemented by 3 other arbitrarily chosen scalar expressions S^I_m $(m = 1 \ldots 3)$ as our 7 independent scalar expressions. Similarly we choose the 2 vectors $[V_a]_\mu$ and $[V_b]_\mu$ supplemented by 3 other arbitrarily chosen expressions $[V^I_m]_\mu$ $(m = 1 \ldots 3)$ as our four independent vector expressions. We also choose $\mathcal{T}_{\mu\nu}$ as one of our two independent tensor expressions We proceed to express $T^{\mu\nu}_{diss}$, J^μ_{diss} and μ_{diss} as the most general linear combinations of all combinations of independent expressions allowed by symmetry

$$
\begin{aligned}
T^{\mu\nu}_{diss} = T^3 \Bigg[&\left(P_a S_a + P_b S_b + P_w S_w + P_{w'} S_{w'} + \sum_{m=1}^{3} P^I_m S^I_m \right) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\
&+ \left(T_a S_a + T_b S_b + T_w S_w + T_{w'} S_{w'} + \sum_{m=1}^{3} T^I_m S^I_m \right) P^{\mu\nu} \\
&+ E_a \left(V^\mu_a n^\nu + V^\nu_a n^\mu \right) + E_b \left(V^\mu_b n^\nu + V^\nu_b n^\mu \right) + \sum_{m=1}^{3} E^I_m \left([V^I_m]^\mu n^\nu + [V^I_m]^\nu n^\mu \right) \\
&+ \tau \mathcal{T}^{\mu\nu} + \tau_2 T^{\mu\nu}_2 \Bigg]
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
J^\mu_{diss} &= T^2 \left[\left(R_a S_a + R_b S_b + R_w S_w + R_{w'} S_{w'} + \sum_{m=1}^{3} R^I_m S^I_m \right) n^\mu + C_a V^\mu_a + C_b V^\mu_b + \sum_{m=1}^{3} C^I_m [V^I_m]^\mu \right] \\
\mu_{diss} &= - \left[Q_a S_a + Q_b S_b + Q_w S_w + Q_{w'} S_{w'} + \sum_{m=1}^{3} Q^I_m S^I_m \right]
\end{aligned}
\tag{25}
$$

Plugging (24) and (25) into (23) we now obtain an explicit expression for the divergence of the entropy current as a quadratic form in first derivative independent data. We wish to enforce the condition that this quadratic form is positive definite. Now the quadratic form from (23) clearly has no terms proportional to $(S^I_m)^2$. It does, however, have terms of the form (for instance) $S_a S^I_m$, and also terms proportional to S_a^2. Now it follows from a moments consideration that no quadratic form of this general structure can be positive unless the coefficient of the $S_a S^I_m$ term vanishes. [7] Using similar reasoning we can immediately conclude that the positive definiteness of (23) requires that

$$
P^I_m = T^I_m = E^I_m = C^I_m = R^I_m = \tau_2 = 0. \tag{26}
$$

(26) is the most important conclusion of this subsubsection. It tells us that a 21 parameter set of first derivative corrections to the constitutive relations are consistent with the positivity of the canonical entropy current.

Of course the remaining 21 parameters are not themselves arbitrary, but are constrained to obey inequalities in order to ensure positivity. In order to derive these conditions we plug (26) into (24) and (25) and use (23) so that the divergence of the entropy current is the linear sum of three different quadratic forms (involving the tensor terms, vector terms and scalar terms respectively)

$$
\partial_\mu J^\mu_s = T^2 \left(Q_s + Q_V + Q_T \right) \tag{27}
$$

where

$$
Q_T = -\tau \mathcal{T}^2
$$

[7] For instance the quadratic form $x^2 + cxy$ (where c is a constant) can be made negative by taking $\frac{y}{x}$ to either positive or negative infinity (depending on the sign of c) unless $c = 0$.

$$
\begin{aligned}
Q_V =& - C_a V_a^2 - (C_b + E_a)\, V_b V_a - E_b V_b^2 \\
=& - C_a \left[V_a + \left(\frac{C_b + E_a}{2C_a} \right) V_b \right]^2 - \left[E_b - \frac{(C_b + E_a)^2}{4C_a} \right] V_5^2
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
Q_S =& - P_w S_w^2 - T_{w'} S_{w'}^2 - Q_a S_a^2 - R_b S_b^2 \\
& - (Q_w + P_a)\, S_w S_a - (Q_{w'} + T_a)\, S_{w'} S_a - (R_w + P_b)\, S_w S_b \\
& - (R_{w'} + T_b)\, S_{w'} S_b - (R_a + Q_b)\, S_a S_b - (T_w + P_{w'})\, S_w S_{w'}
\end{aligned} \tag{29}
$$

Positivity of the entropy current clearly requires that Q_T Q_V and Q_S are separately positive. Let us examine these conditions one at a time. For Q_T to be positive it is necessary and sufficient that $\tau \leq 0$. This is simply the requirement that the normal component of our superfluid have a positive viscosity. In order that Q_V be positive, it is necessary and sufficient that

$$
C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4E_b C_a \geq (C_b + E_a)^2. \tag{30}
$$

Note that this expression involves C_a and E_b on the LHS but the different quantities C_b and E_a on the RHS; the last inequality above is satisfied roughly, when C_b and E_a are larger in modulus than C_a and E_b.

Finally Q_S, listed in (29), is a quadratic form in the the 4 variables S_a, S_b, S_w and $S_{w'}$. We demand that this scalar form be positive. We will not pause here to explicate the precise inequalities that this condition imposes on the coefficients. See below, however, for the special case of a Weyl invariant fluid.

Constraints from the Onsager Relations

After imposing the positivity of the divergence of the entropy current we found that first order dissipative corrections to the equations of perfect superfluid dynamics take the form

$$
\begin{aligned}
T^{\mu\nu}_{diss} =& T^3 \Big[(P_a S_a + P_b S_b + P_w S_w + P_{w'} S_{w'}) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\
& + (T_a S_a + T_b S_b + T_w S_w + T_{w'} S_{w'})\, P^{\mu\nu} \\
& + E_a\, (V_a^\mu n^\nu + V_a^\nu n^\mu) + E_b\, (V_b^\mu n^\nu + V_b^\nu n^\mu) + \tau \mathcal{T}^{\mu\nu} \Big] \\
J^\mu_{diss} =& T^2 \Big[(R_a S_a + R_b S_b + R_w S_w + R_{w'} S_{w'})\, n^\mu + C_a V_a^\mu + C_b V_b^\mu \Big] \\
\mu_{diss} =& - [Q_a S_a + Q_b S_b + Q_w S_w + Q_{w'} S_{w'}]
\end{aligned} \tag{31}
$$

where the coefficients in these equations are constrained by the inequalities listed in the previous subsubsection. The coefficients that appear in these equations are further constrained by the Onsager reciprocity relations (see, for instance, the text book [73], for a discussion). These relations assert, in the present context, that we should equate any two dissipative parameters that multiply the same terms in the formulas (28) and (29) for entropy production. This implies that

$$\begin{gathered} Q_w = P_a, \quad Q_{w'} = T_a, \quad R_w = P_b \\ R_{w'} = T_b, \quad R_a = Q_b, \quad T_w = P_{w'}, \quad C_b = E_a \end{gathered} \tag{32}$$

In summary we are left with a 14 parameter set of equations of first order dissipative superfluid dynamics. The requirement of positivity constrains further these coefficients to obey the inequalities spelt out in the previous subsubsection.

Specializing to the Weyl invariant case in the transverse frame

Let us now specialize these results to the case of super fluid dynamics for a conformal superfluid. The analysis presented above is simplified in this special case by the fact that the trace of the stress tensor vanishes in an arbitrary state (and so in the fluid limit) of a conformal field theory. This fact reduces the number of explicit scalars that appear in (23) from 4 to 3 (the scalar $S_{w'}$ never makes an appearance). It follows that the requirement of Weyl invariance forces $P_{w'} = R_{w'} = T_{w'} = Q_{w'} = 0$. Moreover the requirement that $T^{\mu\nu}_{diss}$ be traceless forces $T_a = T_b = T_w = 0$. It turns out that there are no further constraints from the requirement of Weyl invariance. The expansion of the dissipative part of the stress tensor and charge current for a conformal superfluid is given by

$$\begin{aligned} T^{\mu\nu}_{diss} &= T^3 \bigg[(P_a S_a + P_b S_b + P_w S_w) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\ &\qquad + E_a \left(V_a^\mu n^\nu + V_a^\nu n^\mu \right) + E_b \left(V_b^\mu n^\nu + V_b^\nu n^\mu \right) + \tau \mathcal{T}^{\mu\nu} \bigg] \\ J^\mu_{diss} &= T^2 \bigg[(R_a S_a + R_b S_b + R_w S_w)\, n^\mu + C_a V_a^\mu + C_b V_b^\mu \bigg] \\ \mu_{diss} &= -\left[Q_a S_a + Q_b S_b + Q_w S_w \right] \end{aligned} \tag{33}$$

The entropy production is given by

$$\partial_\mu J^\mu_s = T^2 (Q_s + Q_V + Q_T) \tag{34}$$

where

$$Q_T = -\tau \mathcal{T}^2$$

$$\begin{aligned} Q_V &= - C_a V_a^2 - (C_b + E_a)\, V_b V_a - E_b V_b^2 \\ &= - C_a \left[V_a + \left(\frac{C_b + E_a}{2 C_a} \right) V_b \right]^2 - \left[E_b - \frac{(C_b + E_a)^2}{4 C_a} \right] V_b^2 \end{aligned} \tag{35}$$

$$Q_S = - P_w S_w^2 - Q_a S_a^2 - R_b S_b^2 + (Q_w + P_a)\, S_w S_a - (R_a + Q_b)\, S_a S_b + (R_w + P_b)\, S_w S_b \tag{36}$$

For the entropy current to be positive it is necessary and sufficient that $\tau \leq 0$ and that

$$C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4E_bC_a \geq (C_b + E_a)^2. \tag{37}$$

and that the quadratic form

$$\begin{aligned} Q_S &= a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_2x_3 + b_3x_1x_3 \\ &= a_1\left[x_1 + \left(\frac{b_1}{2a_1}\right)x_2 + \left(\frac{b_3}{2a_1}\right)x_3\right]^2 + \left(a_3 - \frac{b_3^2}{4a_1}\right)\left[x_3 + \left(\frac{2a_1b_2 - b_1b_3}{4a_1a_3 - b_3^2}\right)x_2\right]^2 \\ &\quad + \left[\frac{(4a_1a_2 - b_1^2)(4a_1a_3 - b_3^2) - (2a_1b_2 - b_1b_3)^2}{4a_1(4a_1a_3 - b_3^2)}\right]x_2^2 \end{aligned} \tag{38}$$

is positive with $x_1 = S_w$, $x_2 = S_a$ and $x_3 = S_b$ and

$$a_1 = -P_w, \quad a_2 = -Q_a, \quad a_3 = -R_b, \quad b_1 = Q_w + P_a, \quad b_2 = -(Q_b + R_a), \quad b_3 = R_w + P_b$$

For the last quadratic form to be positive it is necessary and sufficient that

$$\begin{aligned} &a_1 \geq 0 \\ &4a_1a_2 > b_1^2 \\ &(4a_1a_2 - b_1^2)(4a_1a_3 - b_3^2) > (2a_1b_2 - b_1b_3)^2 \end{aligned} \tag{39}$$

By rewriting (38) as a sum of squares in a cyclically permuted manner we can also derive the cyclical permutations of these equations.

In summary, the most general Weyl invariant fluid dynamics consistent with positivity on the entropy current is parameterized by a negative τ_1, 4 parameters in the vector sector constrained by the inequalities (37) and 9 parameters in the scalar sector, subject to the inequalities (39). These 14 dissipative parameters are further constrained by the 4 Onsager relations

$$Q_w = P_a, \qquad R_w = P_b, \qquad R_a = Q_b, \qquad C_b = E_a \tag{40}$$

leaving us with a 10 parameter set of final equations.

S2.2 Gravity derivation of boundary hydrodynamics

As explained before in §S1 the hydrodynamic systems described in §S2.1 has a dual description in terms of a gravitational system through the AdS/CFT correspondence. We exploit this duality not only to compute the transport coefficients of the gravitational fluid (fluid with a gravity dual), but also we use it to verify the general theory of hydrodynamics (developed in §S2.1) for the special case of conformal fluids.

Charged fluids

In this subsection we work with the Einstein Maxwell equations augmented by a Chern Simon's term. This is because the equations of IIB sugra on $\mathrm{AdS}_5 \times \mathrm{S}^5$ (which is conjectured to be dual to $\mathcal{N}=4$ Yang Mills) with the restriction of equal charges for the three natural Cartans, admit a consistent truncation to this system. Under this truncation, we get the following action

$$S=\frac{1}{16\pi G_5}\int\sqrt{-g_5}\left[R+12-F_{AB}F^{AB}-\frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\right] \tag{41}$$

In the above action the size of the S_5 has been set to 1. The value of the parameter κ for $\mathcal{N}=4$ Yang Mills is given by $\kappa=1/(2\sqrt{3})$ - however, with a view to other potential applications we leave κ as a free parameter in all the calculations below. Note in particular that our bulk Lagrangian reduces to the true Einstein Maxwell system at $\kappa=0$.

The equations of motion that follow from the action (41) are given by

$$\begin{aligned} G_{AB}-6g_{AB}+2\left[F_{AC}F^{C}{}_{B}+\frac{1}{4}g_{AB}F_{CD}F^{CD}\right]=0\\ \nabla_B F^{AB}+\kappa\epsilon^{ABCDE}F_{BC}F_{DE}=0 \end{aligned} \tag{42}$$

where g_{AB} is the five-dimensional metric, G_{AB} is the five dimensional Einstein tensor. These equations admit an AdS-Reisner-Nordström black-brane solution

$$\begin{aligned} &ds^2=-2u_\mu dx^\mu dr-r^2V(r,m,q)\ u_\mu u_\nu dx^\mu dx^\nu+r^2P_{\mu\nu}dx^\mu dx^\nu\\ &A=\frac{\sqrt{3}q}{2r^2}u_\mu dx^\mu, \end{aligned} \tag{43}$$

where

$$\begin{aligned} u_\mu dx^\mu=-dv;\qquad V(r,m,q)\equiv 1-\frac{m}{r^4}+\frac{q^2}{r^6};\\ P_{\mu\nu}\equiv\eta_{\mu\nu}+u_\mu u_\nu, \end{aligned} \tag{44}$$

Following the procedure elucidated in [8], we shall take this flat black-brane metric as our zeroth order metric/gauge field ansatz and promote the parameters u_μ, m and q to slowly varying fields depending on the boundary coordinates. Subsequently we shall iteratively correct the metric and the gauge field order by order in a derivative expansion so that they remain a solution to our Maxwell-Einstein system (41). We would find it useful to define the following quantities

$$\rho\equiv\frac{r}{R};\qquad M\equiv\frac{m}{R^4};\qquad Q\equiv\frac{q}{R^3};\qquad Q^2=M-1 \tag{45}$$

The global metric and the gauge field at first order

We solve the Einstein-Maxwell equations (42) using a suitable gauge and implementing suitable boundary conditions (for more details see [2]). Here we report the entire metric and the gauge field accurate up to first order in the derivative expansion. We obtain the metric to be

$$
\begin{aligned}
ds^2 &= g_{AB}dx^A dx^B \\
&= -2u_\mu dx^\mu dr - r^2\ V\ u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu\ r\left[u^\lambda\partial_\lambda u_\nu - \frac{\partial_\lambda u^\lambda}{3}u_\nu\right]dx^\nu + \frac{2r^2}{R}F_2(\rho,M)\sigma_{\mu\nu}dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu\left[\frac{\sqrt{3}\kappa q^3}{mr^4}l_\nu + \frac{6qr^2}{R^7}P_\nu^\lambda\mathcal{D}_\lambda q F_1(\rho,M)\right]dx^\nu + \dots \\
A &= \left[\frac{\sqrt{3}q}{2r^2}u_\mu + \frac{3\kappa q^2}{2mr^2}l_\mu - \frac{\sqrt{3}r^5}{2R^8}P_\mu^\lambda\mathcal{D}_\lambda q F_1^{(1,0)}(\rho,M)\right]dx^\mu + \dots
\end{aligned}
\tag{46}
$$

where $\mathcal{D}_\lambda$ is the Weyl covariant derivative which we now define. The Weyl-covariant derivative acting on a general tensor field $Q^{\mu\cdots}_{\nu\ldots}$ with weight w (by which we mean that the tensor field transforms as $Q^{\mu\cdots}_{\nu\ldots} = e^{-w\phi}\tilde{Q}^{\mu\cdots}_{\nu\ldots}$ under a Weyl transformation of the boundary metric $g_{\mu\nu} = e^{2\phi}g_{\mu\nu}$)

$$
\begin{aligned}
\mathcal{D}_\lambda\ Q^{\mu\cdots}_{\nu\ldots} \equiv\ & \nabla_\lambda\ Q^{\mu\cdots}_{\nu\ldots} + w\ \mathcal{A}_\lambda Q^{\mu\cdots}_{\nu\ldots} \\
& + \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right]Q^{\alpha\cdots}_{\nu\ldots} + \dots \\
& - \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right]Q^{\mu\cdots}_{\alpha\ldots} - \dots
\end{aligned}
\tag{47}
$$

where the Weyl-connection $\mathcal{A}_\mu$ is related to the fluid velocity u^μ via the relation

$$
\mathcal{A}_\mu = u^\lambda\nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{3}u_\mu. \tag{48}
$$

In (46) we also have defined

$$
V \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6}\ ; \qquad P_\mu^\lambda\mathcal{D}_\lambda q \equiv P_\mu^\lambda\partial_\lambda q + 3(u^\lambda\partial_\lambda u_\mu)q; \tag{49}
$$

and

$$
\begin{aligned}
F_1(\rho,M) &\equiv \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right)\int_\rho^\infty dp\frac{1}{\left(1 - \frac{M}{p^4} + \frac{Q^2}{p^6}\right)^2}\left(\frac{1}{p^8} - \frac{3}{4p^7}\left(1 + \frac{1}{M}\right)\right) \\
F_2(\rho,M) &\equiv \int_\rho^\infty \frac{p\left(p^2+p+1\right)}{(p+1)\left(p^4+p^2-M+1\right)}dp\ .
\end{aligned}
\tag{50}
$$

The Stress Tensor and Charge Current at first order

We now obtain the stress tensor and the charge current from the metric and the gauge field. The stress tensor can be obtained from the extrinsic curvature after subtraction of the appropriate counterterms. We get the first order stress tensor as

$$
T_{\mu\nu} = p(\eta_{\mu\nu} + 4u_\mu u_\nu) - 2\eta\sigma_{\mu\nu} + \dots \tag{51}
$$

where the fluid pressure p and the viscosity η are given by the expressions

$$p \equiv \frac{MR^4}{16\pi G_5} \qquad ; \qquad \eta \equiv \frac{R^3}{16\pi G_5} = \frac{s}{4\pi} \tag{52}$$

where s is the entropy density of the fluid obtained from the Bekenstein formula.

To obtain the charge current, we use

$$J_\mu = \lim_{r\to\infty} \frac{r^2 A_\mu}{8\pi G_5} = n\ u_\mu - \mathfrak{D}\ P^\nu_\mu \mathcal{D}_\nu n + \xi\ l_\mu + \dots \tag{53}$$

where the charge density n, the diffusion constant $\mathfrak{D}$ and an additional transport coefficient ξ for the fluid under consideration are given by [8]

$$n \equiv \frac{\sqrt{3}q}{16\pi G_5} \quad ; \quad \mathfrak{D} = \frac{1+M}{4MR} \quad ; \quad \xi \equiv \frac{3\kappa q^2}{16\pi G_5 m} \tag{54}$$

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the charge current which is proportional to the vorticity of the fluid. This is because the boundary equation for charge current conservation that follows from the Maxwell's equations (42) (given the current as defined in (53)) is

$$\partial_\mu J^\mu = \left(-\frac{\kappa}{2\pi G}\right) E.B \tag{55}$$

Thus on comparing the above equations (55) and (1) we conclude $\kappa = -2\pi Gc$. The presence of this non-dissipative term and the value of its coefficient matches the predictions of (12) and (13).

This new term in the constitutive relation was indirectly observed by the authors of [51] where they noted a discrepancy between the thermodynamics of charged rotating AdS black holes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N}=4$ SYM fluid.

The second order charge current and stress tensor

The expression for the metric and the gauge field at second order is very complicated. Here we merely present the boundary charge current and the stress tensor that is read off from the bulk gauge field and the metric respectively.

The second order corrections to the charge current (using the formula (53)) are obtained to be

$$J_i^{(2)} = \left(\frac{1}{8\pi G_5}\right) \sum_{l=1}^{5} \mathcal{C}_l (W_v)^l_i, \tag{56}$$

[8] Here we have taken the chemical potential $\mu = 2\sqrt{3}QR$ which determines the normalization factor of the charge density n (because thermodynamics tells us $n\mu = 4p - Ts$) which in turn determines the normalization of J_μ.

where

$$
\begin{aligned}
(W_v)^1_\mu &= P^\nu_\mu \mathcal{D}_\lambda {\sigma_\nu}^\lambda & \mathcal{C}_1 &= \frac{3\sqrt{3}R\sqrt{M-1}}{8M}, \\
(W_v)^2_\mu &= P^\nu_\mu \mathcal{D}_\lambda {\omega_\nu}^\lambda & \mathcal{C}_2 &= \frac{\sqrt{3}R(M-1)^{3/2}}{4M^2}, \\
(W_v)^3_\mu &= l^\lambda \sigma_{\mu\lambda} & \mathcal{C}_3 &= -\frac{3R\kappa(M-1)}{2M^2}, \\
(W_v)^4_\mu &= n^{-1} {\sigma_\mu}^\lambda \mathcal{D}_\lambda n & \mathcal{C}_4 &= \frac{1}{4}\sqrt{3}R\sqrt{M-1}\log(2) + \mathcal{O}(M-1), \\
(W_v)^5_\mu &= n^{-1} {\omega_\mu}^\lambda \mathcal{D}_\lambda n & \mathcal{C}_5 &= -\frac{\sqrt{3}R\sqrt{M-1}\left(M^2 - 48(M-1)\kappa^2 + 3\right)}{16M^2}.
\end{aligned}
$$

On plugging in the asymptotic solution for the metric in to the formula (51) we obtain

$$
T_{\mu\nu} = \left(\frac{1}{16\pi G_5}\right) \sum_{l=1}^{9} \mathcal{N}_l \; WT^{(l)}_{\mu\nu}. \tag{57}
$$

with $\mathcal{N}_l$ being the transport coefficients at second order in derivative expansion corresponding to the weyl covariant tensors WT^l. They are given by

$$
\begin{aligned}
WT^{(1)}_{\mu\nu} &= u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} & \mathcal{N}_1 &= R^2\left(\frac{M}{\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{3+\sqrt{4M-3}}\right) + 2\right), \\
WT^{(2)}_{\mu\nu} &= -2\left({\omega^\mu}_\lambda \sigma^{\lambda\nu} + {\omega^\nu}_\lambda \sigma^{\lambda\mu}\right) & \mathcal{N}_2 &= -\frac{MR^2}{2\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{\sqrt{4M-3}+3}\right), \\
WT^{(3)}_{\mu\nu} &= {\sigma^\mu}_\lambda \sigma_{\lambda\nu} - \frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta} & \mathcal{N}_3 &= 2R^2, \\
WT^{(4)}_{\mu\nu} &= 4\left({\omega^\mu}_\lambda \omega_{\lambda\nu} + \frac{1}{3}P^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta}\right) & \mathcal{N}_4 &= \frac{R^2}{M}(M-1)\left(12(M-1)\kappa^2 - M\right), \\
WT^{(5)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_\alpha \mathcal{D}_\beta n & \mathcal{N}_5 &= -\frac{(M-1)R^2}{2M}, \\
WT^{(6)}_{\mu\nu} &= n^{-2}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_\alpha n \mathcal{D}_\beta n & \mathcal{N}_6 &= \frac{1}{2}(M-1)R^2\Big(\log(8) - 1\Big) + \mathcal{O}\left((M-1)^2\right), \\
WT^{(7)}_{\mu\nu} &= \mathcal{D}_\mu l_\nu + \mathcal{D}_\nu l_\mu & \mathcal{N}_7 &= \frac{\sqrt{3}(M-1)^{3/2}R^2\kappa}{M}, \\
WT^{(8)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu} l_\alpha \mathcal{D}_\beta n & \mathcal{N}_8 &= 0 \\
WT^{(9)}_{\mu\nu} &= n^{-1}{\epsilon^{\alpha\beta\lambda}}_{(\mu}\sigma_{\nu)\lambda} u_\alpha \mathcal{D}_\beta n & \mathcal{N}_9 &= 0.
\end{aligned}
$$

In the above expressions we have introduced the projection tensor $\Pi^{\alpha\beta}_{\mu\nu}$ which projects out the transverse traceless symmetric part of second rank tensors

$$
\Pi^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}\left[P^\alpha_\mu P^\beta_\nu + P^\alpha_\nu P^\beta_\mu - \frac{2}{3}P^{\alpha\beta}P_{\mu\nu}\right]
$$

and $\mathcal{R}$ which is the Weyl invariant curvature scalar

$$
\mathcal{R} = R + 6\nabla_\lambda \mathcal{A}^\lambda - 6\mathcal{A}_\lambda \mathcal{A}^\lambda.
$$

Finally we have used the usual definition $\omega_{\mu\nu}$

$$\omega_{\mu\nu} = \frac{1}{2}\ P_{\mu\alpha}P_{\nu\beta}(\partial^\alpha u^\beta - \partial^\beta u^\alpha).$$

Superfluids

Following [69] in [3] we consider the system

$$\mathcal{L} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(\mathcal{R} + 12 + \frac{1}{e^2}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|D_\mu\phi|^2 + 2|\phi|^2.\right)\right), \tag{58}$$

Where $D_\mu = \nabla_\mu - iA_\mu$, and ∇_μ is the gravitational covariant derivative. Note that unlike the system in (41) this system is not any obvious truncation of type-IIB sugra. This system is therefore purely phenomenological and has been designed keeping in mind the solvability of the equations that follow from it.

The equation of motion for the scalar field and the gauge field that follows from (58) are respectively

$$D_\mu D^\mu\phi + 4\phi = 0, \tag{59}$$

and

$$D_\mu F^{\mu\nu} = \frac{1}{2}J^\nu, \tag{60}$$

where the current $J_\mu = i\left(\phi^* D\mu\phi - \phi(D_\mu\phi)^*\right)$. The Einstein Equation that follows from (58) is

$$G_{\mu\nu} - 6g_{\mu\nu} = \frac{1}{e^2}\big((T_{\max})_{\mu\nu} + (T_{\mathrm{mat}})_{\mu\nu}\big), \tag{61}$$

where

$$\begin{aligned}(T_{\max})_{\mu\nu} &= -\frac{1}{2}\left(F_{\mu\beta}F^\beta{}_\nu - \frac{1}{4}g_{\mu\nu}F_{\sigma\beta}F^{\beta\sigma}\right),\\ (T_{\mathrm{mat}})_{\mu\nu} &= \frac{1}{4}\left(D_\mu\phi D_\nu\phi^* + D_\nu\phi D_\mu\phi^*\right) - \frac{1}{4}g_{\mu\nu}\left(|D_\beta\phi|^2 - 4|\phi|^2\right).\end{aligned} \tag{62}$$

In [69] it was demonstrated that the system (58) at infinite e undergoes a second order phase transition towards superfluidity whenever $|\frac{\mu}{T}| \geq 2$. The stable gravitational solution, for $|\frac{\mu}{T}|$ just larger that 2, has a background scalar vev. Let ϵ denote the value of this vev. In [69] the authors analytically determined the relevant bulk solutions perturbatively in ϵ and separately in the difference between superfluid and normal velocities.

In [3] we generalize the infinite charge solutions in [69] beyond the strict probe approximation, to first nontrivial order in the $\frac{1}{e^2}$. This generalization is necessary in order to allow for the study of the response of the normal velocity and temperature fields to the dynamics of the superfluid velocity and chemical potential fields. We then proceed to use these solutions as raw ingredients for the fluid gravity correspondence [2, 8, 9, 10, 11, 12, 13, 66].

Following the procedure of the fluid gravity correspondence, we search for solutions of the Einstein Maxwell scalar system that tube wise approximate the stationary solutions described in the previous paragraph. More explicitly, we study a perturbative expansion to the solutions of Einstein's equations whose first term is given by the stationary solutions of the previous paragraph with the parameters of equilibrium superfluid flows replaced by slowly varying functions of spacetime. The configuration described in this paragraph does not obey the bulk equations; however it may sometimes be systematically corrected, order by order in boundary derivatives, to yield a solution to these equations. This procedure works if and only if our eight fields are chosen such that $\xi^\mu(x)$ is curl free, and such that the energy momentum and charge current built out of these fields is conserved. The constitutive relations that allow us to express the stress tensor and charge current in terms of fluid dynamical fields is generated by the perturbative procedure itself. In other words the output of our perturbative procedure is a set of gravitational solutions that are in one to one correspondence with the solutions of superfluid dynamics, with superfluid constitutive relations that are determined by the bulk gravitational equations.

Note that the construction described in the previous paragraph is carried out in a triple expansion. We follow [69] to expand our equilibrium solutions in a power series in the deviations from criticality (let us denote the relevant parameter by ϵ) [9] , and further expand these solutions in a power series in $\frac{1}{e^2}$. We then go on to use the solutions as ingredients in a spacetime derivative expansion.

Again the solutions of the bulk fields are complicated and therefore we do not write then explicitly here. Using the solution of these equations in equilibrium section we evaluate the boundary stress tensor charge current. For this purpose we use the standard AdS/CFT formulas

$$\begin{aligned}
T^\mu_\nu &= \frac{1}{16\pi G}\lim_{r\to\infty} r^4\left(2\left(\delta^\mu_\nu K_{\alpha\beta}\gamma^{\alpha\beta} - K^\mu_\nu\right) - 6\delta^\mu_\nu + \frac{\phi^*\phi}{e^2}\delta^\mu_\nu\right)\\
J^\mu &= \frac{1}{16\pi G\, e^2}\lim_{r\to\infty} r^3 F^{\mu r}\\
s &= \frac{\sqrt{k(1)}}{4G},\\
T &= \frac{f'(1)}{4\pi g(1)}.
\end{aligned} \tag{63}$$

where $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$ are respectively the induced metric and extrinsic curvature of a constant r surface; and T and s are the temperature and entropy density respectively. Using the above expressions and the solutions we can compute the coefficients in (18). In (18) let us define

$$f = \rho_s/\mu_s^2.$$

[9] In our analysis we also treat the difference between superfluid and normal velocities (denoted by ζ) to be small (following [69]). However in constructing the solution at the first derivative order we assume that this small parameter ζ is of the same order of magnitude as ϵ (but the order 1 ratio of the two (denoted by χ) is still kept arbitrary). This assumption is justified because of the presence of a dynamical instability in the system at values of ζ proportional to ϵ (see [3]).

Then we find

$$
\begin{aligned}
16\pi G\,(\rho) &= 3r_c^4+\frac{r_c^4}{e^2}\Big\{\,\left[4+2\zeta^2+\mathcal{O}(\zeta^4)\right]+\epsilon^2\left[-\frac{5}{12}+\mathcal{O}(\zeta^2)\right]+\mathcal{O}(\epsilon^4)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G(\rho_s) &= \frac{r_c^4}{e^2}\Big\{\,\left[\mathcal{O}(\zeta^4)\right]+\epsilon^2\left[1+\mathcal{O}(\zeta^2)\right]+\mathcal{O}(\epsilon^4)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G(P) &= r_c^4+\frac{r_c^4}{e^2}\Big\{\,\left[\frac{4}{3}+\frac{2}{3}\zeta^2+\mathcal{O}(\zeta^4)\right]+\epsilon^2\left[\frac{7}{36}+\mathcal{O}(\zeta^2)\right]+\mathcal{O}(\epsilon^4)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G(q) &= -\frac{r_c^3}{e^2}\Big\{\,\left[4+\zeta^2+\mathcal{O}(\zeta^4)\right]+\epsilon^2\left[-\frac{5}{24}+\mathcal{O}(\zeta^2)\right]+\mathcal{O}(\epsilon^4)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)\\
\mu_s &= \ r_c\Big\{\Big[2+\frac{\zeta^2}{4}+\zeta^4\left(-\frac{13}{64}+\frac{\log(2)}{4}\right)+\mathcal{O}\left(\zeta^6\right)\Big)]\\
&\quad+\epsilon^2\Big[+\frac{1}{48}+\zeta^2\left(-\frac{3\log(2)}{32}+\frac{43}{1152}\right)+\mathcal{O}\left(\zeta^4\right)\Big]\\
&\quad+\epsilon^4\Big[\Big(\left(-\frac{253}{55296}+\frac{7\log(2)}{1152}\right)+O\left(\zeta^2\right)\Big)+\mathcal{O}(\epsilon^6)\Big]\Big\}+\mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned}
\tag{64}
$$

Further the chemical potential of our solution is given by

$$
\begin{aligned}
\mu = u^\mu\xi_\mu = r_c\Big\{\Big[&-2-\frac{\zeta^2}{2}+\zeta^4\left(\frac{1}{4}-\frac{\log(2)}{4}\right)+\mathcal{O}\left(\zeta^6\right)\Big)]\\
&+\epsilon^2\Big[-\frac{1}{48}+\zeta^2\left(\frac{3\log(2)}{32}-\frac{5}{144}\right)+\mathcal{O}\left(\zeta^4\right)\Big]\\
&+\epsilon^4\Big[\Big(\left(\frac{253}{55296}-\frac{7\log(2)}{1152}\right)+O\left(\zeta^2\right)\Big)+\mathcal{O}(\epsilon^6)\Big]\Big\}+\mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned}
\tag{65}
$$

Moreover we find

$$
s=\frac{r_c^3}{4G}\left[1+\frac{1}{e^2}\Big\{\epsilon^2\left[\frac{\log(4)-1}{32}\zeta^2+\mathcal{O}(\zeta^4)\right]+\mathcal{O}(\epsilon^4)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)\right]
\tag{66}
$$

and

$$
\begin{aligned}
T=\frac{r_c}{\pi}+\frac{r_c}{4\pi e^2}\Big\{\Big[&-\frac{8}{3}-\frac{4\zeta^2}{3}+\zeta^4\left(\frac{1}{2}-\frac{2\log(2)}{3}\right)+\mathcal{O}\left(\zeta^6\right)\Big]\\
+\epsilon^2\Big[&\frac{1}{9}+\zeta^2\left(\frac{\log(2)}{4}-\frac{23}{216}\right)+\mathcal{O}\left(\zeta^4\right)\Big]\\
+\epsilon^4\Big[&\left(\frac{91}{20736}-\frac{\log(2)}{108}\right)+\mathcal{O}\left(\zeta^2\right)\Big]+\mathcal{O}(\epsilon^6)\Big\}+\mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{67}
$$

Using these expressions and the quantities obtained in (64) we have verified the relations (20) and (21) to the order to which we have evaluated our solution.

The first order result in the transverse frame from gravity

Using the gravity solution we can compute the undetermined transport coefficients in (33). Note that the boundary condition that we used in our gravity computation, guided by convenience, does not yield the boundary answer in the transverse frame. So, in order to obtain the answer in the transverse frame we had to perform a frame transformation on our gravity answer. We find

$$
\begin{aligned}
Q_b &= \frac{1}{\pi}\left[-\frac{24\chi}{25\epsilon} + \mathcal{O}(\epsilon)^0\right] + \mathcal{O}\left(\frac{1}{e^2}\right) & R_b &= \frac{\pi}{(16\pi G)e^2}\left[\left(-1 - \frac{288}{25}\chi^2\right) + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
P_a &= \left[\frac{24}{25}\chi^2 + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{1}{e^2}\right) & Q_a &= 16\pi G\left(\frac{e^2}{\pi^3}\right)\left[-\frac{52}{25\epsilon^2} + \mathcal{O}(\epsilon)^0\right] + \mathcal{O}\left(\frac{1}{e^2}\right)^0 \\
R_a &= \frac{1}{\pi}\left[-\frac{24\chi}{25\epsilon} + \mathcal{O}(\epsilon)^0\right] + \mathcal{O}\left(\frac{1}{e^2}\right) & P_b &= \frac{\pi^2}{(16\pi G)e^2}\left[\frac{288}{25}\,\epsilon\chi^3 + \mathcal{O}(\epsilon)^2\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
Q_w &= \left[\frac{24}{25}\chi^2 + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{1}{e^4}\right) & R_w &= \frac{\pi^2}{(16\pi G)e^2}\left[\frac{288}{25}\,\epsilon\chi^3 + \mathcal{O}(\epsilon)^2\right] + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
P_w &= -\frac{3\pi^3}{16\pi G} + \frac{\pi^3}{(16\pi G)e^2}\left[-6 - \left(\frac{1}{4} - 3\chi^2 + \frac{288}{25}\,\chi^4\right)\epsilon^2 + \mathcal{O}(\epsilon)^3\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\ E_a &= \frac{\pi^2}{e^2}\left[\mathcal{O}(\epsilon)^3\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\ C_a &= \frac{\pi}{e^2}\left[-1 + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\ E_b &= -2\pi^3 + \frac{\pi^3}{e^2}\left[-4 + \left(\frac{1}{6} - 2\chi^2\right)\epsilon^2 + \mathcal{O}(\epsilon)^4\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\ C_b &= \frac{\pi^2}{e^2}\left[\epsilon^3\chi\left(\frac{-1+\log(4)}{8}\right) + \mathcal{O}(\epsilon)^4\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\ \tau &= -2\pi^3 + \frac{\pi^3}{e^2}\left[-4 + \left(\frac{1}{6} - 2\chi^2\right)\epsilon^2 + \mathcal{O}(\epsilon)^4\right] + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{69}
$$

where the quantities appearing in (68) and (69) are same as the coefficients appearing in (33) Note that in this transverse frame also we have $Q_w = P_a$, $R_w = P_b$, $R_a = Q_b$ and $C_b = E_a$, which constitutes the expected Onsager relations. All the positivity constraints given in (37) and (39) are also obeyed by the above gravity result.

S3 Lumps of plasma dual to exotic black objects

As mentioned in the introduction we may also use the boundary hydrodynamics in a suitable way to infer properties of exotic black objects in the bulk. To this end we study horizon topologies and thermodynamics of black objects in arbitrary high dimensional Scerk-Schwarz compactified AdS spaces (SSAdS). The spectrum of black objects in more than 4 dimensions is extremely rich and consequently has drawn considerable interest recently [75, 83]. As the construction of these exotic horizon topologies directly in gravity turns out to be technically difficult we study them in a somewhat indirect manner using the AdS/CFT correspondence [1, 76, 77].

We consider the field theory obtained by Scherk-Schwarz compactification of this dual CFT, which consequently lives in d dimensions. This field theory has a first order confinement/deconfinement phase transition. This corresponds to a Hawking-Page-like phase transition in the bulk, for which the low temperature phase is the AdS-soliton and the high temperature phase is a large AdS black brane [78].

In the long wavelength limit, this field theory admits a fluid description where the dynamics is governed by the d dimensional relativistic Navier-Stokes equation. The effect of the Scerk-Schwarz compactification is only to introduce a constant additive piece to the free energy of the deconfined fluid [79]. Due to this shift, the pressure can go to zero at finite energy densities, allowing the existence of arbitrarily large finite lumps of deconfined fluid separated from the confined phase by a surface – the plasmaballs of [79]. Now by the AdS/CFT correspondence finite energy localized non-dissipative configurations of the plasma fluid in the deconfined phase is dual to stationary black objects in the bulk. Thus, by studying fluid configurations that solve the d dimensional relativistic Navier-Stokes equation we can infer facts about the black objects in SSAdS_{d+2} [6, 80].

Two important feature of the dual black object that one can infer from the fluid configurations are the horizon topology and the thermodynamics. The thermodynamics of the black object can be studied by simply computing the thermodynamic properties of the fluid configuration – one integrates the energy density, entropy density etc. to compute the total energy, entropy etc. and the rest follows.

The horizon topology can be inferred as follows. Far outside the region corresponding to the plasma, the bulk should look like the AdS-soliton. In this configuration the Scherk-Schwarz circle contracts as one moves away from the boundary, eventually reaching zero size and capping off spacetime smoothly. Deep inside the region corresponding to the plasma, the bulk should look like the black brane. In this configuration the Scherk-Schwarz circles does not contract, it still has non-zero size when one reaches the horizon. It follows that as one moves along the horizon, the Scherk-Schwarz circle must contract as one approaches the edge of the region corresponding to the plasma. The horizon topology is found by looking at the fibration of a circle over a region the same shape as the plasma configuration, contracting the circle at the edges [79, 80]. We have provided a schematic drawing of this in fig.1.

In the fluid description the degrees of freedom includes the velocity field, $u^\mu(x)$, and the temperature field, $\mathcal{T}(x)$, (we consider uncharged fluids dual to uncharged black objects; otherwise the degrees of freedom would also include the chemical potentials for those charges). Now as we seek time-independent solutions, Lorentz symmetry allows us to consider fluid velocities of the form

$$u^\mu = \gamma(\partial_t + \omega_a l_a),$$

where l_a are the killing vectors along the Cartan direction of the spatial rotation group, γ is the normalization and the ω_a are some constants. This along with the fact that our solutions are non-dissipative forces the temperature field to be of the form

$$\mathcal{T} = \gamma T,$$

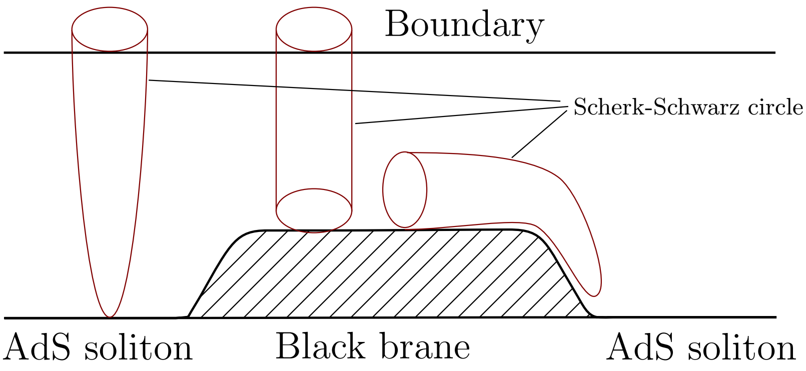

Figure 1: Schematic description of the bulk dual of a plasmaball with some circle fibres indicated.

where T is a constant. With a simple thermodynamic argument we show that T is the overall thermodynamic temperature of the fluid configuration and ω_a are the thermodynamic angular velocities. Further we demonstrate that the equations of motion for non-dissipative time-independent solutions at the surface of the fluid configuration reduce to the condition

$$\mathcal{P}|_{\text{surface}} = \sigma\Theta,$$

where σ is the surface tension and Θ is the trace of the extrinsic curvature of the fluid surface under consideration. The pressure $\mathcal{P}$ is related to the temperature $\mathcal{T}$ by the equation of state, so this provides a differential equation for the position of the surface. These configurations are parameterized by the temperature T and the angular velocities ω_a.

We then proceed to construct a class of fluid configurations whose surface is a solution of the above equation in a certain limit. In d spacetime dimensions the topologies of these configurations are

$$B^{(d-1-n)} \times \underbrace{S^1 \times S^1 \ldots\ldots S^1}_{\text{n times}}, \tag{70}$$

where n satisfies

$$\begin{aligned} &n = 0, \quad \text{for } d = 3. \\ &n \leq \frac{d-1}{2}, \quad \text{for odd } d \text{ greater than 3.} \\ &n \leq \frac{d-2}{2}, \quad \text{for even d.} \end{aligned}$$

These solutions are rotating in the plane in which the S^1s lie, and for simplicity we turn off angular momentum along any other directions. In these configurations pressure in the radial direction of the ball is balanced by the surface tension. While along the radial direction of the S^1s the centrifugal force balances the pressure (therefore rotation is essential in the plane of the S^1s). We refer to the limit in which the (average) radius of the ball is small compared to the (average) radius along the S^1s [10] as 'the generalized thin ring' limit. The ratio of these

[10] when there is more than one S^1 this radius refers to the magnitude of the vector which is obtained by

two radii serves as the small parameter in the problem. To leading order in this parameter we find that the fluid configurations are exactly $B^{(d-1-n)} \times T^n$ (in contrast to merely having the same topology). The force balance conditions then relate the intrinsic fluid parameters (the temperature and the angular velocities) to the parameters of the fluid configuration (the radius of the ball and the radii of the various S^1s). These fluid configurations are dual to black objects with horizon topologies $S^{(d-n)} \times T^n$ and hence this provides an indirect proof of existence of such exotic horizon topologies of black objects in SSAdS_{d+2}. This approach is reminiscent of (and inspired by) the black-fold approach of [81].

It is possible to analytically deduce several properties of these fluid configurations. The configurations in (70) are parameterized by the radius of the ball (R) and the radii of the various S^1s ($\ell_0 P_a$). In the generalized thin ring limit locally these configurations are like filled cylinders with the topology $B^{(d-1-n)} \times \mathbb{R}^n$. Then we can bend the different directions in $\mathbb{R}^n$ into S^1s in a controlled way with a perturbation expansion in ϵ. Now the intrinsic fluid parameters (namely the temperature (T) and the angular velocities (ω_a)) are related to the parameters of the fluid configuration (R and $\ell_0 P_a$) by the force balance conditions. The pressure along the radial direction of the ball is balanced by the surface tension. This condition yields

$$T^{d+1} = \left(\frac{(d-n-2)+R}{R}\right)\left(1 - \sum_a (\ell_0 P_a w_a)^2\right)^{\left(\frac{d+1}{2}\right)}$$

On the other hand the pressure along the radial direction of the S^1s is balanced by the centrifugal force. In order to obtain this force balance we require these configurations to be rotating (at least) in the planes in which the S^1s lie. For the sake of simplicity we have turned off angular velocity along any other direction. This force balance determines the angular velocities to be

$$w_a^2 = \frac{1}{(\ell_0 P_a)^2\left(((d-n-2)+R)(d+1)+n\right)}.$$

Note that the angular velocities has an upper bound in the limit $R \to 0$ when it goes as $1/d$, for large d and small n. Although this limit is outside the validity of our hydrodynamic approximation, it is fascinating to note that such an upper bound to the angular velocity even exists for asymptotically flat rings [83].

Further it is possible to construct a well controlled perturbation theory about these generalized thin ring solution. This we demonstrate by explicitly computing the leading order corrections to the thin ring solutions in specific examples, namely the ring in 4 dimensions and the ring and the 'torus' (the one with the topology $B^2 \times T^2$) in 5 dimensions. We find that the leading order correction only appear at the second order in the expansion parameter (the small parameter described above). Also in these cases we explicitly compute the thermodynamic quantities (which are again correct up to second order in the expansion parameter) with which we construct the phase diagrams of these solutions within appropriate validity regimes.

the vector sum of the radii of the various S^1s

Then we go on to perform a detailed numerical study of the black objects occurring in 6 dimensional SS compactified AdS space. First we perform a thorough numerical scan to demonstrate that the rotating fluid configurations with the topology of a ball and that of a solid torus (previously obtained in [80]) are the only stationary rotating solutions of the relevant Navier-Stokes equations. Second we determine the thermodynamic properties and the phase diagram of these solutions.

The thermodynamic properties of the ball and ring solutions of [80] turn out to be very similar to the properties of the analogous solutions in one lower dimension (discussed in detail in [80]). In fig:2 we present a plot of the entropy versus the angular momentum of the relevant solution, at a fixed particular value of the energy. As is apparent from fig:2 there we find at least one rotating fluid solution for every value of the angular momentum. However in a particular window of angular momentum - in the range (L_B, L_C) - there exist three solutions which have the same energy and angular momentum. These three solutions may be thought of as a ball a thick ring and a thin ring respectively of rotating fluid. The ball solution is entropically dominant for $L < L_P$ while the thin ring dominates for $L > L_P$. At angular momentum L_P (which lies in the range (L_B, L_C) the system (in the microcanonical ensemble) consequently undergoes a 'first order phase transition' from the ball to the ring. It follows that the dual gravitational system must exhibit a dual phase transition from a black hole to a black ring at the same angular momentum.

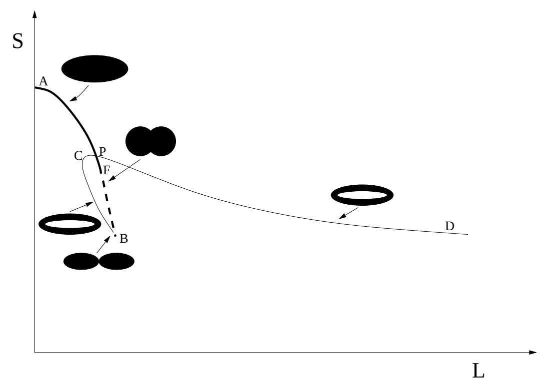

Figure 2: Schematic plot of the phase diagram for the various plasma configurations which by AdS/CFT correspondence gives the phase structure of black holes with various horizon topologies in Scherk-Schwarz compactified AdS_6.

The phase diagram depicted in fig:2 has some similarities, but several qualitative points of difference from a conjectured phase diagram for on the solution space of rotating black holes and black rings in 6 flat spacetime directions. This suggests that the properties of black holes and black rings in 6 dimensional AdS space are rather different from those of the corresponding objects in flat six dimensional space. This is a bit of a surprise, as black holes and rings in Scherk-Schwarz compactified AdS_5 appear to have properties that are qualitatively similar to their flat space counterparts [80, 82].

S4 Discussions

In §S2 we first presented a theory to describe the dynamics of a charged fluid up to first order in derivatives based on simple principles like the second law of thermodynamics and the Onsager's relations. We then went on to use the metric dual to a fluid with a globally conserved charge to find the energy-momentum tensor and the charge current in arbitrary fluid configurations to second order in the boundary derivative expansion.

We have seen that a nonzero value for the coefficient of the Chern-Simons in the bulk leads to an interesting dual hydrodynamic effect (note that this coefficient is indeed nonzero in strongly coupled $\mathcal{N} = 4$ Yang Mills). At first order in the derivative expansion we find that the charge current has a term proportional to $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha} u_\mu \partial_\nu u_\lambda$ in addition to the more familiar Fick type diffusive term. After the discovery of this term in [2, 66] in the context of conformal fluids, it was argued in [68] that the presence of this term was required by a local form of the second law of thermodynamics (which we have briefly reviewed in §S2.1) for any fluids (not necessarily conformal) with an anomalous $U(1)$ current. Thus this term which does not find any mention in earlier hydrodynamic literature (known to us) can be crucial for real fluids if such a fluid suffers from an $U(1)$ anomaly. Also since anomalies are essential quantum phenomenon, it is fascinating to note that the hydrodynamic transport phenomenon associated with this special term is a macroscopic manifestation of underlying quantum mechanics.

On a similar vein, we went on to construct a theory of first order superfluid dynamics and obtained its dual gravity configuration in a particular convenient corner of parameter space. Here again we found a new transport coefficient which to our knowledge was not considered earlier in the superfluid literature (classic references on the subject like [71, 72] miss this term). This new term in the constitutive relations indicates the presence of interesting transport phenomenon (not studied till date) which may even be observable in real superfluids like liquid helium. However the observation of such phenomenon may be experimentally challenging as it is observable only for finite superfluid velocities and most superfluids are unstable beyond a particular superfluid velocity (which may be quite small for real superfluids).

In §S3 we went on to study exotic black objects in SS compactified (higher dimensional) AdS spaces, in an indirect way using boundary fluid configurations. It will be fascinating to construct these solutions directly in gravity and compare their properties against our predictions. This investigation is primarily obstructed by the fact that the domainwall solution in the bulk separating the confined and deconfined phase in the boundary at the transition temperature is only known numerically. Further, the fluid configurations that we study have boundaries which plays a crucial role in their dynamics. These boundaries should support local fluctuation which is expected to interact non-trivially with the bulk sheer and density waves. A detailed study of these fluctuations which forms an important part of the dynamical perturbations of our static configurations may throw light on the stability properties of these objects.

The investigations referred to in this synopsis opens up several interesting questions that require future investigation. One question is the study of superfluid dynamics and its gravity duals in the presence of anomaly. This questions has been addressed in [4] where a consistent theory of anomalous superfluids has been developed upto first order in derivatives. The authors of [4] verifies some of the predictions of this theory through a dual gravitational construction which focuses on a particular collinear $SO(3)$ invariant limit. The most general gravitational construction away from this special limit remains a open challenge.

The exotic black objects discussed in this synopsis makes us wonder if it is possible to add scalar hairs to these objects. From the existence of soliton solutions in AdS reported in [84, 85], we may conclude that such non-trivial scalar field configurations would also exist for the SS compactified AdS. Like their AdS counterparts these solutions would be expected to exist at all temperatures but only beyond a certain chemical potential. Thus it is natural to wonder whether such superfluid configurations remain in equilibrium with finite lumps of deconfined plasma at the same temperature and chemical potential.

In developing the theories of hydrodynamics both in the presence and absence of superfluidity we have found that the principle of local increase of entropy was extremely powerful. For example in the case of parity even superfluids considered here this principle cuts down the total number of allowed constitutive parameters from 50 (which are allowed by symmetry) to 21 (the Onsager's relations bringing it down further to 14). This throws open the question whether such principles may be used to constrain (higher derivative) corrections to the theory of gravity. If the answer turns out to be affirmative then it will have the potential to throw enormous light into quantum gravity.

List of Publications

Papers Relevant to the Thesis Work

- [A1] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, "A theory of dissipative superfluid hydrodynamics," arXiv:1105.3733 [hep-th].
- [A2] J. Bhattacharya, S. Bhattacharyya, and S. Minwalla, "Dissipative superfluid hydrodynamics from gravity ," JHEP **1104**, 125 (2011), arXiv:1101.3332 [hep-th].
- [A3] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," JHEP **1101**, 094 (2011), arXiv:0809.2596 [hep-th].
- [A4] J. Bhattacharya, and S. Lahiri, "Lumps of plasma in arbitrary dimensions," JHEP **1008**, 073 (2010), arXiv:0903.4734 [hep-th].
- [A5] S. Bhardwaj and J. Bhattacharya, "Thermodynamics of Plasmaballs and Plasmarings in 3+1 Dimensions," JHEP **0903** 101 (2009), arXiv:0806.1897 [hep-th].

Other Papers

- [B1] P. Basu, J. Bhattacharya, S. Bhattacharyya, R. Loganayagam, S. Minwalla and V. Umesh, "Small Hairy Black Holes in Global AdS Spacetime," JHEP **1010**, 045 (2010), arXiv:1003.3232 [hep-th].
- [B2] J. Bhattacharya and S. Minwalla, "Superconformal Indices for Chern Simons Theories," JHEP **0901**, 014 (2009), arXiv:0806.3251 [hep-th].
- [B3]J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju, "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions," JHEP **0802**, 064 (2008), arXiv:0801.1435 [hep-th].

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Chapter 1

Introduction

In this thesis we focus on several applications of the gauge/gravity duality, which is one of the most prominent outcomes of the study of string theory in the last decade. The gauge/gravity duality is an equivalence between two theories: a quantum gauge theory in d dimensional space time and a theory of gravity in $d+1$ dimensional spacetime. The best understood examples of this duality involve AdS spaces as the bulk and a conformal field theory living on the boundary. In those contexts the gauge gravity duality is refered to as the AdS/CFT correspondance. The $d+1$ dimensional theory of gravity in asymptotic AdS space is refered to as the bulk theory. The $d+1$ AdS space has a conformal boundary which is d dimensional and the dual gauge theory lives on this boundary. The fact that the entire information of the bulk theory can be encoded in a theory in one less dimension is analogous to an optical hologram, and so the study of this correspondence is also sometimes referred to as holography. The gauge/gravity duality, in its strongest form, is presently at the level of a conjecture. But for certain specific examples there is overwhelming non-trivial evidence suggesting its correctness. Since this duality will be the central theme of this thesis we shall now provide a brief introduction to the AdS/CFT correspondence.

1.1 The AdS/CFT correspondence

The AdS/CFT correspondence [1, 86, 92] postulates that all the physics of any consistent quantum theory of gravity in AdS space can be described by a local conformal quantum gauge theory living in its boundary. The metric of AdS space is given by

$$ds^2_{AdS_{d+1}} = R^2\bigg(-(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega^2_{d-1}\bigg), \tag{1.1}$$

where $d\Omega^2_{d-1}$ is the metric on a unit sphere, S^{d-1}, and R is the radius of curvature of AdS. This space is conformally equivalent to a solid cylinder. The boundary of AdS is obtained by taking the limit $r \to \infty$, and is given by $R \times S^{d-1}$. The bosonic isometries of $d+1$ dimensional AdS is $SO(2,d)$ which is the also the d dimensional conformal group. Thus the action of the isometries in AdS on its boundary is simply a conformal transformation in the boundary. Hence, on symmetry grounds it is plausible that the dynamcis in AdS space is captured by a CFT living on the boundary $R \times S^{d-1}$.

In some applications (especially the ones that we shall consider in this thesis) it is useful to consider a small patch of the boundary and treat it as $R^{1,d}$. There exist a choice of coordinates where a patch of AdS space has the following metric

$$ds^2 = R^2 \left(\frac{dz^2 - dt^2 + d\vec{x}_{d-1}^2}{z^2} \right). \tag{1.2}$$

In this coordinates the boundary is at $z = 0$ and constant z slices possesses d dimensional Poincare symmetry. This patch coordinates are convenient when we want to consider a CFT living in d dimensional Minkowski space, $R^{1,d}$.

Since the correspondence relates theories in different dimensions it is necessary to understand the matching of number of degrees of freedom on both sides. Let us consider that the boundary theory which is a CFT on $R \times S^{d-1}$ is at a temperature T. Then for temperatures (T) large compared to the radius of S^{d-1}, its entropy density should scale with T as

$$s \sim cT^{d-1}, \tag{1.3}$$

where c is a dimensionless constant that measures the effective number of degrees of freedom (fields) in the boundary theory. The bulk theory, however, is a theory of gravity which admits black hole solutions of the form

$$ds_{bh}^2 = R^2 \left(-\left(r^2 + 1 - \frac{gm}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\left(r^2 + 1 - \frac{gm}{r^{d-2}}\right)} + r^2 d\Omega_{d-1}^2 \right), \tag{1.4}$$

where g is related to the Newton constant in $d+1$ dimensions $G_{(d+1)}$

$$g \sim \frac{G_{(d+1)}}{R^{d+1}}. \tag{1.5}$$

It is the effective gravitational coupling at the AdS scale. Now at large enough energies the thermodynamics is entirely dominated by the black hole phase which can be used to put an upper bound on the entropy of the system. The entropy in this phase is given by the area of the event horizon and for large temperatures it is given by

$$s \sim \frac{r_s^{d-1}}{g} \sim \frac{T^{d-1}}{g}, \tag{1.6}$$

where we have used the fact that the Hawking temperature of the black hole (1.4), $T \sim r_s$, r_s being the Schwarzchild radius. Thus we see that the degrees of freedom in the bulk and boundary match provided we identify

$$c \sim \frac{1}{g} \sim \frac{R^{d-1}}{G_{(d+1)}} \tag{1.7}$$

Thus we are led to conclude that the effective number of fields in the boundary CFT should scale inversely as the bulk Newton constant. This fact is particularly important as it implies that if we are interested in a weakly coupled bulk theory (for better control) we must have a large number of fields in the boundary. This fact is very nicely incorporated in large N gauge theories.

In the better understood examples of the correspondence the rank of the gauge group of the boundary gauge theory is taken to be N which is considered to be large so that the bulk gravity is weakly coupled. In a conformal field theory, in general, there exists a map between the states and the operators of the theory. Both the states and the operators of the CFT organize themselves into unitary representation of the conformal group. These representations are characterized by the spin and scaling dimension of the operator at the head of the representation. Since we put the CFT on the boundary of AdS which is $R \times S^{d-1}$ gauss law constraints forces us to project onto the gauge singlet states. Thus the operators of the CFT may be generally classified as single trace operators and multitrace operators (product of single trace operators). From the general structure of Feynman diagrams in the CFT it is possible to argue that in the large N limit the scaling dimensions of the multitrace operators are simply given by the sum of scaling dimensions of each of its individual single trace components. Thus there is no new information in the multitrace operators in the large N limit and we can legitimately focus on the single trace operators only. Also it was noted by 't Hooft [7] that in the large N limit we could organize the Feynman diagrams of scattering amplitudes of a large N gauge theories in a way such that it resembled the perturbative expansion of string scattering amplitudes, higher genus diagrams being suppressed by a factor of $1/N^2$. This analogy is made concrete in the context of the AdS/CFT correspondence where a string theory with string coupling $g_s \sim 1/N$ is dual to the boundary gauge theory. Note that in the strict large N limit we are left with only the planar diagrams on the gauge theory side and only the tree level diagrams on the string theory side. It is in this sense that the gauge theory is said to admit a classical description in the large N limit. Note that it is different from the usual classical description (where $\hbar$ is set to zero) because in the planar diagrams that survive this large N limit there are non-trivial loop diagrams.

The graviton is one of the lowest (massless) modes of a vibrating string. By considering the large N limit we have we have essentially made the stings non-interacting. But the oscillating sting would in general have other massive states of higher spins. If we want to obtain ordinary gravity with the spin 2 graviton being the highest spin state then we require a seperation of scales between the massless states and the massive states. This is achieved by the condition

$$R \gg \ell_s, \tag{1.8}$$

ℓ_s being the string length. This statement also has a gauge theory analogue. In the large N limit we can have single trace operators with spin grater than 2, which in general can have small scaling dimensions in the weak coupling limit. For the classical gravity approximation to work these operators must be lifted by gaining large anomalous dimensions which is only possible in the strongly coupled regime of the gauge theory. Thus if we are interested in the boundary description of a two derivative theory of gravity with the graviton being the highest spin object it should be given by a large N strongly coupled gauge theory. The fact that the gauge theory is strongly coupled makes it vary intractable and this one of the main stumbling block towards constructing a direct proof of the correspondence.

One of the most prominent and well understood example of this correspondence is the conjecture that type-IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills theory in four dimension with $SU(N)$ gauge group[1]. In the 't Hooft limit (large N limit) this field theory has two parameters - N and the 't Hooft coupling constant λ [2]. These two parameters are related to the radius of curvature R and the string coupling constant g_s of the bulk theory in the following way

$$g_s = \frac{\lambda}{4\pi N}; R = \lambda^{\frac{1}{4}} \ell_s. \tag{1.9}$$

Therefore if we take the large N limit, the string loops are suppressed and further if we take the large λ limit the string theory geometry is weakly curved and we are left with a ordinary two derivative theory of gravity in the bulk.

1.2 Applications of the AdS/CFT correspondence in the long wavelength limit

This correspondence, at the first glance, seems to have enormous applicability as one can use one side of it to gain structural and conceptual insights about the other. As was emphasized in the previous section this possibility is hindered by the fact that in the limit when we have ordinary gravity in the bulk we have a strongly coupled field theory in the boundary which is not tractable. In several applications this fact is considered in positive light as this can be then used to understand strongly coupled field theories using gravity computations. However, there exists one limit in which even the strongly coupled field theory becomes tractable. Any quantum field theory, no matter what its coupling, is believed to admit a hydrodynamic description in the long wavelength limit. In most of the discussions in this thesis we shall focus on this limit. As emphasized in the previous section, we consider the large N limit followed by the strong coupling limit so that we have ordinary two derivative gravity in the bulk. Also in most of our discussions we will be focusing on the phase of the boundary field theory where the degrees of freedom scales as some positive powers of N (for example, N^2 for $\mathcal{N} = 4$ Yang-Mills theory in 4 dimensions) so that the field theory is deconfined. Hence our fluid is that of a deconfined Yang-Mills plasma. This in the bulk would correspond to a black hole (or a black object) whose degrees of freedom also scales as the same power of N.

In the long wavelength limit [3], the near equilibrium dynamics of gauge theories is captured by a few effective degrees of freedom which constitutes a hydrodynamic description. The variables of such a description are the local densities of all conserved charges and the local

[1]Note that $\mathcal{N}$ refers to the amount of supersymmetry in the theory while N refers to the rank of the gauge group.

[2]Here λ is related to the Yang-Mills coupling constant by the relation $\lambda = g_{YM}^2 N$ and in the 't Hooft limit N is taken to infinity keeping λ fixed. Also note that this theory is a conformal field theory and so its β-function vanishes. For this reason λ is a parameter of the theory unlike non-conformal theories where its runs with the energy scale.

[3]By the long wavelength limit we mean that we focus on modes that vary over length scales which is large compared to the mean free path which constitutes a consistent truncation of the system.

fluid velocities. If there is a spontaneously broken symmetry then we should augment this list of variables to include the corresponding Goldstone boson (as it is a low energy mode and hence must be accounted for in all the low energy effective descriptions). The equations of fluid dynamics are the equations of local conservation of charge currents. These equations have to be supplemented by constitutive relations expressing these currents in terms of the fluid variables. Using the fact that fluid dynamics is a long wavelength effective theory these constitutive relations are specified in a derivative expansion. At any given order in derivatives, symmetries plus a few other canonical principles (like the second law of thermodynamics and time reversal invariance) determine the form of this expansion upto a finite number of undetermined coefficients. These coefficients may then be obtained either from experiments or from microscopic computations.

Using the AdS/CFT correspondence it is possible to capture the boundary hydrodynamics in a bulk system with gravity. In fact it has recently been demonstrated that a class of long distance, regular, locally asymptotically AdS_{d+1} solutions to Einstein's equations with a negative cosmological constant is in one to one correspondence with solutions to the charge free Navier Stokes equations in d dimensions [8, 9, 10, 11, 12, 13, 14, 16, 93, 94]. The most striking feature of these bulk solutions is the fact that it is possible to foliate them into a collection of tubes, each of which is centered around a radial ingoing null geodesic emanating from the AdS boundary. The congruence of these null geodesics provides us with a natural map between the boundary and the event horizon. Locally these tubes are well approximated by a uniform black brane solution which corresponds to local equilibrium in the boundary fluid. In these solutions there is a singularity in the bulk (just as the one in case of a uniform black brane), but it is shielded from the boundary by a regular event horizon. In the picture that emerges if we stitch these tubes together, the event horizon behaves as a membrane whose 'vibrations' capture the boundary fluid dynamics. In this thesis, we generalize this construction to include fluctuations around a charged black brane which provides a bulk description of a fluid with a conserved global charge (see chapter 3 for more details). A global $U(1)$ symmetry in the boundary maps to a $U(1)$ gauge symmetry in the bulk. We carried out our analysis in five dimensional bulk space time and included a parity violating Chern-Simons term beside the standard kinetic term for the gauge fields.

There are several applications of such holographic constructions. Firstly, it gives us a systematic procedure to test and verify the general structure of hydrodynamics. For example, through such holographic methods we were able to predict the existence of a new term in the first order hydrodynamics of fluids with a global conserved charge (when the charge current suffered from an anomaly). It was later shown in [68] that the consistency of hydrodynamics with the second law of thermodynamics forced such a term to be present. Secondly, as we pointed out in the previous section, the boundary gauge theory (with the gravity dual) is strongly coupled. Therefore a microscopic derivation of coefficients in the constitutive relations is not possible within the field theory itself. However for field theories with a bulk description we can compute these coefficients from the dual gravity solution. Such computations may be used to throw light on the transport properties of certain exotic (strongly coupled) phases of matter like the quark gluon plasma. Besides, a general understanding of fluid dynamics from a different angle might help us gain insight into outstanding problems in fluid dynamics like turbulence.

In this thesis we also extend the fluid gravity map into the domain of parity invariant superfluid hydrodynamics (see chapter 3 for details). This is achieved by introducing a non-trivial charged scalar field in the bulk which corresponds to turning on a charged scalar operator in the boundary. The non-zero expectation value of this charged scalar operator spontaneously breaks the global $U(1)$ symmetry in the boundary and hence gives rise to a superfluid phase. In this context the gravitational computation of the constitutive relations were chiefly used to derive and verify the structure of parity invariant superfluid hydrodynamics. We found that the first order constitutive relations derived from gravity did not fit into the general structure of superfluid hydrodynamics that existed in the literature [71, 72] (which claimed to have a 13 undetermined parameters at first order). We could however, generalize that existing structure (by increasing the undetermined parameters to 14) and found that our gravity computation was perfectly consistent with this new generalization (see chapter 2 for details of this generalization). Thus in the context of superfluids, our holographic methods guided us to discover yet another previously unexplored transport phenomenon.

Till now we primarily discussed instances in which the fluid gravity map was used to learn new physics about fluids in general. However, we can use this correspondence in an inverse fashion to explore new interesting solutions in the bulk (see chapter 4 for more details). Here we constructed plasma configuration in the boundary that solves the Navier Stokes equation. Through a technical trick of Scherk-Schwarz compactification it is possible to create interesting configuration that are confined to finite regions of space. These fluid configurations can be then mapped to horizon topologies in the bulk [4]. Exotic configurations in the boundary provide us with new and interesting horizon topologies (in five or more bulk dimensions). This method not only serves as a proof of existence of such new horizons but also have the prower to predict their thermodynamic properties (as there is a direct map between the bulk and boundary thermodynamics).

The thesis is organized as follows. In chapter 2 we present a complete theory of relativistic hydrodynamics upto first order in derivative expansion. Here we initially discuss fluids with a conserved global $U(1)$ charge which may be anomalous. Then we proceed to discuss superfluid hydrodynamics in a parity invariant situation. In chapter 3 we provide the a specific bulk construction which aids us to test and verify the general structure of first order fluid dynamcis presented in chapter 2. Finally in chapter 4 we study the properties a completely new class of black-objects in higher dimensional AdS space with the aid of localized plasma configurations in the boundary.

[4] This map is not precisely known as in the previous case as here we only have a knowledge of the map between the topologies. Constructing a one-to-one map between the horizon and the boundary fluid configurations, in case of the exotic topologies discussed in this thesis, constitutes an interesting open problem.

Chapter 2

A theory of dissipative hydodynamics

In this chapter, we shall present a theory of most general dissipative relativistic hydrodynamics upto first order in derivatives in two cases. First we consider fluids with a global $U(1)$ charge than may be anamolous. We then go on to study the fluid dynamics of parity even superfluids. We should emphasize that this section is completely independant of any reference to gravity (or string theory or the AdS/CFT correspondance) and is based on general thermodynamic principles applicable to the long wavelenth limit of any reasonable quantum field theory.

2.1 The theory of charged fluid dynamics

In this section we construct the most general equations of Lorentz invariant charged fluid dynamics consistent with the second law of thermodynamics. Our goal is to illustrate our method for determining the most general form of fluid-dynamical equations of motion in a simple and familiar context before tackling the slightly more complicated case of superfluids. The final results of this section are well known; the novelty of this section lies in our method of computation.

The long-wavelength degrees of freedom of a locally equilibrated system with a single global $U(1)$ charge can be taken to be the velocity field $u_\mu(x)$ (normalized so that $u^\mu u_\mu = -1$), the temperature field $T(x)$ and a chemical potential field $\mu(x)$. Both the energy momentum tensor and the charged current can be written in terms of these five fields and their gradients. The equations of motion of charged fluid dynamics are the conservation of the stress tensor and charge current

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu \\
\nabla_\mu J^\mu &= -\frac{c}{8}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}
\end{aligned}
\tag{2.1}
$$

which provides the five equations for the five hydrodynamic fields. In these equations we have allowed for the possibility that the current in question has a $U(1)^3$ anomaly. We call the coefficient c the anomaly coefficient. We have also allowed the current to be coupled to an external source with field strength $F_{\mu\nu}$. To completely determine the equations of motion it remains to determine the dependence of $T^{\mu\nu}$ and J^μ on the fields $u^\mu(x)$, $T(x)$, $\mu(x)$ and their derivatives.

By considering a stationary fluid for which $u^\mu = (1, 0, 0, 0)$ and using boost invariance one can argue that the stress tensor and charge current take the form

$$\begin{aligned} T^{\mu\nu} &= (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} + T^{\mu\nu}_{diss} \\ J^\mu &= qu^\mu + J^\mu_{diss} \end{aligned} \tag{2.2}$$

where $T^{\mu\nu}_{diss}$ and J^μ_{diss} are the contributions to the stress tensor and charge current that involve derivatives of μ, T and u^μ. The equations that express $T^{\mu\nu}_{diss}$ and J^μ_{diss} in terms of fluid dynamical fields and their derivatives are termed constitutive relations. In the long wavelength fluid dynamical limit it is sensible to expand the constitutive relations in powers of derivatives of the fluid dynamical fields u^μ, T and μ. We will refer to such an expansion as a derivative expansion and refer to the terms which are linear in gradients as first order terms. In this paper we work only to first order in the derivative expansion. The electromagnetic source term $F^{\mu\nu}$ is taken to be of first order in derivatives in this counting.

Field Redefinitions and frame choices

Note that the fluid temperature T, chemical potential μ and velocity u^μ are thermodynamical concepts that are well defined in equilibrium but have no microscopic definitions in dynamical situations. In other words, we are always free to redefine the thermodynamic variables into primed ones according to the equations

$$\begin{aligned} u^\mu &= u'^\mu + \delta u^\mu \\ T &= T' + \delta T \\ \mu &= \mu' + \delta\mu \end{aligned} \tag{2.3}$$

where δu^μ is an arbitrary one derivative vector that obeys $\delta u^\mu u_\mu = \delta u^\mu u'_\mu = 0$ and δT and $\delta\mu$ are arbitrary one derivative scalars. The primed and unprimed fields are each equally good definitions of the velocity, temperature and chemical potential fields. Physically meaningful assertions, such as the constitutive relations for $T^{\mu\nu}_{diss}$ and J^μ_{diss}, must only involve field redefinition invariant quantities.

Thus, let us determine the field redefinition invariant combinations of $T^{\mu\nu}_{diss}$ and J^μ_{diss}. Under the field redefinition (2.3),

$$\begin{aligned} \delta T^{\mu\nu}_{diss} &= (u^\mu \delta u^\nu + u^\nu \delta u^\mu)(P + \rho) + u^\mu u^\nu d(P + \rho) + \eta^{\mu\nu} dP \\ \delta J^\mu_{diss} &= q\delta u^\mu + dqu^\mu \end{aligned} \tag{2.4}$$

where

$$\delta T^{\mu\nu}_{diss} = T'^{\mu\nu}_{diss} - T^{\mu\nu}_{diss}$$

$$\delta J^\mu_{diss} = J'^\mu_{diss} - J^\mu_{diss}$$

and $df(\mu, T)$ represents the change in the function f under the first order variable change (2.3). It useful to decompose $T^{\mu\nu}_{diss}$ and J^μ_{diss} into $SO(3)$ invariant tensors, vectors and scalars The $SO(3)$ that we are referring to is the group of rotations orthogonal to u^μ. To this end we introduce the projection matrix

$$P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu \,. \tag{2.5}$$

We find that there is one tensor, two vectors and three scalars. The unique tensor

$$P^\mu{}_\alpha P^\nu{}_\beta T^{\alpha\beta}_{diss} - \frac{P^{\mu\nu}}{3} P_{\alpha\beta} T^{\alpha\beta}_{diss} \tag{2.6}$$

is automatically field redefinition invariant. The two vectors $P^\mu{}_\alpha T^{\alpha\beta}_{diss} u_\beta$ and $P^\mu{}_\alpha J^\alpha$ transform under field redefinitions as

$$\begin{aligned} \delta\left(P^\mu{}_\alpha T^{\alpha\beta}_{diss} u_\beta\right) &= -(P+\rho)\delta u^\mu \\ \delta\left(P^\mu{}_\alpha J^\alpha\right) &= q\delta u^\mu \end{aligned} \tag{2.7}$$

so that the unique invariant combination of vectors is given by

$$P^\mu{}_\alpha J^\alpha + \frac{q}{P+\rho}\left(P^\mu{}_\alpha T^{\alpha\beta}_{diss} u_\beta\right) \,. \tag{2.8}$$

The three scalars transform under field redefinitions as

$$\begin{aligned} \delta\left(P_{\alpha\beta} T^{\alpha\beta}_{diss}\right) &= 3dP \\ \delta\left(u_\alpha T^{\alpha\beta}_{diss} u_\beta\right) &= d\rho \\ \delta\left(u_\alpha J^\alpha\right) &= -dq \end{aligned} \tag{2.9}$$

so that the unique invariant scalar is given by

$$\frac{1}{3}\left(P_{\alpha\beta} T^{\alpha\beta}_{diss}\right) - \frac{\partial P}{\partial \rho}\left(u_\alpha T^{\alpha\beta}_{diss} u_\beta\right) + \frac{\partial P}{\partial q}\left(u_\alpha J^\alpha\right) \tag{2.10}$$

where $\frac{\partial P}{\partial \rho}$ is taken at constant q and $\frac{\partial P}{\partial q}$ is taken at constant ρ.

Instead of working in a manifest field redefinition invariant manner, it is sometimes convenient to 'fix' the field redefinition ambiguity by imposing five additional conditions on the thermodynamic fields so that they are well defined. Different choices of fixing the ambiguity are referred to as frames. One often used frame is the so called Landau frame, in which the velocity and temperature fields are defined to obey the conditions $T^{\mu\nu}_{diss} u_\nu = 0$ and $J^\mu_{diss} u_\nu = 0$. This gives one vector and two scalar conditions, matching the field redefinition degrees of freedom. Another choice of frame is the Eckart frame which is defined by the conditions $J^\mu_{diss} = 0$ together with $u_\mu T^{\mu\nu}_{diss} u_\nu = 0$. The expressions for the invariant vector (2.8) and the invariant scalar (2.10) greatly simplify in either of these frames. In this paper we adopt no such 'gauge' choice but work in a fully field redefinition invariant manner.

The strategy for the rest of this section

In order to complete our specification of the equations of charged fluid dynamics, we need to specify $T^{\mu\nu}_{diss}$ and J^{μ}_{diss} (or more precisely the field redefinition invariant parts (2.6), (2.8) and (2.10) of these expressions) as a function of first derivatives of fluid dynamical fields. Of course, in any particular dynamical system, the explicit form of the constitutive relations for $T^{\mu\nu}_{diss}$ and J^{μ}_{diss} can be determined only by a detailed dynamical computation. In this paper we will be interested not in computing the precise form of these quantities in any particular system, but in parameterizing the most general form that the constitutive relationship can take in any system. As we will see below, it will prove possible to completely determine the form of the first order constitutive relations up to three undetermined dissipative parameters, each of which is an arbitrary function of T and μ.

We proceed as follows. As in any effective field theory, we start by writing down all possible expressions which may contribute to $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}. We then eliminate those that do not satisfy the symmetries of the theory, Lorentz invariance in this case. In addition, since we are dealing with a hydrodynamic theory, we must ensure that the second law of thermodynamics is satisfied. We demand the existence of an entropy current of positive semi-definite divergence even when the theory is formulated on a curved background. The entropy current is defined to be a four vector J^{μ}_{S} satisfying two requirements. The first is that in a configuration where the fluid is in uniform motion,[1]

$$J^{\mu}_{S} = su^{\mu} \quad \text{(for a spacetime independent configuration)} \tag{2.11}$$

with s the entropy density which is related to ρ, P, q, μ and T through

$$\rho + P = sT + \mu q\,. \tag{2.12}$$

Our second requirement of the entropy current is that its divergence is positive semi-definite in an arbitrary curved background,

$$\nabla_{\mu} J^{\mu}_{S} \geq 0 \tag{2.13}$$

implying that the entropy increase in any region is always greater than the entropy inflow into that region.

For a perfect fluid level (i.e. a fluid in which all gradient terms have been neglected—$T^{\mu\nu}_{diss} = J^{\mu}_{diss} = 0$) the entropy current is given by (2.11). At this order it is not difficult to verify that $\nabla_{\mu}(su^{\mu}) = 0$ using (2.12) and $dP = sdT + qd\mu$.

Once the gradients of u^{μ}, T and μ/T are non vanishing the divergence of the entropy current no longer vanishes. Indeed, the divergence of the entropy current at the one derivative level will be the focus of much of the rest of this chapter. We will demand, on physical grounds, that it is possible to modify (2.11) by first order corrections so that (2.13) will be satisfied. This requirement will turn out to constrain the possible forms of $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}.

We start our analysis in section 2.1.1 by considering parity conserving charged fluids. In section 2.1.2 we move on to describe parity-violating fluid dynamics.

[1] Such a configuration is a stationary, dissipation free solution to the equations of fluid dynamics. Indeed it may be obtained by boosting a uniform fluid at rest (by which we mean a uniform fluid with velocity field $u^{\mu} = (1, 0, 0, 0)$).

2.1.1 Parity Invariant Charged Fluid Dynamics

Consider a hydrodynamic theory in the presence of external electromagnetic fields satisfying (2.1) with $c = 0$. Following the general prescription described at the beginning of this section, we would like to write the most general parity-invariant and Lorentz invariant contributions to J^μ_{diss}, $T^{\mu\nu}_{diss}$ and J^μ_S which involves a single derivative of the hydrodynamic fields u^μ, T and μ. We then work out the the restrictions on these terms by requiring that the entropy current has positive semi-definite divergence.

Classification of one and two derivative data

We begin our analysis on a technical point. The tangent space about any point in our spacetime manifold has an $SO(3,1)$ rotational invariance. However, the fluid velocity vector, $u^\mu(x)$, takes a definite value at that point and breaks this rotational group down to $SO(3)$. It is useful to decompose all derivatives of fluid dynamical fields, at any given spacetime point, into representations of this residual $SO(3)$ rotational group.

In the first column of table 2.1 we have classified all expressions formed from a single derivative of any of $u^\mu(x)$, $T(x)$ and $\mu(x)$ according to their $SO(3)$ and parity transformation properties. We refer to these expressions as one derivative fluid dynamical data. We have also classified one derivative expressions constructed out of the background electromagnetic fields according to their $SO(3)$ and parity transformation properties. We will refer to these as background data. As fluid and background field data enter our analysis on an even footing, we have listed these expressions together in the first column of table 2.1.[2]

Not all the expressions in the first column of table 2.1 are independent under the equations of motion. The equations of motion can be used to solve for some pieces of data in terms of other data. The classification of the equations of motion according to their $SO(3)$ and parity transformation properties can be found in the middle column of table 2.1. Note that there are no tensor equations of motion.

In the last column of table 2.1 we have listed a choice of independent data. By this we mean a choice of independent one derivative fluid dynamical expressions and one derivative field expressions in terms of which all others can be solved for.

While some of the expressions used in table 2.1 such as

$$\sigma_{\mu\nu} = \frac{1}{2}P^{\mu\alpha}P^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - P_{\alpha\beta}\left(\nabla_\lambda u^\lambda\right)\right) \tag{2.14}$$

and

$$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \tag{2.15}$$

are standard, some of our notation isn't. The new notation has been introduced in order to prepare the reader for later sections. In particular V_3 is the electric field in the rest frame

[2]Since all curvature invariants built out of the background metric have at least two derivatives, there is no one derivative data associated with the metric.

Table 2.1: One-derivative expressions classified according to their transformation laws under the $SO(3)$ residual symmetry and parity. The first column lists all one derivative data. The second column lists the equations of motion. The last column lists a choice of independent data. See (2.14) and (2.15) for the definition of $\sigma_{\mu\nu}$ and $P_{\mu\nu}$ respectively.

$SO(3)$ and P classification	All data	Equations of motion	Independent data
Scalars	$u^\mu\partial_\mu T$ $u^\mu\nabla_\mu\frac{\mu}{T}$ $\partial_\mu u^\mu$	$u_\mu\nabla_\nu T^{\mu\nu}=0$ $\nabla_\mu J^\mu=0$	$S_1=\partial_\mu u^\mu$
Vectors	$P^{\mu\nu}\partial_\nu T$ $u^\nu\partial_\nu u^\mu$ $P^{\mu\nu}\partial_\nu\frac{\mu}{T}$ $F^{\mu\nu}u_\nu$	$P^\mu{}_\nu\partial_\rho T^{\rho\nu}=0$	$V_1^\mu=-P^{\mu\nu}\partial_\nu\frac{\mu}{T}+\frac{F^{\mu\nu}u_\nu}{T}$ $V_2^\mu=u^\nu\nabla_\nu u^\mu$ $V_3^\mu=F^{\mu\nu}u_\nu$
Tensors	$\sigma_{\mu\nu}$	–	$T_1=\sigma_{\mu\nu}$
Pseudo vectors	$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$ $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$	–	$\omega^\mu=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$ $B^\mu=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$

of a fluid element. In the conventions of Son and Surówka [68] we have

$$(V_3)_\mu = E_\mu\,. \tag{2.16}$$

We will soon construct an entropy current that includes terms which are first order in derivatives. The divergence of such an entropy current is of second order in derivatives and includes terms quadratic in first order fluid (and background field) data plus expressions built out of two derivatives acting on fluid fields or single derivatives of electromagnetic field strengths. We refer to the second class of expressions as two derivative scalar data. When studying the divergence of the entropy current it is useful to have a listing of independent scalar two derivative data.

In the first column of table 2.2 we list the most general fluid and background field (but not curvature related) two derivative data that transforms as an $SO(3)$ scalar. More explicitly, we list all scalar expressions formed by acting with two derivatives on $u^\mu(x)$, $T(x)$ and $\mu(x)$ together with all scalars formed from the action of a single derivative on electromagnetic field strengths.[3] In the second column of the same table we list all scalar two derivative equations of motion. In the last column of the same table we list our choice of independent two derivative scalar data (in terms of which we have solved for all the other two derivative scalars).

The general entropy current and its divergence

Armed with the listings in tables 2.1 and 2.2 we now proceed with our analysis. Traditional studies of first order charged fluid dynamics (see, for example, [73]) assume that the entropy

[3]It is also easy to list two derivative fluid data in the 3, 5 and 7 dimensional representations of $SO(3)$, but that will not be required in what follows, so we do not present such a listing.

Table 2.2: Parity even two derivative scalar data for charged fluids. The first column lists all six second order scalars constructed from two derivatives of the hydrodynamic variables and background field strengths. The second column lists the three scalar two derivative equations of motion. The last column lists one choice of a $6-3=3$ dimensional basis for the independent two derivative scalar data.

All data	Equations of motion	Independent data
$u^\mu u^\nu \partial_\mu \partial_\nu T$		
$P^{\mu\nu}\partial_\mu\partial_\nu T$	$\nabla_\mu \nabla_\nu T^{\mu\nu}=0$	$P^{\mu\nu}\nabla_\mu \partial_\nu \frac{\mu}{T}$
$u^\mu u^\nu \partial_\mu \partial_\nu \left(\frac{\mu}{T}\right)$	$P^\rho_\nu \nabla_\rho \nabla^\mu T^{\mu\nu}=0$	$u^\mu \nabla_\mu \partial_\nu u^\nu$
$P^{\mu\nu}\partial_\mu\partial_\nu \left(\frac{\mu}{T}\right)$	$u^\nu \nabla_\nu \nabla^\mu J^\mu=0$	$\nabla_\mu (F^{\mu\nu}u_\nu)$
$u^\mu \partial_\mu \partial_\nu u^\nu$		
$\partial_\mu (F^{\mu\nu}u_\nu)$		

current takes a canonical form, [4]

$$J^\mu_{S\,\mathrm{canon}} = s u^\mu - \frac{1}{T} u_\mu T^{\mu\nu}_{diss} - \frac{\mu}{T} J^\mu_{diss}\,. \tag{2.17}$$

As we explained in the introduction, in this work we will not make any prior assumption about the form of the entropy current. According to the analysis of section 2.1.1 the most general parity even first order entropy current is given by

$$J^\mu_S = J^\mu_{S\,\mathrm{canon}} + s_1\, S_1 u^\mu + \sum_{i=1}^{3} v_i V_i^\mu \tag{2.18}$$

where S_1 and V_i are defined in the last column of table 2.1, and s_1 and the v_i's are arbitrary functions of $\frac{\mu}{T}$ and T.

We now explore the constraints obtained by enforcing the positivity of the divergence of the entropy current (2.18). It is easily demonstrated (see, for instance, [3, 73]) that the divergence of the canonical part of the entropy current is given by

$$\nabla_\mu J^\mu_{S\,\mathrm{canon}} = -\nabla_\mu \left(\frac{u_\nu}{T}\right) T^{\mu\nu}_{diss} - \left(\partial_\mu \left(\frac{\mu}{T}\right) - \frac{F_{\mu\nu}u^\nu}{T}\right) J^\mu_{diss}. \tag{2.19}$$

The right hand side of (2.19) is a quadratic form in one derivative fluid and background electromagnetic field data. The divergence of the non canonical part of the entropy current in (2.18) is also a two derivative expression but is composed of two kinds of terms. The first set of terms are linear in independent two derivative and curvature data. Such terms are always inconsistent with the positivity of the entropy current, and so we must choose s_1 and v_i so that these terms vanish. The second set of terms contains products of one derivative terms. Such terms would modify the quadratic form on the right hand side of (2.19) and do not necessarily vanish. Schematically, we have

$$\partial_\mu J^\mu_S = \begin{pmatrix}\text{independent two} \\ \text{derivative and curvature data}\end{pmatrix} + \begin{pmatrix}\text{quadratic form in} \\ \text{first order data}\end{pmatrix}. \tag{2.20}$$

[4] As explained in [3] the expression in (2.17) is frame invariant, i.e. invariant under a first order field redefinition of T, u^μ and μ. Note that the second term on the right hand side vanishes in the Landau frame while the third term vanishes in the Eckart frame.

The first term on the right hand side of (2.20) must vanish while the second term must be tuned to be positive.

Constraints from positivity of the divergence of the entropy current

We will first explore the constraints that follows from the requirement that no two derivative data appears in the divergence of the entropy current. As explained previously, this implies that the first set of terms on the right hand side of (2.20) must vanish. We will implement this condition separately for two derivative and curvature terms.

Constraints from the vanishing of 2 derivative terms

The two derivative part of the divergence of the entropy is given by

$$-v_1 P^{\mu\nu}\nabla_\mu \partial_\nu \frac{\mu}{T} + (s_1 + v_2) u^\mu \nabla_\mu \partial_\nu u^\nu + \left(v_3 + \frac{v_1}{T}\right) \nabla_\mu (F^{\mu\nu} u_\nu)\,.$$

This expression is a linear combination of the three independent two derivative pieces of data (see Table 2.2). It follows that the vanishing of two derivative terms requires us to set the coefficients of each of these terms to zero, i.e. to set $v_1 = v_3 = 0$ and $v_2 = -s_1$. Thus the vanishing of two derivative terms in the divergence of the entropy current restricts the entropy current (2.18) to take the form

$$J^\mu_S = J^\mu_{S\,\mathrm{canon}} + s_1 \left(S_1 u^\mu - V_2^\mu\right)\,. \tag{2.21}$$

where s_1 is still an arbitrary function of T and μ.

Constraints from vanishing of curvature terms

According to (2.21) the entropy current has a one parameter ambiguity, s_1. Were we to restrict our attention to a flat space background we would not have been able resolve this ambiguity. Consider a charged fluid propagating on an arbitrary curved background. The cancellation of two derivative terms proportional to s_1 is now incomplete; it is not difficult to check that there is an additional, curvature dependent term in the divergence of the entropy current proportional to $s_1 R_{\alpha\beta} u^\alpha u^\beta$ with $R_{\alpha\beta}$ the Ricci tensor. This term is inconsistent with positivity of the divergence of J^μ_S. Thus, we are forced to set $s_1 = 0$.

We conclude that the requirement that the divergence of the entropy current is positive *in an arbitrary curved background* forces the entropy current to take its canonical form, justifying the assumptions of standard treatments of fluid dynamics e.g. [73].

Constraints on dissipative terms

We have demonstrated that the entropy current takes its canonical form and consequently that its divergence is given by (2.19). It is now not difficult to work out the constraints on dissipative terms that ensure the positivity of the quadratic form on the right hand side of (2.19). We outline the calculation here.

Consider the expansion of $\nabla_\mu\left(\frac{u_\nu}{T}\right)$ and $-V_1 = \partial_\mu \frac{\mu}{T} - \frac{E_\mu}{T}$ which appear on the right hand side of (2.19) into $SO(3)$ invariant tensors vectors and scalars. We find a single tensor, $\sigma_{\mu\nu}$, two vectors,

$$V_1^\mu, \quad \text{and} \quad \left(P^{\mu\nu}\frac{\partial_\nu T}{T} + (u.\partial)u^\mu\right)$$

(see table 2.1 for a definition of the vector V_1^μ) and three scalars,

$$\frac{(u\cdot\partial)T}{T}, \quad (u\cdot\partial)\nu, \quad \text{and} \quad (\nabla\cdot u)\,.$$

While the two vectors are completely distinct off-shell, it turns out that the equations of motion imply that they are proportional to each other on-shell. Similarly, the equations of motion imply that the three scalars are also proportional to each other on-shell. The explicit relations are

$$\begin{aligned}
\frac{(u.\partial)T}{T} &= -\left[\frac{\partial P}{\partial \rho}\right]_q (\nabla.u) \\
(u.\partial)\nu &= -\frac{1}{T}\left[\frac{\partial P}{\partial q}\right]_\rho (\nabla.u) \\
P^{\mu\nu}\frac{\partial_\nu T}{T} + (u.\partial)u^\mu &= \frac{qT}{\rho+P}V_1^\mu\,.
\end{aligned} \tag{2.22}$$

Plugging these relations into (2.19), we can rewrite the divergence of the entropy current in the form

$$\begin{aligned}
\nabla_\mu J_S^\mu = &-\frac{(\nabla_\mu u^\mu)}{T}\left[\frac{(T_{diss})_{ab}P^{ab}}{3} - \frac{\partial P}{\partial \rho}(u_\mu u_\nu T^{\mu\nu}_{diss}) + \frac{\partial P}{\partial q}(u_\mu J^\mu_{diss})\right] \\
&+ V_{1\,\mu}\left[J^\mu_{diss} + \frac{q}{\rho+P}(u_\nu T^{\mu\nu}_{diss})\right] - \frac{T^{\mu\nu}_{diss}\sigma_{\mu\nu}}{T}
\end{aligned} \tag{2.23}$$

where V_1^μ, B^μ and E^μ were defined in Table 2.1 and (2.16). We collect their definitions here for convenience:

$$\begin{aligned}
E_\mu &= F_{\mu\nu}u^\nu \\
B_\mu &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta} \\
V_{1\,\mu} &= \frac{E_\mu}{T} - P_\mu^\theta \partial_\theta \nu\,.
\end{aligned}$$

We will now use (2.23) to constrain the constitutive relations of fluid dynamics, i.e. the expressions for $T^{\mu\nu}_{diss}$ and J^μ_{diss} as a linear expansion in first order scalars, vectors and tensors. To first order in gradients there is only one independent scalar data so the scalar parts of $T^{\mu\nu}_{diss}$ and J^μ_{diss} are necessarily proportional to $\nabla\cdot u$. The vector parts of $T^{\mu\nu}_{diss}$ and J^μ_{diss} must each be expanded as a linear sum of the three independent vectors listed in Table 2.1. The tensor in Table 2.1 is proportional to $\sigma_{\mu\nu}$ since there is only one $SO(3)$ invariant tensor. It follows from group theory that positivity of the divergence of the entropy current implies positivity of the scalar, vector and tensor components separately. Thus, we have

$$\begin{aligned} P^{\mu}_{\alpha}P^{\nu}_{\beta}T^{\alpha\beta}_{diss} - \frac{P^{\mu\nu}}{3}P_{\alpha\beta}T^{\alpha\beta}_{diss} &= -\eta\sigma^{\mu\nu} \\ P^{\mu}_{\alpha}\left(J^{\alpha}_{diss} + \frac{q}{\rho+P}(u_\nu T^{\alpha\nu}_{diss})\right) &= \kappa V^{\mu}_{1} \\ \frac{(T_{diss})_{ab}P^{ab}}{3} - \frac{\partial P}{\partial\rho}(u_\mu u_\nu T^{\mu\nu}_{diss}) + \frac{\partial P}{\partial q}(u_\mu j^{\mu}_{diss}) &= -\beta\partial_\alpha u^\alpha \end{aligned} \tag{2.24}$$

where

$$\eta \geq 0, \quad \kappa \geq 0, \quad \beta \geq 0.$$

These three coefficients are the shear viscosity, η, the heat conductivity, κ, and the bulk viscosity, β. The bulk viscosity is traditionally denoted by ζ but in this work we reserve ζ for different use.[5]

Several aspects of (2.24) deserve comment. First, the requirement of positivity does not individually constrain the three scalar and two vector pieces in $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}, but only constrains the combinations that appear in (2.10) and (2.8). This is exactly as we would expect: only field redefinition invariant data can be constrained in a physical way. The vectors and scalars that are left undetermined are unphysical; they can be changed, or chosen arbitrarily, by a field redefinition. Despite appearances, (2.24) constitutes a complete determination of the constitutive relations of our system.

We also note that we could have used the fact that the divergence of the entropy current is frame invariant (see [3]) to determine the frame invariant scalar, vector and tensor combinations in (2.6), (2.8) and (2.10); the expression on the right hand side of (2.23) must arrange itself into such frame invariant combinations.

The third aspect to note is that the constraints of positivity are relatively mild in the scalar and tensor sector. The expansion of scalars and tensors is the most general one permitted by symmetry; the requirement of positivity merely imposes inequalities in the coefficients of this expansion. However, the constraint on vectors is much stronger. Symmetry alone would have allowed the expansion of the second line in (2.24) as an arbitrary linear combination of the 3 vectors V_1, V_2 and V_3. However the requirement of positivity sets the coefficients V_2 and V_3 to zero,[6] apart from imposing an inequality on the coefficient of the third. We will see this pattern repeated and magnified in the study of superfluid dynamics in section 2.2 in the scalar, vector and tensor sector.

2.1.2 Parity non invariant charged fluid dynamics

Let us now turn to the dynamics of fluids that are not invariant under parity transformations. According to table 2.1 we should allow the entropy current to depend on an additional

[5]Note that the speed of sound, c_s, is related to the variation of the pressure with respect to energy density through $\frac{\partial P}{\partial \rho} = c_s^2$. Using dimensional analysis one can conclude that $\frac{\partial P}{\partial q} = 0$ in a scale invariant theory. It then becomes clear that in a conformal theory the left hand side of the last equality in (2.24) vanishes as it should.

[6]The origin of this constraint is the observation that the quadratic form $ax^2 + bxy + cxz$ is positive only when $b = c = 0$ and $a \geq 0$. The role of x is played by the vector V, while the roles of y and z are played by the other two vectors

arbitrary pseudo vector. Thus, the most general entropy current for such a fluid takes the form

$$J_S^\mu = J^\mu_{S\,\text{canon}} + s_1\, S_1 u^\mu + \sum_{i=1}^{3} v_i V_i^\mu + \sigma_\omega \omega^\mu + \sigma_B B^\mu\,. \tag{2.25}$$

In the parity even sector the divergence of this entropy current is identical to the one discussed in subsection 2.1.1; the arguments in 2.1.1 go through unchanged and in particular the cancellation of two derivative and scalar terms set $s_1 = v_i = 0$. In the parity odd sector the divergence of the entropy current receives contributions involving the dot product of the pseudo vectors ω^μ and B^μ with ordinary vectors. Positivity of the divergence of the entropy current implies that such products vanish.[7] This restriction was analyzed in detail by Son and Surówka [68] who found that it leads to

$$\begin{aligned}
P^\mu_\alpha P^\nu_\beta T^{\alpha\beta}_{diss} - \frac{P^{\mu\nu}}{3} P_{\alpha\beta} T^{\alpha\beta}_{diss} &= -\eta\sigma^{\mu\nu} \\
P^\mu_\alpha \left(J^\alpha_{diss} + \frac{q}{\rho+P}(u_\nu T^{\alpha\nu}_{diss}) \right) &= \kappa V^\mu + \tilde\kappa_\omega \omega^\mu + \tilde\kappa_B B^\mu \\
\frac{(T_{diss})_{ab}P^{ab}}{3} - \frac{\partial P}{\partial \rho}(u_\mu u_\nu T^{\mu\nu}_{diss}) + \frac{\partial P}{\partial q}(u_\mu j^\mu_{diss}) &= -\beta \partial_\alpha u^\alpha
\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
\sigma_\omega &= c\frac{\mu^3}{3T} + T\mu k_2 + T^2 k_1 \\
\sigma_B &= c\frac{\mu^2}{2T} + \frac{T}{2}k_2 \\
\tilde\kappa_\omega &= c\left(\mu^2 - \frac{2}{3}\frac{q}{\rho+P}\mu^3\right) + T^2\left(1 - \frac{2q}{\rho+P}\mu\right)k_2 - \frac{2q}{\rho+P}k_1 \\
\tilde\kappa_B &= c\left(\mu - \frac{1}{2}\frac{q}{\rho_n+P}\mu^2\right) - \frac{T^2}{2}\frac{q}{\rho+P}k_2
\end{aligned} \tag{2.27}$$

and k_1 and k_2 are integration constants. We will now argue that the requirement of CPT invariance forces k_2 to vanish. [8] The argument goes as follows. Consider the CPT transformation $x^\mu \to -x^\mu$ $q \to -q$ (and so $\mu \to -\mu$). Under this transformation $T^{\mu\nu}_{diss} \to T^{\mu\nu}_{diss}$ and $J^\mu_{diss} \to -J^\mu_{diss}$. Also $u^\mu \to u^\mu$ so that $\omega^\mu \to -\omega^\mu$ and $B^\mu \to B^\mu$. Thus under a CPT transformation it must be that $\tilde\kappa_\omega \to \tilde\kappa_\omega$ while $\tilde{\kappa_B} \to -\tilde{\kappa_B}$. Consistency of this requirement with (2.27) sets $k_2 = 0$. Nothing in our argument requires that k_1 vanish (although it would be intersting to find a specific system with $k_1 \neq 0$; k_1 vanishes in all AdS/CFT computations performed so far).

The results (2.26) and (2.27) have several interesting features. First, the presence of an anomaly forces the entropy current to depart from the canonical form (i.e. σ_B and σ_ω are never zero if c is nonzero). Second, it induces new terms in the vector part of the constitutive relations, proportional to the vorticity and the magnetic field. Third, the new contributions to both the entropy current and the vector part of the constitutive relations are completely

[7] We will see later that products of vectors and pseudo vectors do not necessarily need to vanish in the case of superfluid dynamics. In the current setup vanishing of such bilinear terms follows from the fact that the divergence has no squares of pseudo vectors and contains only a single squared vector.

[8] We thank D. Son for pointing this out to us.

determined (up to an integration constant that is independent of T and μ) by the anomaly. In other words, although the constitutive relations take a different form from the parity even case, this change in form is completely determined by the anomaly, and we have no new free parameters apart from the integration constant k_2.

2.2 The thoery of parity invariant superfluid hydrodynamics

By definition, a superfluid is a fluid phase of a system with a spontaneously broken global symmetry. When discussing superfluids this forces us to consider the gradient of the Goldstone boson as an extra hydrodynamical degrees of freedom in addition to the standard variables u^μ, T and μ. More precisely, if we denote the Goldstone Boson by ψ (ψ is the phase of the condensate of the charged scalar operator) and we also wish turn on a background gauge field A_μ then

$$\xi_\mu = -\partial_\mu \psi + A_\mu \tag{2.28}$$

represents the covariant derivative of the Goldstone Boson and is an extra hydrodynamic degree of freedom. According to the Landau-Tisza two fluid model the superfluid should be thought of as a two component fluid: a condensed component and a non condensed or normal component. The velocity field of the normal fluid is given by u_μ and the velocity of the condensed phase is proportional to ξ_μ. It is often convenient to define the component of ξ orthogonal to u,

$$\zeta^\mu = P^{\mu\nu}\xi_\nu \, . \tag{2.29}$$

The equations of motion of the superfluid are given by

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu \\ \partial_\mu J^\mu &= c E_\mu B^\mu \\ \partial_\mu \xi_\nu - \partial_\nu \xi_\mu &= F_{\mu\nu} \end{aligned} \tag{2.30}$$

together with the constitutive relations

$$\begin{aligned} T^{\mu\nu} &= (\rho + P) u^\mu u^\nu + P\eta^{\mu\nu} + f\xi^\mu \xi^\nu + T^{\mu\nu}_{diss} \\ J^\mu &= q u^\mu - f\xi^\mu + J^\mu_{diss} \\ u\cdot\xi &= \mu + \mu_{diss} \end{aligned} \tag{2.31}$$

Note that the first and higher order quantities in (2.31) are ambiguous upto field redefinition ambiguities of the fluid fields. The quantity ξ_μ is microscopically defined and therefore we do not allow any field redefinitions of this quantities (such a choice was refred to as the fluid frame in [3] where a more detailed discussion about the various choice of frames has been provided). The rest of the redefinition ambiguities are fixed by imposing certain conditions on the first (or higher) order quantities $T^{\mu\nu}_{diss}$, J^μ_{diss} and μ_{diss}. In all our discussions in this thesis whenever we need to make a frame choice we will work in the transverse frame [9]

[9] In the gravity calculations, however we had to make a different choice for convinience, but we report

which is the choice

$$u_\mu T^{\mu\nu}_{diss} = 0; u_\mu J^\mu_{diss} = 0. \tag{2.32}$$

As was the case for the theory of charged fluids which we described in the previous section, superfluids also allow for a simple 'canonical' entropy current [3]

$$J^\mu_{S\,\text{canon}} = su^\mu - \frac{\mu}{T} J^\mu_{diss} - \frac{u_\nu T^{\mu\nu}_{diss}}{T} \tag{2.33}$$

where s is the thermodynamical entropy density of our fluid and is related to ρ and P through the Gibbs-Duhem relation

$$\rho + P = sT + \mu q \tag{2.34}$$

and

$$dP = sdT + qd\mu + \frac{1}{2} f d\xi^2 \tag{2.35}$$

where

$$\xi = \sqrt{-\xi^\mu \xi_\mu}\,. \tag{2.36}$$

It can be demonstrated (see [3]) that the entropy current (2.33) is invariant under field redefinitions. The divergence of this entropy current is given by

$$\partial_\mu J^\mu_{S_canon} = -\partial_\mu \left(\frac{u_\nu}{T}\right) T^{\mu\nu}_{diss} - \left(\partial_\mu \left(\frac{\mu}{T}\right) - \frac{E_\mu}{T}\right) J^\mu_{diss} + \frac{\mu_{diss}}{T} \partial_\mu \left(f \xi^\mu\right) \tag{2.37}$$

The rest of this section closely follows section 2.1. In 2.2.1 we list the independent first order data and second order scalar data, in section 2.2.2 we construct the most general positive divergence parity conserving entropy current consistent with Lorentz invariance. We find that up to a certain ambiguous term which is physically trivial, the entropy current agrees with its canonical form (2.33).

2.2.1 Onshell inequivalent First order independent data

In the case of superfluid dynamics, the $SO(3,1)$ tangent space symmetry at any point is generically broken down to $SO(2)$ by the nonzero velocity fields u^μ and ξ^μ. In the special case that u^μ and ξ^μ are collinear, $SO(2)$ is enhanced to $SO(3)$. This special case is physically interesting since it implies that the superfluid component is motionless relative to the normal component—once the superfluid velocity is too large superfluidity breaks down. We will find it convenient to decompose all first order fluid dynamical data into representations of $SO(2)$ and treat the collinear limit as a special point in parameter space.

Representations of $SO(2)$ are all one dimensional. We refer to fluid dynamical data that is invariant under $SO(2)$ as scalar data. All other fluid data has charge $\pm m$ under $SO(2)$, where m is an integer. There is always as much $+m$ as $-m$ data. We will find it useful to group together $+1$ and -1 charge data into a two column which we refer to as vector data;

the final answer in this transverse frame after performing suitable field redefinitions (see chapter 3 for more details).

Table 2.3: One derivative data for superfluids. The first column lists all quantities formed from the action of a single derivative on fluid and background fields. The second column lists all one derivative equations of motion. The last columns lists a choice of independent data. The tensors $\sigma^{\xi}_{\mu\nu}$ and $\sigma^{u}_{\mu\nu}$ are defined in (2.39)-(2.38). We also used $e^{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}u_\mu\xi_\nu$.

Classification	All data	Equations of motion	Independent data
Scalars (set 1)	$\partial_\mu u^\mu$ $\partial_\mu(T\xi^\mu)$ $\xi^\mu\xi^\nu\partial_\mu u_\nu$ $\xi^\mu\partial_\mu\left(\frac{\xi}{T}\right)$ $\xi^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $\xi^\mu\partial_\mu T$ $E\cdot\xi$ $u^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $u^\mu\partial_\mu T$ $u^\mu\partial_\mu\left(\frac{\xi}{T}\right)$ $\xi^\mu u^\nu\partial_\nu u_\nu$	$\partial_\mu J^\mu = cE^\mu B_\mu$ $\xi^\mu u^\nu\left(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu\right) = \xi\cdot E$ $\xi_\mu\partial_\nu T^{\mu\nu} = \xi^\mu F_{\mu\nu}J^\nu$ $u_\mu\partial_\nu T^{\mu\nu} = -E_\mu J^\mu$	$\tilde{P}^{\mu\nu}\partial_\mu u_\nu$ $\tilde{P}^{\mu\nu}\partial_\mu(T\xi_\nu)$ $\xi^\mu\xi^\nu\partial_\mu u_\nu$ $\xi^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $\xi^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $\xi^\mu\partial_\mu T$ $E\cdot\xi$
Scalars (set 2)	$\partial_\mu u^\mu$ $\partial_\mu(T\xi^\mu)$ $\xi^\mu\xi^\nu\partial_\mu u_\nu$ $\xi^\mu\partial_\mu\left(\frac{\xi}{T}\right)$ $\xi^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $\xi^\mu\partial_\mu T$ $E\cdot\xi$ $u^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $u^\mu\partial_\mu T$ $u^\mu\partial_\mu\left(\frac{\xi}{T}\right)$ $\xi^\mu u^\nu\partial_\nu u_\nu$	$\partial_\mu J^\mu = cE^\mu B_\mu$ $\xi^\mu u^\nu\left(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu\right) = \xi\cdot E$ $\xi_\mu\partial_\nu T^{\mu\nu} = \xi^\mu F_{\mu\nu}J^\nu$ $u_\mu\partial_\nu T^{\mu\nu} = -E_\mu J^\mu$	$\tilde{P}^{\mu\nu}\partial_\mu u_\nu$ $\tilde{P}^{\mu\nu}\partial_\mu(T\xi_\nu)$ $u^\mu\xi^\nu\partial_\mu u_\nu$ $u^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $u^\mu\partial_\mu\left(\frac{\mu}{T}\right)$ $u^\mu\partial_\mu T$ $E\cdot\xi$
Pseudo scalars	$\omega\cdot\xi$ $B\cdot\xi$ $\epsilon^{\mu\nu\alpha\beta}u_\mu\xi_\nu\partial_\alpha\xi_\beta$	$e^{\alpha\beta}(\partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha) = e^{\alpha\beta}F_{\alpha\beta}$	$\omega\cdot\xi$ $B\cdot\xi$
Vectors	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho u_\nu$ $\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\xi_\nu$ $\tilde{P}^{\mu\nu}E_\nu$ $\tilde{P}^{\mu\nu}F_{\nu\beta}\xi^\beta$ $\tilde{P}^{\mu\nu}\partial_\nu\frac{\mu}{T}$ $\tilde{P}^{\mu\nu}\partial_\nu\frac{\xi}{T}$ $\tilde{P}^{\mu\nu}\partial_\nu T$ $\tilde{P}^{\mu\nu}\xi^\alpha\partial_\nu u_\alpha$ $\tilde{P}^{\mu\nu}u^\rho\partial_\rho u_\nu$ $\tilde{P}^{\mu\nu}u^\rho\partial_\rho\xi_\nu$	$\tilde{P}^{\mu\nu}\partial_\beta T^\beta_{\ \nu} = \tilde{P}^{\mu\nu}F_{\nu\beta}J^\beta$ $\tilde{P}^{\alpha\mu}u^\nu\left(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu\right) = \tilde{P}^{\alpha\mu}E_\mu$ $\tilde{P}^{\alpha\mu}\xi^\nu\left(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu\right) = \tilde{P}^{\alpha\mu}F_{\mu\nu}\xi^\nu$	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho u_\nu$ $\tilde{P}^{\mu\nu}u^\rho\partial_\rho\xi_\nu$ $\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho u_\nu$ $\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\xi_\nu$ $\tilde{P}^{\mu\nu}E_\nu$ $\tilde{P}^{\mu\nu}F_{\nu\beta}\xi^\beta$ $\tilde{P}^{\mu\nu}\partial_\nu\frac{\mu}{T}$
Tensors	$\sigma^u_{\mu\nu}$ $\sigma^\xi_{\mu\nu}$	– –	$\sigma^u_{\mu\nu}$ $\sigma^\xi_{\mu\nu}$

Table 2.4: Labels for the two sets of independent one derivative scalars and one set of independent vectors.

i	1	2	3	4	5	6	7
$\mathcal{S}^a_i$	$\tilde{P}^{\mu\nu}\partial_\mu u_\nu$	$\tilde{P}^{\mu\nu}\partial_\mu(T\xi_\nu)$	$\xi^\mu\xi^\nu\partial_\mu u_\nu$	$\xi^\mu\partial_\mu\left(\frac{\xi}{T}\right)$	$\xi^\mu\partial_\mu\left(\frac{\mu}{T}\right)$	$\xi^\mu\partial_\mu T$	$E.\xi$
$\mathcal{S}^b_i$	$\tilde{P}^{\mu\nu}\partial_\mu u_\nu$	$\tilde{P}^{\mu\nu}\partial_\mu(T\xi_\nu)$	$u^\mu\xi^\nu\partial_\mu u_\nu$	$u^\mu\partial_\mu\left(\frac{\xi}{T}\right)$	$u^\mu\partial_\mu\left(\frac{\mu}{T}\right)$	$u^\mu\partial_\mu T$	$E.\xi$
$\mathcal{V}^a_i$	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho u_\nu$	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho\xi_\nu$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho u_\nu$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\xi_\nu$	$\tilde{P}^{\mu\nu}\partial_\nu\frac{\mu}{T}$	$\tilde{P}^{\mu\nu}\partial_\nu E_\mu$	$\tilde{P}^{\mu\nu}F_{\nu\beta}\xi^\beta$

similarly we group $+2$ and -2 data together into tensor data.

We now turn to a listing of the one derivative fluid dynamical and field data for superfluids. In Table 2.3 we explicitly list all one derivative data, one derivative equations of motion, and then eventually independent one derivative data. The scalar ξ used in this table is given by (2.36). We do not list pseudo vectors and pseudo tensors independently from vectors and tensors as they are isomorphic and contain the same data. In Table 2.4 we assign labels to our independent data. In the same table we also present a second listing of a basis for independent scalar data which will be more convenient at places. In Appendix B we demonstrate that both sets of seven scalars and the seven vectors listed are independent data, i.e. that we can solve for all other scalars and all other vectors in terms of the chosen basis.

As can be seen from Tables 2.3 and 2.4, after imposing the equations of motion we have six first order scalars and one first order pseudo scalar built out of fluid data, one first order

Table 2.5: Two derivative scalar data. The first row gives all two derivative scalar data, the second row lists all the equations of motion. The third row represents a particular choice of independent second order data.

All data	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho\partial_\mu u_\nu$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\partial_\mu u_\mu$	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho\partial_\mu(T\xi_\nu)$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\partial_\mu(T\xi_\nu)$
	$\xi^\mu\xi^\nu u^\rho\partial_\rho\partial_\mu u_\nu$	$u^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\xi}{T}\right)$	$u^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\mu}{T}\right)$	$u^\mu\xi^\rho\partial_\rho\partial_\mu T$
	$\tilde{P}^{\mu\nu}\partial_\mu\partial_\nu\left(\frac{\mu}{T}\right)$	$\tilde{P}^{\mu\nu}\partial_\mu\partial_\nu\left(\frac{\xi}{T}\right)$	$\tilde{P}^{\mu\nu}\partial_\mu\partial_\nu T$	$u^\mu u^\rho\partial_\rho\partial_\mu\left(\frac{\xi}{T}\right)$
	$\xi^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\xi}{T}\right)$	$\xi^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\mu}{T}\right)$	$\xi^\mu\xi^\rho\partial_\rho\partial_\mu T$	$u^\mu u^\rho\partial_\rho\partial_\mu\left(\frac{\mu}{T}\right)$
	$u^\mu u^\rho\partial_\rho\partial_\mu T$	$\xi^\mu\xi^\nu\xi^\rho\partial_\rho\partial_\mu u_\nu$	$u^\mu\xi^\nu u^\rho\partial_\rho\partial_\mu u_\nu$	$\xi^\nu\tilde{P}^{\rho\mu}\partial_\rho\partial_\mu u_\nu$
	$u^\mu\xi^\nu\partial_\mu E_\nu$	$\xi^\mu\xi^\nu\partial_\mu E_\nu$	$\tilde{P}^{\mu\nu}\partial_\mu E_\nu$	$\epsilon^{\mu\nu\lambda\sigma}\xi_\mu u_\lambda\partial_\mu B_\sigma$
Equations of motion	$u^\beta\partial_\beta\left(\partial_\mu J^\mu\right)=u^\beta\partial_\beta\left(cE^\mu B_\mu\right)$		$\xi^\beta\partial_\beta\left(\partial_\mu J^\mu\right)=\xi^\beta\partial_\beta\left(cE^\mu B_\mu\right)$	
	$u^\beta\partial_\beta\left(\xi_\mu\partial_\nu T^{\mu\nu}\right)=u^\beta\partial_\beta\left(\xi^\mu F_{\mu\nu}J^\nu\right)$		$\xi^\beta\partial_\beta\left(\xi_\mu\partial_\nu T^{\mu\nu}\right)=\xi^\beta\partial_\beta\left(\xi^\mu F_{\mu\nu}J^\nu\right)$	
	$u^\beta\partial_\beta\left(u_\mu\partial_\nu T^{\mu\nu}\right)=u^\beta\partial_\beta\left(-E_\mu J^\mu\right)$		$\xi^\beta\partial_\beta\left(u_\mu\partial_\nu T^{\mu\nu}\right)=\xi^\beta\partial_\beta\left(-E_\mu J^\mu\right)$	
	$u^\beta\partial_\beta\left(\xi^\mu u^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=u^\beta\partial_\beta\left(\xi^\mu E_\mu\right)$		$\xi^\beta\partial_\beta\left(\xi^\mu u^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=\xi^\beta\partial_\beta\left(\xi^\mu E_\mu\right)$	
	$\partial_\alpha\left(\tilde{P}^{\alpha\mu}u^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=\partial_\alpha\left(\tilde{P}^{\alpha\mu}E_\mu\right)$		$\partial_\mu\left(\tilde{P}^{\mu\nu}\partial_\beta T^\beta_{\ \nu}\right)=\partial_\mu\left(\tilde{P}^{\mu\nu}F_{\nu\beta}J^\beta\right)$	
	$\partial_\alpha\left(\tilde{P}^{\alpha\mu}\xi^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=\partial_\alpha\left(\tilde{P}^{\alpha\mu}F_{\mu\nu}\xi^\nu\right)$			
Independent data	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho\partial_\mu u_\nu$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\partial_\mu u_\mu$	$\tilde{P}^{\mu\nu}u^\rho\partial_\rho\partial_\mu(T\xi_\nu)$	$\tilde{P}^{\mu\nu}\xi^\rho\partial_\rho\partial_\mu(T\xi_\nu)$
	$\xi^\mu\xi^\nu u^\rho\partial_\rho\partial_\mu u_\nu$	$u^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\xi}{T}\right)$	$u^\mu\xi^\rho\partial_\rho\partial_\mu\left(\frac{\mu}{T}\right)$	$u^\mu\xi^\rho\partial_\rho\partial_\mu T$
	$\tilde{P}^{\mu\nu}\partial_\mu\partial_\nu\left(\frac{\mu}{T}\right)$	$u^\mu\xi^\nu\partial_\mu E_\nu$	$\xi^\mu\xi^\nu\partial_\mu E_\nu$	$\tilde{P}^{\mu\nu}\partial_\mu E_\nu$
	$\tilde{P}^{\mu\nu}\left(\partial_\mu F_{\nu\beta}\right)\xi^\beta$			

scalar and one first order pseudo scalar built out of background field strengths, five first order vectors built out of fluid data, two first order vectors built from background fields and two independent tensors. The first tensor is simply the usual shear tensor $\sigma^{\mu\nu}$ projected orthogonal to the plane formed by the two fluid velocities.

$$\sigma^u_{\mu\nu}=\tilde{P}^{\mu\alpha}\tilde{P}^{\nu\beta}\left(\sigma_{\alpha\beta}-\frac{1}{2}\eta_{\alpha\beta}\tilde{P}_{\gamma\delta}\sigma^{\gamma\delta}\right)\,. \tag{2.38}$$

The second tensor $\sigma^\xi_{\mu\nu}$ is defined by

$$\sigma^\xi_{\mu\nu}=\frac{1}{2}\tilde{P}^{\mu\alpha}\tilde{P}^{\nu\beta}\left(\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha-\tilde{P}_{\alpha\beta}\tilde{P}^{\gamma\delta}\partial_\gamma\xi_\delta\right)\,. \tag{2.39}$$

The counting of data in the absence of background fields agrees with [3].

As was the case for normal fluids, the divergence of the first order superfluid entropy current is a sum over quadratic one derivative terms and two derivative pieces of data. In order to assist the analysis of the positivity of the divergence of the entropy current we list all the scalar two derivative data, the two derivative equations of motion and a basis for onshell independent two derivative scalars in Table 2.5. Note that we have nine independent pieces of two derivative fluid dynamical data together with four additional pieces of two derivative data from background field strengths. In Appendix B demonstrate that the scalars listed in the last column of table 2.5 form a basis of onshell independent scalars.

2.2.2 Constructing the entropy current

With the independent data at hand we proceed with our analysis. The most general entropy current allowed by symmetries takes the form

$$J^\mu_S=J^\mu_{S\,\mathrm{canon}}+u^\mu\sum_{i=1}^{7}s^a_i\mathcal{S}^a_i+\xi^\mu\sum_{i=1}^{7}s^b_i\mathcal{S}^b_i+\sum_{i=1}^{7}v_i\mathcal{V}^{a\,\mu}_i\,, \tag{2.40}$$

where the coefficients s_i^a, s_i^b, and v_i are, at the moment, arbitrary functions of $\frac{\mu}{T}$, T, and ξ^2. Note that we have chosen to expand the terms proportional to u^μ in the basis $\mathcal{S}_i^a$ while terms proportional to ξ^μ are expanded in the basis $\mathcal{S}_i^b$. This choice will prove algebraically convenient below. In total we start with twenty one free parameters in the entropy current.

The two derivative terms in the divergence of the entropy current (2.40) are given by

$$
\begin{aligned}
\partial_\mu J_S^\mu = & \left(s_1^a + v_1\right) \tilde{P}^{\mu\nu} u^\rho \partial_\rho \partial_\mu u_\nu + \left(s_2^a + v_2\right) \tilde{P}^{\mu\nu} u^\rho \partial_\rho \partial_\mu (T\xi_\nu) + \left(s_1^b + v_3\right) \tilde{P}^{\mu\nu} \xi^\rho \partial_\rho \partial_\mu u_\nu \\
& + \left(s_2^b + v_4\right) \tilde{P}^{\mu\nu} \xi^\rho \partial_\rho \partial_\mu (T\xi_\nu) + \left(s_3^a + s_3^b\right) \xi^\mu \xi^\nu u^\rho \partial_\rho \partial_\mu u_\nu \\
& + \left(s_3^a + s_3^b\right) u^\mu \xi^\rho \partial_\rho \partial_\mu \left(\frac{\xi}{T}\right) + \left(s_3^a + s_3^b\right) u^\mu \xi^\rho \partial_\rho \partial_\mu \left(\frac{\mu}{T}\right) \\
& + \left(s_3^a + s_3^b\right) u^\mu \xi^\rho \partial_\rho \partial_\mu T + v_5 \; \tilde{P}^{\mu\nu} \partial_\mu \partial_\nu \left(\frac{\mu}{T}\right) + s_7^a \; u^\mu \xi^\nu \partial_\mu E_\nu \\
& + s_7^b \; \xi^\mu \xi^\nu \partial_\mu E_\nu + v_6 \; \tilde{P}^{\mu\nu} \partial_\mu E_\nu + v_7 \; \tilde{P}^{\mu\nu} \left(\partial_\mu F_{\nu\beta}\right) \xi^\beta + \ldots .
\end{aligned}
\tag{2.41}
$$

Following the algorithm of the previous section, we first set the coefficient of each of the thirteen independent two derivative terms listed in Table 2.5, which appear in the divergence of the entropy current, to zero. The vanishing of the nine fluid dynamical two derivative terms yields the following nine relations between the s^a's s^b's and v's

$$
\begin{array}{lllll}
v_1 = -s_1^a & v_2 = -s_2^a & v_3 = -s_1^b & v_4 = -s_2^b & s_3^b = -s_3^a \\
s_4^b = -s_4^a & s_5^b = -s_5^a & s_6^b = -s_6^a & v_5 = 0 \, . &
\end{array}
\tag{2.42}
$$

The vanishing of the four electromagnetic field related two derivative scalars yields the additional four relations

$$
s_7^a = s_7^b = v_6 = v_7 = 0 \, . \tag{2.43}
$$

Apart from the two derivative fluid dynamical and background electromagnetic field data, there are four nontrivial curvature invariants one can form out of the contractions of u^μ, ξ^μ and $g^{\mu\nu}$ with the the Reimann tensor $R_{\alpha\beta\mu\nu}$.[10] After plugging in the constraints in (2.42) and (2.43) into the expression for the entropy current (2.40) we find

$$
\begin{aligned}
\partial_\mu J_S^\mu = & s_1^a \tilde{P}^{\alpha\beta} u^\mu u^\lambda R_{\lambda\beta\alpha\mu} + s_2^b \tilde{P}^{\alpha\beta} \xi^\mu \xi^\lambda R_{\lambda\beta\alpha\mu} \\
& + \left(T s_2^a + s_1^b\right) \xi^\mu u^\lambda R_{\lambda\beta\alpha\mu} + s_3^a u^\alpha \xi^\beta u^\gamma \xi^\delta R_{\alpha\beta\gamma\delta} + \ldots .
\end{aligned}
\tag{2.44}
$$

Each of the terms in (2.44) is of indefinite sign. Thus, the coefficients of these four terms must vanish. This implies

$$
s_1^a = 0 \, , \quad s_1^b = -T s_2^a \, , \quad s_2^b = 0 \, , \quad s_3 = 0. \tag{2.45}
$$

To summarize, by setting the two derivative and curvature terms that appear in the divergence of the entropy current to zero we have eliminated $9 + 4 + 4 = 17$ of the original 21

[10] We omit the curvature scalar R in this listing since it is a pure gravitational term and therefore never appears in the divergence of fluid dynamical entropy current.

coefficients and are left with an entropy current with four undetermined coefficients,

$$J^\mu_S = J^\mu_{S\,\text{canon}} + s^a_2 \left(u^\mu \tilde{P}^{\alpha\beta}\partial_\alpha(T\xi_\beta) - \tilde{P}^{\mu\beta}u^\nu\partial_\nu(T\xi_\beta) - T\xi^\mu\tilde{P}^{\alpha\beta}\partial_\alpha u_\beta + T\tilde{P}^{\mu\beta}\xi^\nu\partial_\nu u_\beta \right)$$
$$+ s^a_4 \left(u^\mu\xi^\nu\partial_\nu\frac{\xi}{T} - \xi^\mu u^\nu\partial_\nu\frac{\xi}{T} \right) + s^a_5 \left(u^\mu\xi^\nu\partial_\nu\frac{\mu}{T} - \xi^\mu u^\nu\partial_\nu\frac{\mu}{T} \right) + s^a_6 \left(u^\mu\xi^\nu\partial_\nu T - \xi^\mu u^\mu\partial_\nu T \right). \tag{2.46}$$

The entropy current (2.46) can be rewritten in a simpler form by introducing the antisymmetric tensor

$$\mathcal{Q}_{\mu\nu} = T(\xi_\mu u_\nu - \xi_\nu u_\mu)\,, \tag{2.47}$$

introducing a unified notation for the three thermodynamical scalar fields

$$\Sigma_i = \left\{ \frac{\mu}{T}, \frac{\xi}{T}, T \right\} \qquad i = 1, 2, 3\,,$$

and also redefining our coefficient functions

$$c_0 = s^a_2 \quad c_1 = s_5 - \frac{T\mu}{\mu^2 - \xi^2} \quad c_2 = s_4 + \frac{T\xi}{\mu^2 - \xi^2} \quad c_3 = s_6 - \frac{2}{T}.$$

Then (2.46) takes the form

$$J^\mu_S = J^\mu_{S\,\text{canon}} + c_0\partial_\nu\mathcal{Q}^{\nu\mu} + \sum_{i=1}^{3} c_i\mathcal{Q}^{\mu\nu}\partial_\nu\Sigma_i\,. \tag{2.48}$$

The divergence of the entropy current (2.48) is given by

$$\begin{aligned}\partial_\mu J^\mu_S =& -\partial_\mu\left(\frac{u_\nu}{T}\right)T^{\mu\nu}_{diss} + V_{1\,\mu}J^\mu_{diss} + \frac{\mu_{diss}}{T}\partial_\mu\left(f\xi_\nu\right) \\ &+ \left(\partial_{\Sigma_i}c_0\right)\left(\partial_\mu\Sigma_i\right)\partial_\nu\mathcal{Q}^{\nu\mu} + c_i\partial_\mu\mathcal{Q}^{\mu\nu}\partial_\nu\Sigma_i + \left(\partial_{\Sigma_j}c_i\right)\left(\partial_\mu\Sigma_j\right)\mathcal{Q}^{\mu\nu}\left(\partial_\nu\Sigma_i\right)\,.\end{aligned} \tag{2.49}$$

The first line of the right hand side of (2.49) corresponds to the divergence of the canonical entropy current and the second line corresponds to the divergence of the new terms. The right hand side of (2.49) can be written as a sum of three classes of quadratic forms: a quadratic form in one derivative scalar data, a quadratic form in one derivative vector data and a quadratic form in one derivative tensor data. Let us first focus our attention on the quadratic form in the vector data. All vector terms that appear on the right hand side of (2.49) are linear combination of the six independent vector pieces of data,

$$\tilde{P}^{\alpha\nu}\xi^\mu\sigma_{\mu\nu}\,, \quad \tilde{P}^\alpha_{\ \mu}\partial_\nu\mathcal{Q}^{\mu\nu}\,, \quad \tilde{P}^\alpha_{\ \mu}V^\mu_1\,, \quad \tilde{P}^{\alpha\nu}\left(\partial_\nu\Sigma_i\right)\ \ (i = 1, \ldots, 3)\,.$$

Independent of the structure of the (as yet undetermined) dissipative terms $T^{\mu\nu}_{diss}$ and J^μ_{diss}, the quadratic form in one derivative vectors does not contain squares of $\partial_\mu\Sigma_i$, and $\tilde{P}^\alpha_{\ \mu}\partial_\nu\mathcal{Q}^{\mu\nu}$.[11] It is an easily verified fact that a quadratic form that does not contain the square of any given variable is positive if an only if it is independent of that variable. Thus,

[11]This follows because these terms do not appear explicitly on the right hand side of the divergence of the canonical entropy current.

positivity of the divergence of the entropy current requires that the right hand side of (2.49) does not contain any term proportional to $P^{\alpha}_{\mu}\partial_\alpha \Sigma_i$ and $\tilde{P}^{\alpha}_{\mu}\partial_\nu \mathcal{Q}^{\mu\nu}$.

The term in (2.49) proportional to $\tilde{P}^{\alpha}_{\mu}\partial_\alpha \Sigma_i \times \partial_\nu \mathcal{Q}^{\mu\nu}$ appears with a coefficient $(c_i + \partial_{\Sigma_i} c_0)$. Setting this coefficient to zero yields an expression for the c_i's in terms of c_0

$$c_i = -\partial_{\Sigma_i} c_0 \,. \tag{2.50}$$

Inserting (2.50) into (2.48) we find

$$J^\mu_S = J^\mu_{S\,\mathrm{canon}} + \partial_\nu \left(c_0 \mathcal{Q}^{\nu\mu}\right) \,. \tag{2.51}$$

We have ended up with a one parameter set of entropy currents (2.51) specified by the undetermined coefficient c_0. The divergence of these entropy currents is, however, independent of c_0. In fact, due to the antisymmetry properties of $\mathcal{Q}^{\mu\nu}$

$$\partial_\mu J^\mu_S = \partial_\mu J^\mu_{S\,\mathrm{canon}} \,. \tag{2.52}$$

The parameter c_0 is essentially trivial and is related to a pullback ambiguity as we now explain. It was pointed out in [9] that the following set of operations maps one positive divergence entropy current J^μ_S to another

1. Dualize J^μ to a three-form.
2. Shift this three-form by its Lie derivative with respect to any vector field V^μ
3. Dualize the resultant form back to a current.

The end result of this operation is a shift in the entropy current given by (see eq. 6.6 in [9])

$$\delta J^\mu_S = \nabla_\nu (J^\nu_S V^\mu - V^\nu J^\mu_S) + V^\mu \nabla_\nu J^\nu_S \,. \tag{2.53}$$

In the current setup we are interested in first order corrections to the entropy current. The right hand side of (2.53) has an explicit derivative. Therefore, the entropy current on the right hand side should be replaced by the perfect fluid entropy current $J^\mu_S = s u^\mu$. This implies that the second term on the right hand side of (2.53) is zero (recall that the perfect fluid entropy current is divergence free). Moreover V^μ must be a derivative free vector field.

In ordinary (non superfluid) fluid dynamics there is a unique vector at the zero derivative order—the fluid velocity u^μ. Since $J^\mu \propto u^\mu$ then $V^\mu \propto u^\mu$ implies that the first term on the right hand side of (2.53) also vanishes, and so (2.53) leads to no ambiguity in the entropy current at the first derivative order.

In superfluid dynamics there exist two zero derivative vectors, u^μ and ξ^μ. Consequently (2.53) can be used to generate a shift in the current proportional to $\partial_\nu \left(c_0 \mathcal{Q}^{\nu\mu}\right)$. We conclude that the freedom to add the total derivative term $\partial_\nu \left(c_0 \mathcal{Q}^{\nu\mu}\right)$ is precisely the 'pullback ambiguity' freeedom described in [9].

Going back to (2.52), it follows that the constraints on the dissipative terms $T^{\mu\nu}_{diss}$, J^{μ}_{diss} and μ_{diss} from demanding the positivity of the divergence of J_S are identical to the constraints from the positivity of the divergence of the canonical entropy current $J^{\mu}_{S\,\text{canon}}$.

2.2.3 Constraints from positive divergence of cannonical entropy current

In this subsection we will present constranints on the constitutive relations by considering posidetive semidefinite divergence of the cannonical entropy current. We will parameterize the allowed forms of $T^{\mu\nu}_{diss}$ and J^{μ}_{diss} at first order in the derivative expansion. Our parameterization will be in terms of a certain number of undetermined functions of T μ and ξ. One of these functions is the viscosity of the normal part of the superfluid. Following standard (but slightly misleading) usage, we will refer to these functions as dissipative parameters of the superfluid.

2.2.4 Summary of arguments and results

As the analysis of this subsection will be rather lengthy, we first present a summary of our logic and our procedure. To start with we simply classify all onshell inequivalent one derivative contributions to $T^{\mu\nu}_{diss}$, J^{μ}_{diss} and μ_{diss}. It is not difficult to establish that, in any given frame there exists a 36 parameter space of inequivalent first derivative corrections to the equations of superfluid dynamics (assuming parity invariance).

In order to cut down the set of possibilities we now demand that the canonical entropy current should have positive semi-defintie divergence. Using the expression (2.37) we find that this requirement cuts down the 36 parameter space of possible one derivative corrections to the entropy current to a 21 parameter space of possibilities. The coefficients in this 21 parameter space are further constrained by a complicated set of inequalities that ensure positivity of the entropy production. (One of these inequalities, for instance, asserts the positivity of the normal viscosity). Finally, the Onsager reciprocity relations relate 7 of the remaining parameters to each other, leaving us with a 14 parameter space of dissipative coefficients. As mentioned above, these 14 parameters (each of which is a function of T, μ and ξ) are further constrained to obey a set of inequalities. As far as we are aware, there are no further restrictions on this 14 parameter space from general principles.

To end this summary we explain how the framework presented in this subsection relates to previous work. The programme outlined in the paragraph above was implemented by Landau and Lifshitz [73] for the special case of flows with zero (or negligibly small) superfluid velocities. Landau and Lifshitz found a set of equations with 5 first order dissipative parameters. Our 14 parameter set of equations indeed reduce to the Landau Lifshitz form upon setting the superfluid velocity to equal the normal velocity; consequently our framework agrees with that of Landau and Lifshitz within its domain of validity.

Clark and Putterman [71, 72] extended the Landau Lifshitz programme to the case of nonzero superfluid velocities. The end result of their analysis was a thirteen parameter set of equations. We believe that Clark and Putterman overlooked one allowed parameter [12]

Reinstating that parameter yields our 14 parameter set of equations. Thus Clark and Putterman's equations are a special case of our 14 parameter equations with one parameter set to zero.

2.2.5 Counting of parameters

As we have seen, there are 6 onshell inequivalent parity even scalars, 5 onshell inequivalent parity even vector and 2 onshell inequivalent parity even tensor first derivatives of fluid dynamical fields. [13]

In order to be specific we will assume in the rest of this subsection that we are working in the transverse frame (2.32). In this frame, $T^{\mu\nu}_{diss}$ has two inequivalent scalar components $\xi^\mu\xi^\nu\pi_{\mu\nu}$ and π^μ_μ, one vector component $\tilde{P}^\mu_\alpha\pi^{\alpha\nu}\xi_\nu$ and a single tensor component $\tilde{P}_{\alpha\mu}\tilde{P}_{\beta\nu}\pi^{\alpha\beta}$, where $\tilde{P}_{\alpha\beta}$ is the projector orthogonal to the u^μ, ξ^μ plane. On the other hand J^μ_{diss} has one scalar component $J^\mu_{diss}\xi_\mu$ and one vector component $\tilde{P}_{\alpha\mu}J^\alpha_{diss}$. Finally μ_{diss} has one scalar component. It follows that the total number of undetermined parameters in the arbitrary expansion of $T^{\mu\nu}_{diss}$, J^μ_{diss} and μ_{diss} in terms of first derivatives of the fluid dynamical fields (assuming parity invariance) is given by

$$4 \times 6 + 2 \times 5 + 2 = 36$$

where the three terms above originate in the scalar, vector and tensor sector respectively. [14]

2.2.6 Constraints from positivity of entropy production and Onsager relations in the transverse frame

In this subsection we will explore the constraints on dissipative coefficients from the physical requirements of positivity of entropy production and the Onsager reciprocity relations. We will find these requirements cut down the 36 parameter set of possible dissipative coefficients (assuming parity invariance) to a 14 parameter set of coefficients that are further constrained by positivity requirements. For concreteness we present our analysis in the transverse frame (2.32).

[12]Specifically, the traceless symmetric 3 index tensor listed in equation (A VI-9) of Putterman's book is not unique. Another such tensor is given by

$$Y_{ijk} = w_i(w_j w_k - (1/3)w^2\delta_{jk}).$$

[13]In addition we have one additional parity odd scalar field. Further, every vector V_μ can be transformed to a psuedo vector $\tilde{V}_\mu$ according to the formula $\tilde{V}_\mu = \epsilon_{\mu\nu\alpha\beta}V^\nu n^\alpha u^\beta$.

[14]Dropping the assumption of parity invariance we have $4 \times 7 + 2 \times 10 + 2 = 50$ independent dissipative coefficients.

Constraints from positivity of entropy production

The divergence of the 'canonical' entropy current, given by (2.37), involves only terms proportional to $\partial_\mu u_\nu T^{\mu\nu}_{diss}$, $\partial_\nu(\mu/T)J^\nu_{diss}$ and $\mu_{diss}\partial_\mu(f\xi^\mu)$. Let us examine these terms one by one. In the transverse frame

$$\partial_\mu u_\nu T^{\mu\nu}_{diss} = \sigma_{\mu\nu}T^{\mu\nu}_{diss} + \left(\frac{\partial_\mu u^\mu}{3}\right)\pi^\theta_\theta$$

where $\sigma_{\mu\nu}$ is the traceless symmetric part of $\partial_\mu u_\nu$, projected in the direction perpendicular to u^μ.

$$\begin{aligned} &\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\frac{[\partial.u]}{3}\right) \\ &\text{and} \\ &P_{\mu\nu} = \text{The projector} = \eta_{\mu\nu} + u_\mu u_\nu \end{aligned} \tag{2.54}$$

Now the field $\sigma_{\mu\nu}$ has one scalar piece of data [15]

$$S_w = n^\mu n^\nu \sigma_{\mu\nu}$$

one vector piece of data [16]

$$[V_b]_\mu = \tilde{P}^\nu_\mu n^\alpha \sigma_{\nu\alpha}$$

and a tensor piece of data [17]

$$\mathcal{T}_{\mu\nu} = \tilde{P}^\alpha_\mu \tilde{P}^\beta_\nu \sigma_{\alpha\beta}$$

The trace of $T^{\mu\nu}_{diss}$ couples to another scalar piece of data [18]

$$S_{w'} = \partial_\mu u^\mu.$$

In the expressions above n^μ is the unique normal vector in the plane spanned by u^μ and ξ^μ that is orthogonal to u^μ and is given by

$$n_\mu \equiv \frac{w_\mu}{w},$$

with w_μ being the component of ξ_μ projected orthogonal to u_μ and w being the norm of the w_μ vector.

Similarly, in the transverse gauge

$$\partial_\nu(\mu/T)J^\nu_{diss} = P^\nu_\alpha \partial_\nu(\mu/T)J^\alpha_{diss},$$

[15] In the terminology of §3.3.2 below, $S_w = \frac{2S_4 - S_6}{3}$
[16] In the terminology of §3.3.2 below, $[V_b]_\mu = \frac{1}{2}[V_5]_\mu$
[17] In the terminology of §3.3.2 below, $\mathcal{T}_{\mu\nu} = \frac{1}{2}[T_1]_{\mu\nu}$
[18] In the terminology of §3.3.2 below, $S_{w'} = S_4 + S_6$.

where $P_{\mu\nu}$ is the projection operator (defined in (2.54)) that projects orthogonal to u_μ only . The quantity $P^\nu_\alpha \partial_\nu(\mu/T)$ has one scalar piece of data

$$S_b = (n^\mu \partial_\mu)\,(\mu/T)$$

and one vector piece of data

$$[V_a]_\mu = \tilde{P}^\sigma_\mu \partial_\sigma\,(\mu/T)$$

Finally

$$S_a = \frac{\partial_\mu(f\xi^\mu)}{T^3}$$

is itself a scalar piece of data.

In other words we conclude that the expression for the divergence of the entropy current, (2.37), depends explicitly (i.e. apart from the dependence of $T^{\mu\nu}_{diss}$ J^μ_{diss} and μ_{diss} on these terms) only on 4 scalar expressions, 2 vector expressions and one tensor expression. Let us choose these 4 vectors scalars S_a, S_b, S_w and $S_{w'}$, supplemented by 3 other arbitrarily chosen scalar expressions S^I_m $(m = 1 \dots 3)$ as our 7 independent scalar expressions. Similarly we choose the 2 vectors $[V_a]_\mu$ and $[V_b]_\mu$ supplemented by 3 other arbitrarily chosen expressions $[V^I_m]_\mu$ $(m = 1 \dots 3)$ as our four independent vector expressions. We also choose $\mathcal{T}_{\mu\nu}$ as one of our two independent tensor expressions [19]. We proceed to express $T^{\mu\nu}_{diss}$, J^μ_{diss} and μ_{diss} as the most general linear combinations of all combinations of independent expressions allowed by symmetry

$$\begin{aligned}
T^{\mu\nu}_{diss} = T^3 \Bigg[&\left(P_a S_a + P_b S_b + P_w S_w + P_{w'} S_{w'} + \sum_{m=1}^{3} P^I_m S^I_m \right) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\
&+ \left(T_a S_a + T_b S_b + T_w S_w + T_{w'} S_{w'} + \sum_{m=1}^{3} T^I_m S^I_m \right) P^{\mu\nu} \\
&+ E_a \left(V^\mu_a n^\nu + V^\nu_a n^\mu \right) + E_b \left(V^\mu_b n^\nu + V^\nu_b n^\mu \right) + \sum_{m=1}^{3} E^I_m \left([V^I_m]^\mu n^\nu + [V^I_m]^\nu n^\mu \right) \\
&+ \tau \mathcal{T}^{\mu\nu} + \tau_2 T^{\mu\nu}_2 \Bigg] \\
J^\mu_{diss} = T^2 \Bigg[&\left(R_a S_a + R_b S_b + R_w S_w + R_{w'} S_{w'} + \sum_{m=1}^{3} R^I_m S^I_m \right) n^\mu \\
&+ C_a V^\mu_a + C_b V^\mu_b + \sum_{m=1}^{3} C^I_m [V^I_m]^\mu \Bigg] \\
\mu_{diss} = - &\left[Q_a S_a + Q_b S_b + Q_w S_w + Q_{w'} S_{w'} + \sum_{m=1}^{3} Q^I_m S^I_m \right]
\end{aligned} \tag{2.55}$$

Plugging this into (2.37) we now obtain an explicit expression for the divergence of the

[19] We could now go ahead and use the perfect fluid equations to solve for for the dependent data in terms of independent data; however we will not need the explicit form of this solution in this subsection.

entropy current as a quadratic form in first derivative independent data. We wish to enforce the condition that this quadratic form is positive definite. Now the quadratic form from (2.37) clearly has no terms proportional to $(S_m^I)^2$. It does, however, have terms of the form (for instance) $S_a S_m^I$, and also terms proportional to S_a^2. Now it follows from a moments consideration that no quadratic form of this general structure can be positive unless the coefficient of the $S_a S_m^I$ term vanishes. [20] Using similar reasoning we can immediately conclude that the positive definiteness of (2.37) requires that

$$P_m^I = T_m^I = E_m^I = C_m^I = R_m^I = \tau_2 = 0. \tag{2.56}$$

(2.56) is the most important conclusion of this subsubsection. It tells us that a 21 parameter set of first derivative corrections to the constitutive relations are consistent with the positivity of the canonical entropy current.

Of course the remaining 21 parameters are not themselves arbitrary, but are constrained to obey inequalities in order to ensure positivity. In order to derive these conditions we plug (2.56) into (2.55) and use (2.37) so that the divergence of the entropy current is the linear sum of three different quadratic forms (involving the tensor terms, vector terms and scalar terms respectively)

$$\partial_\mu J_s^\mu = T^2 \left(Q_s + Q_V + Q_T\right) \tag{2.57}$$

where

$$Q_T = -\tau \mathcal{T}^2$$

$$\begin{aligned} Q_V =& -C_a V_a^2 - (C_b + E_a) V_b V_a - E_b V_b^2 \\ =& -C_a \left[V_a + \left(\frac{C_b + E_a}{2C_a}\right) V_b\right]^2 - \left[E_b - \frac{(C_b+E_a)^2}{4C_a}\right] V_5^2 \end{aligned} \tag{2.58}$$

$$\begin{aligned} Q_S =& -P_w S_w^2 - T_{w'} S_{w'}^2 - Q_a S_a^2 - R_b S_b^2 \\ & - (Q_w + P_a) S_w S_a - (Q_{w'} + T_a) S_{w'} S_a - (R_w + P_b) S_w S_b \\ & - (R_{w'} + T_b) S_{w'} S_b - (R_a + Q_b) S_a S_b - (T_w + P_{w'}) S_w S_{w'} \end{aligned} \tag{2.59}$$

Positivity of the entropy current clearly requires that Q_T Q_V and Q_S are separately positive. Let us examine these conditions one at a time. For Q_T to be positive it is necessary and sufficient that $\tau \leq 0$. This is simply the requirement that the normal component of our superfluid have a positive viscosity. In order that Q_V be positive, it is necessary and sufficient that

$$C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4E_b C_a \geq (C_b + E_a)^2. \tag{2.60}$$

Note that this expression involves C_a and E_b on the LHS but the different quantities C_b and E_a on the RHS; the last inequality above is satisfied roughly, when C_b and E_a are larger in modulus than C_a and E_b.

Finally Q_S, listed in (2.59), is a quadratic form in the the 4 variables S_a, S_b, S_w and $S_{w'}$. We demand that this scalar form be positive. We will not pause here to explicate the

[20] For instance the quadratic form $x^2 + cxy$ (where c is a constant) can be made negative by taking $\frac{y}{x}$ to either positive or negative infinity (depending on the sign of c) unless $c = 0$.

precise inequalities that this condition imposes on the coefficients. See below, however, for the special case of a Weyl invariant fluid.

Constraints from the Onsager Relations

In the previous subsection we found that first order dissipative corrections to the equations of perfect superfluid dynamics take the form

$$\begin{aligned}
T^{\mu\nu}_{diss} &= T^3 \bigg[(P_a S_a + P_b S_b + P_w S_w + P_{w'} S_{w'}) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\
&\quad + (T_a S_a + T_b S_b + T_w S_w + T_{w'} S_{w'}) \, P^{\mu\nu} \\
&\quad + E_a \left(V_a^\mu n^\nu + V_a^\nu n^\mu \right) + E_b \left(V_b^\mu n^\nu + V_b^\nu n^\mu \right) \\
&\quad + \tau \mathcal{T}^{\mu\nu} \bigg] \\
J^\mu_{diss} &= T^2 \bigg[(R_a S_a + R_b S_b + R_w S_w + R_{w'} S_{w'}) \, n^\mu \\
&\quad + C_a V_a^\mu + C_b V_b^\mu \bigg] \\
\mu_{diss} &= - \left[Q_a S_a + Q_b S_b + Q_w S_w + Q_{w'} S_{w'} \right]
\end{aligned} \tag{2.61}$$

where the coefficients in these equations are constrained by the inequalities listed in the previous subsubsection. The coefficients that appear in these equations are further constrained by the Onsager reciprocity relations (see, for instance, the text book [73], for a discussion). These relations assert, in the present context, that we should equate any two dissipative parameters that multiply the same terms in the formulas (2.58) and (2.59) for entropy production. This implies that

$$\begin{aligned}
& Q_w = P_a, \quad Q_{w'} = T_a, \quad R_w = P_b \\
& R_{w'} = T_b, \quad R_a = Q_b, \quad T_w = P_{w'} \\
& \text{and} \\
& C_b = E_a
\end{aligned} \tag{2.62}$$

In summary we are left with a 14 parameter set of equations of first order dissipative superfluid dynamics. The requirement of positivity constrains further these coefficients to obey the inequalities spelt out in the previous subsubsection.

2.2.7 Weyl Invariant Superfluid Dynamics in the transverse frame

Let us now specialize the results of the previous subsection to the case of super fluid dynamics for a conformal superfluid. The analysis is simplified in this special case by the fact that the trace of the stress tensor vanishes in an arbitrary state (and so in the fluid limit) of a conformal field theory. This fact reduces the number of explicit scalars that appear in (2.37) from 4 to 3 (the scalar $S_{w'}$ never makes an appearance). It follows that the requirement of Weyl invariance forces $P_{w'} = R_{w'} = T_{w'} = Q_{w'} = 0$. Moreover the requirement that $T^{\mu\nu}_{diss}$

be traceless forces $T_a = T_b = T_w = 0$. It turns out that there are no further constraints from the requirement of Weyl invariance. The expansion of the dissipative part of the stress tensor and charge current for a conformal superfluid is given by

$$\begin{aligned}
T^{\mu\nu}_{diss} =& T^3 \Bigg[(P_a S_a + P_b S_b + P_w S_w) \left(n_\mu n_\nu - \frac{P_{\mu\nu}}{3} \right) \\
&+ E_a (V_a^\mu n^\nu + V_a^\nu n^\mu) + E_b (V_b^\mu n^\nu + V_b^\nu n^\mu) \\
&+ \tau \mathcal{T}^{\mu\nu} \Bigg] \\
J^\mu_{diss} =& T^2 \Bigg[(R_a S_a + R_b S_b + R_w S_w) n^\mu \\
&+ C_a V_a^\mu + C_b V_b^\mu \Bigg] \\
\mu_{diss} =& - [Q_a S_a + Q_b S_b + Q_w S_w]
\end{aligned} \tag{2.63}$$

The entropy production is given by

$$\partial_\mu J^\mu_s = T^2 (Q_s + Q_V + Q_T) \tag{2.64}$$

where

$$Q_T = -\tau \mathcal{T}^2$$

$$\begin{aligned}
Q_V =& - C_a V_a^2 - (C_b + E_a) V_b V_a - E_b V_b^2 \\
=& - C_a \left[V_a + \left(\frac{C_b + E_a}{2C_a} \right) V_b \right]^2 - \left[E_b - \frac{(C_b + E_a)^2}{4C_a} \right] V_b^2
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
Q_S =& - P_w S_w^2 - Q_a S_a^2 - R_b S_b^2 \\
&+ (Q_w + P_a) S_w S_a - (R_a + Q_b) S_a S_b + (R_w + P_b) S_w S_b
\end{aligned} \tag{2.66}$$

For the entropy current to be positive it is necessary and sufficient that $\tau \leq 0$ and that

$$C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4E_b C_a \geq (C_b + E_a)^2. \tag{2.67}$$

and that the quadratic form

$$\begin{aligned}
Q_S =& a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + b_1 x_1 x_2 + b_2 x_2 x_3 + b_3 x_1 x_3 \\
=& a_1 \left[x_1 + \left(\frac{b_1}{2a_1} \right) x_2 + \left(\frac{b_3}{2a_1} \right) x_3 \right]^2 \\
&+ \left(a_3 - \frac{b_3^2}{4a_1} \right) \left[x_3 + \left(\frac{2a_1 b_2 - b_1 b_3}{4a_1 a_3 - b_3^2} \right) x_2 \right]^2 \\
&+ \left[\frac{(4a_1 a_2 - b_1^2)(4a_1 a_3 - b_3^2) - (2a_1 b_2 - b_1 b_3)^2}{4a_1 (4a_1 a_3 - b_3^2)} \right] x_2^2
\end{aligned} \tag{2.68}$$

is positive with $x_1 = S_w$, $x_2 = S_a$ and $x_3 = S_b$ and

$$a_1 = -P_w, \quad a_2 = -Q_a, \quad a_3 = -R_b, \quad b_1 = Q_w + P_a, \quad b_2 = -(Q_b + R_a), \quad b_3 = R_w + P_b$$

For the last quadratic form to be positive it is necessary and sufficient that

$$\begin{aligned} &a_1 \geq 0 \\ &4a_1a_2 > b_1^2 \\ &(4a_1a_2 - b_1^2)(4a_1a_3 - b_3^2) > (2a_1b_2 - b_1b_3)^2 \end{aligned} \tag{2.69}$$

By rewriting (2.68) as a sum of squares in a cyclically permuted manner we can also derive the cyclical permutations of these equations.

In summary, the most general Weyl invariant fluid dynamics consistent with positivity on the entropy current is parameterized by a negative τ_1, 4 parameters in the vector sector constrained by the inequalities (2.67) and 9 parameters in the scalar sector, subject to the inequalities (2.69). These 14 dissipative parameters are further constrained by the 4 Onsager relations

$$Q_w = P_a, \quad R_w = P_b, \quad R_a = Q_b, \quad C_b = E_a \tag{2.70}$$

leaving us with a 10 parameter set of final equations.

Chapter 3

Boundary hydrodynamcis from bulk gravity

In this chapter we shall test and verify the theories of hydrodynamics developed in the previous chapter by constructing bulk duals. We use the AdS/CFT correspondance to relate our bulk solutions to the boundary fluid dynamics as discussed in chapter 1. First we shall consider fluctualtions about an electrically charged black brane solution in 5 dimensional AdS space which would be dual to a fluid in the boundary with a globally conserved charge. Then we shall consider fluctuations about hairy charged black branes to capture superfluid hydrodynamics in the boundary.

3.1 Hydrodynamics from charged black branes

3.1.1 Notations and Conventions

In this section. we will establish the basic conventions and notations that we will use in the rest of the paper. We start with the five-dimensional action[1]

$$S = \frac{1}{16\pi G_5}\int\sqrt{-g_5}\left[R + 12 - F_{AB}F^{AB} - \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\right] \tag{3.1.1}$$

which is a consistent truncation of IIB SUGRA Lagrangian on $AdS_5 \times S^5$ [93] background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = 1/(2\sqrt{3})$ (See for example, [2, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64]). However, for the sake of generality (and to keep track of the effects of the Chern-Simons term), we will work with an arbitrary value of κ in the following. In particular, $\kappa = 0$ corresponds to a pure Maxwell theory with no Chern-Simons type interactions.

[1] We use Latin letters $A, B \in \{r, v, x, y, z\}$ to denote the bulk indices and $\mu, \nu \in \{v, x, y, z\}$ to denote the boundary indices.

The field equations corresponding to the above action are

$$\begin{aligned} G_{AB} - 6g_{AB} + 2\left[F_{AC}F^{C}{}_{B} + \frac{1}{4}g_{AB}F_{CD}F^{CD}\right] = 0 \\ \nabla_B F^{AB} + \kappa\epsilon^{ABCDE}F_{BC}F_{DE} = 0 \end{aligned} \tag{3.1.2}$$

where g_{AB} is the five-dimensional metric, G_{AB} is the five dimensional Einstein tensor. These equations admit an AdS-Reisner-Nordström black-brane solution

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 V(r,m,q)\ u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ A &= \frac{\sqrt{3}q}{2r^2} u_\mu dx^\mu, \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} u_\mu dx^\mu = -dv; \qquad V(r,m,q) \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6}; \\ P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu, \end{aligned} \tag{3.1.4}$$

with $\eta_{\mu\nu} = \text{diag}(-+++)$ being the Minkowski-metric. Following the procedure elucidated in [8], we shall take this flat black-brane metric as our zeroth order metric/gauge field ansatz and promote the parameters u_μ, m and q to slowly varying fields.

In the course of our calculations, we will often find it convenient to use the following 'rescaled' variables

$$\rho \equiv \frac{r}{R}; \qquad M \equiv \frac{m}{R^4}; \qquad Q \equiv \frac{q}{R^3}; \qquad Q^2 = M - 1 \tag{3.1.5}$$

where R is the radius of the outer horizon,i.e., the largest positive root of the equation $V = 0$. The Hawking temperature, chemical potential and the charge density of this black-brane are given by[2]

$$T \equiv \frac{R}{2\pi}(2 - Q^2)\ , \qquad \mu \equiv \frac{2\sqrt{3}q}{R^2} = 2\sqrt{3}QR \qquad \text{and} \qquad n \equiv \frac{\sqrt{3}q}{16\pi G_5}. \tag{3.1.6}$$

In terms of the rescaled variables, the outer and the inner horizon are given by

$$\rho_+ \equiv 1 \qquad \text{and} \qquad \rho_- \equiv \left[\left(Q^2 + 1/4\right)^{1/2} - 1/2\right]^{1/2}$$

and the extremality condition $\rho_+ = \rho_-$ corresponds to $(Q^2 = 2, M = 3)$. We shall assume the black-branes and the corresponding fluids to be non-extremal unless otherwise specified - this corresponds to the regime $0 < Q^2 < 2$ or $0 < M < 3$ which we will assume henceforth.

Using the flat black-brane solutions with slowly varying velocity, temperature and charge fields, our intention is to systematically determine the corrections to the metric and the gauge field in a derivative expansion. More precisely, we expand the metric and the gauge

[2] In much of the literature the chemical potential μ is taken to be the potential difference between the boundary and the horizon. However we have chosen a different normalization for μ (and hence the charge density n).

field in terms of derivatives of velocity, temperature and charge fields of the fluid as

$$\begin{aligned} g_{AB} &= g^{(0)}_{AB} + g^{(1)}_{AB} + g^{(2)}_{AB} + \dots \\ A_M &= A^{(0)}_M + A^{(1)}_M + A^{(2)}_M + \dots \end{aligned} \tag{3.1.7}$$

where $g^{(k)}_{AB}$ and $A^{(k)}_M$ contain the k-th derivatives of the velocity, temperature and the charge fields with

$$\begin{aligned} g^{(0)}_{AB}dx^A dx^B &= -2u_\mu(x)dx^\mu dr - r^2 V(r, m(x), q(x))\ u_\mu(x)u_\nu(x)dx^\mu dx^\nu + r^2 P_{\mu\nu}(x)dx^\mu dx^\nu \\ A^{(0)}_M dx^M &= \frac{\sqrt{3}q(x)}{2r^2}u_\mu(x)dx^\mu . \end{aligned} \tag{3.1.8}$$

In order to solve the Einstein-Maxwell-Chern-Simons system of equations, it is necessary to work in a particular gauge for the metric and the gauge fields. Following [8], we choose our gauge to be

$$g_{rr} = 0; \quad g_{r\mu} \propto u_\mu\ ; \quad A_r = 0; \quad Tr[(g^{(0)})^{-1} g^{(k)}] = 0. \tag{3.1.9}$$

Further, in order to relate the bulk dynamics to boundary hydrodynamics, it is useful to parameterise the fluid dynamics in the boundary in terms of a 'fluid velocity' u_μ. In case of relativistic fluids with conserved charges, there are two widely used conventions of how the fluid velocity should be defined. In this paper, we will work with the Landau frame velocity where the fluid velocity is defined with reference to the energy transport. In a more practical sense working in the Landau frame amounts to taking the unit time-like eigenvector of the energy-momentum tensor at a point to be the fluid velocity at that point.

Alternatively, one could work in the 'Eckart frame' where the fluid velocity is defined with reference to the charge transport where the unit time-like vector along the charged current to be the definition of fluid velocity. Though the later is often the more natural convention in the context of charged fluids, we choose to use the Landau's convention for the ease of comparison with the other literature.

In the next two subsections, we will report in some detail the calculations leading to the determination of the metric and the gauge field up to second order in the derivative expansion. This will enable us to determine the boundary stress tensor and charge current up to the second order.

3.1.2 First Order Hydrodynamics

In this subsection, we present the computation of the metric and the gauge field up to first order in derivative expansion, the derivative being taken with respect to the boundary coordinates. We choose the boundary coordinates such that $u^\mu = (1, 0, 0, 0)$ at x^μ. Since our procedure is ultra local therefore we intend to solve for the metric and the gauge field at first order about this special point x^μ. We shall then write the result thus obtained in a covariant form which will be valid for arbitrary choice of boundary coordinates.

In order to implement this procedure we require the zeroth order metric and gauge field expanded up to first order. For this we recall that the parameters m, q and the velocities (β_i) are functions of the boundary coordinates and therefore admit an expansion in terms of the boundary derivatives. These parameters expanded up to first order is given by

$$\begin{aligned} m =& m_0 + x^\mu \partial_\mu m^{(0)} + \dots \\ q =& q_0 + x^\mu \partial_\mu q^{(0)} + \dots \\ \beta_i =& x^\mu \partial_\mu \beta_i^{(0)} + \dots \end{aligned} \tag{3.1.10}$$

Here $m^{(i)}$, $q^{(i)}$, $\beta^{(i)}$ refers to the i-th order correction to mass, charge and velocities respectively.

The zeroth order metric expanded about x^μ up to first order in boundary coordinates is given by

$$\begin{aligned} {ds^{(0)}}^2 =& \, 2\, dv\, dr - r^2 V^{(0)}(r)\, dv^2 + r^2\, dx_i\, dx^i \\ & - 2\, x^\mu\, \partial_\mu \beta_i^{(0)}\, dx^i\, dr - 2\, x^\mu \partial_\mu \beta_i^{(0)}\, r^2 (1 - V^{(0)}(r))\, dx^i\, dv \\ & - \left(\frac{-x^\mu \partial_\mu\, m^{(0)}}{r^2} + \frac{2 q_0 x^\mu \partial_\mu\, q^{(0)}}{r^4} \right) dv^2, \end{aligned} \tag{3.1.11}$$

where m_0 and q_0 are related to the mass and charge of the background blackbrane respectively and

$$V^{(0)} = 1 - \frac{m_0}{r^4} + \frac{q_0^2}{r^6}.$$

Similarly the zeroth order gauge fields expanded about x^μ up to first order is given by

$$A = -\frac{\sqrt{3}}{2} \left[\left(\frac{q_0 + x^\mu \partial_\mu\, q^{(0)}}{r^2} \right) dv - \frac{q_0}{r^2} x^\mu \partial_\mu\, \beta_i^{(0)}\, dx^i \right] \tag{3.1.12}$$

Since the background black brane metric preserves an $SO(3)$ symmetry [3], the Einstein-Maxwell equations separate into equations in scalar, transverse vector and the symmetric traceless transverse tensor sectors. This in turn allows us to solve separately for $SO(3)$ scalar, vector and symmetric traceless tensor components of the metric and the gauge field.

[3] Here we are referring to the $SO(3)$ rotational symmetry in the boundary spatial coordinates.

Scalars Of $SO(3)$ at first order

The scalar components of first order metric and gauge field perturbations ($g^{(1)}$ and $A^{(1)}$ respectively) are parameterized by the functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ as follows [4]

$$\begin{aligned} \sum_i g^{(1)}_{ii}(r) &= 3r^2 h_1(r), \\ g^{(1)}_{vv}(r) &= \frac{k_1(r)}{r^2} \\ g^{(1)}_{vr}(r) &= -\frac{3}{2}h_1(r) \\ A^{(1)}_v(r) &= -\frac{\sqrt{3}w_1(r)}{2r^2} \end{aligned} \tag{3.1.13}$$

Note that $g^{(1)}_{ii}(r)$ and $g^{(1)}_{vr}(r)$ are related to each other by the gauge choice $Tr[(g^{(0)})^{-1}g^{(1)}] = 0$.

Constraint equations

We begin by finding the constraint equations that constrain various derivatives velocity,temperature and charge that appear in the first order scalar sector.The constraint equations are obtained by taking a dot of the Einstein and Maxwell equations with the vector dual to the one form dr. If we denote the Einstein and the Maxwell equations by $E_{AB} = 0$ and $M_{AB} = 0$, then there are three constraint relations.

Two of them come from Einstein equations. They are given by

$$g^{rr}E_{vr} + g^{rv}E_{vv} = 0 \ , \tag{3.1.14}$$

and

$$g^{rr}E_{rr} + g^{rv}E_{vr} = 0 \ , \tag{3.1.15}$$

and the third constraint relation comes from Maxwell equations and is given by

$$g^{rr}M_r + g^{rv}M_v = 0 \ . \tag{3.1.16}$$

Equation (3.1.14) reduces to

$$\partial_v m^{(0)} = -\frac{4}{3}m_0\partial_i\beta^{(0)}_i \ . \tag{3.1.17}$$

which is same as the conservation of energy in the boundary at the first order in the derivative expansion, i.e., the above equation is identical to the constraint (scalar component of the constraint in this case)

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ . \tag{3.1.18}$$

on the allowed boundary data.

The second constraint equation (3.1.15) in scalar sector implies a relation between $h_1(r)$

[4] here i runs over the boundary spatial coordinates, v is the boundary time coordinate and r is the radial coordinate in the bulk

and $k_1(r)$.

$$2\partial_i\beta_i^{(0)}r^5+12r^6h_1(r)+4q_0w_1(r)-m_0r^3h_1'(r)+3r^7h_1'(r)-r^3k_1'(r)-2q_0rw_1'(r)=0. \quad (3.1.19)$$

The constraint relation coming from Maxwell equation (See Eq. (3.1.16)) gives

$$\partial_v\ q^{(0)} = -q_0\ \partial_i\beta_i^{(0)}\ . \quad (3.1.20)$$

This equation can be interpreted as the conservation of boundary current density at the first order in the derivative expansion.

$$\partial_\mu J^\mu_{(0)} = 0. \quad (3.1.21)$$

We now proceed to find the scalar part of the metric dual to a fluid configuration which obeys the above constraints.

Dynamical equations and their solutions

Among the Einstein equations four are $SO(3)$ scalars (namely the vv, rv, rr components and the trace over the boundary spatial part). Further the r and v-components of the Maxwell equations constitute two other equations in this sector. Two specific linear combination of the rr and vv components of the Einstein equations constitute the two constraint equations in (3.1.17). Further, a linear combination of the r and v-components of the Maxwell equations appear as a constraint equation in (3.1.20). Now among the six equations in the scalar sector we can use any three to solve for the unknown functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ and we must make sure that the solution satisfies the rest. The simplest two equations among these dynamical equations are

$$5h_1'(r) + rh_1''(r) = 0. \quad (3.1.22)$$

which comes from the rr-component of the Einstein equation and

$$6q_0h_1'(r) + w_1'(r) - rw_1''(r) = 0. \quad (3.1.23)$$

which comes from the r-components of the Maxwell equation. We intend to use these dynamical equations (3.1.22), (3.1.23) along with one of the constraint equations in (3.1.17) to solve for the unknown functions $h_1(r)$, $k_1(r)$ and $w_1(r)$.

Solving (3.1.22) we get

$$h_1(r) = \frac{C^1_{h_1}}{r^4} + C^2_{h_1}, \quad (3.1.24)$$

where $C^1_{h_1}$ and $C^2_{h_1}$ are constants to be determined. We can set $C^2_{h_1}$ to zero as it will lead to a non-normalizable mode of the metric. We then substitute the solution for $h_1(r)$ from (3.1.24) into (3.1.23) and solve the resultant equation for $w_1(r)$. The solution that we obtain is given by

$$w_1(r) = C^1_{w_1}r^2 + C^2_{w_1} - q_0\frac{C^1_{h_1}}{r^4}. \quad (3.1.25)$$

Here again $C^1_{w_1}$, $C^2_{w_2}$ are constants to be determined. Again $C^1_{w_1}$ corresponds to a non-normalizable mode of the gauge field and therefore can be set to zero.

Finally plugging in these solutions for $h_1(r)$ and $w_1(r)$ into one of the constraint equations in (3.1.17) and then solving the subsequent equation we obtain

$$k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} + C_{k_1} - \frac{2q0}{r^2} C_{w_1}^2 + \left(\frac{2{q_0}^2}{r^6} - \frac{m_0}{r^4} \right) C_{h_1}^1 \tag{3.1.26}$$

Now the constants C_{k_1} and $C_{w_1}^2$ may be absorbed into redefinitions of mass (m_0) and charge (q_0) respectively and hence may be set to zero. Further we can gauge away the constant $C_{h_1}^1$ by the following redefinition of the r coordinate

$$r \to r \left(1 + \frac{C}{r^4} \right),$$

C being a suitably chosen constant.

Thus we conclude that all the arbitrary constants in this sector can be set to zero and therefore our solutions may be summarized as

$$h_1(r) = 0, \quad w_1(r) = 0, \quad k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} \ . \tag{3.1.27}$$

In terms of the first order metric and gauge field this result reduces to

$$\begin{aligned} \sum_i g_{ii}^{(1)}(r) &= 0, \\ g_{vv}^{(1)}(r) &= \frac{2}{3} r \partial_i \beta_i^{(0)}, \\ g_{vr}^{(1)}(r) &= 0, \\ A_v^{(1)}(r) &= 0 \ . \end{aligned} \tag{3.1.28}$$

Now, we proceed to solving the equations in the vector sector.

Vectors Of $SO(3)$ at first order

The vector components of metric and gauge field $g^{(1)}$ and $A^{(1)}$ are parameterized by the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ as follows

$$\begin{aligned} g_{vi}^{(1)}(r) &= \left(\frac{m_0}{r^2} - \frac{q_0^2}{r^4} \right) j_i^{(1)}(r) \\ A_i^{(1)}(r) &= -\left(\frac{\sqrt{3} q_0}{2r^2} \right) j_i^{(1)}(r) + g_i^{(1)}(r) \end{aligned} \tag{3.1.29}$$

Now we intend to solve for the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$.

Constraint equations

The constraint equations in the vector sector comes only from the Einstein equation. So there is only one constraint equation in this sector. It is given by

$$g^{rr}E_{ri} + g^{rv}E_{vi} = 0 \tag{3.1.30}$$

which implies

$$\partial_i m^{(0)} = -4m_0 \partial_v \beta_i^{(0)}. \tag{3.1.31}$$

These equations also follow from the conservation of boundary stress tensor at first order. We shall use this constraint equation to simplify the dynamical equations in the vector sector.

Dynamical equations and their solutions

In the vector sector we have two equations from Einstein equations (the ri and vi-components) and one from Maxwell equations (the ith-component) [5].

The dynamical equation obtained from the vi-component of the Einstein equations is given by

$$\left(q_0^2 - 3m_0r^2\right)\frac{dj_i^{(1)}(r)}{dr} + 4\sqrt{3}q_0r^2\frac{dg_i(1)(r)}{dr} + \left(m_0r^2 - q_0^2\right)r\frac{d^2j_i^{(1)}(r)}{dr^2} = -3r^4\partial_v\beta_i^{(0)}. \tag{3.1.32}$$

Also the dynamical equation from the ith-component of the Maxwell equation is given by

$$\begin{aligned} & r\left[2\left(r^6 - m_0r^2 + q_0^2\right)\frac{d^2g_i^{(1)}}{dr^2}r^2 + \left(6r^7 + 2m_0r^3 - 6q_0^2r\right)\frac{dg_i^{(1)}(r)}{dr}\right] \\ & -\sqrt{3}q_0r\left(r^6 - m_0r^2 + q_0^2\right)\frac{d^2j_i^{(1)}(r)}{dr^2} + \sqrt{3}q_0\left(r^6 - 3m_0r^2 + 5q_0^2\right)\frac{dj_i^{(1)}(r)}{dr} \\ & \quad = \sqrt{3}(q_0\partial_v\beta_i^{(0)} + \partial_i q^{(0)})r^3 - 24q_0^2\kappa r l_i^{(0)}, \end{aligned} \tag{3.1.33}$$

where l_i is defined as

$$l_i \equiv \epsilon_{ijk}\partial_j\beta_k. \tag{3.1.34}$$

Now in order to solve this coupled set of differential equations (3.1.32) and (3.1.33) we shall substitute $g_i^{(1)}(r)$ obtained from (3.1.32) into (3.1.33) and solve the resultant equation for $j_i^{(1)}(r)$. For any function $j_i^{(1)}(r)$, using (3.1.32) $g_i^{(1)}(r)$ may be expressed as

$$g_i^{(1)}(r) = (C_g)_i + \frac{1}{4\sqrt{3}q_0}\left(-\partial_v\beta_i^{(0)}r^3 + 4m_0j_i^{(1)}(r) - \frac{\left(m_0r^2 - q_0^2\right)\frac{dj_i^{(1)}(r)}{dr}}{r}\right). \tag{3.1.35}$$

Here $(C_g)_i$ is an arbitrary constant. It corresponds to non normalizable mode of the gauge field and hence may be set to zero.

Substituting this expression for $g_i^{(1)}(r)$ into (3.1.33) we obtain the following differential

[5] Note that a linear combination of the ri and vi-components of the Einstein equation appear as the constraint equation in (3.1.31).

equation for $j_i^{(1)}(r)$

$$\begin{aligned}
&\left(35q_0^4+5r^2\left(r^4-6m_0\right)q_0^2+3m_0r^4\left(3r^4+m_0\right)\right)\frac{dj_i^{(1)}(r)}{dr}\\
&r\left(-11q_0^4-\left(5r^6-14m_0r^2\right)q_0^2-m_0r^4\left(r^4+3m_0\right)\right)\frac{d^2j_i^{(1)}(r)}{dr^2}\\
&+r^2\left(q_0^2-m_0r^2\right)\left(r^6-m_0r^2+q_0^2\right)\frac{d^3j_i^{(1)}(r)}{dr^3}\\
&=\frac{1}{\sqrt{3}}\left(6\sqrt{3}q_0\partial_iq^{(0)}r^4+3\sqrt{3}\partial_v\beta_i^{(0)}\left(5r^6-m_0r^2+q_0^2\right)r^4-144\ r\ l_i^{(0)}q_0^3\kappa\right)
\end{aligned} \tag{3.1.36}$$

The solution to this equation is given by,

$$\begin{aligned}
j_i^{(1)}(r)=(C_j^1)_i+\frac{(C_j^2)_ir^2}{\frac{m_0}{r^2}-\frac{q_0^2}{r^4}}+\frac{r\ \partial_v\beta_i^{(0)}}{\frac{m_0}{r^2}-\frac{q_0^2}{r^4}}\\
+\frac{\sqrt{3}\ l_i^{(0)}\ q_0^3\kappa}{m_0\left(\frac{m_0}{r^2}-\frac{q_0^2}{r^4}\right)r^4}+\frac{6r^2q_0(\partial_iq^{(0)}+3q_0\partial_v\beta_i^{(0)})}{R^7\left(\frac{m_0}{r^2}-\frac{q_0^2}{r^4}\right)}F_1(\frac{r}{R},\frac{m_0}{R^4}),
\end{aligned} \tag{3.1.37}$$

where again $(C_j^1)_i$ and $(C_j^2)_i$ are arbitrary constants. $(C_j^2)_i$ corresponds to a non-normalizable mode of the metric and so is set to zero. $(C_j^1)_i$ can be absorbed into a redefinition of the velocities and hence is also set to zero.

Here the function $F_1(\frac{r}{R},\frac{m_0}{R^4})$ is given by[6]

$$F_1(\rho,M)\equiv\frac{1}{3}\left(1-\frac{M}{\rho^4}+\frac{Q^2}{\rho^6}\right)\int_\rho^\infty dp\frac{1}{\left(1-\frac{M}{p^4}+\frac{Q^2}{p^6}\right)^2}\left(\frac{1}{p^8}-\frac{3}{4p^7}\left(1+\frac{1}{M}\right)\right), \tag{3.1.38}$$

where $Q^2=M-1$.

Substituting this result for $j_i^{(1)}(r)$ into (3.1.35) we obtain the following expression for $g_i^{(1)}(r)$

$$\begin{aligned}
g_i^{(1)}(r)=\frac{\sqrt{3}r^3\sqrt{R^2\left(m_0-R^4\right)}}{2\left(m_0(r-R)(r+R)+R^6\right)}(\partial_v\beta_i^{(0)})+\frac{3R^2\kappa(m_0-R^4)}{2\left(m_0\left(r^2-R^2\right)+R^6\right)}l_i\\
-\frac{\sqrt{3}r^4\left(r\left(m_0\left(r^2-R^2\right)+R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+\left(6R^7-6m_0R^3\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{2R^8\left(m_0\left(r^2-R^2\right)+R^6\right)}(\partial_iq^{(0)}+3q_0\partial_v\beta_i^{(0)})
\end{aligned} \tag{3.1.39}$$

where we use the notation $f^{(i,j)}(\alpha,\beta)$ to denote the partial derivative $\partial^{i+j}f/\partial\alpha^i\partial\beta^j$ of the function f.

Plugging back $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ back into (3.1.29) we conclude that the first order metric

[6] Although the expression for $F_1(\frac{r}{R},\frac{m_0}{R^4})$ is very complicated but it satisfies some identities. One can use those identities to perform practical calculations with this function.

and gauge field in the vector sector is given by

$$
\begin{aligned}
g_{vi}^{(1)}(r) &= r\partial_v\beta_i^{(0)} + \frac{\sqrt{3}\; l_i^{(0)} q_0^3\kappa}{m_0 r^4} + \frac{6r^2}{R^7} q_0(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)})F_1(\frac{r}{R},\frac{m_0}{R^4}) \\
A_i^{(1)}(r) &= -\frac{\sqrt{3}r^5 F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{2R^8}(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)}) + \frac{3R\kappa\sqrt{m_0-R^4}\sqrt{R^2\left(m_0-R^4\right)}}{2m_0 r^2} l_i
\end{aligned}
\tag{3.1.40}
$$

Tensors Of $SO(3)$ at first order

The tensor components of the first order metric is parameterized by the function $\alpha_{ij}^{(1)}(r)$ such that

$$
g_{ij}^{(1)} = r^2\alpha_{ij}^{(1)}. \tag{3.1.41}
$$

The gauge field does not have any tensor components therefore in this sector there is only one unknown function to be determined.

There are no constraint equations in this sector and the only dynamical equation is obtained from the ij-component of the Einstein equation. This equation is given by

$$
r\left(r^6 - m_0 r^2 + q_0^2\right)\frac{d^2\alpha_{ij}(r)}{dr^2} - \left(-5r^6 + m_0 r^2 + q_0^2\right)\frac{d\alpha_{ij}(r)}{dr} = -6\sigma_{ij}^{(0)} r^4 \tag{3.1.42}
$$

where σ_{ij} is given by

$$
\sigma_{ij}^{(0)} = \frac{1}{2}\left(\partial_i\beta_j^{(0)} + \partial_j\beta_i^{(0)}\right) - \frac{1}{3}\partial_k\beta_k^{(0)}\delta_{ij}. \tag{3.1.43}
$$

The solution to equation (3.1.42) obtained by demanding regularity at the future event horizon and appropriate normalizability at infinity. The solution is given by

$$
\alpha_{ij}^{(1)} = \frac{2}{R}\sigma_{ij}F_2(\frac{r}{R},\frac{m_0}{R^4}), \tag{3.1.44}
$$

where the function $F_2(\rho, M)$ is given by

$$
F_2(\rho, M) \equiv \int_\rho^\infty \frac{p\left(p^2+p+1\right)}{(p+1)\left(p^4+p^2-M+1\right)}dp \tag{3.1.45}
$$

with $M \equiv m/R^4$ as before.

Thus the tensor part of the first order metric is determined to be

$$
g_{ij}^{(1)} = \frac{2r^2}{R}\sigma_{ij}F_2(\frac{r}{R},\frac{m_0}{R^4}). \tag{3.1.46}
$$

The global metric and the gauge field at first order

In this subsection, we gather the results of our previous sections to write down the entire metric and the gauge field accurate up to first order in the derivative expansion.

We obtain the metric as

$$
\begin{aligned}
ds^2 &= g_{AB}dx^A dx^B \\
&= -2u_\mu dx^\mu dr - r^2\ V\ u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu\ r\left[u^\lambda \partial_\lambda u_\nu - \frac{\partial_\lambda u^\lambda}{3} u_\nu\right] dx^\nu + \frac{2r^2}{R} F_2(\rho, M)\sigma_{\mu\nu} dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu \left[\frac{\sqrt{3}\kappa q^3}{m r^4} l_\nu + \frac{6qr^2}{R^7} P^\lambda_\nu \mathcal{D}_\lambda q F_1(\rho, M)\right] dx^\nu + \ldots \\
A &= \left[\frac{\sqrt{3}q}{2r^2} u_\mu + \frac{3\kappa q^2}{2mr^2} l_\mu - \frac{\sqrt{3}r^5}{2R^8} P^\lambda_\mu \mathcal{D}_\lambda q F_1^{(1,0)}(\rho, M)\right] dx^\mu + \ldots
\end{aligned}
\tag{3.1.47}
$$

where $\mathcal{D}_\lambda$ is the weyl covarient derivative defined in appendix A.1. We also have defined

$$
\begin{aligned}
&V \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6}\ ; \qquad l^\mu \equiv \epsilon^{\nu\lambda\sigma\mu} u_\nu \partial_\lambda u_\sigma; \qquad P^\lambda_\mu \mathcal{D}_\lambda q \equiv P^\lambda_\mu \partial_\lambda q + 3(u^\lambda \partial_\lambda u_\mu) q; \qquad \rho \equiv \frac{r}{R} \\
&\sigma^{\mu\nu} \equiv P^{\mu\alpha} P^{\nu\beta} \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P^{\mu\nu} \partial_\alpha u_\alpha; \qquad M \equiv \frac{m}{R^4}; \qquad Q \equiv \frac{q}{R^3}; \qquad Q^2 = M - 1
\end{aligned}
\tag{3.1.48}
$$

and

$$
\begin{aligned}
F_1(\rho, M) &\equiv \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right) \int_\rho^\infty dp \frac{1}{\left(1 - \frac{M}{p^4} + \frac{Q^2}{p^6}\right)^2} \left(\frac{1}{p^8} - \frac{3}{4p^7}\left(1 + \frac{1}{M}\right)\right) \\
F_2(\rho, M) &\equiv \int_\rho^\infty \frac{p\left(p^2 + p + 1\right)}{(p+1)\left(p^4 + p^2 - M + 1\right)} dp\ .
\end{aligned}
\tag{3.1.49}
$$

The Stress Tensor and Charge Current at first order

In this section, we obtain the stress tensor and the charge current from the metric and the gauge field. The stress tensor can be obtained from the extrinsic curvature after subtraction of the appropriate counterterms. We get the first order stress tensor as

$$
T_{\mu\nu} = p(\eta_{\mu\nu} + 4u_\mu u_\nu) - 2\eta\sigma_{\mu\nu} + \ldots \tag{3.1.50}
$$

where the fluid pressure p and the viscosity η are given by the expressions

$$
p \equiv \frac{MR^4}{16\pi G_5} \qquad ; \qquad \eta \equiv \frac{R^3}{16\pi G_5} = \frac{s}{4\pi} \tag{3.1.51}
$$

where s is the entropy density of the fluid obtained from the Bekenstein formula.

To obtain the charge current, we use

$$
J_\mu = \lim_{r\to\infty} \frac{r^2 A_\mu}{8\pi G_5} = n\ u_\mu - \mathfrak{D}\ P^\nu_\mu \mathcal{D}_\nu n + \xi\ l_\mu + \ldots \tag{3.1.52}
$$

where the charge density n, the diffusion constant $\mathfrak{D}$ and an additional transport coefficient

ξ for the fluid under consideration are given by [7]

$$n \equiv \frac{\sqrt{3}q}{16\pi G_5} \quad ; \quad \mathfrak{D} = \frac{1+M}{4MR} \quad ; \quad \xi \equiv \frac{3\kappa q^2}{16\pi G_5 m} \tag{3.1.53}$$

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the charge current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid made of $\mathcal{N} = 4$ SYM matter. It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation and we hope to return to this issue in future.

The presence of such an effect was indirectly observed by the authors of [51] where they noted a discrepancy between the thermodynamics of charged rotating AdS black holes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N} = 4$ SYM fluid. In fact, one could go further and compare the first order metric that we have obtained with rotating black hole metrics written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.

3.1.3 Second Order Hydrodynamics

In this section we will find out the metric, stress tensor and charge current at second order in derivative expansion. We will follow the same procedure as in [8] but in presence charge parameter q. Note we have not performed a general constuction of second order hydrodynamics (as done for first order in Chapter 2) but the result of this section provides a definite prediction of the form of second order charged hydrodynamics for conformal fluids.

The metric and gauge field perturbations at second order that we consider are

$$g^{(2)}_{\alpha\beta}dx^\alpha dx^\beta = -3h_2(r)dvdr + r^2 h_2(r)dx^i dx_i + \frac{k_2(r)}{r^2}dv^2 + 12r^2 j^{(2)}_i dvdx^i + r^2\alpha^{(2)}_{ij}dx^i dx^j \tag{3.1.54}$$

and

$$\begin{aligned} A^{(2)}_v &= -\frac{\sqrt{3}}{2r^2}w_2(r) \\ A^{(2)}_i &= \frac{\sqrt{3}}{2}r^5 g^{(2)}_i(r)dx^i \ . \end{aligned} \tag{3.1.55}$$

Here we have used a little different parameterizations (from first order) for metric and gauge field perturbations in the vector sector. We found that this aids in writting the corresponding dynamical equations for $j^{(2)}_i(r)$ and $g^{(2)}_i(r)$ in a more tractable form (as we will see later).

[7] Here we have taken the chemical potential $\mu = 2\sqrt{3}QR$ which determines the normalization factor of the charge density n (because thermodynamics tells us $n\mu = 4p - Ts$) which in turn determines the normalization of J_μ. Note that due to the difference in μ with [66], our normalization of J_μ is different from that in [66].

Like neutral black brane case, here also we will list all the source terms (second order in derivative expansion) which will appear on the right hand side of the constraint dynamical equations in scalar, vector and tensor sectors. These source terms are built out of second derivatives of m, q and β or square of first derivatives of these three fields. We can group these source terms according to their transformation properties under $\mathbf{SO(3)}$ group. A complete list has been provided in table 3.1. In the table the quantities l_i and σ_{ij} are defined to be

$$l_i = \epsilon_{ijk}\partial_j\beta_k \ , \qquad \sigma_{ij} = \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_k) - \frac{1}{3}\delta_{ij}\partial_k\beta_k \ . \tag{3.1.56}$$

In table 3.1 we have already employed the first order conservation relations i.e. equation 3.1.18 and 3.1.19. Using these two relations we have eliminated the first derivatives of m and q. However at second order in derivative expansion we also have the relations

$$\partial_\mu\partial_\nu T^{\mu\nu}_{(0)} = 0 \ , \tag{3.1.57}$$

and

$$\partial_\lambda\partial_\mu J^{\mu}_{(0)} = 0 \ . \tag{3.1.58}$$

The equations (3.1.57) and (3.1.58) imply some relations between the second order source terms which are listed in table 3.1. These relations are

$$\begin{aligned}
\mathrm{S1} &= \frac{\mathrm{S3}}{3} - \frac{8m}{3}\mathrm{ST1} + \frac{16m}{9}\mathrm{ST3} - \frac{2m}{3}\mathrm{ST4} + \frac{4m}{3}\mathrm{ST5} \\
\mathrm{S2} &= -\frac{1}{4m}\mathrm{S3} + 4\mathrm{ST1} + \frac{1}{2}\mathrm{ST4} - \mathrm{ST5} \\
\mathrm{QS1} &= q\,(-\mathrm{ST1} - \mathrm{S2} + \mathrm{ST3}) - \mathrm{QS5} \\
\mathrm{V1}_i &= m\left(-\frac{40}{9}\mathrm{V4}_i - \frac{4}{9}\mathrm{V5}_i + \frac{56}{3}\mathrm{VT1}_i + \frac{4}{3}\mathrm{VT2}_i + \frac{8}{3}\mathrm{VT3}_i\right) \\
\mathrm{V2}_i &= \frac{10}{9}\mathrm{V4}_i + \frac{1}{9}\mathrm{V5}_i - \frac{2}{3}\mathrm{VT1}_i + \frac{1}{6}\mathrm{VT2}_i - \frac{5}{3}\mathrm{VT3}_i \\
\mathrm{V3}_i &= -\frac{1}{3}\mathrm{VT4}_i + \mathrm{VT5}_i \\
\mathrm{QV1}_i &= -q\left(\frac{10}{3}\mathrm{V4}_i + \frac{1}{2}(\mathrm{VT2}_i + 2\mathrm{VT1}_i + 2\mathrm{VT3}_i) + \frac{1}{3}\mathrm{V5}_i\right) \\
&\quad -\mathrm{QV2}_i - \frac{1}{2}\left(2\mathrm{QV4}_i + \mathrm{QV3}_i + \frac{2}{3}\mathrm{QV2}_i\right) \\
\mathrm{T1}_{ij} &= -4m\left(\mathrm{T3}_{ij} + \frac{1}{4}\mathrm{TT5}_{ij} - 4\mathrm{TT1}_{ij} + \frac{1}{3}\mathrm{TT4}_{ij} + \mathrm{TT6}_{ij}\right)
\end{aligned} \tag{3.1.59}$$

With these relation between the source terms we will now solve the Einstein equations and Maxwell equations to find out the constraint and dynamical equations at second order in derivative expansion. As in the first order calculations we shall perform this seperately in various sectors denoting different representation of the boundary rotation group $SO(3)$.

Table 3.1: An exhaustive list of two derivative terms in made up from the mass, charge and velocity fields. In order to present the results economically, we have dropped the superscript on the velocities β_i charge q and the mass m, leaving it implicit that these expressions are only valid at second order in the derivative expansion.

1 of $SO(3)$	**3** of $SO(3)$	**5** of $SO(3)$
$\text{S1} = \partial_v^2 m$	$\text{V1}_i = \partial_i \partial_v m$	$\text{T1}_{ij} = \partial_i \partial_j m - \frac{1}{3}\ \text{s3}\ \delta_{ij}$
$\text{S2} = \partial_v \partial_i \beta_i$	$\text{V2}_i = \partial_v^2 \beta_i$	$\text{T2}_{ij} = \partial_{(i} l_{j)}$
$\text{S3} = \partial^2 m$	$\text{V3}_i = \partial_v l_i$	$\text{T3}_{ij} = \partial_v \sigma_{ij}$
$\text{ST1} = \partial_v \beta_i\, \partial_v \beta_i$	$\text{V4}_i = \frac{9}{5} \partial_j \sigma_{ji} - \partial^2 \beta_i$	$\text{TT1}_{ij} = \partial_v \beta_i\, \partial_v \beta_j - \frac{1}{3}\ \text{ST1}\ \delta_{ij}$
$\text{ST2} = l_i\, \partial_v \beta_i$	$\text{V5}_i = \partial^2 \beta_i$	$\text{TT2}_{ij} = l_{(i}\, \partial_v \beta_{j)} - \frac{1}{3}\ \text{ST2}\ \delta_{ij}$
$\text{ST3} = (\partial_i \beta_i)^2$	$\text{VT1}_i = \frac{1}{3} (\partial_v \beta_i)(\partial_j \beta^j)$	$\text{TT3}_{ij} = 2\, \epsilon_{kl(i}\, \partial_v \beta^k\, \partial_{j)} \beta^l + \frac{2}{3}\ \text{ST2}\ \delta_{ij}$
$\text{ST4} = l_i\, l^i$	$\text{VT2}_i = -\epsilon_{ijk}\, l^j\, \partial_v \beta^k$	$\text{TT4}_{ij} = \partial_k \beta^k\, \sigma_{ij}$
$\text{ST5} = \sigma_{ij}\, \sigma^{ij}$	$\text{VT3}_i = \sigma_{ij}\, \partial_v \beta^j$	$\text{TT5}_{ij} = l_i\, l_j - \frac{1}{3}\ \text{ST4}\ \delta_{ij}$
$\text{QS1} = \partial_v^2 q$	$\text{VT4}_i = l_i\, \partial_j \beta^j$	$\text{TT6}_{ij} = \sigma_{ik}\, \sigma^k_j - \frac{1}{3}\ \text{ST5}\ \delta_{ij}$
$\text{QS2} = \partial_i \partial_i q$	$\text{VT5}_i = \sigma_{ij}\, l^j$	$\text{TT7}_{ij} = 2\, \epsilon_{mn(i}\, l^m\, \sigma^n_{j)}$
$\text{QS3} = l_i \partial_i q$	$\text{QV1}_i = \partial_i \partial_v q$	$\text{QT1}_{ij} = \partial_i \partial_j q - \frac{1}{3}\ \text{QS2}\ \delta_{ij}$
$\text{QS4} = (\partial_i q)^2$	$\text{QV2}_i = \partial_i q \partial_k \beta^k$	$\text{QT2}_{ij} = \partial_{(i} q l_{j)} - \frac{1}{3}\ \text{QS3}\ \delta_{ij}$
$\text{QS5} = (\partial_i q)(\partial_v \beta_i)$	$\text{QV3}_i = \epsilon_{ijk} \partial_j l_k$	$\text{QT3}_{ij} = \partial_{(i} q \partial_{j)} q - \frac{1}{3}\ \text{QS4}\ \delta_{ij}$
	$\text{QV4}_i = \sigma_{ij} \partial_j q$	$\text{QT4}_{ij} = \partial_{(i} q \partial_v \beta_{j)} - \frac{1}{3}\ \text{QS5}\ \delta_{ij}$
	$\text{QV5}_i = \epsilon_{ijk} \partial_v \beta_j \partial_k q$	$\text{QT5}_{ij} = \epsilon_{(ikm} \partial_k q\ \sigma_{mj)}$

Scalars of $SO(3)$ at second order

We parametrise the metric and the gauge field as follows

$$\begin{aligned}\sum_i g^{(2)}_{ii}(r) &= 3r^2 h_2(r),\\ g^{(2)}_{vv}(r) &= \frac{k_2(r)}{r^2}\\ g^{(2)}_{vr}(r) &= -\frac{3}{2}h_2(r)\\ A^{(2)}_v(r) &= -\frac{\sqrt{3}w_2(r)}{2r^2}.\end{aligned} \tag{3.1.60}$$

Now we intend to solve for the functions $h_2(r), k_2(r)$ and $w_2(r)$.

Constraint Equations

As we have already explained, there are three constraint equations. First two come from Einstein equations (Eq. 3.1.14 and 3.1.14) and the third one comes from Maxwell equations (Eq. 3.1.16). The first constrain from Einstein equations gives

$$\partial_v m^{(1)} = \frac{2}{3}R^3\ \mathrm{ST5} \tag{3.1.61}$$

Second constraint implies relation between $k_2(r)$ and $h_2(r)$. This constraint equation is given by

$$-m_0 h_2'(r) + 3r^4 h_2'(r) + 12r^3 h_2(r) - k_2'(r) + \frac{4q_0 w_2(r)}{r^3} - \frac{2q_0 w_2'(r)}{r^2} = S_C, \tag{3.1.62}$$

where the source term S_C is given in appendix A.2.

The constraint relation coming from Maxwell equations is given by

$$\begin{aligned}\partial_v q^{(1)} \;&=\; -\frac{3q_0\left(R^4+m_0\right)}{16{m_0}^2 R}\mathrm{S3} + \frac{\left(R^4+m_0\right)}{4m_0 R}\mathrm{QS2} - \frac{6\sqrt{3}{q_0}^2\kappa}{m_0}\mathrm{ST2}\\ &-\; \frac{\left(m_0-11R^4\right)}{4m_0 R}\mathrm{QS5} - \frac{2\sqrt{3}q_0\kappa}{m_0}\mathrm{QS3} - \frac{q_0}{4m_0R^3}\mathrm{QS4}\\ &+\; \frac{9q_0\left(3R^4+m_0\right)}{4m_0 R}\mathrm{ST1}\end{aligned} \tag{3.1.63}$$

Dynamical Equations and their solutions

The Dynamical Equations in the scalar sector (coming from the Einstein equation $E_{rr}=0$) is given by

$$r h_2''(r) + 5h_2'(r) = S_h\ . \tag{3.1.64}$$

The source term S_h is explicitly given in appendix A.2.

The second dynamical scalar equation, which comes form the Maxwell equations ($M(r)=0$), is given by

$$-6q_0 h_2'(r) + r w_2''(r) - w_2'(r) = S_M(r). \tag{3.1.65}$$

The explicit form of the source term $S_M(r)$ is again given in appendix A.2.

The source terms have the same large r behavior as uncharged case (see [8]) because the charge dependent terms (leading) are more suppressed than that of charge independent terms. So one can follow the same procedure to obtain the solution for $h_2(r)$ and $k_2(r)$. Here we present the result schematically. Firstly, we solve equation (3.1.64) for the function $h_2(r)$; we obtain

$$h_2(r) = \int \left(\frac{1}{r^5} \left(\int \left(r^4 S_h(r) \right) dr + C_h^{(1)} \right) \right) dr + C_h^{(2)}, \tag{3.1.66}$$

where $C_h^{(1)}$ and $C_h^{(2)}$ are the constants of integration. We then plug in this solution for $h_2(r)$ in to (3.1.65). Solving the resultant equation for the w_2 we obtain,

$$w_2(r) = \int \left(r \left(\int \left(\frac{1}{r^2} S_w(r) \right) dr + C_w^{(1)} \right) \right) dr + C_w^{(2)}, \tag{3.1.67}$$

where again $C_w^{(1)}$ and $C_w^{(2)}$ are integration constants, and the function $S_w(r)$ is

$$S_w(r) = S_M(r) + 6q_0 h_2'(r).$$

Finally, we substitute the functions $h_2(r)$ and $w_2(r)$ solved above, in to (3.3.2) to obtain the following equation for $k_2(r)$

$$k_2'(r) = (3r^4 - m_0)h_2'(r) + 12r^3 h_2(r) + \frac{4q_0}{r^3} w_2(r) - \frac{2q_0}{r^2} w_2'(r) - S_C \equiv S_k(r). \tag{3.1.68}$$

This equation can be easily integrated to obtain

$$k_2(r) = \int S_k(r) dr + C_k, \tag{3.1.69}$$

C_k being the integration constant. All the integration constants in the above solutions are obtained by imposing regularity at the horizon and normalizability of the functions, just as in the first order computation.

Vectors of $SO(3)$ at second order

As given in (3.1.54) and (3.1.55), in this sector we parametrize[8] the metric, and the gauge field respectively in the following way

$$\begin{aligned} g_{vi} &= 6r^2 j_i^{(2)}(r) \\ A_i^{(2)} &= \frac{\sqrt{3}}{2} r^2 g_i^{(2)}(r). \end{aligned} \tag{3.1.70}$$

Constraint Equations

In this sector, the constraint equation comes only from the Einstein equations (3.1.30). This

[8]Note that the parametrization of the gauge field at this order is different from the one used for the scalar sector.

constraint relation is give by

$$\begin{aligned}\partial_i m^{(1)} &= \frac{10R^3}{9}\mathrm{V4}_i + \frac{10R^3}{9}\mathrm{V5}_i + \frac{10R^3}{3}\mathrm{VT1}_i - \frac{5R^3}{6}\mathrm{VT2}_i \\ &+ \frac{6q_0 R}{m_0 - 3R^4}\mathrm{QV4}_i - \frac{\left(21R^7 - 43m_0R^3\right)}{3\left(m_0 - 3R^4\right)}\mathrm{VT3}_i \,. \end{aligned} \tag{3.1.71}$$

Dynamical Equations and their solutions

There are two vector dynamical equations. The first equation comes from Einstein equation and is given by

$$q_0 r g_i^{(2)'}(r) + 5q_0 g_i^{(2)}(r) + r j_i^{(2)''}(r) + 5 j_i^{(2)'}(r) = (S_E^{\mathrm{vec}})_i(r), \tag{3.1.72}$$

where $(S_E^{\mathrm{vec}})_i(r)$ is the source terms given in the appendix A.3. The second dynamical equation comes from Maxwell equation and is given by

$$\begin{aligned}&\sqrt{3}\Big(- m_0 r^4 g_i^{(2)''}(r) + q_0^2 r^2 g_i^{(2)''}(r) + r^8 g_i^{(2)''}(r) + g_i^{(2)'}(r)\left(-9m_0r^3 + 7q_0^2 r + 13r^7\right) \\ &\quad + 5g_i^{(2)}(r)\left(-3m_0r^2 + q_0^2 + 7r^6\right) + 12q_0 j_i^{(2)'}(r)\Big) = (S_M^{\mathrm{vec}})_i(r)\end{aligned} \tag{3.1.73}$$

where $(S_M^{\mathrm{vec}})_i(r)$ is the other source term the explicit form of which is also given in the appendix A.3. The sources $(S_M^{\mathrm{vec}})_i(r)$ and $(S_E^{\mathrm{vec}})_i(r)$ are expressed in terms of the weyl invariant quantities $(W_v)_i^m$ which are defined in appendix A.1. We can now solve equation (3.1.72) for the function $g_i^{(2)}(r)$ to obtain

$$g_i^{(2)}(r) = -\frac{j_i^{(2)'}(r)}{q_0} + \frac{(W_v)_i^1 + (W_v)_i^2}{6q_0r^3} - \left(\frac{1}{q_0r^5}\right)\int_r^\infty x^4\left((S_E^{\mathrm{vec}})_i(r) - \frac{(W_v)_i^1 + (W_v)_i^2}{3x^3}\right)dx, \tag{3.1.74}$$

where the integrating constant has been chosen by the normalizability condition. Plugging in this solution in to (3.1.73) we obtain the following effective equation for $j_i^{(2)}(r)$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^7\left(V^{(0)}(r)\right)^2\frac{d}{dr}\left(\frac{1}{V^{(0)}(r)}j_i^{(2)}(r)\right)\right)\right) + S_i(r) = 0, \tag{3.1.75}$$

where

$$\begin{aligned}S_i(r) = \left(-\frac{1}{\sqrt{3}r^2}\right)\Bigg(&\sqrt{3}\left(r\left(m_0\left(R^2 - r^2\right) + r^6 - R^6\right)(S_E^{\mathrm{vec}})_i'(r)\right. \\ &\left. + (S_E^{\mathrm{vec}})_i(r)\left(m_0\left(R^2 - 3r^2\right) + 7r^6 - R^6\right)\right) - \sqrt{R^2\left(m_0 - R^4\right)}(S_M^{\mathrm{vec}})_i(r)\Bigg).\end{aligned} \tag{3.1.76}$$

Finally, the solution to the equation (3.1.75) is given by

$$
\begin{aligned}
j_i^{(2)}(r) = &-V^{(0)}(r)\int_r^\infty \frac{1}{x^7\left(V^{(0)}(x)\right)^2}\left(\int_x^\infty y\int_y^\infty S_i^{reg}(z)dzdy\right)dx \\
&- V^{(0)}(r)\int_r^\infty \frac{1}{x^7\left(V^{(0)}(x)\right)^2}\Bigg[C_i^{(j)} - \frac{1}{3(m_0 3R^2)}3R^7\Bigg(\left((W_v)_i^1+(W_v)_i^4\right)x \\
&- m_0R^3\left((W_v)_i^1+3(W_v)_i^4\right)x - \frac{1}{2}m_0\left((W_v)_i^1+(W_v)_i^2\right)x^4 + \frac{3}{2}R^4\left((W_v)_i^1+(W_v)_i^2\right)x^4\Bigg)\Bigg]dx,
\end{aligned}
\tag{3.1.77}
$$

where again for convenience we have defined

$$
S_i^{reg}(z) = \frac{R^3\left(m_0((W_v)_i^1+3(W_v)_i^4)-3R^4((W_v)_i^1+(W_v)_i^4)\right)}{3z^2\left(m_0-3R^4\right)} - S_i(z) - \frac{4}{3}z((W_v)_i^1+(W_v)_i^2).
\tag{3.1.78}
$$

The constant $C_i^{(j)}$ is determined by the regularity at horizon and is given by

$$
\begin{aligned}
C_i^{(j)} = &-\frac{1}{12m_0\left(m_0-3R^4\right)}\Bigg(R^4\Big(m_0^2(9(W_v)_i^1+4(W_v)_i^2+15(W_v)_i^4) \\
&- 6m_0R^4(6(W_v)_i^1+3(W_v)_i^2+4(W_v)_i^4)+9R^8(3(W_v)_i^1+2(W_v)_i^2+(W_v)_i^4)\Big) \\
&- 9R^2\left(m_0^2-4m_0R^4+3R^8\right)\left(\int_R^\infty S_i^{reg}(x)\,dx\right)+6m_0\left(m_0-3R^4\right)\int_R^\infty y^2S_i^{reg}(y)\,dy\Bigg),
\end{aligned}
\tag{3.1.79}
$$

We now have to plug in the source terms (given in Appendix A.3) and perform the integrals to write the solutions explicitly. Since such explicit solution would be very complicated, we do not provide it here. Nevertheless, from the above solution we extract the boundary charge current as we explicate in the following section.

Boundary Charge Current at second order

The charge current at second order in derivative expansion is given by

$$
J_\mu^{(2)} = \lim_{r\to\infty}\frac{r^2A_\mu^{(2)}}{8\pi G_5}\ .
\tag{3.1.80}
$$

The gauge field perturbation at this order is parametrised by the function $g_i^{(2)}(r)$. Thus to obtain the charge current density we have to consider the asymptotic limit (i.e. the $r\to\infty$ limit) of the function $g_i^{(2)}(r)$. This function is given by (3.1.74). The function $j_i^{(2)}(r)$ in that equation is in turn given by (3.1.77).

If we carefully extract the coefficient of the $1/r^2$ term in the $r\to\infty$ limit of the gauge field (using the equation referred to in the last paragraph) we find that the charge current is given by

$$J_i^{(2)} = \frac{m_0 (W_v)_i^2 - 6C_i^{(j)}}{4\sqrt{3}\sqrt{R^2\left(m_0 - R^4\right)}}, \tag{3.1.81}$$

the constant $C_i^{(j)}$ being given by the equation (3.1.79). Plugging in the sources in to equation (3.1.79) and performing the integrations we find

$$J_i^{(2)} = \left(\frac{1}{8\pi G_5}\right)\sum_{l=1}^{5} \mathcal{C}_l (W_v)_i^l, \tag{3.1.82}$$

where the coefficients of the Weyl invariant terms $(W_v)_i^l$ are given by [9]

$$\begin{aligned}
\mathcal{C}_1 &= \frac{3\sqrt{3}R\sqrt{M-1}}{8M},\\
\mathcal{C}_2 &= \frac{\sqrt{3}R(M-1)^{3/2}}{4M^2},\\
\mathcal{C}_3 &= -\frac{3R\kappa(M-1)}{2M^2},\\
\mathcal{C}_4 &= \frac{1}{4}\sqrt{3}R\sqrt{M-1}\log(2) + \mathcal{O}(M-1),\\
\mathcal{C}_5 &= -\frac{\sqrt{3}R\sqrt{M-1}\left(M^2 - 48(M-1)\kappa^2 + 3\right)}{16M^2}.
\end{aligned} \tag{3.1.83}$$

We have expressed the above results in terms of the parameters M and R with $M = m_0/R^4$.

Tensors Of $SO(3)$ at second order

We now consider the tensor modes at second order. Following the first order calculations we pametrize the traceless symmetric tensor components of the second order metric by the function $\alpha_{ij}^{(2)}(r)$ such that

$$g_{ij}^{(2)} = r^2 \alpha_{ij}^{(2)}(r). \tag{3.1.84}$$

In this sector there are no constraint equations. However, there is a dynamical equation which we solve in the following subsection.

Dynamical equations and their solutions

The ij-component of the Einstein equation gives the dynamical equation for $\alpha_{ij}^{(2)}(r)$ which is similar to (3.1.42). However the source term of the differential equation is modified in the second order. Thus, at second order this equation is given by

$$-\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{r}\left(q_0^2 - m_0 r^2 + r^6\right)\frac{d}{dr}\alpha_{ij}^{(2)}(r)\right) = \mathbf{T}_{ij}(r), \tag{3.1.85}$$

where we write the source in terms of weyl-covariant quantities as follows

$$\mathbf{T}_{ij}(r) = \sum_{l=1}^{9} \tau_l(r)\; WT_{ij}^{(l)}. \tag{3.1.86}$$

[9] All these coefficients match with the corresponding coefficients in [66] except $\mathcal{C}_2$ and $\mathcal{C}_5$ which differ by a overall sign

We define the weyl-covariant terms $WT_{ij}^{(l)}$ in appendix A.1. The coefficients $\tau_l(r)$ of these weyl-covariant terms are given in appendix A.4.

The solution to (3.1.85) which is regular at the outer horizon and normalizable at infinity is given by

$$\alpha_{ij}^{(2)}(r) = \int_r^\infty \left(\left(\frac{\xi}{q_0^2 - m_0\xi^2 + \xi^6} \right) \int_1^\xi \left(\, 2\ \zeta\ \mathbf{T}_{ij}(\zeta) \right) d\zeta \right) d\xi. \tag{3.1.87}$$

We need to plug in the source from appendix A.4 in to the above equation and perform the integrals to obtain an explicit answer. However, as in the second order vector sector this turns out to be very complicated in general and therefore we do not produce it here. The transport coefficients, however, of the boundary stress tensor at second order in derivative expansion may be obtained only by knowing the function $\alpha_{ij}^{(2)}(r)$ asymptotically (near the boundary). In the next subsection, we compute this boundary stress tensor.

Boundary Stress Tensor at second order

As mentioned earlier, the AdS/CFT prescription for obtaining the boundary stress tensor from the bulk metric is given by

$$T_\nu^\mu = -\frac{1}{8\pi G_5} \lim_{r\to\infty} \left(r^4 \left(K_\nu^\mu - \delta_\nu^\mu \right) \right), \tag{3.1.88}$$

where K_ν^μ is the extrinsic curvature normal to the constant r surface. Now, as is apparent from the formula, we need to know the asymptotic expansion of the metric perturbation $\alpha_{ij}^{(2)}(\rho)$ in order to obtain the stress tensor. The asymptotic expansion of the solution (3.1.87) for $\alpha_{ij}^{(2)}(\rho)$ is given by

$$\alpha_{ij}^{(2)}(\rho) = \frac{1}{r^2}\left(WT_{ij}^{(3)} - \frac{1}{2} WT_{ij}^{(2)} - \frac{1}{4} WT_{ij}^{(4)} \right) + \frac{1}{4r^4} \sum_{l=1}^{9} \mathcal{N}_l\ WT_{ij}^{(l)} + \mathcal{O}\left(\frac{1}{r^5}\right), \tag{3.1.89}$$

The leading term of this asymptotic expansion gives divergent contributions to the stress tensor which are canceled by divergence arising from the expansion of $g^{(0)}+g^{(1)}$ up to second order.

On plugging in this asymptotic solution for the metric in to the formula (3.1.88) we obtain

$$T_{\mu\nu} = \left(\frac{1}{16\pi G_5} \right) \sum_{l=1}^{9} \mathcal{N}_l\ WT_{\mu\nu}^{(l)}. \tag{3.1.90}$$

with $\mathcal{N}_l$ being the transport coefficients at second order in derivative expansion. These transport coefficients are given by

$$\begin{aligned} \mathcal{N}_1 &= R^2 \left(\frac{M}{\sqrt{4M-3}} \log\left(\frac{3-\sqrt{4M-3}}{3+\sqrt{4M-3}} \right) + 2 \right), \\ \mathcal{N}_2 &= -\frac{MR^2}{2\sqrt{4M-3}} \log\left(\frac{3-\sqrt{4M-3}}{\sqrt{4M-3}+3} \right), \end{aligned} \tag{3.1.91}$$

and

$$
\begin{aligned}
\mathcal{N}_3 &= 2R^2, \\
\mathcal{N}_4 &= \frac{R^2}{M}(M-1)\left(12(M-1)\kappa^2 - M\right), \\
\mathcal{N}_5 &= -\frac{(M-1)R^2}{2M}, \\
\mathcal{N}_6 &= \frac{1}{2}(M-1)R^2\Big(\log(8)-1\Big) + \mathcal{O}\left((M-1)^2\right), \\
\mathcal{N}_7 &= \frac{\sqrt{3}(M-1)^{3/2}R^2\kappa}{M}, \\
\mathcal{N}_8 &= 0, \quad \mathcal{N}_9 = 0.
\end{aligned}
\tag{3.1.92}
$$

3.2 An analytically tractable limit of hairy black branes

In this section we shall write down static hairy balck brane solution perturbatively in certain small parameters. In particular, we shall demonstrate that the thermodynamics of these solutions is given by the Landau-Tiza two fluid model which directly related them to the boundary superfluids. In order to explicitly determine the thermodynamics of any particular gravitational system, however, we need to explicitly determine the solutions dual to uniform superfluid flows. Unfortunately, the ordinary differential equations that arise in this attempt have proved so complicated that it has not proved possible to analytically solve for hairy black branes (the gravitational duals to superfluids) in any reasonable gravitational system [10]. The only analytic results that we are aware of, for hairy black brane solutions, are those of Herzog [69]. Herzog considered a very special model, the model of a charged scalar field of $m^2 = -4$ and infinite charge e (i.e. a model in the so called probe limit). He demonstrated that this model displays a second order phase transition towards superfluidity whenever $|\frac{\mu}{T}| \geq 2$.

When $|\frac{\mu}{T}|$ is just larger that 2, the stable gravitational solutions develop a scalar vev. Let ϵ denote the value of this vev. Herzog was able to generate the relevant gravitational solutions perturbatively in ϵ and also perturbatively in the difference between superfluid and normal velocities.

In this paper we will be interested in probing the structure of viscous superfluid dynamics from gravity. In the infinite charge or probe limit of [69] scalar and gauge dynamics do not back react on spacetime. In order to probe the dynamics of the interaction between the stress tensor and the charge current we need to go beyond the infinite charge probe limit. In this section we generalize Herzog's perturbative construction of gravitational solutions to go beyond the probe limit. In other words we generalize Herzog's infinite e solutions to retain the first nontrivial correction in a $(\frac{1}{e})$ expansion.

In the next section we will use the results of this section as an input into the fluid gravity map, in order to generate gravitational solutions dual to viscous superfluid flows.

[10] Of course much attention has been focused on the numerical solutions of the relevant equations in several models.

3.2.1 The bulk system and the equations of motion

Following Herzog [69] we consider the system

$$\mathcal{L} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(\mathcal{R} + 12 + \frac{1}{e^2}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|D_\mu\phi|^2 + 2|\phi|^2.\right)\right), \tag{3.2.93}$$

Where $D_\mu = \nabla_\mu - iA_\mu$, and ∇_μ is the gravitational covariant derivative.

The equation of motion for the scalar field and the gauge field that follows from (3.2.93) are respectively

$$D_\mu D^\mu \phi + 4\phi = 0, \tag{3.2.94}$$

and

$$D_\mu F^{\mu\nu} = \frac{1}{2}J^\nu, \tag{3.2.95}$$

where the current $J_\mu = i\left(\phi^* D\mu\phi - \phi(D_\mu\phi)^*\right)$. The Einstein Equation that follows from (3.2.93) is

$$G_{\mu\nu} - 6g_{\mu\nu} = \frac{1}{e^2}\big((T_{\text{max}})_{\mu\nu} + (T_{\text{mat}})_{\mu\nu}\big), \tag{3.2.96}$$

where

$$\begin{aligned}(T_{\text{max}})_{\mu\nu} &= -\frac{1}{2}\left(F_{\mu\beta}F^{\beta}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\sigma\beta}F^{\beta\sigma}\right),\\ (T_{\text{mat}})_{\mu\nu} &= \frac{1}{4}\left(D_\mu\phi D_\nu\phi^* + D_\nu\phi D_\mu\phi^*\right) - \frac{1}{4}g_{\mu\nu}\left(|D_\beta\phi|^2 - 4|\phi|^2\right).\end{aligned} \tag{3.2.97}$$

We assume that this system admits a homogeneous stationary asymptotically AdS family of solutions - dual to homogeneous stationary superfluid flows - that take the form

$$\begin{aligned}\text{Metric}: \; ds^2 &= -2g(\frac{r}{r_c})u_\mu dx^\mu dr - r_c^2 f(\frac{r}{r_c})u_\mu u_\nu dx^\mu dx^\nu + r_c^2\, k(\frac{r}{r_c})n_\mu n_\nu dx^\mu dx^\nu\\ &\quad + r_c^2\, j(\frac{r}{r_c})\left(n_\mu u_\nu + u_\mu n_\nu\right)dx^\mu dx^\nu + r^2\tilde{P}_{\mu\nu}dx^\mu dx^\nu,\\ \text{Gauge field}: \; r_c A &= H(\frac{r}{r_c})\; u^\mu\partial_\mu + L(\frac{r}{r_c})\; n^\mu\partial_\mu\\ \text{Bulk scalar field} &= \phi\left(\frac{r}{r_c}\right)\end{aligned} \tag{3.2.98}$$

where

$$\tilde{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu. \tag{3.2.99}$$

Here u_μ and n_μ are two arbitrary constant vectors obeying

$$u^\mu n_\mu = 0; \quad u^\mu u_\mu = -1; \quad n^\mu n_\mu = 1. \tag{3.2.100}$$

We work in a gauge such that the scalar field is real $\phi^* = \phi$ (this implies $A^r = 0$), so that the boundary value of the gauge field gives the superfluid velocity. We choose the constant vector u_μ so as to ensure that the killing vector coincides with the generators of the event horizon of our solution. n^μ is then uniquely determined by (3.2.100) together with the requirement that A_μ at infinity (i.e. ξ_μ) can be written as a linear combination of u^μ and n^μ.

We now choose coordinates so that the killing vector ∂_v points along the direction of the u^μ and the vector ∂_x points in the direction of n^μ. Our solution retains rotational invariance in the remaining two spatial directions. Here r_c is a parameter of our solution and corresponds to the position of the horizon. We also work in the rescaled variables $\frac{r}{r_c}$ and $r_c x^\mu$ in terms of which (3.2.98) reduces to

$$\begin{aligned}
&ds^2 = 2g(r)\ dv\ dr - f(r)dv^2 - 2j(r)\ dv\ dx + k(r)\ dx^2 + r^2\left(\sum dy_i^2\right) \\
&A^r = 0, \quad A^v = H(r), \quad A^x = L(r), \quad A^y = 0, \quad A^z = 0 \\
&\text{Bulk scalar field} = \phi(r)
\end{aligned} \tag{3.2.101}$$

3.2.2 Boundary Conditions and Solutions

We search for solutions of the form (3.2.101). The 4-vectors n^μ, defined in the previous subsection, may be computed as follows. Let

$$r_c\zeta_\mu = (\eta_{\mu\nu} + u_\mu u_\nu)\,\xi^\mu.$$

It follows that n_μ is given by

$$n_\mu = \zeta_\mu/|\zeta|.$$

We search for solutions that obey the following large r boundary conditions

$$\begin{aligned}
k(r) &= r^2 + \frac{k_2}{r^2} \\
f(r) &= r^2 + \frac{f_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
j(r) &= \frac{j_2}{r^2} + \dots \\
L(r) &= \frac{\zeta}{r^2} + \dots \\
\phi(r) &= \frac{\epsilon}{r^2} + \dots
\end{aligned} \tag{3.2.102}$$

It turns out that the conditions above, together with the equations of motion, automatically ensure

$$\lim_{r\to\infty} g(r) = 1$$

so that this condition, while true, does not have to be additionally imposed. Also, it turns out that the coefficient of $1/r^2$ term in the asymptotic expansion of $H(r)$ is fixed by equations of motion and the requirement that ϕ be regular at the horizon.

Our functions are also constrained at $r = 1$ as follows

$$j(1) = f(1) = 0 \tag{3.2.103}$$

On the other hand the functions $H(r)$, $k(r)$, $L(r)$ and $\phi(r)$ are required only to be regular at $r = 1$.

It is possible to argue that there exists an 8 parameter class of solutions of the form (3.2.98), to the system (3.2.93), subject to the boundary conditions listed above. One of these parameters is r_c in (3.2.98). The three normal velocity parameters can be set to zero by a boost, and rotations can be used to point the superfluid velocity in the x direction, as in the previous section. This leaves us with a two parameter set of solutions, parameterized by ϵ and ζ.

3.2.3 Perturbative Solutions

In this subsection we will generalize the work out in [69] to the hairy black branes of our system, as a function of ϵ and ζ at small values of those parameters. Our starting point is Herzog's observation that, at $e = \infty$, the linearized equations of motion about the Reissner Nordstrom black brane at $|\frac{\mu}{T}| = 2$ admit a regular static solution scalar solution proportional to $\frac{\epsilon}{1+r^2}$. As was explained in [69] this solution can be taken to be the starting point for a perturbative expansion of hairy black brane solutions in a power series in ϵ. The solutions of [69] were further generalized to nonzero ζ.

In this subsection we generalize Herzog's solutions away from the infinite charge limit, to first order in a power series expansion in $\mathcal{O}(\frac{1}{e^2})$, i.e to first order in deviations away from the probe approximation. This generalization will prove crucial for generating the equations of superfluid dynamics including effects of back reaction of the superfluid on the normal fluid.

The techniques for obtaining this perturbative expansion are standard. We do not pause to explain our computations in detail; in the rest of this section we simply present the results of our calculations. As a function of ϵ and ξ (with both taken to be small) we find that the scalar field is given by

$$
\begin{aligned}
\phi(r) =& \Bigg\{ \epsilon \left[\frac{1}{r^2+1} + \frac{\zeta^2 \left(2\log(r) - \log\left(r^2+1\right)\right)}{4r^2+4} + \mathcal{O}\left(\zeta^4\right) \right] \\
&+ \epsilon^3 \left[\frac{-2\left(r^2+1\right)\log(r) + \left(r^2+1\right)\log\left(r^2+1\right) - 2}{48\left(r^2+1\right)^2} + \mathcal{O}\left(\zeta^2\right) \right] + \mathcal{O}(\epsilon^5) \Bigg\} \\
&+ \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned}
\tag{3.2.104}
$$

The functions in the gauge field in (3.2.98) are given by

$$
\begin{aligned}
H(r) &= \left(H_0(r) + H_1(r)\epsilon^2 + H_2(r)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) + \mathcal{O}(1/e^2), \\
L(r) &= \left(L_0(r) + L_1(r)\epsilon^2 + L_2(r)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) + \mathcal{O}(1/e^2)
\end{aligned}
\tag{3.2.105}
$$

where

$$
H_0(r) = \frac{2}{r^2+1} + \frac{\zeta^2}{2\left(r^2+1\right)} - \frac{\zeta^4(1-\log(2))}{4\left(r^2+1\right)} + O\left(\zeta^6\right), \tag{3.2.106}
$$

and

$$H_1(r) = \left(\frac{r^2-5}{48\left(r^2+1\right)^2} + \frac{\zeta^2}{288\left(r^2-1\right)\left(r^2+1\right)^2}\left(10r^4 + 72r^4\log(r) - 27r^4\log(2) + 18r^2 + 18r^2\log(2)\right.\right.$$
$$\left.\left. - 36r^4\log\left(r^2+1\right) - 28 + 45\log(2)\right)\right) + O\left(\zeta^4\right),$$
$$H_2(r) = -\frac{1}{55296\left(\left(r^2-1\right)\left(r^2+1\right)^3\right)}\left(253r^6 + 589r^4 - 589r^2 + 48\left(r^2+1\right)^2\log(64)\right.$$
$$\left. - 336\left(r^2-1\right)\left(r^2+1\right)^2\log(2) + 576\left(r^6+r^4\right)\log\left(\frac{r^2}{r^2+1}\right) - 253\right) + O\left(\zeta^2\right), \tag{3.2.107}$$

and also

$$L_0(r) = \frac{\zeta}{r^2}, \quad L_1(r) = -\frac{\zeta}{8r^2(1+r^2)} + \mathcal{O}(\zeta^3), \quad L_2(r) = \mathcal{O}(\zeta). \tag{3.2.108}$$

The functions in the metric in (3.2.98) are given by

$$\begin{aligned} f(r) &= \left(r^2 - \frac{1}{r^2}\right) + \frac{1}{e^2}\left(f_0(r) + f_1(r)\epsilon^2 + f_2(r)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) + \mathcal{O}\left(\frac{1}{e^4}\right), \\ g(r) &= 1 + \frac{1}{e^2}\left(g_0(r) + g_1(r)\epsilon^2 + g_2(r)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) + \mathcal{O}\left(\frac{1}{e^4}\right), \\ j(r) &= 0 + \frac{1}{e^2}\left(j_0(r) + j_1(r)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) + \mathcal{O}\left(\frac{1}{e^4}\right), \\ k(r) &= r^2 + \frac{1}{e^2}\left(k_0(r) + k_1(r)\epsilon^2 + k_2(r)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) + \mathcal{O}\left(\frac{1}{e^4}\right). \end{aligned} \tag{3.2.109}$$

where

$$f_0(r) = -\frac{4\left(r^2-1\right)}{3r^4} - \frac{2\left(r^2-1\right)\zeta^2}{3r^4} + \frac{\zeta^4\left(3r^2 + r^2(-\log(16)) - 3 + \log(16)\right)}{12r^4} + O\left(\zeta^6\right),$$
$$f_1(r) = \frac{-7r^4 + 12r^2 - 5}{36r^4\left(r^2+1\right)}$$
$$+ \frac{\zeta^2}{432r^2\left(r^2+1\right)}\left(\frac{54r^6 + r^4(54\log(2) - 23) - 36r^2(2+\log(2)) + 41 - 90\log(2)}{r^2}\right.$$
$$\left. + 18\left(3r^6 + 3r^4 - 9r^2 - 1\right)\left(2\log(r) - \log\left(r^2+1\right)\right)\right) + O\left(\zeta^4\right),$$
$$f_2(r) = \frac{1}{48r^2}\left(- \frac{2\left(r^6 + r^4 - 2r^2\right)\left(2\log(r) - \log\left(r^2+1\right)\right)}{3\left(r^2+1\right)}\right.$$
$$- \frac{1}{864r^2\left(r^2+1\right)^3}\left(576r^{10} + 989r^8 + 624r^8\log(2) - 1538r^6 + 1248r^6\log(2) - 1044r^4\right.$$
$$\left.\left. + 914r^2 - 1248r^2\log(2) + 103 - 624\log(2)\right)\right), \tag{3.2.110}$$

$$\begin{aligned}
g_0(r) &= \mathcal{O}\left(\zeta^6\right), \\
g_1(r) &= -\frac{1}{6\left(r^2+1\right)^2} \\
&\quad + \frac{\zeta^2\left(54\left(r^2+1\right)^3 - 6\left(r^2+1\right)^2\left(-9r^4-18r^2+3\right)\left(2\log(r)-\log\left(r^2+1\right)\right)\right)}{864\left(r^2+1\right)^4} + O\left(\zeta^4\right), \\
g_2(r) &= \frac{-6r^6-21r^4-14r^2-6\left(r^2+1\right)^2\left(r^4+2r^2\right)\left(2\log(r)-\log\left(r^2+1\right)\right)+4}{864\left(r^2+1\right)^4} + O\left(\zeta^2\right),
\end{aligned} \tag{3.2.111}$$

$$\begin{aligned}
j_0(r) &= \mathcal{O}\left(\zeta^6\right), \\
j_1(r) &= \frac{\left(r^2-1\right)\zeta}{8\left(r^2+1\right)} + \mathcal{O}\left(\zeta^3\right),
\end{aligned} \tag{3.2.112}$$

and

$$\begin{aligned}
k_0(r) &= \mathcal{O}\left(\zeta^6\right), \\
k_1(r) &= \frac{r^2\zeta^2\left(-2\left(r^2+1\right)\log(r)+\left(r^2+1\right)\log\left(r^2+1\right)-1\right)}{8\left(r^2+1\right)} + O\left(\zeta^4\right), \\
k_2(r) &= \mathcal{O}\left(\zeta^2\right),
\end{aligned} \tag{3.2.113}$$

Upon setting $\frac{1}{e^2} = 0$, our result exactly matches with the equations 2.30, 2.31, 2.32 in [69],if we replace $u = 1/r$ in those equations.

3.2.4 Boundary Thermodynamics

Using the solution obtained in the previous subsection we evaluate the boundary stress tensor charge current. For this purpose we use the standard AdS/CFT formulas

$$\begin{aligned}
\text{Boundary stress tensor} = T^\mu_\nu &= \frac{1}{16\pi G}\lim_{r\to\infty} r^4\left(2\left(\delta^\mu_\nu K_{\alpha\beta}\gamma^{\alpha\beta} - K^\mu_\nu\right) - 6\delta^\mu_\nu + \frac{\phi^*\phi}{e^2}\delta^\mu_\nu\right) \\
\text{Boundary charge current} = j^\mu &= \frac{1}{16\pi G\ e^2}\lim_{r\to\infty} r^3 F^{\mu r} \\
\text{Entropy density} = s &= \frac{\sqrt{k(1)}}{4G}, \\
\text{Temperature} = T &= \frac{f'(1)}{4\pi g(1)}.
\end{aligned} \tag{3.2.114}$$

where $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$ are respectively the induced metric and extrinsic curvature of a constant r surface.

The result for the stress tensor and current can be parameterized as the form

$$
\begin{aligned}
T^{\mu\nu} &= \frac{1}{16\pi G}\left[Au^{\mu}u^{\nu} + Bn^{\mu}n^{\nu} + C\left(n^{\mu}u^{\nu} + u^{\mu}n^{\nu}\right) + \left(\frac{A-B}{2}\right)\tilde{P}^{\mu\nu}\right] \\
j^{\mu} &= \frac{1}{16\pi G}\left[Q_1 u^{\mu} + Q_2 n^{\mu}\right]
\end{aligned}
\tag{3.2.115}
$$

$A,\ B$ and C are given by the following expressions.

$$
\begin{aligned}
A = 3r_c^4 + \frac{r_c^4}{e^2}\Bigg\{ &\left[4 + 2\zeta^2 + \zeta^4\left(\log(2) - \frac{3}{4}\right) + \mathcal{O}\left(\zeta^6\right)\right] \\
&+ \epsilon^2\left[\frac{7}{12} + \zeta^2\left(\frac{59}{144} - \frac{3\log(2)}{8}\right) + \mathcal{O}\left(\zeta^4\right)\right] \\
&+ \epsilon^4\left[\frac{624\log(2) - 451}{13824} + \mathcal{O}\left(\zeta^2\right)\right] + \mathcal{O}(\epsilon^6)\Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
B = r_c^4 + \frac{r_c^4}{e^2}\Bigg\{ &\left[\frac{4}{3} + \frac{2\zeta^2}{3} + \zeta^4\left(\frac{\log(2)}{3} - \frac{1}{4}\right) + \mathcal{O}\left(\zeta^6\right)\right] \\
&+ \epsilon^2\left[\frac{7}{36} + \zeta^2\left(\frac{131}{432} - \frac{\log(2)}{8}\right) + \mathcal{O}\left(\zeta^4\right)\right] \\
&+ \epsilon^4\left[\frac{624\log(2) - 451}{41472} + \mathcal{O}\left(\zeta^2\right)\right] + \mathcal{O}(\epsilon^6)\Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
C = \frac{r_c^4}{e^2}&\left[\epsilon^2\left[\frac{\zeta}{2} + \mathcal{O}\left(\zeta^3\right)\right] + \mathcal{O}(\epsilon^4)\right] + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{3.2.116}
$$

While Q_1 and Q_2 are given by the following expressions

$$
\begin{aligned}
Q_1 = -\frac{r_c^3}{e^2}\Bigg\{&\left[4 + \zeta^2 + \frac{1}{2}\zeta^4(\log(2) - 1) + \mathcal{O}\left(\zeta^6\right)\right] + \epsilon^2\left[\frac{7}{24} + \zeta^2\left(\frac{7}{36} - \frac{5\log(2)}{16}\right) + \mathcal{O}\left(\zeta^4\right)\right] \\
&+ \epsilon^4\left[\left(\frac{13\log(2)}{576} - \frac{493}{27648}\right) + \mathcal{O}\left(\zeta^2\right)\right] + \mathcal{O}(\epsilon^6)\Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
Q_2 = \frac{r_c^3}{e^2}\Bigg\{&\epsilon^2\left[-\frac{\zeta}{4} + \mathcal{O}(\zeta^3)\right] + \mathcal{O}(\epsilon^4)\Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{3.2.117}
$$

Now the relations in (2.31) in equilibrium may be casted in the form

$$\begin{aligned}
T^{\mu\nu} &= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu\nu} + \frac{\rho_s}{\xi^2}\xi^\mu\xi^\nu \\
&= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu\nu} + \rho_s u_s^\mu u_s^\nu \\
J^\mu &= q_n u^\mu - q_s\frac{\xi^\mu}{\xi} \\
&= q_n u^\mu + q_s u_s^\mu \\
u^\mu \xi_\mu &= \mu
\end{aligned} \tag{3.2.118}$$

where the following thermodynamical relations are obeyed

$$\begin{aligned}
\rho_n + P &= q_n\mu + Ts \\
\rho_s &= \mu_s q_s \\
\mu_s &= \xi = \xi_\mu u_s^\mu \\
dP &= sdT + q_s d\mu_s + q_n d\mu \\
&= sdT + q_s d\xi + q_n d\mu
\end{aligned} \tag{3.2.119}$$

In our gravity system the parametes in (3.2.118) are found to be

$$\begin{aligned}
16\pi G\left(\rho_n\right) &= 3r_c^4 + \frac{r_c^4}{e^2}\left\{\left[4 + 2\zeta^2 + \mathcal{O}(\zeta^4)\right] + \epsilon^2\left[-\frac{5}{12} + \mathcal{O}(\zeta^2)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G(\rho_s) &= \frac{r_c^4}{e^2}\left\{\left[\mathcal{O}(\zeta^4)\right] + \epsilon^2\left[1 + \mathcal{O}(\zeta^2)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G(P) &= r_c^4 + \frac{r_c^4}{e^2}\left\{\left[\frac{4}{3} + \frac{2}{3}\zeta^2 + \mathcal{O}(\zeta^4)\right] + \epsilon^2\left[\frac{7}{36} + \mathcal{O}(\zeta^2)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G(q_n) &= -\frac{r_c^3}{e^2}\left\{\left[4 + \zeta^2 + \mathcal{O}(\zeta^4)\right] + \epsilon^2\left[-\frac{5}{24} + \mathcal{O}(\zeta^2)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G(q_s) &= \frac{r_c^3}{e^2}\left\{\left[\mathcal{O}(\zeta^4)\right] + \epsilon^2\left[\frac{1}{2} + \mathcal{O}(\zeta^2)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned} \tag{3.2.120}$$

Further the chemical potential and μ_s of our solution are given by

$$\begin{aligned}
\mu = u^\mu\xi_\mu = r_c\Bigg\{&\left[-2 - \frac{\zeta^2}{2} + \zeta^4\left(\frac{1}{4} - \frac{\log(2)}{4}\right) + \mathcal{O}\left(\zeta^6\right)\right)] \\
&+ \epsilon^2\left[-\frac{1}{48} + \zeta^2\left(\frac{3\log(2)}{32} - \frac{5}{144}\right) + \mathcal{O}\left(\zeta^4\right)\right] \\
&+ \epsilon^4\left[\left(\left(\frac{253}{55296} - \frac{7\log(2)}{1152}\right) + O\left(\zeta^2\right)\right) + \mathcal{O}(\epsilon^6)\right]\Bigg\} + \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned} \tag{3.2.121}$$

and

$$\begin{aligned}
\mu_s = u_s^\mu \xi_\mu = r_c \Bigg\{ & \Bigg[2 + \frac{\zeta^2}{4} + \zeta^4 \left(-\frac{13}{64} + \frac{\log(2)}{4} \right) + \mathcal{O}\left(\zeta^6\right) \Bigg)] \\
& + \epsilon^2 \Bigg[+ \frac{1}{48} + \zeta^2 \left(-\frac{3\log(2)}{32} + \frac{43}{1152} \right) + \mathcal{O}\left(\zeta^4\right) \Bigg] \\
& + \epsilon^4 \Bigg[\Bigg(\left(-\frac{253}{55296} + \frac{7\log(2)}{1152} \right) + \mathcal{O}\left(\zeta^2\right) \Bigg) + \mathcal{O}(\epsilon^6) \Bigg] \Bigg\} + \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned} \tag{3.2.122}$$

Moreover we find

$$s = \frac{r_c^3}{4G} \left[1 + \frac{1}{e^2} \left\{ \epsilon^2 \left[\frac{\log(4) - 1}{32} \zeta^2 + \mathcal{O}(\zeta^4) \right] + \mathcal{O}(\epsilon^4) \right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \right] \tag{3.2.123}$$

$$\begin{aligned}
T = \frac{r_c}{\pi} + \frac{r_c}{4\pi e^2} \Bigg\{ & \Bigg[-\frac{8}{3} - \frac{4\zeta^2}{3} + \zeta^4 \left(\frac{1}{2} - \frac{2\log(2)}{3} \right) + \mathcal{O}\left(\zeta^6\right) \Bigg] \\
& + \epsilon^2 \Bigg[\frac{1}{9} + \zeta^2 \left(\frac{\log(2)}{4} - \frac{23}{216} \right) + \mathcal{O}\left(\zeta^4\right) \Bigg] \\
& + \epsilon^4 \Bigg[\left(\frac{91}{20736} - \frac{\log(2)}{108} \right) + \mathcal{O}\left(\zeta^2\right) \Bigg] + \mathcal{O}(\epsilon^6) \Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned} \tag{3.2.124}$$

Using these expressions and the quantities obtained in (3.2.120) we have verified all the relations (3.2.119) to the order to which we have evaluated our solution.

3.3 Superfluid dynamics to first order in the derivative expansion

In the previous section we have determined the equilibrium solutions for hairy black branes, perturbatively in ϵ and the superfluid velocity, and separately in an expansion in $\frac{1}{e^2}$. In this section we use the results of the previous subsection as an input into the fluid gravity map.

The basic idea here is a simple generalization of the ideas spelt out in [2, 8, 9, 10, 11, 12, 13, 66]. We search for gravitational solutions that tube wise approximate the 8 parameter hairy black brane solutions described in the previous section, with values of the temperature, the chemical potential, ζ^μ and u^μ varying in space and time. The tubes in question run along null ingoing geodesics, and foliate our spacetime. Technically, this programme is implemented by working in ingoing Eddington Finklestein coordinates (as we have been through this paper) but promoting the parameters of our solutions to fields that vary in spacetime.

The fluid gravity map generates the gravitational solutions dual to fluid flows perturbatively in a boundary derivative expansion. The zero order ansatz for such a solution is simply the solution (3.2.98) with ϵ, r_c, ζ^μ and u^μ promoted to arbitrary slowly varying functions of spacetime. This ansatz of course solves the equations of motion (3.2.94), (3.2.95) and (3.2.96), when all parameters are constant, but does not solve these equations when these parameter vary in spacetime. As in [2, 8, 11, 66] this ansatz may be corrected to obtain a true solution (systematically in a derivative expansion) provided the eight fluid fields that parameterize our ansatz obey certain constraint equations. These constraint equations are simply the fluid equations with holographically generated constitutive relations for the stress tensor, the charge current, and a holographically generated correction to the Josephson equation.

In this section we implement this programme to first order in the derivative expansion.

3.3.1 The method

As we have explained above, we will search for gravity solutions that tube wise approximate the equilibrium solutions of the previous section. In principle our solutions could be labeled by a temperature and a chemical potential field in addition to the normal and superfluid velocity fields. However, for calculation purposes we will find it convenient to trade chemical potential for $\epsilon(x)$, the local expectation value of the operator O, and a temperature like variable $r_c(x)$, together with $u^\mu(x)$ and $\zeta^\mu(x)$. The precise definitions of our field variables is given by the equations

$$\begin{aligned}
&\phi(r,x) = \frac{r_c^2(x)\epsilon(x)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
&u^\mu T_\mu^\nu(x) = -\rho_n(x)u^\nu + \frac{\rho_s(x)}{\mu(x)^2 - r_c^2(x)\zeta(x)^2}\left[-\mu(x)u^\nu + r_c(x)\zeta^\nu(x)\right] \\
&P^{\mu\nu}\xi_\nu(x) = r_c(x)\zeta^\mu(x), \quad \zeta = \sqrt{\zeta^\mu\zeta_\mu}
\end{aligned} \tag{3.3.125}$$

where $\phi(r,x)$ is the slowly varying bulk scalar field and $T^{\mu\nu}(x)$ is the boundary stress tensor. The functions $\rho_s(x)$, $\rho_n(x)$ and $\mu(x)$ are given in terms of $r_c(x)$, $\epsilon(x)$ and $\zeta(x)$ determined by thermodynamics (i.e. from previous sections). As usual, $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$.

The fluid gravity map is generated by solving Einstein's equations tube wise, point by point on the boundary. At any given boundary point we can always boost and rotate coordinates so that

$$u_\mu = (-1,0,0,0), \qquad n_\mu = (0,1,0,0)$$

In the neighborhood of our special point, however,

$$\begin{aligned}
&u_\mu = \gamma_u(-1,\beta_1,\beta_2,\beta_3), \quad n_\mu = \gamma_n(-n_v,1,n_2,n_3), \quad n_v = \beta_1 + n_2\beta_2 + n_3\beta_3 \\
&\gamma_u = \frac{1}{\sqrt{1-\beta_1^2-\beta_2^2-\beta_3^2}}, \quad \gamma_n = \frac{1}{\sqrt{n_v^2-1-n_2^2-n_3^2}}
\end{aligned} \tag{3.3.126}$$

where β_i and n_i are of first or higher order in derivatives of fluid fields at the special point.

In this paper we will work only to first order in the derivative expansion. At this order we are sensitive only to first derivatives of β_1 β_2, β_3, n_2 and n_3 along with the first derivatives of ξ, r_c and ϵ.

The solution at our special point preserves an $SO(2)$ symmetry (of rotations in a plane perpendicular to u_μ and n_μ; the yz plane in our coordinates). This symmetry will help us organize our calculation. To start with it will prove useful to organize first derivative 'fluid data', i.e. all the first derivatives of the fluid fields at our special point, in terms of their $SO(2)$ transformation properties. We list our results

- First order derivative excitations with spin 0 (scalars):
  $S_1 = \frac{1}{\epsilon}\partial_1\zeta,\ S_2 = \frac{1}{\epsilon}\partial_1\epsilon,\ S_3 = \partial_0\beta_1,\ S_4 = \partial_1\beta_1,\ S_5 = \partial_i n_i,$
  $S_6 = \partial_i\beta_i,\ S_7 = \epsilon_{ij}\partial_i n_j,$
  $S_8 = \partial_0 r_c,\ S_9 = \partial_1 r_c,\ S_{10} = \frac{1}{\epsilon}\partial_0\zeta,\ ,S_{11} = \frac{1}{\epsilon}\partial_0\epsilon,\ S_{12} = \epsilon_{ij}\partial_i\beta_j$
- First order derivative excitations with spin ± 1 (vectors):
  $[V_1]_i = \frac{1}{\epsilon}\partial_i\epsilon,\ [V_2]_i = \frac{1}{\epsilon}\partial_i\zeta,\ [V_3]_i = \partial_1 n_i,\ [V_4]_i = \partial_0\beta_i,$
  $[V_5]_i = \partial_i\beta_1 + \partial_1\beta_i\ [V_6]_i = \partial_i r_c,\ [V_7]_i = \partial_0 n_i,\ [V_8]_i = \partial_i\beta_1 - \partial_1\beta_i$
- First order derivative excitations with spin ± 2 (traceless symmetric tensors):
  $[T_1]_{ij} = \partial_i\beta_j + \partial_j\beta_i - (\partial_k\beta_k)\delta_{ij},\quad [T_2]_{ij} = \partial_i n_j + \partial_j n_i - (\partial_k n_k)\delta_{ij}$
  Here $\{i,j\} = \{2,3\}$.

Following the methods of [2, 8, 11], in order to derive the metric dual to a fluid flow we need to solve the equations of motion, order by order, in the derivative expansion. That is we set the metric g of our solution to $g_0 + \epsilon g_1 \ldots$ (and similarly for the gauge fields and the scalars) and solve the bulk equations of motion at first order in ϵ. As explained in [2, 8, 11], the resulting equations are of two sorts. The Einstein and Maxwell constraint equations reduce simply to the equations of energy momentum and current conservation, and do not involve the unknown fields g_1 etc. These equations relate some of the independent derivatives listed above to others. On the other hand the dynamical Einstein and Maxwell equations allow you us compute the unknown fields g_1 etc in terms of the constrained derivative data listed above.

3.3.2 The constraint equations

We will now first describe the solution of the constraint equations, before turning to the dynamical equations.

In addition to the conservation equations described above, there is one additional source of constraints on the derivative data given in §3.3.1. Our demand that our solution be asymptotically AdS requires, in particular that the boundary field strength vanishes, implying $\partial_\mu\xi_\nu - \partial_\nu\xi_\nu$ vanishes. We must add this equation to the list of equations that constrain independent data.

It is convenient to decompose the constraint equations according to the its quantum numbers under the preserved $SO(2)$. We now perform the relevant decompositions, and state which pieces of data we use these constraints to solve for.

- *Current conservation:* It is a spin-0 constraint. Using this we shall solve for S_{11}.
- *Stress-tensor conservation:* It is effectively four equations. Among them two are spin-0 constraints and one spin-1 constraint. Using this we shall solve for S_8, S_9 and V_6,.

- *Curl-free condition on ξ_μ:* This imposes a set of 6 equations.

  Two of them transform in spin-0 ($[\partial_0\xi_1 - \partial_1\xi_0]$ and $\epsilon_{ij}\partial_i\xi_j$). Using these we solve for S_{10} and S_{12} respectively.

  Four of them transform in two separate spin-1 ($[\partial_i\xi_1 - \partial_1\xi_i]$ and $[\partial_i\xi_0 - \partial_0\xi_i]$). Using these we solve for V_7 and V_8.

After solving for dependent data [11], the remaining independent one derivative pieces of data are given as follows. We have seven spin-0 ($S_1, \cdots, S_7$), five spin-1 ($V_1, \cdots, V_5$) and two spin-2 (T_1, T_2) boundary data.

For later use we will find it useful to list covariant expressions for the independent data. These expressions are most usefully written in terms of the projector normal to the velocity/ superfluid velocity frame

$$\tilde{P}_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu} - n_\mu n_\nu$$

Using this projector one can write the following covariant expressions for our choices of independent boundary data as follows:

Spin-0

$$\begin{aligned} &S_1 = \frac{1}{\epsilon}(n^\mu\partial_\mu)\zeta, \quad S_2 = \frac{1}{\epsilon}(n^\mu\partial_\mu)\epsilon, \quad S_3 = u^\mu n^\nu\partial_\mu u_\nu, \quad S_4 = n^\mu n^\nu\partial_\mu u_\nu, \\ &S_5 = \tilde{P}^{\mu\nu}\partial_\mu n_\nu, \quad S_6 = \tilde{P}^{\mu\nu}\partial_\mu u_\nu, \quad S_7 = \epsilon^{\mu\nu\rho\sigma} n_\mu u_\mu \partial_\rho n_\sigma \end{aligned} \tag{3.3.127}$$

Spin-1

$$\begin{aligned} &[V_1]_\mu = \frac{1}{\epsilon}\tilde{P}^\sigma_\mu\partial_\sigma\epsilon, \quad [V_2]_\mu = \frac{1}{\epsilon}\tilde{P}^\sigma_\mu\partial_\sigma\zeta, \quad [V_3]_\mu = \tilde{P}^\nu_\mu n^\sigma\partial_\sigma n_\nu, \\ &[V_4]_\mu = \tilde{P}^\nu_\mu u^\sigma\partial_\sigma u_\nu, \quad [V_5]_\mu = \tilde{P}^\nu_\mu n^\sigma\left(\partial_\nu u_\sigma + \partial_\sigma u_\nu\right) \end{aligned} \tag{3.3.128}$$

Spin-2

$$\begin{aligned} &[T_1]_{\mu\nu} = \tilde{P}^\sigma_\mu\tilde{P}^\rho_\nu\left[\partial_\sigma u_\rho + \partial_\rho u_\sigma\right] - S_6\tilde{P}_{\mu\nu}, \\ &[T_2]_{\mu\nu} = \tilde{P}^\sigma_\mu\tilde{P}^\rho_\nu\left[\partial_\sigma n_\rho + \partial_\rho n_\sigma\right] - S_5\tilde{P}_{\mu\nu} \end{aligned} \tag{3.3.129}$$

3.3.3 The dynamical equations

Following earlier work on the fluid gravity correspondence [2, 8, 11, 66] , we work in the gravitational gauge $g_{rr} = 0$ and $g_{r\mu} = u_\mu$. For the $U(1)$ field we continue to demand that the scalar field be real. With derivatives taken into account this requirement no longer sets

[11] As we have indicated above, we solve for some first derivatives of fluid fields in terms of other derivatives. The relevant equations are linear and easy to solve; the solutions are explicit but lengthy and we do not present them here.

A^r to zero, but allows us to determine A^r rather simply, by demanding the consistency of the equations for ϕ and ϕ^*.

We will now solve for the first derivative corrections about the basic fluid gravity ansatz. As we have determined the equilibrium solutions, in the previous section, only to order $\frac{1}{e^2}$, we can of course compute the metric dual to fluid flows only at the same order in $\frac{1}{e^2}$.

We now describe in rough terms how we determine the deviations away from the zero order fluid ansatz. Let us start with the gauge field and scalars. At leading order in $\frac{1}{e^2}$ we take derivative corrections to the gauge field and the scalar field to have the form (δA^M is the derivative correction for the gauge field)

$$\begin{aligned}
\delta A^r &= \sum_{i=1}^{7} \delta A_i\left(\frac{r}{r_c}\right) S_i + \mathcal{O}\left(\frac{1}{e^2}\right) \\
\delta A^\mu &= \frac{1}{r_c^2}\left[u^\mu \sum_{i=1}^{7} \delta H_i\left(\frac{r}{r_c}\right) S_i \;+\; n^\mu \sum_{i=1}^{7} \delta L_i\left(\frac{r}{r_c}\right) S_i + \sum_{i=1}^{5} X_i\left(\frac{r}{r_c}\right)\,[V_i]^\mu\right] + \mathcal{O}\left(\frac{1}{e^2}\right) \\
\delta\phi &= \frac{1}{r_c}\left[\sum_{i=1}^{7} \delta\phi_i\left(\frac{r}{r_c}\right) S_i\right] + \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned} \tag{3.3.130}$$

We now describe in structural terms how we have solved for these functions, emphasizing boundary conditions

1. It turns out that $\delta A_i(r)$ obeys a first order differential equation in r. The general solution of δA^r diverges linearly in r while expanded around $r = \infty$. We fix the constant of integration (coefficient of the homogeneous solution in this equation) by setting the coefficient of the linear term in r to zero. This choice of boundary conditions is forced on us by the requirement that the bulk current goes to zero at the boundary so that the boundary current is really conserved.

2. $\delta H_i(r)$ obeys a second order differential equation (arising from the r component of the Maxwell equation). The two integration constants for this equation are fixed as follows. One of the integration constant is determined from the requirement of regularity at the horizon. The other integration constant is obtained from the requirement that there exist a regular scalar field solution (see below).

3. $\delta L_i(r)$ obeys a second order differential equation given by the x component of the Maxwell equation. Here one of the integration constant is determined imposing the regularity of the solution at the horizon. The other integration constant is fixed using the fact that according to equation (3.3.125) ζ^μ does not receive any derivative correction. A generic solution of $\delta L_i(r)$ dies of at infinity like $\frac{1}{r^2}$; the coefficient of this $\frac{1}{r^2}$ must be set to zero.

4. The equation for $X_i(r)$ comes from the y or z component of the Maxwell equation. This is also a second order differential equation and its integration constants are determined in a similar way as in $\delta L_i(r)$.

5. The equation for the scalar field determines $\delta\phi_i(r)$. Normalizability and the definition of $\epsilon(x)$ as given in equation (3.3.125) fixes the two integration constants here. More specifically, an expansion about infinity of a generic solution to the scalar field equation takes the form
$$\delta\phi_i(r) = a_i \frac{\ln(r)}{r^2} + \frac{b_i}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)$$
Our boundary conditions are that both a_i vanishes (from the requirement of normalizability) and that b_i vanishes (from our definition of ϵ). These two requirements completely fix the scalar fluctuation. As described above, the further requirement that the scalar fluctuation be regular at the horizon yields a boundary condition on $\delta H_i(r)$ (see above).

Let us now turn to the metric field. In the strict limit of $\frac{1}{e^2} \to 0$ the scalar and gauge field do not back react on the metric. The derivative expansion of the metric in this limit is thus that of uncharged fluid dynamics and was determined in [8] to be

$$\begin{aligned} ds^2 =& -2u_\mu dx^\mu dr + r^2\left[-\left(1-\frac{r_c^4}{r^4}\right)u_\mu u_\nu + +P_{\mu\nu}\right]dx^\mu dx^\nu \\ & - dx^\mu dx^\nu\left[2ru_\mu\left((u.\partial)\,u_\nu - \frac{1}{3}(\partial.u)\,u_\nu\right) + r_c F\left(\frac{r}{r_c}\right)\sigma_{\mu\nu}\right] \end{aligned} \tag{3.3.131}$$

Where

$$F(r) = -\frac{r^2}{2}\left[-\log\left(r^2+1\right) + 4\log(r) - 2\log(r+1) + 2\tan^{-1}(r) - \pi\right]$$

and

$$\sigma_{\mu\nu} = P_\mu^\alpha P_\nu^\beta\left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \frac{\partial.u}{3}\eta_{\alpha\beta}\right)$$

The new results of this paper are for the derivative correction to the metric at order $\mathcal{O}\left(\frac{1}{e^2}\right)$. We parameterize the corrections to the metric as

$$\begin{aligned} &\delta(ds^2) \\ &= -\frac{2}{e^2}\left[\sum_{i=1}^{7}\frac{S_i}{r_c}\,\delta g_i\left(\frac{r}{r_c}\right)\right]u_\mu dx^\mu dr \\ &+ \frac{r_c}{e^2}dx^\mu dx^\nu\left\{\left[\sum_{i=1}^{7}S_i\,\delta f_i\left(\frac{r}{r_c}\right)\right]u_\mu u_\nu + \left[\sum_{i=1}^{7}S_i\,\delta K_i\left(\frac{r}{r_c}\right)\right]n_\mu n_\nu \right. \\ &\qquad \left. + \left[\sum_{i=1}^{7}S_i\,\delta J_i\left(\frac{r}{r_c}\right)\right](n_\mu u_\nu + n_\nu u_\mu)\right\} \\ &+ \frac{r_c}{e^2}dx^\mu dx^\nu\left\{\sum_{i=1}^{5}\left[Y_i\left(\frac{r}{r_c}\right)(u_\mu[V_i]_\nu + u_\nu[V_i]_\mu) + W_i\left(\frac{r}{r_c}\right)(n_\mu[V_i]_\nu + n_\nu[V_i]_\mu)\right]\right. \\ &\qquad \left. + \sum_{i=1}^{2}Z_i\left(\frac{r}{r_c}\right)[T_i]_{\mu\nu}\right\} + \mathcal{O}\left(\frac{1}{e^4}\right) \end{aligned} \tag{3.3.132}$$

We now describe, very qualitatively, how we have solved for these functions.

1. $\delta g_i(r)$, $\delta f_i(r)$ and $\delta K_i(r)$ are determined solving three coupled equations obtained from the (rr) (rv) and (xx) component of the Einstein equations. Once decoupled using appropriate combination of these functions, two of the equations become first order and the third one is a second order differential equation. Two of the four integration constants are determined using the asymptotic AdS condition of the metric. [12] A third integration constant is fixed by demanding the regularity of the function $\delta K_i(r)$ at $r = r_c$. The last integration constant is fixed to ensure that the vv component of the boundary stress tensor receives no derivative corrections so that the equation (3.3.125) is satisfied.

2. The function $\delta J_i(r)$ is determined using the (rx) or (vx) component of the Einstein equation. This is a second order differential equation in r. The two integration constants are determined using the normalizability and the definition of boundary stress tensor (according to (3.3.125) the vx component of the boundary stress tensor should not receive any derivative correction). The general solution for $\delta J_i(r)$ has the following expansion around $r = \infty$.

$$\lim_{r\to\infty} \delta J_i(r) = j_0 \; r^2 + \frac{j_1}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

   Our boundary condition is that j_0 and j_1 both vanish.

3. $Y_i(r)$ is determined from the (vy) or (vz) component of the Einstein equation . This is a second order differential equation in r. The two integration constants are determined using normalizability of the metric and the definition of boundary stress tensor (the (vy) or (vz) component of the stress tensor should not receive any derivative corrections). This condition is exactly same as that of $\delta J_i(r)$ in terms of the coefficients of $\frac{1}{r}$ expansion.

4. $W_i(r)$ is determined from the (xy) or (xz) component of the Einstein equation . This is a second order differential equation in r. The two integration constants are determined using normalizability and regularity of the metric respectively. A generic solution of W_i behaves like r^2 at large r. Our boundary conditions are that the leading coefficient of this leading r^2 piece vanish.

5. $Z_i(r)$ is determined from the (yz) component of the Einstein equation . This is a second order differential equation in r. The two integration constants are determined exactly the same way as for $W_i(r)$.

[12] After this condition is imposed $\frac{1}{r}$ expansion of the functions $\delta g_i(r)$ and $\delta K_i(r)$ take the form

$$\lim_{r\to\infty} \delta g_i(r) = \mathcal{O}\left(\frac{1}{r^4}\right), \quad \lim_{r\to\infty} \delta K_i(r) = \mathcal{O}\left(\frac{1}{r^2}\right)$$

3.3.4 Results for Bulk Fields

Without further ado in this subsection we simply present our final results for all the fields defined in the previous subsection.

We have performed all our computations in this section using Mathematica. In several instances we have carried out calculations to higher order, in the Mathematica file, than we have presented below, mainly to avoid burdening the reader with very lengthy expressions.

The solutions presented in this subsection determine the full first order correction to the gauge field, scalar field and metric to the relevant order in an expansion in ϵ and $\frac{1}{e^2}$. Now we choose to scale ζ like ϵ. We present our results below in terms of the order one field

$$\chi = \frac{\zeta}{\epsilon}$$

Recall that, a supefluid in general becomes unstable for high values of superfluid velocities. In the particlular in the perturbation theory that we are considering this instability set in whenever χ exceeds a number of order unity (see [3] for more details). So while χ can be arbitrarily small, it is unphysical for χ to be made arbitrarily large.

Results for the gauge field and scalar field

$$\begin{aligned}
\delta A_1(r) &= \epsilon\left[\frac{r^2\left(96\chi^2-5\right)+48\chi^2+1}{14r^3}\right]+\mathcal{O}(\epsilon^3)\\
\delta A_2(r) &= \epsilon\left[\frac{\left(2-3r^2\right)\chi}{7r^3}\right]+\mathcal{O}(\epsilon^3)\\
\delta A_3(r) &= -\epsilon\left[\frac{\left(2r^2+1\right)\chi}{7r^3}\right]+\mathcal{O}(\epsilon^3)\\
\delta A_4(r) &= \left[\frac{16\left(2r^2+1\right)\chi^2}{7r^3}-\frac{2r}{3\left(r^2+1\right)}\right]+\mathcal{O}(\epsilon^2)\\
\delta A_5(r) &= \epsilon\left[\frac{\left(1-5r^2\right)\chi}{14r^3}\right]+\mathcal{O}(\epsilon^3)\\
\delta A_6(r) &= -\frac{2r}{3\left(r^2+1\right)}-\frac{8\left(2r^2+1\right)\chi^2}{7r^3}+\mathcal{O}(\epsilon^2)
\end{aligned} \tag{3.3.133}$$

$$\begin{aligned}
\delta H_1(r) &= \epsilon\left[\frac{r(r+2)\left(96\chi^2-5\right)-48\chi^2-1}{14r(r+1)\left(r^2+1\right)}\right]+\mathcal{O}(\epsilon^3)\\
\delta H_2(r) &= \epsilon\left[-\frac{\left(3r^2+6r+2\right)\chi}{7r\left(r^3+r^2+r+1\right)}\right]+\mathcal{O}(\epsilon^3)\\
\delta H_3(r) &= \epsilon\left[\frac{\left(5r^2+10r+1\right)\chi}{7r\left(r^3+r^2+r+1\right)}\right]+\mathcal{O}(\epsilon^3)
\end{aligned} \tag{3.3.134}$$

$$\begin{aligned}
\delta H_4(r) &= \frac{16\left(2r^2+4r-1\right)\chi^2}{7r\left(r^3+r^2+r+1\right)} + \mathcal{O}(\epsilon^2) \\
\delta H_5(r) &= \epsilon\left[-\frac{\left(5r^2+10r+1\right)\chi}{14r\left(r^3+r^2+r+1\right)}\right] + \mathcal{O}(\epsilon^3) \\
\delta H_6(r) &= -\frac{8\left[2r(r+2)-1\right]\chi^2}{7r(r+1)\left(r^2+1\right)} + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{3.3.135}$$

$$\begin{aligned}
\delta L_1(r) &= \epsilon^2\left[\frac{\chi\left(\log\left(r^2+1\right)-2\log(r+1)+2\tan^{-1}(r)-\pi\right)}{4r^2}\right] + \mathcal{O}(\epsilon^4) \\
\delta L_2(r) &= \epsilon^2\left[\frac{\log\left(r^2+1\right)-2\log(r+1)+2\tan^{-1}(r)-\pi}{96r^2}\right] + \mathcal{O}(\epsilon^4) \\
\delta L_3(r) &= \mathcal{O}(\epsilon^4) \\
\delta L_4(r) &= -\epsilon\left[\frac{\chi\left(\log\left(r^2+1\right)-4\log(r)+2\log(r+1)-2\tan^{-1}(r)+\pi\right)}{3r^2}\right] \\
&\quad + \mathcal{O}(\epsilon^3) \\
\delta L_5(r) &= \mathcal{O}(\epsilon^4) \\
\delta L_6(r) &= -\epsilon\left[\frac{\chi\left(\log\left(r^2+1\right)-4\log(r)+2\log(r+1)-2\tan^{-1}(r)+\pi\right)}{6r^2}\right] \\
&\quad + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{3.3.136}$$

$$\begin{aligned}
X_1(r) &= \epsilon^2\left[\frac{\log\left(r^2+1\right)-2\log(r+1)+2\tan^{-1}(r)-\pi}{96r^2}\right] + \mathcal{O}(\epsilon^4) \\
X_2(r) &= \epsilon^2\left[\frac{\chi\left(\log\left[\frac{r^2+1}{(r+1)^2}\right]+2\tan^{-1}(r)-\pi\right)}{4r^2}\right] + \mathcal{O}(\epsilon^4) \\
X_3(r) &= \mathcal{O}(\epsilon^4) \\
X_4(r) &= \mathcal{O}(\epsilon^4) \\
X_5(r) &= \epsilon\left[-\frac{\chi\left(\log\left(r^2+1\right)-4\log(r)+2\log(r+1)-2\tan^{-1}(r)+\pi\right)}{4r^2}\right] \\
&\quad + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{3.3.137}$$

$$
\begin{aligned}
\delta\phi_1(r) &= \epsilon^2\Bigg[\frac{3}{14}\left(1-8\chi^2\right)\left(\tan^{-1}(r)-\log(1+r)-\frac{\pi}{2}\right)\\
&\quad+\frac{2}{7}\left(36\chi^2-1\right)\log(r)+\left(\frac{1}{4}-6\chi^2\right)\log\left(r^2+1\right)\Bigg]+\mathcal{O}(\epsilon^4)\\
\delta\phi_2(r) &= \epsilon^2\left[-\frac{1}{28}\chi\left(-7\log\left(r^2+1\right)+4\log(r)+10\log(r+1)-10\tan^{-1}(r)+5\pi\right)\right]\\
&\quad+\mathcal{O}(\epsilon^4)\\
\delta\phi_3(r) &= \epsilon^2\left[\frac{\chi\left(-7\log\left(r^2+1\right)+8\log(r)+6\log(r+1)-6\tan^{-1}(r)+3\pi\right)}{14\left(r^2+1\right)}\right]\\
&\quad+\mathcal{O}(\epsilon^4)\\
\delta\phi_4(r) &= \epsilon\left[\frac{4\chi^2\left(-7\log\left(r^2+1\right)+12\log(r)+2\log(r+1)-2\tan^{-1}(r)+\pi\right)}{7\left(r^2+1\right)}\right]\\
&\quad+\mathcal{O}(\epsilon^3)\\
\delta\phi_5(r) &= \epsilon^2\left[-\frac{\chi}{28}\left(-7\log\left(r^2+1\right)+8\log(r)+6\log(r+1)-6\tan^{-1}(r)+3\pi\right)\right]\\
&\quad+\mathcal{O}(\epsilon^4)\\
\delta\phi_6(r) &= \epsilon\left[-\frac{2}{7}\chi^2\left(-7\log\left(r^2+1\right)+12\log(r)+2\log(r+1)-2\tan^{-1}(r)+\pi\right)\right]\\
&\quad+\mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.138}
$$

Results for the metric

$$
\begin{aligned}
\delta f_1(r) &= \epsilon\left[\frac{-2\left(80\chi^2-3\right)}{7r^4}\right]+\mathcal{O}(\epsilon^3)\\
\delta f_2(r) &= \epsilon\left[\frac{16\chi}{21r^4}\right]+\mathcal{O}(\epsilon^3)\\
\delta f_3(r) &= -\epsilon\left[\frac{12\chi}{7r^4}\right]+\mathcal{O}(\epsilon^3)\\
\delta f_4(r) &= \Bigg[-\frac{320\chi^2}{21r^4}-\frac{\left(r^4+1\right)\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)\right)}{9r^2}\\
&\quad+\frac{2\left(r^4+1\right)\left(r^2\left(\pi r^2-r+\pi-3\right)-2\right)}{9\left(r^6+r^4\right)}\Bigg]+\mathcal{O}(\epsilon^2)\\
\delta f_5(r) &= \epsilon\left[\frac{6\chi}{7r^4}\right]+\mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.139}
$$

$$\delta f_6(r) = -\frac{1}{2}\bigg[-\frac{320\chi^2}{21r^4} - \frac{\left(r^4+1\right)\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)\right)}{9r^2} + \frac{2\left(r^4+1\right)\left(r^2\left(\pi r^2-r+\pi-3\right)-2\right)}{9\left(r^6+r^4\right)}\bigg] + \mathcal{O}(\epsilon^2) \tag{3.3.140}$$

$$\delta g_1(r) = \mathcal{O}(\epsilon^3)$$
$$\delta g_2(r) = \mathcal{O}(\epsilon^3)$$
$$\delta g_3(r) = \mathcal{O}(\epsilon^3)$$

$$\delta g_4(r) = \bigg[\frac{1}{18}\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)-2\pi\right) + \frac{r^6+4r^5+4r^4+6r^3+r^2-2r-2}{9(r+1)\left(r^3+r\right)^2}\bigg] + \mathcal{O}(\epsilon^2) \tag{3.3.141}$$

$$\delta g_5(r) = \mathcal{O}(\epsilon^3)$$
$$\delta g_6(r) = -\frac{1}{2}\bigg[\frac{1}{18}\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)-2\pi\right) + \frac{r^6+4r^5+4r^4+6r^3+r^2-2r-2}{9(r+1)\left(r^3+r\right)^2}\bigg] + \mathcal{O}(\epsilon^2)$$

$$\delta K_1(r) = \mathcal{O}(\epsilon^3)$$
$$\delta K_2(r) = \mathcal{O}(\epsilon^3)$$
$$\delta K_3(r) = \mathcal{O}(\epsilon^4)$$

$$\delta K_4(r) = \bigg[\frac{r^2}{3}\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)\right) + \frac{4-2r^2\left(\pi r^2-r+\pi-3\right)}{3\left(r^2+1\right)}\bigg] + \mathcal{O}(\epsilon^2) \tag{3.3.142}$$

$$\delta K_5(r) = \mathcal{O}(\epsilon^3)$$
$$\delta K_6(r) = -\frac{1}{2}\bigg[\frac{r^2}{3}\left(-5\log\left(r^2+1\right)+8\log(r)+2\log(r+1)+4\tan^{-1}(r)\right) + \frac{4-2r^2\left(\pi r^2-r+\pi-3\right)}{3\left(r^2+1\right)}\bigg] + \mathcal{O}(\epsilon^2)$$

$$
\begin{aligned}
\delta J_1(r) &= -\epsilon^2\Bigg[\frac{\left(r^4-1\right)\chi}{4r^2}\left(\log\left(r^2+1\right)-2\log(r+1)-2\tan^{-1}(r)+\pi\right)\\
&\quad+\frac{\chi}{6r^2}(2-3r)\Bigg]+\mathcal{O}(\epsilon^4)\\
\delta J_2(r) &= -\epsilon^2\Bigg[-\frac{\left(r^4-1\right)\left(-\log\left(r^2+1\right)+2\log(r+1)-16\tan^{-1}(r)+8\pi\right)}{96r^2}\\
&\quad+\frac{27r^4-3r^3+20r^2-3r-19}{144\left(r^3+r\right)}\Bigg]+\mathcal{O}(\epsilon^4)\\
\delta J_3(r) &= -\epsilon^2\left(-\frac{r}{6\left(r^2+1\right)^2}\right)+\mathcal{O}(\epsilon^3)\\
\delta J_4(r) &= \mathcal{O}(\epsilon^3)\\
\delta J_5(r) &= \mathcal{O}(\epsilon^3)\\
\delta J_6(r) &= \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.143}
$$

$$
\begin{aligned}
Y_1(r) &= -\epsilon^2\Bigg[\frac{\left(r^4-1\right)}{96\ r^2}\left[-\log\left(r^2+1\right)+2\log(r+1)-16\tan^{-1}(r)+8\pi\right]\\
&\quad+\frac{27r^4-3r^3+20r^2-3r-19}{144r\left(r^2+1\right)}\Bigg]+\mathcal{O}(\epsilon^4)\\
Y_2(r) &= -\epsilon^2\Bigg[\frac{\left(r^4-1\right)\chi}{4r^2}\left(\log\left(r^2+1\right)-2\log(r+1)-2\tan^{-1}(r)+\pi\right]\\
&\quad+\frac{\chi}{6\ r}(2-3r)\Bigg]+\mathcal{O}(\epsilon^4)\\
Y_3(r) &= \mathcal{O}(\epsilon^4)\\
Y_4(r) &= \epsilon^2\left[\frac{r}{6\left(r^2+1\right)^2}\right]+\mathcal{O}(\epsilon^4)\\
Y_5(r) &= \ \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.144}
$$

$$
\begin{aligned}
W_1(r) &= \epsilon^3\ \chi\left[\frac{-3\pi\left(r^3+r\right)+6\left(r^3+r\right)\tan^{-1}(r)+6r^2+4}{16\left(r^3+r\right)}\right]+\mathcal{O}(\epsilon^5)\\
W_2(r) &= \epsilon^3\left[\frac{-3\pi\left(r^3+r\right)+6\left(r^3+r\right)\tan^{-1}(r)+6r^2+4}{32\left(r^3+r\right)}\right]+\mathcal{O}(\epsilon^5)\\
W_3(r) &= \epsilon^3\left[\frac{\chi\left[-3\pi\left(r^3+r\right)+6\left(r^3+r\right)\tan^{-1}(r)+6r^2+4\right]}{32\left(r^3+r\right)}\right]+\mathcal{O}(\epsilon^4)\\
W_4(r) &= \epsilon^3\left[\frac{\chi\left[-3\pi\left(r^3+r\right)+6\left(r^3+r\right)\tan^{-1}(r)+6r^2+4\right]}{32\left(r^3+r\right)}\right]+\mathcal{O}(\epsilon^4)\\
W_5(r) &= -\frac{r^2}{6}\Bigg[-\frac{2(r+1)}{r^2+1}-\frac{4}{r^2}+5\log\left(r^2+1\right)-8\log(r)-2\log(r+1)\\
&\quad-4\tan^{-1}(r)+2\pi\Bigg]+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{3.3.145}
$$

$$
\begin{aligned}
Z_1(r) = & \frac{r^2}{3}\left[\frac{2(r+1)}{r^2+1} + \frac{4}{r^2} - 5\log\left(r^2+1\right) + 8\log(r)\right. \\
& \left. + 2\log(r+1) + 4\tan^{-1}(r) - 2\pi\right] + \mathcal{O}(\epsilon^2) \\
Z_2(r) = & \ \epsilon^3\ \chi\left[\frac{-3\pi\left(r^3+r\right) + 6\left(r^3+r\right)\tan^{-1}(r) + 6r^2+4}{16\left(r^3+r\right)}\right] + \mathcal{O}(\epsilon^4)
\end{aligned}
\tag{3.3.146}
$$

3.3.5 The $\zeta \to 0$ limit

The gravitational solutions presented above are complicated largely because they possess very little rotational symmetry. At any given spacetime point we have a normal fluid velocity and an independent superfluid velocity. These two velocities together break the local Lorentz group at a point down to the abelian group $SO(2)$. While we have usefully organized the results of our gravitational calculation in representations of $SO(2)$, as representations of $SO(2)$ are all one dimensional, our solutions admit several different functions of r.

In the special case that $\zeta = 0$, however, the residual symmetry group about a point is $SO(3)$. $SO(3)$ representation theory is considerably more constraining than $SO(2)$ representation theory. This implies that the gravitational dual to superfluid dynamics should be considerably simpler in the special limit $\zeta \to 0$ than in the generic case.

Let us first present a brief ab initio analysis of the nature of the gravitational solution when $\zeta = 0$. All independent first derivative data may be organized into $SO(3)$ scalars, vectors and tensors. These may be chosen as follows

Scalar

$$\partial_\mu u^\mu \ \text{ and } \ P^{\mu\nu}\partial_\mu\zeta_\nu$$

Vector

$$u^\mu\partial_\mu u^\nu, \ \ P^{\mu\nu}\partial_\nu\epsilon \ \text{ and } \ \epsilon^{\mu\nu\lambda\sigma}u_\nu\partial_\lambda\zeta_\sigma$$

Tensor

$$\sigma_{\mu\nu} \text{ and } \sigma^{(\zeta)}_{\mu\nu} = P^\alpha_\mu P^\beta_\nu\left(\frac{\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha}{2} - \left[\frac{P^{\theta_1\theta_2}\partial_{\theta_1}\zeta_{\theta_2}}{3}\right]\eta_{\alpha\beta}\right)$$

Note of course that an $SO(3)$ vector or an $SO(3)$ may be decomposed into an $SO(2)$ vector and a scalar, while an $SO(3)$ tensor is composed of an $SO(2)$ tensor, vector and scalar. In $SO(2)$ terms, therefore, the data listed above totals to 7 scalars, 5 vectors and two tensor.

It follows from symmetry considerations (and the fact that our parity conserving gravitational system will never generate a parity violating vector term, so we can ignore the third vector above) that it must be possible to write the metric and gauge field, in the $\zeta \to 0$

limit, in the form

$$
\begin{aligned}
ds^2 =& -2g\left(\frac{r}{r_c}\right)u^\mu dx^\mu dr + \left[-r_c^2 f\left(\frac{r}{r_c}\right)u_\mu u_\nu + r^2 P_{\mu\nu}\right]dx^\mu dx^\nu \\
&+ r_c^2 F\left(\frac{r}{r_c}\right)\sigma_{\mu\nu}dx^\mu dx^\nu \\
&+ \frac{1}{e^2}\Bigg\{-2\left[\mathcal{G}_1\left(\frac{r}{r_c}\right)(\partial_\mu u^\mu) + \mathcal{G}_2\left(\frac{r}{r_c}\right)(P^{\mu\nu}\partial_\mu\zeta_\nu)\right]u_\mu dx^\mu dr \\
&+ r_c^2\left[\mathcal{F}_1\left(\frac{r}{r_c}\right)(\partial_\mu u^\mu) + \mathcal{F}_2\left(\frac{r}{r_c}\right)(P^{\mu\nu}\partial_\mu\zeta_\nu)\right]u_\mu u_\nu dx^\mu dx^\nu \\
&+ r_c^2\left[\mathcal{V}_1\left(\frac{r}{r_c}\right)(u.\partial)\,u_\nu + \mathcal{V}_2\left(\frac{r}{r_c}\right)P_\nu^\alpha\partial_\alpha\epsilon\right]u_\mu dx^\mu dx^\nu \\
&+ r_c^2\left[\mathcal{T}_1\left(\frac{r}{r_c}\right)\sigma_{\mu\nu} + \mathcal{T}_2\left(\frac{r}{r_c}\right)\sigma_{\mu\nu}^{(\zeta)}\right]dx^\mu dx^\nu\Bigg\} + \mathcal{O}\left(\frac{1}{e^4}\right) \\
A =& \frac{1}{r_c}H\left(\frac{r}{r_c}\right)u^\mu\partial_\mu \\
&+ \left[\mathcal{A}_1\left(\frac{r}{r_c}\right)(\partial_\mu u^\mu) + \mathcal{A}_2\left(\frac{r}{r_c}\right)(P^{\mu\nu}\partial_\mu\zeta_\nu)\right]\partial_r \\
&+ \frac{1}{r_c^2}\left[\mathcal{H}_1\left(\frac{r}{r_c}\right)(\partial_\mu u^\mu) + \mathcal{H}_2\left(\frac{r}{r_c}\right)(P^{\mu\nu}\partial_\mu\zeta_\nu)\right]u^\mu\partial_\mu \\
&+ \frac{1}{r_c^2}\mathcal{L}_1\left(\frac{r}{r_c}\right)(u.\partial)\,u^\mu\partial_\mu + \frac{1}{r_c^2}\mathcal{L}_2\left(\frac{r}{r_c}\right)P^{\mu\nu}\partial_\nu\epsilon\partial_\mu + \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned}
\tag{3.3.147}
$$

The results of the previous subsection must obey several relations in the limit $\zeta \to 0$ for them to agree with the form presented in (3.3.147). [13]. We have explicitly verified that each required relation is indeed obeyed. Our gravity solution is consistent with the form (3.3.147) once we make the identifications

$$
\begin{aligned}
\mathcal{V}_1(r) &= \epsilon^2\left[\frac{r}{3\left(r^2+1\right)^2}\right] + \mathcal{O}(\epsilon^4) \\
\mathcal{V}_2(r) &= \mathcal{O}(\epsilon^4) \\
\mathcal{T}_1(r) &= -\frac{r^2}{3}\Bigg[-\frac{2(r+1)}{r^2+1} - \frac{4}{r^2} + 5\log\left(r^2+1\right) - 8\log(r) - 2\log(r+1) \\
&\qquad - 4\tan^{-1}(r) + 2\pi\Bigg] + \mathcal{O}(\epsilon^2) \\
\mathcal{T}_2(r) &= \mathcal{O}(\epsilon^3) \\
\mathcal{G}_1(r) &= 0, \quad \text{(Required by Weyl invariance)} \\
\mathcal{F}_1(r) &= \mathcal{O}(\epsilon^2) \\
\mathcal{G}_2(r) &= \mathcal{O}(\epsilon^3) \\
\mathcal{F}_2(r) &= \frac{6\epsilon}{7r^4} + \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.148}
$$

[13] A direct comparison between these two forms is complicated by an irritating feature; the coordinate choice of the previous subsection differs from the one above (it breaks manifest $SO(3)$ invariance) even in the limit $\zeta \to 0$. We have explicitly performed the coordinate change that allows one to transform the results between coordinates.

and

$$
\begin{aligned}
\mathcal{A}_1(r) &= -\frac{2r}{3(r^2+1)} + \mathcal{O}(\epsilon^2) \\
\mathcal{A}_2(r) &= \frac{\epsilon}{14r^3} + \mathcal{O}(\epsilon^3) \\
\mathcal{H}_1(r) &= \mathcal{O}(\epsilon^2) \\
\mathcal{H}_2(r) &= \epsilon\left[\frac{-5r(r+2)-1}{14r(r+1)\left(r^2+1\right)}\right] + \mathcal{O}(\epsilon^3) \\
\mathcal{L}_1(r) &= \mathcal{O}(\epsilon^4) \\
\mathcal{L}_2(r) &= \epsilon^2\left[\frac{\log\left(r^2+1\right) - 2\log(r+1) + 2\tan^{-1}(r) - \pi}{96r^2}\right] + \mathcal{O}(\epsilon^4)
\end{aligned}
\tag{3.3.149}
$$

3.3.6 Stress Tensor, Charge current and the Josephson Equation

The results of the previous subsection may be used to read off the values of the boundary stress tensor, the boundary current and the correction to the Josephson equation at first order in the derivative expansion. Like all the calculations in this paper our results are obtained in a power series expansion in ϵ and $\frac{1}{e^2}$.

We parameterize our boundary stress tensor and current as

$$
\begin{aligned}
T^{\mu\nu} &= \frac{1}{16\pi G}\left[Au^\mu u^\nu + Bn^\mu n^\nu + C\left(n^\mu u^\nu + u^\mu n^\nu\right) + \left(\frac{A-B}{2}\right)\tilde{P}^{\mu\nu}\right] + \tilde{T}^{\mu\nu}_{diss} \\
J^\mu &= \frac{1}{16\pi G}\left[Q_1 u^\mu + Q_2 n^\mu\right] + \tilde{J}^\mu_{diss}
\end{aligned}
\tag{3.3.150}
$$

where A, B, C, Q_1 and Q_2 are functions of $\epsilon(x)$, $\zeta(x)$ and $r_c(x)$ as given in equations (3.2.116) and (3.2.117). We further expand the corrections to the perfect fluid stress tensor and current as

$$
\begin{aligned}
16\pi G(\tilde{T}_{diss})_{\mu\nu} &= -2r_c^3\,\sigma_{\mu\nu} + \frac{1}{e^2}\Bigg[r_c^3\sum_{i=1}^{7} S_i P_i\left(\,n_\mu n_\nu - \frac{1}{2}\tilde{P}_{\mu\nu}\right) \\
&\quad + r_c^3\sum_{i=1}^{5} v_i\left(n_\mu [V_i]_\nu + n_\nu [V_i]_\mu\right) + r_c^3\sum_{i=1}^{2} t_i\ [T_i]_{\mu\nu}\Bigg] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
16\pi G\tilde{J}^\mu_{diss} &= \frac{r_c^2}{e^2}\sum_{i=1}^{7} S_i\left(a_i u^\mu + b_i n^\mu\right) + \frac{r_c^2}{e^2}\sum_{i=1}^{5} c_i\ [V_i]^\mu + \mathcal{O}\left(\frac{1}{e^4}\right) \\
\mu_{diss} &= \sum_{i=1}^{7}\delta\mu_i\ S_i + \mathcal{O}\left(\frac{1}{e^2}\right)
\end{aligned}
\tag{3.3.151}
$$

Our results are given as follows.

Results for stress tensor:

$$\begin{aligned}
&P_1 = \mathcal{O}(\epsilon^4), \quad P_2 = \mathcal{O}(\epsilon^4), \quad P_3 = \mathcal{O}(\epsilon^4), \quad P_4 = \mathcal{O}(\epsilon^3), \quad P_5 = \mathcal{O}(\epsilon^4), \quad P_6 = \mathcal{O}(\epsilon^3)\\
&v_1 = \mathcal{O}(\epsilon^5), \quad v_2 = \mathcal{O}(\epsilon^5), \quad v_3 = \mathcal{O}(\epsilon^5), \quad v_4 = \mathcal{O}(\epsilon^5), \quad v_5 = \mathcal{O}(\epsilon^4)\\
&t_1 = \mathcal{O}(\epsilon^4), \quad t_2 = \mathcal{O}(\epsilon^4)
\end{aligned} \tag{3.3.152}$$

Results for current:

$$\begin{aligned}
a_1 &= \epsilon \left[\frac{3}{7}\left(3 - 80\ \chi^2\right)\right] + \mathcal{O}(\epsilon^3), \quad b_1 = \epsilon^2\ \chi + \mathcal{O}(\epsilon^4)\\
a_2 &= \epsilon \left(\frac{8\ \chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad b_2 = -\epsilon^2 \left[-\frac{1}{24}\right] + \mathcal{O}(\epsilon^4)\\
a_3 &= -\epsilon \left(\frac{18\chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad b_3 = \mathcal{O}(\epsilon^4)\\
a_4 &= -\left(\frac{160\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2), \quad b_4 = \mathcal{O}(\epsilon^4)\\
a_5 &= \epsilon \left(\frac{9\ \chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad b_5 = \mathcal{O}(\epsilon^4)\\
a_6 &= \left(\frac{80\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2), \quad b_6 = \mathcal{O}(\epsilon^4)
\end{aligned} \tag{3.3.153}$$

$$\begin{aligned}
c_1 &= \frac{\epsilon^2}{24} + \mathcal{O}(\epsilon^4)\\
c_2 &= \epsilon^2\ \chi + \mathcal{O}(\epsilon^4)\\
c_3 &= \mathcal{O}(\epsilon^4)\\
c_4 &= \mathcal{O}(\epsilon^4)\\
c_5 &= \epsilon^3\ \chi \left(\frac{-1 + 2\log(2)}{16}\right) + \mathcal{O}(\epsilon^4)
\end{aligned}$$

Results for the correction to the Josephson equation:

$$\begin{aligned}
\delta\mu_1 &= \epsilon \left[\frac{1}{14}\left(5 - 96\ \chi^2\right)\right] + \mathcal{O}(\epsilon^3)\\
\delta\mu_2 &= \epsilon \left(\frac{3\ \chi}{7}\right) + \mathcal{O}(\epsilon^3)\\
\delta\mu_3 &= -\epsilon \left(\frac{5\chi}{7}\right) + \mathcal{O}(\epsilon^3)\\
\delta\mu_4 &= -\left(\frac{32\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2)\\
\delta\mu_5 &= \epsilon \left(\frac{5\ \chi}{14}\right) + \mathcal{O}(\epsilon^3)\\
\delta\mu_6 &= \left(\frac{16\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{3.3.154}$$

In $\zeta \to 0$ limit this derivative corrections to stress tensor, charge current and the phase equation take the following form

$$\begin{aligned}
&\lim_{\zeta\to 0} \tilde{T}^{\mu\nu}_{diss} = -2r_c^3\sigma^{\mu\nu} + \mathcal{O}(\epsilon^3)\\
&\lim_{\zeta\to 0} \tilde{J}^{\mu}_{diss} = r_c^2\left\{\alpha_1 u^\mu\left[P^{ab}\partial_a\zeta_b\right] + \alpha_2 P^{\mu\nu}\partial_\nu\epsilon\right\}\\
&\lim_{\zeta\to 0} \tilde{\mu}_{diss} = \alpha_3\left[P^{ab}\partial_a\zeta_b\right]
\end{aligned} \tag{3.3.155}$$

where

$$\alpha_1 = \frac{9}{7} + \mathcal{O}(\epsilon^3), \quad \alpha_2 = \frac{\epsilon}{24} + \mathcal{O}(\epsilon^4), \quad \alpha_3 = \frac{5}{14} + \mathcal{O}(\epsilon^3)$$

3.3.7 Weyl Covariance of our bulk fields and boundary currents

In this section we will demonstrate that our fluid dynamical solutions must, on general grounds, obey certain constraints that follow from the requirement of Weyl invariance. We then verify that our explicit solution does indeed obey these constraints, providing a nontrivial check on these solutions.

All computations reported in this paper have been performed for superfluid motion on a flat boundary metric. However our final results must be the restriction to a flat boundary of results that apply in a general weakly curved space. The (boundary) generally covariant version of our final bulk metric, stress tensor etc are all given simply by promoting all derivatives to covariant derivatives (ambiguities in this procedure and boundary curvature terms all show up only at second order in the derivative expansion).

Given these results in a general boundary spacetime, it follows on general grounds (see [10]) that our bulk metric, gauge field and scalar fields must enjoy invariance under the following spacetime dependent Weyl transformations and coordinate redefinitions.

$$\begin{aligned}
&\tilde{r} = r\mathrm{e}^{\psi(v,x_i)}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu}\mathrm{e}^{-2\psi(v,x_i)}\\
&\tilde{u}_\mu = u_\mu\mathrm{e}^{-\psi(v,x_i)}, \quad \tilde{n}_\mu = n_\mu\mathrm{e}^{-\psi(v,x_i)}, \quad \tilde{\zeta} = \zeta, \quad \tilde{\epsilon} = \epsilon
\end{aligned}$$

Note that the Weyl transformed metric $\tilde{g}_{\mu\nu}$ is, in general, not flat even if the original metric is. Let us work in the special case that the original metric $g_{\mu\nu}$ is taken to be flat. The boundary connection with respect to $\tilde{g}_{\mu\nu}$ is non zero and is given by

$$\tilde{\Gamma}^\sigma_{\mu\nu} = -\left(\delta^\sigma_\mu\partial_\nu\psi + \delta^\sigma_\nu\partial_\mu\psi - \eta_{\mu\nu}\partial^\sigma\psi\right)$$

The new frame covariant derivatives of u_μ and n_μ are given by

$$\begin{aligned}
\tilde{\nabla}_\mu\tilde{u}_\nu &= \mathrm{e}^{-\psi}\left[\partial_\mu u_\nu + u_\mu\partial_\nu\psi - \eta_{\mu\nu}(u.\partial)\psi\right]\\
\tilde{\nabla}_\mu\tilde{n}_\nu &= \mathrm{e}^{-\psi}\left[\partial_\mu n_\nu + n_\mu\partial_\nu\psi - \eta_{\mu\nu}(n.\partial)\psi\right]
\end{aligned}$$

Using these expressions one can deduce the transformation properties of the scalar, vector and the tensor forms appearing in the bulk solution

$$
\begin{aligned}
&\tilde{S}_1 = \mathrm{e}^{\psi} S_1, \quad \tilde{S}_2 = \mathrm{e}^{\psi} S_2 \\
&\tilde{S}_3 = \mathrm{e}^{\psi}\left[S_3 - (n.\partial)\psi\right], \quad \tilde{S}_4 = \mathrm{e}^{\psi}\left[S_4 - (u.\partial)\psi\right], \\
&\tilde{S}_5 = \mathrm{e}^{\psi}\left[S_5 - 2(n.\partial)\psi\right], \quad \tilde{S}_6 = \mathrm{e}^{\psi}\left[S_6 - 2(u.\partial)\psi\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left[\tilde{V}_1\right]_\mu = [V_1]_\mu, \quad \left[\tilde{V}_2\right]_\mu = [V_2]_\mu \\
&\left[\tilde{V}_3\right]_\mu = [V_3]_\mu + \tilde{P}^\nu_\mu \partial_\nu \psi, \quad \left[\tilde{V}_4\right]_\mu = [V_4]_\mu - \tilde{P}^\nu_\mu \partial_\nu \psi, \quad \left[\tilde{V}_5\right]_\mu = [V_5]_\mu
\end{aligned}
\tag{3.3.156}
$$

$$
\left[\tilde{T}_1\right]_{\mu\nu} = \mathrm{e}^{-\psi}\ [T_1]_{\mu\nu}, \quad \left[\tilde{T}_2\right]_{\mu\nu} = \mathrm{e}^{-\psi}\ [T_2]_{\mu\nu}
$$

If we transform the gauge field and the metric from the new Weyl frame (the frame with tilde variables) to the old Weyl frame (the frame where the variables are denoted without tilde), the equilibrium solution itself generates some new terms due to the r coordinate redefinition. In the new frame the coordinates are $\tilde{r} = r\mathrm{e}^{\psi(v,x_i)}$ and $\tilde{x}^\mu = x^\mu$. This implies the following transformation rule for the differentials.

$$
\begin{aligned}
d\tilde{r} &= \mathrm{e}^{\psi(v,x_i)}\left(dr + r dx^\mu \partial_\mu \psi\right) \\
d\tilde{x}^\mu &= dx^\mu \\
\tilde{\partial}_\mu &= \frac{\partial r}{\partial \tilde{x}_\mu}\partial_r + \partial_\mu \\
&= \left[r\mathrm{e}^{\psi}\partial_\mu \psi\right]\partial_r + \partial_\mu
\end{aligned}
\tag{3.3.157}
$$

This induces the following transformations on gauge field

$$
\begin{aligned}
\tilde{A} &= \frac{1}{\tilde{r}_c}\left[H\left(\frac{\tilde{r}}{\tilde{r}_c}\right)(\tilde{u}.\tilde{\partial}) + L\left(\frac{\tilde{r}}{\tilde{r}_c}\right)(\tilde{n}.\tilde{\partial})\right] \\
&= -\frac{r\mathrm{e}^{\psi}}{r_c}\left[H\left(\frac{r}{r_c}\right)(u.\partial\psi) + L\left(\frac{r}{r_c}\right)(n.\partial\psi)\right]\partial_r \\
&+ \frac{1}{r_c}\left[H\left(\frac{r}{r_c}\right)(u.\partial) + L\left(\frac{r}{r_c}\right)(n.\partial)\right]
\end{aligned}
\tag{3.3.158}
$$

$$
= -r\mathrm{e}^{\psi}\left[H(r)(u.\partial\psi) + L(r)(n.\partial\psi)\right]\partial_r + r\left[H(r)(u.\partial) + L(r)(n.\partial)\right]
$$

In the last line we have used the scaling symmetry to set $r_c = 1$.

Similarly the equilibrium metric also transforms and the nontrivial transformation is generated due to the term $dx^\mu dr$.

$$
\begin{aligned}
&-2g\left(\frac{\tilde{r}}{\tilde{r}_c}\right)\tilde{u}_\mu dx^\mu d\tilde{r} = -2g\left(\frac{r}{r_c}\right)u_\mu\; dx^\mu\;\; (dr + r\partial_\nu\psi dx^\nu)\\
&= -2g\left(\frac{r}{r_c}\right)u_\mu dx^\mu dr - 2rg\left(\frac{r}{r_c}\right)u_\mu u_\nu (u.\partial)\psi dx^\mu dx^\nu\\
&- rg\left(\frac{r}{r_c}\right)\left[(u_\mu n_\nu + u_\nu n_\mu)(n.\partial)\psi + \left(u_\mu \tilde{P}^\sigma_\nu + u_\nu \tilde{P}^\sigma_\mu\right)\partial_\sigma\psi\right]dx^\mu dx^\nu
\end{aligned}
\tag{3.3.159}
$$

$$
\begin{aligned}
&= -2g\,(r)\,u_\mu dx^\mu dr - 2rg\,(r)\,u_\mu u_\nu (u.\partial)\psi dx^\mu dx^\nu\\
&- rg\,(r)\left[(u_\mu n_\nu + u_\nu n_\mu)(n.\partial)\psi + \left(u_\mu \tilde{P}^\sigma_\nu + u_\nu \tilde{P}^\sigma_\mu\right)\partial_\sigma\psi\right]dx^\mu dx^\nu
\end{aligned}
$$

Here also in the last step the scaling symmetry is used to set r_c =1

Combining these transformations we find the transformed metric, gauge field and scalar have the expected form (expected according to (3.3.7)) together with some additional pieces that multiply a single derivative of ψ. The coefficients of these unwanted pieces themselves have no derivatives, and must vanish in order that our result respect Weyl invariance. This requirement imposes the following simple algebraic conditions on the fields in the metric, scalar and gauge field:

$$
\begin{aligned}
&\delta A_3(r) + 2\;\delta A_5(r) - rL(r) = 0\\
&\delta A_4(r) + 2\;\delta A_6(r) - rH(r) = 0\\
&\delta H_3(r) + 2\;\delta H_5(r) = 0, \quad \delta H_4(r) + 2\;\delta H_6(r) = 0\\
&\delta L_3(r) + 2\;\delta L_5(r) = 0, \quad \delta L_4(r) + 2\;\delta L_6(r) = 0\\
&\delta\phi_3(r) + 2\;\delta\phi_5(r) = 0, \quad \delta\phi_4(r) + 2\;\delta\phi_6(r) = 0\\
&X_3(r) - X_4(r) = 0
\end{aligned}
\tag{3.3.160}
$$

$$
\begin{aligned}
&\delta f_3(r) + 2\;\delta f_5(r) = 0\\
&\delta f_4(r) + 2\;\delta f_6(r) + 2rg(r) = 0\\
&\delta J_3(r) + 2\;\delta J_5(r) + rg(r) = 0, \quad \delta J_4(r) + 2\;\delta J_6(r) = 0\\
&\delta K_3(r) + 2\;\delta K_5(r) = 0, \quad \delta K_4(r) + 2\;\delta K_6(r) = 0\\
&Y_3(r) - Y_4(r) - rg(r) = 0, \quad W_3(r) - W_4(r) = 0
\end{aligned}
\tag{3.3.161}
$$

We also require that the stress tensor, charge current and Josephson equation in our model are invariant under Weyl transformations. As these boundary quantities are all independent of r, the redefinition of r is irrelevant to the study of Weyl transformations of these quantities. Using only the equations (3.3.156) we find the following constraints on the coefficients in

(3.3.151)

$$
\begin{aligned}
&P_3 + 2\ P_5 = P_4 + 2\ P_6 = 0 \\
&v_3 - v_4 = 0 \\
\\
&a_3 + 2\ a_5 = a_4 + 2\ a_6 = 0 \\
&b_3 + 2\ b_5 = b_4 + 2\ b_6 = 0 \\
&c_3 - c_4 = 0 \\
&\delta\mu_3 + 2\ \delta\mu_5 = \delta\mu_4 + 2\ \delta\mu_6 = 0
\end{aligned}
\tag{3.3.162}
$$

The equations (3.3.161), (3.3.160) and (3.3.162) must apply to any consistent asymptotically AdS solution of gravitational equations. In particular these equations must apply to the results of this paper, and constitute a nontrivial consistency check on our algebra. We have explicitly checked that the results of our solutions obey these constraints, to the calculated order in ϵ and $\frac{1}{e^2}$.

3.3.8 Entropy Current from Gravity

Fluid flows obtained from the fluid gravity correspondence are automatically equipped with families of local entropy currents of positive divergence. A particularly natural choice for this entropy current was presented in equation 3.11 of [9]. Using this formula for our solution we have computed the entropy current dual to our fluid flow. This entropy current has a piece at $\mathcal{O}(1)$ and a piece at $\mathcal{O}(1/e^2)$, and takes the form

$$
4GJ_s^\mu = r_c^3 u^\mu + \frac{r_c^2}{e^2}\sum_{i=1}^{7} S_i \left(\kappa_i^{(u)} u_\mu + \kappa_i^{(n)} n_\mu\right) + \frac{r_c^2}{e^2}\sum_{i=1}^{5} \kappa_i^{(v)}\ [V_i]_\mu + \mathcal{O}\left(\frac{1}{e^4}\right) \tag{3.3.163}
$$

where

$$
\begin{aligned}
\kappa_1^{(u)} &= \frac{1}{2}\epsilon\left[\frac{3}{7}\left(3 - 80\ \chi^2\right)\right] + \mathcal{O}(\epsilon^3), \quad \kappa_1^{(n)} = \frac{1}{2}\epsilon^2\ \chi + \mathcal{O}(\epsilon^3) \\
\kappa_2^{(u)} &= \frac{1}{2}\epsilon\left(\frac{8\ \chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad \kappa_2^{(n)} = -\frac{1}{2}\epsilon^2\left[-\frac{1}{24}\right] + \mathcal{O}(\epsilon^3) \\
\kappa_3^{(u)} &= -\frac{1}{2}\epsilon\left(\frac{18\chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad \kappa_3^{(n)} = \mathcal{O}(\epsilon^3) \\
\kappa_4^{(u)} &= -\frac{1}{2}\left(\frac{160\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2), \quad \kappa_4^{(n)} = \mathcal{O}(\epsilon^3) \\
\kappa_5^{(u)} &= \frac{1}{2}\epsilon\left(\frac{9\ \chi}{7}\right) + \mathcal{O}(\epsilon^3), \quad \kappa_5^{(n)} = \mathcal{O}(\epsilon^3) \\
\kappa_6^{(u)} &= \frac{1}{2}\left(\frac{80\ \chi^2}{7}\right) + \mathcal{O}(\epsilon^2), \quad \kappa_6^{(n)} = \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{3.3.164}
$$

$$\begin{aligned}
\kappa_1^{(v)} &= \frac{1}{2}\frac{\epsilon^2}{24} + \mathcal{O}(\epsilon^3) \\
\kappa_2^{(v)} &= \frac{1}{2}\epsilon^2\ \chi + \mathcal{O}(\epsilon^3) \\
\kappa_3^{(v)} &= \mathcal{O}(\epsilon^3) \\
\kappa_4^{(v)} &= \mathcal{O}(\epsilon^3) \\
\kappa_5^{(v)} &= \mathcal{O}(\epsilon^3)
\end{aligned} \tag{3.3.165}$$

Quite remarkably this gravitational entropy current agrees exactly with the simple fluid dynamical current described in (2.37) (to the order at which we have done the calculation).

3.3.9 Transformation to transverse frame and identification of the dissipative parameters

The equations of gravitational super fluid dynamics, derived in the previous subsection are presented in a frame (which is not transverse frame) that is adapted to the expectation value of the operatore $\epsilon(x)$, and is not particularly natural from a fluid dynamical point of view.

In this subssection we will transform our results to the transverse frame. We will then compare these results with the general 'theory' of dissipative dynamics presented in chapter 2. We will find perfect agreement with that general structure, and so be able to read off the values of all 10 nonzero dissipative fluid parameters in (2.63).

The basis of first derivative quantities with non zero coefficients most suitable for this frame is given by

$$\begin{aligned}
&S_a = \partial_\mu\left(\frac{q_s}{\xi}\xi^\mu\right); \quad S_b = (n^\mu\partial_\mu)\left(\frac{\mu}{T}\right); \quad S_w = n^\mu n^\nu \sigma_{\mu\nu}; \\
&V_a^\mu = \tilde{P}^{\sigma\mu}\partial_\sigma\left(\frac{\mu}{T}\right); \quad V_b^\mu = \tilde{P}^{\alpha\mu}\sigma_{\alpha\nu}n^\nu; \quad \mathcal{T}^{\mu\nu} = \tilde{P}^{\alpha\mu}\tilde{P}^{\beta\nu}\sigma_{\mu\nu}
\end{aligned} \tag{3.3.166}$$

These quantities may be expressed in term of the quantities (defined in (3.3.127)) used for the gravity calculation as follows

$$\begin{aligned}
S_a &= \frac{1}{16\pi G\ e^2}\left[\epsilon^2\frac{8\chi^2}{7}(2S_4 - S_6) + \epsilon^3\left(\frac{-5+96\chi^2}{28}\right)S_1 - \epsilon^3\frac{3\chi}{14}S_2 + \epsilon^3\frac{5\chi}{28}(2S_3 - S_5) + \mathcal{O}(\epsilon^4)\right]; \\
S_b &= -\epsilon^2\chi S_1 - \frac{\epsilon^2}{24}S_2 + \mathcal{O}(\epsilon^4); \quad S_w = \frac{2S_4 - S_6}{3}; \\
V_a^\mu &= -\epsilon^2\chi V_2^\mu - \frac{\epsilon^2}{24}V_1^\mu + \mathcal{O}(\epsilon^4); \quad V_b^\mu = \frac{V_5^\mu}{2}; \quad \mathcal{T}^{\mu\nu} = \frac{T_1^{\mu\nu}}{2}
\end{aligned} \tag{3.3.167}$$

Let us rewrite the first derivative corrections to charge current obtained from gravity (given in (3.3.151) and (3.3.153)) in the following schematic form

$$\tilde{J}^{\mu}_{diss} = \left(\frac{1}{16\pi G}\right)\frac{r_c^2}{e^2}\left(\tilde{j}_u u^{\mu} + \tilde{j}_n n^{\mu} + \sum_i c_i [V_i]^{\mu}\right). \tag{3.3.168}$$

In the gravity solution the stress tensor ($\tilde{T}^{\mu\nu}_{diss}$) is given as the following (see (3.3.152)).

$$16\pi G\ \tilde{T}^{\mu\nu}_{diss} = \left[-2r_c^3\sigma^{\mu\nu} + \mathcal{O}(\epsilon^3)\right] + \mathcal{O}\left(\frac{1}{e^4}\right)$$

We can now compute first derivative corrections to stress tensor, charge current and chemical potential in the transverse frame (which we denote by $(T_{diss})^{(T)}_{\mu\nu}$, $(J^{(T)}_{\text{diss}})_{\mu}$ and $\mu^{(T)}_{\text{diss}}$ respectively). We find

$$\begin{aligned}
16\pi G\ ((T_{diss})^{(T)})^{\mu\nu} &= \frac{6}{5}\left[\tilde{j}_u - 4\mu_{diss}\right]\zeta^2\left(n^{\mu}n^{\nu} - \frac{P^{\mu\nu}}{3}\right) + \tilde{T}^{\mu\nu}_{diss},\\
16\pi G\ (J^{(T)}_{\text{diss}})^{\mu} &= \frac{r_c^2}{e^2}\left(\left(\tilde{j}_n - \frac{6\zeta}{5}\left[\tilde{j}_u - 4\mu_{diss}\right]\right)n^{\mu} + \sum_i c_i[V_i]^{\mu}\right),\\
\mu^{(T)}_{\text{diss}} &= \frac{1}{10}\left[\tilde{j}_u - 14\ \mu_{diss}\right],
\end{aligned} \tag{3.3.169}$$

where μ_{diss} is the first derivative correction to the chemical potential obtained from gravity given in (3.3.151) and in (3.3.154). Here the formulas presented in (3.3.169) are valid only at the leading order in ϵ for each independent data at one derivative order.

As in the previous subsection, we then consider the expected standard form fluid expression which is given by

$$\begin{aligned}
T^{\mu\nu}_{diss} = T^3\Big[&(P_a S_a + P_b S_b + P_w S_w)\left(n_{\mu}n_{\nu} - \frac{P_{\mu\nu}}{3}\right)\\
&+ E_a\left(V_a^{\mu}n^{\nu} + V_a^{\nu}n^{\mu}\right) + E_b\left(V_b^{\mu}n^{\nu} + V_b^{\nu}n^{\mu}\right)\\
&+ \tau\mathcal{T}^{\mu\nu}\Big]\\
J^{\mu}_{diss} = T^2\Big[&(R_a S_a + R_b S_b + R_w S_w)\,n^{\mu}\\
&+ C_a V_a^{\mu} + C_b V_b^{\mu}\Big]\\
\mu_{diss} = -&\left[Q_a S_a + Q_b S_b + Q_w S_w\right]
\end{aligned} \tag{3.3.170}$$

The gravity result after the frame transformation (3.3.169) perfectly fits into the above form provided we identify *equation 8.22*

$$
\begin{aligned}
Q_a &= 16\pi G\left(\frac{e^2}{\pi^3}\right)\left[-\frac{52}{25\epsilon^2}+\mathcal{O}(\epsilon)^0\right]+\mathcal{O}\left(\frac{1}{e^2}\right)^0\\
R_a &= \frac{1}{\pi}\left[-\frac{24\chi}{25\epsilon}+\mathcal{O}(\epsilon)^0\right]+\mathcal{O}\left(\frac{1}{e^2}\right)\\
P_a &= \left[\frac{24}{25}\chi^2+\mathcal{O}(\epsilon)\right]+\mathcal{O}\left(\frac{1}{e^2}\right)\\
Q_b &= \frac{1}{\pi}\left[-\frac{24\chi}{25\epsilon}+\mathcal{O}(\epsilon)^0\right]+\mathcal{O}\left(\frac{1}{e^2}\right)\\
R_b &= \frac{\pi}{(16\pi G)e^2}\left[\left(-1-\frac{288}{25}\chi^2\right)+\mathcal{O}(\epsilon)\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
P_b &= \frac{\pi^2}{(16\pi G)e^2}\left[\frac{288}{25}\,\epsilon\chi^3+\mathcal{O}(\epsilon)^2\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
Q_w &= \left[\frac{24}{25}\chi^2+\mathcal{O}(\epsilon)\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
R_w &= \frac{\pi^2}{(16\pi G)e^2}\left[\frac{288}{25}\,\epsilon\chi^3+\mathcal{O}(\epsilon)^2\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
P_w &= -\frac{3\pi^3}{16\pi G}+\frac{\pi^3}{(16\pi G)e^2}\left[-6-\left(\frac{1}{4}-3\chi^2+\frac{288}{25}\,\chi^4\right)\epsilon^2+\mathcal{O}(\epsilon)^3\right]\\
&\quad+\mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{3.3.171}
$$

and

$$
\begin{aligned}
16\pi G\ E_a &= \frac{\pi^2}{e^2}\left[\mathcal{O}(\epsilon)^3\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G\ C_a &= \frac{\pi}{e^2}\left[-1+\mathcal{O}(\epsilon)\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G\ E_b &= -2\pi^3+\frac{\pi^3}{e^2}\left[-4+\left(\frac{1}{6}-2\chi^2\right)\epsilon^2+\mathcal{O}(\epsilon)^4\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G\ C_b &= \frac{\pi^2}{e^2}\left[\epsilon^3\chi\left(\frac{-1+\log(4)}{8}\right)+\mathcal{O}(\epsilon)^4\right]+\mathcal{O}\left(\frac{1}{e^4}\right)\\
16\pi G\ \tau &= -2\pi^3+\frac{\pi^3}{e^2}\left[-4+\left(\frac{1}{6}-2\chi^2\right)\epsilon^2+\mathcal{O}(\epsilon)^4\right]+\mathcal{O}\left(\frac{1}{e^4}\right)
\end{aligned}
\tag{3.3.172}
$$

Note that in this transverse frame also we have $Q_w = P_a$, $R_w = P_b$, $R_a = Q_b$ and $C_b = E_a$, which constitutes the expected Onsager relations. All the positivity constraints given in §2.2.7 are also obeyed in this frame. In $\zeta \to 0$ limit derivative corrections to stress tensor, charge current and the phase equation in transeverse frame take the following form

$$
\begin{aligned}
\lim_{\zeta\to 0}\left[T^{(T)}_{diss}\right]^{\mu\nu} &= T^3\beta_1\sigma^{\mu\nu}\\
\lim_{\zeta\to 0}\left[J^{(T)}_{diss}\right]^{\mu} &= T^2\beta_2 P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right)\\
\lim_{\zeta\to 0}\mu^{(T)}_{diss} &= \beta_3\partial_\mu\left(\frac{q_s}{\xi}\xi^\mu\right)
\end{aligned}
\tag{3.3.173}
$$

where

$$\begin{aligned}
\beta_1 &= -2\pi^3 + \frac{\pi^3}{e^2}\left[-4 + \frac{\epsilon^2}{6} + \mathcal{O}(\epsilon)^4\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
\beta_2 &= \frac{1}{16\pi G}\left[-\frac{\pi}{e^2} + \mathcal{O}(\epsilon^3)\right] + \mathcal{O}\left(\frac{1}{e^4}\right) \\
\beta_3 &= 16\pi G\left(\frac{e^2}{\pi^3 T^3}\right)\left[-\frac{52}{25\epsilon^2} + \mathcal{O}(\epsilon)^0\right] + \mathcal{O}\left(\frac{1}{e^2}\right)^0
\end{aligned} \tag{3.3.174}$$

Chapter 4

Lumps of plasma as duals to exotic black objects

In this chapter we study horizon topologies and thermodynamic properties of black objects in arbitrary dimensions greater than 5 with the assymptotic space being Scerk-Schwarz compactified AdS space. The spectrum of black obejcts in dimensions greater than equal to 5 is extremely rich with interesting phase diagrams. As the construction of these exohotic horizon topologies directly in gravity turns out to be difficult we employ an indirect method to study them which uses the AdS/CFT correspondance in the long wavelength limit in an essential way.

In the long wavelength limit, this field theory admits a fluid description where the dynamics is governed by the d dimensional relativistic Navier-Stokes equation. The effect of the Scerk-Schwarz compactification is only to introduce a constant additive piece to the free energy of the deconfined fluid [79]. Due to this shift, the pressure can go to zero at finite energy densities, allowing the existence of arbitrarily large finite lumps of deconfined fluid separated from the confined phase by a surface – the plasmaballs of [79]. Now by the AdS/CFT correspondence finite energy localized non-dissipative configurations of the plasma fluid in the deconfined phase is dual to stationary black objects in the bulk. Thus, by studying fluid configurations that solve the d dimensional relativistic Navier-Stokes equation we can infer facts about the black objects in SSAdS$_{d+2}$ [6, 80]. We shall conduct this study by explicitly constructing a non-trivial class of fluid configurations in a perturbative expansion.

Two important feature of the dual black object that one can infer from the fluid configurations are the horizon topology and the thermodynamics. The thermodynamics of the black object can be studied by simply computing the thermodynamic properties of the fluid configuration – one integrates the energy density, entropy density etc. to compute the total energy, entropy etc. and the rest follows.

The horizon topology can be inferred as follows. Far outside the region corresponding to the plasma, the bulk should look like the AdS-soliton. In this configuration the Scherk-Schwarz circle contracts as one moves away from the boundary, eventually reaching zero size and capping off spacetime smoothly. Deep inside the region corresponding to the plasma, the

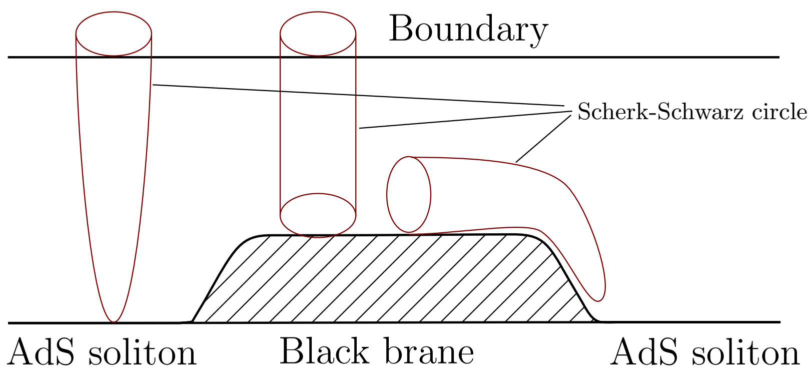


Figure 4.1: Schematic description of the bulk dual of a plasmaball with some circle fibres indicated.

bulk should look like the black brane. In this configuration the Scherk-Schwarz circles does not contract, it still has non-zero size when one reaches the horizon. It follows that as one moves along the horizon, the Scherk-Schwarz circle must contract as one approaches the edge of the region corresponding to the plasma. The horizon topology is found by looking at the fibration of a circle over a region the same shape as the plasma configuration, contracting the circle at the edges [79, 80]. We have provided a schematic drawing of this in fig.4.1.

4.1 General construction and thermodynamics of plasma-lumps

In this section we review the general formalism we will use in this chapter. In §4.1.1 we discuss the thermodynamic properties of the fluids we consider here, in §4.1.2 we review relativistic fluid mechanics, in §4.1.3 we review the relativistic treatment of surface tension and in §4.1.4 we describe the general construction of equilibrium configurations.

4.1.1 Thermodynamics

A fluid with all conserved charges and chemical potentials set to zero satisfies

$$\begin{aligned} \rho + \mathcal{P} &= s\mathcal{T}, \\ \mathrm{d}\rho &= \mathcal{T}\mathrm{d}s, \\ \mathrm{d}\mathcal{P} &= s\,\mathrm{d}\mathcal{T}, \end{aligned} \tag{4.1.1}$$

where ρ, $\mathcal{P}$, s and $\mathcal{T}$ are the local density, pressure, entropy density and temperature as measured in the rest frame of the fluid. Note that all intensive thermodynamic quantities can be written as functions of one variable which we will usually choose to be the temperature. Once we are given the pressure as a function of temperature, we can use (4.1.1) to determine the other quantities.

The fluids that we will consider here – those obtained by the compactification of a conformal theory with a gravity dual on a Scherk-Schwarz circle – have an equation of state of the form

$$\mathcal{P} = \frac{\alpha}{\mathcal{T}_\mathrm{c}} \left(\mathcal{T}^{d+1} - {\mathcal{T}_\mathrm{c}}^{d+1} \right). \tag{4.1.2}$$

Note that this notation is slightly different from that used in [6, 80]. The quantity α here differs from the one used previously by a factor of $\mathcal{T}_\mathrm{c}$. We also have $\rho_0 = \alpha {\mathcal{T}_\mathrm{c}}^d$.

When the parent conformal theory is $\mathcal{N} = 4$ super Yang-Mills, we have $\alpha = \frac{\pi^2 N^2}{8}$.

As the confined phase has a free energy $\sim \mathcal{O}(N^0)$, to leading order at large N we can treat it as the vacuum. The confining/deconfining phase transition occurs when the two phases have the same free energy. In our case, this is approximately at $\mathcal{T} = \mathcal{T}_\mathrm{c}$. At this temperature, the density is given by $\rho_\mathrm{c} = (d+1)\alpha {\mathcal{T}_\mathrm{c}}^d$. Note that ρ_0 is not the critical density.

4.1.2 Fluid mechanics

Provided all length scales are large compared to the thermalisation scale of the fluid (which we call l_mfp), each patch of the fluid is well described by equilibrium thermodynamics in its rest frame. The fluid is characterised by the velocity of these patches — described by a vector $u^\mu = \gamma(1, \vec{v})$ — and the intensive thermodynamic quantities in their rest frames — which can all be computed from the proper temperature $\mathcal{T}$ using the equation of state and the first law of thermodynamics, as in §4.1.1.

The equations of fluid dynamics are simply a statement of the conservation of the stress tensor $T^{\mu\nu}$

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.1.3}$$

This provides d equations for the evolution of for the d quantities $\vec{v}$ and $\mathcal{T}$ once we have expressed the stress tensor as a function of these quantities.

Perfect fluid stress tensor

The dynamics of a fluid is completely specified once the stress tensor and charge currents are given as functions of $\mathcal{T}$ and u^μ. As we have explained in the introduction, fluid mechanics is an effective description at long distances (i.e, it is valid only when the fluid variables vary on distance scales that are large compared to the mean free path l_mfp). As a consequence it is natural to expand the stress tensor in powers of derivatives. In this subsection we briefly review the leading (i.e. zeroth) order terms in this expansion.

It is convenient to define a projection tensor

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{4.1.4}$$

$P^{\mu\nu}$ projects vectors onto the $(d-1)$ dimensional submanifold orthogonal to u^μ. In other words, $P^{\mu\nu}$ may be thought of as a projector onto spatial coordinates in the rest frame of the fluid. In a similar fashion, $-u^\mu u^\nu$ projects vectors onto the time direction in the fluid rest frame.

To zeroth order in the derivative expansion, Lorentz invariance and the correct static limit uniquely determine the stress tensor, charge and the entropy currents in terms of the thermodynamic variables. We have

$$\begin{aligned} T^{\mu\nu}_{\text{perfect}} &= \rho u^\mu u^\nu + \mathcal{P} P^{\mu\nu}, \\ (J^\mu_S)_{\text{perfect}} &= s u^\mu, \end{aligned} \tag{4.1.5}$$

where all thermodynamic quantities are measured in the local rest frame of the fluid, so that they are Lorentz scalars. It is not difficult to verify that in this zero-derivative (or perfect fluid) approximation, the entropy current is conserved. Entropy production (associated with dissipation) occurs only at the first subleading order in the derivative expansion, as we will see in the next subsection.

Dissipation

Now, we proceed to examine the first subleading order in the derivative expansion. In the first subleading order, Lorentz invariance and the physical requirement that entropy be non-decreasing determine the form of the stress tensor and the current to be (see, for example, §§14.1 of [87])

$$\begin{aligned} T^{\mu\nu}_{\text{dissipative}} &= -\zeta \vartheta P^{\mu\nu} - 2\eta \sigma^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu, \\ (J^\mu_S)_{\text{dissipative}} &= \frac{q^\mu}{\mathcal{T}}\,. \end{aligned} \tag{4.1.6}$$

where

$$\begin{aligned} a^\mu &= u^\nu \nabla_\nu u^\mu, \\ \vartheta &= \nabla_\mu u^\mu, \\ \sigma^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta P^{\mu\nu}, \\ q^\mu &= -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T})\,, \end{aligned} \tag{4.1.7}$$

are the acceleration, expansion, shear tensor and heat flux respectively.

These equations define a set of new fluid dynamical parameters in addition to those of the previous subsection: ζ is the bulk viscosity, η is the shear viscosity and κ is the thermal conductivity. Fourier's law of heat conduction $\vec{q} = -\kappa \vec{\nabla} \mathcal{T}$ has been relativistically modified to

$$q^\mu = -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T})\,, \tag{4.1.8}$$

with an extra term that is related to the redshifting of the temperature.

At this order in the derivative expansion, the entropy current is no longer conserved;

however, it may be checked [87] that

$$\mathcal{T}\nabla_\mu J_S^\mu = \frac{q^\mu q_\mu}{\kappa\mathcal{T}} + \zeta\vartheta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{4.1.9}$$

As q^μ and $\sigma^{\mu\nu}$ are all spacelike vectors and tensors, the RHS of (4.1.9) is positive provided η, ζ, κ are positive parameters, a condition we further assume. This establishes that (even locally) entropy can only be produced but never destroyed. In equilibrium, $\nabla_\mu J_S^\mu$ must vanish. It follows that, q^μ, ϑ and $\sigma^{\mu\nu}$ each individually vanish in equilibrium.

For fluids with gravity duals, the shear viscosity is given by $\eta = \frac{s}{4\pi}$ [45]. We can estimate the thermalisation length of the fluid by comparing coefficients at different orders in the derivative expansion

$$l_{\text{mfp}} \sim \frac{\eta}{\rho} = \frac{s}{4\pi\rho}. \tag{4.1.10}$$

This length scale may plausibly be identified with the thermalisation length scale of the fluid. This may be demonstrated within the kinetic theory, where l_{mfp} is simply the mean free path of colliding molecules, but is expected to apply to more generally to any fluid with short range interactions.

With the equation of state (4.1.2), this is given by

$$l_{\text{mfp}} \sim \frac{\mathcal{T}^d}{\pi(d\mathcal{T}^{d+1} + {\mathcal{T}_{\text{c}}}^{d+1})}. \tag{4.1.11}$$

As we will be restricting attention to temperatures close to $\mathcal{T}_{\text{c}}$, we have $l_{\text{mfp}} \sim 1/\mathcal{T}_{\text{c}}$.

4.1.3 Surfaces

The plasma ball configurations we consider have a domain wall separating a bubble of the deconfined phase from the confined phase. As the density, pressure, etc. of the deconfined phase are a factor of N^2 larger than the confined phase, we can treat the confined phase as the vacuum and the domain wall as a surface bounding the deconfined fluid.

At surfaces, the density of the fluid changes too rapidly to be described by fluid mechanics. However, provided that we look at length scales much larger than the thickness of the surface, we can replace this region by a delta function localised piece of the stress tensor.

At these length scales, this stress tensor will depend on the direction of the surface, with dependence on its curvature being suppressed.

In general, introducing a surface energy density σ_E, a surface entropy density σ_S and a surface tension σ, considerations similar to those leading to (4.1.1) lead to

$$\begin{aligned} \sigma_E &= \sigma + \mathcal{T}\sigma_S, \\ \mathrm{d}\sigma &= -\sigma_S\,\mathrm{d}\mathcal{T}. \end{aligned}$$

However, the surface tension was only computed at $\mathcal{T} = \mathcal{T}_{\text{c}}$ in [79], so we will have to ignore its temperature dependence. As we can see above, this is equivalent to setting $\sigma_S = 0$ and $\sigma_E = \sigma$.

Let's describe the location of the surface by a function $f(x)$ that is positive inside the fluid and has a first order zero on the surface:

$$T^{\mu\nu} = \theta(f) T^{\mu\nu}_{\text{fluid}} + \delta(f) T^{\mu\nu}_{\text{surface}}. \tag{4.1.12}$$

At large length scales, as mentioned above, $T^{\mu\nu}_{\text{surface}}$ will only depend on the first derivative of f and no higher derivatives.

If we demand invariance under reparameterisations of the function $f(x) \to g(x) f(x)$, where $g(x) > 0$, and that the surface moves at the velocity of the fluid

$$u^\mu \partial_\mu f|_{f=0} = 0, \tag{4.1.13}$$

the surface stress tensor is (see §2.3 of [80])

$$T^{\mu\nu}_{\text{surface}} = \sqrt{\partial f \cdot \partial f}\left[\sigma_E u^\mu u^\nu - \sigma(g^{\mu\nu} - n^\mu n^\nu + u^\mu u^\nu)\right], \tag{4.1.14}$$

where $n_\mu = -\partial_\mu f / \sqrt{\partial f \cdot \partial f}$ is the normal to the surface. Note that $(\partial_\mu f) T^{\mu\nu}_{\text{surface}} = 0$. If we take the surface tension to be constant, as above, we get

$$T^{\mu\nu}_{\text{surface}} = -\sigma h^{\mu\nu} \sqrt{\partial f \cdot \partial f}, \tag{4.1.15}$$

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric of the surface. The factor of $\sqrt{\partial f \cdot \partial f}$ also has a simple interpretation: suppose we use a coordinate system where f is one of the coordinates. Then

$$\sqrt{\partial f \cdot \partial f} = \sqrt{g^{ff}} = \sqrt{\frac{\det h}{\det g}}\,, \tag{4.1.16}$$

which provides the correct change of integration measure for localisation to the surface. If we used some other coordinates, there'd be an extra Jacobian factor.

We have

$$\nabla_\mu T^{\mu\nu} = \theta(f) \nabla_\mu T^{\mu\nu}_{\text{fluid}} + \delta(f)(\partial_\mu f) T^{\mu\nu}_{\text{fluid}} + \delta(f) \nabla_\mu T^{\mu\nu}_{\text{surface}}. \tag{4.1.17}$$

So, in addition to (4.1.3), we have the boundary conditions

$$\left.(\partial_\mu f) T^{\mu\nu}_{\text{fluid}} + \nabla_\mu T^{\mu\nu}_{\text{surface}}\right|_{f=0} = 0. \tag{4.1.18}$$

Also, when we take the surface tension to be constant:

$$\nabla_\mu T^{\mu\nu}_{\text{surface}} = \sigma \left[\frac{\Box f}{(\partial f \cdot \partial f)^{1/2}} - \frac{(\partial^\mu f)(\partial^\lambda f) \nabla_\mu \partial_\lambda f}{(\partial f \cdot \partial f)^{3/2}}\right] \partial^\nu f = -\sigma\, \Theta\, \partial^\nu f, \tag{4.1.19}$$

where Θ is the trace of the extrinsic curvature of the surface, as seen from outside the fluid (see Appendix C.1).

If we have several disconnected surfaces, it is convenient to make the separation $f = \prod_i f_i$. As the surfaces are disconnected, the zero sets of the f_i do not intersect. Also, the f_i are all positive inside the fluid. Therefore, whenever one of the f_i is negative or zero, all the others are positive. Luckily, (4.1.17) splits nicely

$$\nabla_\mu T^{\mu\nu} = \prod_i \theta\,(f_i)\,\nabla_\mu T^{\mu\nu}_{\text{fluid}} + \sum_i \delta(f_i)\,[(\partial_\mu f_i)T^{\mu\nu}_{\text{fluid}} + \nabla_\mu T^{\mu\nu}_{\text{surface}}(f_i)]\,.$$

From the form of the gravity solution, we would expect σ_E/ρ to be similar to the thickness of the surface. We can estimate it using

$$\xi = \frac{\sigma}{\rho_{\text{c}}} = \frac{\sigma}{(d+1)\alpha \mathcal{T}_{\text{c}}{}^d}\,. \tag{4.1.20}$$

In general, it will be of order N^0 and is similar to the surface thickness and l_{mfp} (if 8π can be considered similar to 1).

For the domain wall of [79] in $d = 2+1$ dimensions, the thickness and surface tension are approximately $6 \times \frac{1}{2\pi \mathcal{T}_{\text{c}}}$ and $\sigma = 2.0 \times \frac{\rho_{\text{c}}}{\mathcal{T}_{\text{c}}}$ respectively. This gives $\xi = \frac{2.0}{\mathcal{T}_{\text{c}}}$, which is pretty close to the thickness.

In $d = 3+1$, the domain wall of [79] has thickness and surface tension approximately equal to $5 \times \frac{1}{2\pi \mathcal{T}_{\text{c}}}$ and $\sigma = 1.7 \times \frac{\rho_{\text{c}}}{\mathcal{T}_{\text{c}}}$ respectively. This gives $\xi = \frac{1.7}{\mathcal{T}_{\text{c}}}$, which is also pretty close to the thickness.

For our purposes, it is more convenient to talk about the length scale

$$\xi' = (d+1)\xi = \frac{\sigma}{\rho_0} = \frac{\sigma}{\alpha \mathcal{T}_{\text{c}}{}^d}\,. \tag{4.1.21}$$

4.1.4 Equilibrium configurations

In this subsection, we will specialise the general discussion above to the construction of equilibrium configurations of fluids with surfaces. We will also derive a simple approach to studying the thermodynamic properties of these configurations.

Solutions for the interior

We want to find solutions of (4.1.3) that are independent of time, which means we need to set (4.1.9) to zero. This means we need velocity configurations that have zero expansion and shear. In general, this would be a combination of a uniform boost and rigid rotation. We can always boost to a frame where the centre of rotation is static and the rotation lies in the Cartan directions of the rotation group. This gives

$$u = \gamma(\partial_t + \omega_a l_a), \tag{4.1.22}$$

where ω_a are the angular velocities and l_a are a set of commuting rotational Killing vectors. The important feature is that the velocity is a normalisation factor times a Killing vector (see §2.2 of [88]):

$$u^\mu = \gamma K^\mu, \qquad \gamma^2 K^\mu K_\mu = -1, \qquad \nabla_{(\mu} K_{\nu)} = 0. \tag{4.1.23}$$

One can deduce that

$$\vartheta = \sigma^{\mu\nu} = 0, \qquad u^\mu \partial_\mu \gamma = 0, \qquad a_\mu = -\frac{\partial_\mu \gamma}{\gamma}.$$

Which leads to

$$q^\mu = -\kappa \gamma P^{\mu\nu} \partial_\nu \left[\frac{\mathcal{T}}{\gamma}\right].$$

One can also show that

$$\nabla_\mu T^{\mu\nu}_{\text{perfect}} = \gamma \left(s P^{\nu\mu} + \mathcal{T} \frac{\partial s}{\partial \mathcal{T}} u^\nu u^\mu \right) \partial_\mu \left[\frac{\mathcal{T}}{\gamma}\right]$$

So the velocity configuration (4.1.22) will be an equilibrium solution to the equations of motion provided that

$$\frac{\mathcal{T}}{\gamma} = T = \text{constant}. \tag{4.1.24}$$

Using the equation of state and (4.1.1), this determines all of the intensive thermodynamic quantities in the fluid.

Solutions for surfaces

The fluid configurations described in the previous subsection have $T^{\mu\nu}_{\text{dissipative}} = 0$. Therefore

$$(\partial_\mu f) T^{\mu\nu}_{\text{fluid}} = (\partial_\mu f) T^{\mu\nu}_{\text{perfect}} = \mathcal{P} \partial^\nu f.$$

This means that (4.1.18) and (4.1.19) reduce to

$$\mathcal{P}|_{f=0} = \sigma \Theta. \tag{4.1.25}$$

As the pressure is determined by (4.1.24), this provides a differential equation that determines allowed positions of surfaces. Demanding that the surface has no conical singularities turns out to provide enough boundary conditions to determine the position of the surface completely (up to discrete choices) in terms of the parameters Ω_a, T.

Thermodynamics of solutions

We compute the extensive thermodynamic properties of these solutions by integrating the time components of the corresponding currents (noting that the current associated with a Killing vector ζ^μ is $J^\mu_\zeta = T^{\mu\nu} \zeta_\nu$):

$$Q_X = \int \mathrm{d}V J^0_X. \tag{4.1.26}$$

We are assuming that the space-time in consideration is static, so it can be foliated by space-like surfaces $t = $ constant with normal ∂_t. In fact, here we will only consider fluids in flat space.

In particular, also noting that for equilibrium configurations $\partial^0 f = 0$,

$$Q_\zeta = \int \mathrm{d}V \theta(f) \left[(\rho + \mathcal{P})\gamma^2 K^0 K\cdot\zeta + \mathcal{P}\zeta^0\right] - \int \mathrm{d}V \delta(f)\sqrt{\partial f\cdot\partial f}\sigma\zeta^0. \tag{4.1.27}$$

Noting that $K^0 = (\partial_t)^0 = 1$ and $l_a^0 = 0$, this gives

$$\begin{aligned} E = -Q_{\partial_t} \quad &= -\int \mathrm{d}V \theta(f) \left[(\rho + \mathcal{P})\gamma^2 K\cdot\partial_t + \mathcal{P}\right] + \int \mathrm{d}V \delta(f)\sqrt{\partial f\cdot\partial f}\sigma, \\ L_a = Q_{l_a} \quad &= \int \mathrm{d}V \theta(f) \left[(\rho + \mathcal{P})\gamma^2 K\cdot l_a\right], \\ S = Q_S \quad &= \int \mathrm{d}V \theta(f) \left[\gamma s\right]. \end{aligned} \tag{4.1.28}$$

From these quantities, we can compute overall angular velocities ω_a and temperature T thermodynamically

$$\mathrm{d}E = \omega_a \,\mathrm{d}L_a + T\,\mathrm{d}S. \tag{4.1.29}$$

A priori, it may not seems that these quantities have to be the same as ω_a, T from (4.1.22) and (4.1.24). However, we can show that they are the same by checking that (4.1.29) holds with ω_a, T taken from (4.1.22) and (4.1.24). In practice, it is easier to verify the equivalent statement

$$\mathrm{d}(E - \omega_a L_a - TS) = -L_a\,\mathrm{d}\omega_a - S\,\mathrm{d}T. \tag{4.1.30}$$

First, making use of (4.1.1), we see that

$$E - \omega_a L_a - TS = -Q_K - TQ_S = -\int \mathrm{d}V \theta(f)\mathcal{P} + \int \mathrm{d}V \delta(f)\sqrt{\partial f\cdot\partial f}\sigma. \tag{4.1.31}$$

Note that the second integral is simply σ times the surface area: as we saw in (4.1.16) the factor of $\sqrt{\partial f\cdot\partial f}$ provides the correct change of measure for the delta function to localise the integral to the surface.

Consider an infinitesimal change of ω_a, T. We have

$$\begin{aligned} \mathrm{d}\mathcal{P} = s\,\mathrm{d}(\gamma T) = \frac{\rho + \mathcal{P}}{\gamma}\,\mathrm{d}\gamma + \gamma s\,\mathrm{d}T. \\ \gamma^{-3}\,\mathrm{d}\gamma = K\cdot\mathrm{d}K = K\cdot l_a\,\mathrm{d}\omega_a. \end{aligned}$$

From this, we see that (4.1.30) is satisfied by the contributions from the interior. As the right hand side of (4.1.30) has no contributions from the surface, we need to check that the surface contributions of the variation of (4.1.31) cancel.

The change in the surface area can be written as

$$\mathrm{d}\mathcal{A} = \oint \mathrm{d}A\,\vec{n}\cdot\vec{w},$$

where the integral is performed over the union of the initial and final surfaces, $\vec{n}$ is a unit normal vector pointing into the initial fluid and out of the final fluid and $\vec{w}$ is some vector field that is equal to the outward pointing normal at both the initial and final surfaces. By

Gauss' theorem, this can be written as

$$\mathrm{d}\mathcal{A} = \int \mathrm{d}V \, \nabla \cdot \vec{w},$$

with the integral performed over the region between the two surfaces. The volume element can be written as $\int \mathrm{d}V = \int \mathrm{d}A \, (\vec{n} \cdot \Delta x)$, with $\vec{n}$ pointing outwards. As the volume element is already infinitesimal, we can replace $\vec{w}$ with the vector field described in (C.1.5), as the difference would be infinitesimal, i.e. $\nabla \cdot \vec{w} \to \Theta$. Also, as $f = 0$ on the initial surface, and $f + \mathrm{d}f = 0$ on the final surface ($\mathrm{d}f$ refers to the change in f due to the change in Ω_a, T), we have

$$\begin{aligned} \partial_\mu f \Delta x^\mu + \frac{\partial f}{\partial \Omega_a} \mathrm{d}\Omega_a + \frac{\partial f}{\partial T} \mathrm{d}T = 0, \\ \implies \vec{n} \cdot \Delta x = \frac{\mathrm{d}f}{\sqrt{\partial f \cdot \partial f}}. \end{aligned}$$

Therefore

$$\mathrm{d}\mathcal{A} = \int \mathrm{d}V \delta(f) \Theta \, \mathrm{d}f.$$

So, we can write the surface contribution to the variation of (4.1.31) as

$$\mathrm{d}(E - \omega_a L_a - TS)_{\text{surface}} = -\int \mathrm{d}V \delta(f) \, \mathcal{P} \, \mathrm{d}f + \int \mathrm{d}V \delta(f) \, \sigma \Theta \, \mathrm{d}f,$$

which vanishes due to (4.1.37).

The thermodynamics of the solution can be summarised by defining a grand partition function

$$\mathcal{Z}_{\text{gc}} = \operatorname{Tr} \exp \left(-\frac{E - \omega_a L_a}{T} \right). \tag{4.1.32}$$

In the thermodynamic limit,

$$\begin{aligned} -T \ln \mathcal{Z}_{\text{gc}} &= E - \omega_a L_a - TS, \\ \mathrm{d}(T \ln \mathcal{Z}_{\text{gc}}) &= L_a \, \mathrm{d}\omega_a + S \, \mathrm{d}T. \end{aligned} \tag{4.1.33}$$

We have seen that

$$T \ln \mathcal{Z}_{\text{gc}} = \int_{f>0} \mathrm{d}V \, \mathcal{P} - \int_{f=0} \mathrm{d}A \, \sigma \tag{4.1.34}$$

and the ω_a, T are the same as those given by (4.1.22) and (4.1.24).

Validity

We are making several approximations in this paper. First of all, the Navier-Stokes equations are merely a long wavelength approximation to the full dynamics of the gauge theory. This will be a good approximation provided that the length scale of variation of $\mathcal{T}$ and u^μ is small compared to the thermalisation scale of the fluid (4.1.10).

Second, we have treated the surface of the plasma as sharp; in reality this surface has a thickness of order ξ (see (4.1.20)). Consequently, our treatment of the surface is valid only when its curvature is small compared to $1/\xi \sim 1/\xi'$ (higher derivative contributions to the surface stress tensor, which we have ignored in our treatment, would become important if this were not the case); further we must also require that only a small fraction of the fluid should reside in surfaces. This boils down to demanding that all sizes are much larger than ξ'.

Thirdly, we have ignored the fact that the surface tension is a function of the fluid temperature at the surface, and simply set $\sigma = \sigma(\mathcal{T}_\mathrm{c})$. This is valid provided that $\mathcal{T}/\mathcal{T}_\mathrm{c} \approx 1$ at all surfaces. When this is the case, the pressure will be small compared to ρ_c (see (4.1.2)). Then, (4.1.37) tells us that the extrinsic curvature of the surface must be small compared to $1/\xi$, which is the same as the previous condition.

We can estimate the scale over which thermodynamic quantities vary as the distance over which the fractional change in the temperature is one. As the temperature is proportional to γ, we should demand (schematically)

$$\frac{1}{\|\nabla \ln \gamma\|} \sim \frac{1-v^2}{\|\omega v\|} \gg l_\mathrm{mfp} \,.$$

At temperatures close to $\mathcal{T}_\mathrm{c}$, where our other approximations are valid, we have $l_\mathrm{mfp} \sim \xi'$. Therefore, we require that that

$$\frac{1-v^2}{\|\omega v\|} \gg \xi'.$$

This will be true if the speed of the fluid is much less than the speed of light and the angular velocities are much less that $1/\xi'$.

Dimensionless variables

It is convenient to rescale the variables as follows. First we rescale all lengths and times by ξ' (i.e. working in units where $\xi' = 1$)

$$x = \xi' \tilde{x}, \qquad \omega_a = \frac{\widetilde{\omega}_a}{\xi'} \,. \tag{4.1.35}$$

We measure temperature in units of $\mathcal{T}_\mathrm{c}$ and further rescale extensive thermodynamic quantities by $\alpha \mathcal{T}_\mathrm{c}{}^d$.

$$\begin{aligned} &T = \mathcal{T}_\mathrm{c}\widetilde{T}, &&\ln \mathcal{Z}_\mathrm{gc} = (\alpha \mathcal{T}_\mathrm{c}{}^{d-1}\xi'^{d-1}) \ln \widetilde{\mathcal{Z}}_\mathrm{gc}, &&E = (\alpha \mathcal{T}_\mathrm{c}{}^{d}\xi'^{d-1})\widetilde{E}, \\ &S = (\alpha \mathcal{T}_\mathrm{c}{}^{d-1}\xi'^{d-1})\widetilde{S}, &&L = (\alpha \mathcal{T}_\mathrm{c}{}^{d}\xi'^{d})\widetilde{L}. \end{aligned} \tag{4.1.36}$$

Then, (4.1.41) becomes

$$\left[(\gamma \widetilde{T})^{d+1} - 1\right]_\mathrm{surface} = \widetilde{\Theta}, \tag{4.1.37}$$

and (4.1.42) becomes

$$
\begin{aligned}
\widetilde{T}\ln\widetilde{\mathcal{Z}}_{\text{gc}} &= \int_{f>0}\mathrm{d}\widetilde{V}\left[(\gamma\widetilde{T})^{d+1}-1\right]-\int_{f=0}\mathrm{d}\widetilde{A},\\
-\widetilde{T}\ln\widetilde{\mathcal{Z}}_{\text{gc}} &= \widetilde{E}-\widetilde{\omega}_a\widetilde{L}_a-\widetilde{T}\widetilde{S},\\
\mathrm{d}(\widetilde{T}\ln\widetilde{\mathcal{Z}}_{\text{gc}}) &= \widetilde{L}_a\,\mathrm{d}\widetilde{\omega}_a+\widetilde{S}\,\mathrm{d}\widetilde{T}.
\end{aligned}
\tag{4.1.38}
$$

From now on, we will suppress all tildes and work entirely with the new variables.

Summary

We can summarise the construction of equilibrium solutions as follows:

The fluid velocity is given by

$$
u^\mu = \gamma K^\mu, \qquad \text{where} \quad \begin{aligned} K &= \partial_t+\omega_a\partial_{\phi_a},\\ \gamma &= (-K^\mu K_\mu)^{-\frac{1}{2}}, \end{aligned}
\tag{4.1.39}
$$

with ϕ_a being a set of angular coordinates such that $\phi_a \to \phi_a + c_a$ are a set of commuting isometries and ω_a the angular velocities.

The thermodynamic properties of the fluid are specified by

$$
\mathcal{T} = \gamma T. \tag{4.1.40}
$$

All other local thermodynamic properties can be computed from the equation of state (4.1.2) and the relations (4.1.1).

The position of the surface is specified by a function f that is positive inside the fluid and negative outside. It is determined by

$$
\left[(\gamma T)^{d+1}-1\right]_{\text{surface}} = \Theta, \tag{4.1.41}
$$

with Θ given by (C.1.7), and the condition that the surface is closed without conical singularities.

The overall thermodynamic properties of the solution can be computed from

$$
\begin{aligned}
T\ln\mathcal{Z}_{\text{gc}} &= \int_{f>0}\mathrm{d}V\left[(\gamma T)^{d+1}-1\right]-\int_{f=0}\mathrm{d}A\,,\\
-T\ln\mathcal{Z}_{\text{gc}} &= E-\omega_a L_a-TS,\\
\mathrm{d}(T\ln\mathcal{Z}_{\text{gc}}) &= L_a\,\mathrm{d}\omega_a+S\,\mathrm{d}T.
\end{aligned}
\tag{4.1.42}
$$

4.2 Black objects in arbitrary dimensions

As mentioned in the introduction we study the horizon topologies of black objects in SSAdS$_{d+2}$ through the dual fluid configurations which solve the d dimensional relativistic

Navier-Stokes equation. In [80] exact disc (B^2) like plasma configurations were obtained in 2+1 dimensions. In one higher dimension (i.e. in 3+1 dimensions) we can expect the existence of the solution $B^2 \times \mathbb{R}^1$, which is topologically a cylinder. Now we can bend the cylinder into a ring. The fact that such a ring solution exists were shown (numerically) in [80] and its thermodynamics were explored in [6]. This ring solution can be constructed perturbatively from the cylinder (a suitable expansion parameter being the ratio $\epsilon = \frac{R}{\mathcal{L}}$, R being the radius of the cylinder and $\mathcal{L}$ being the distance of the cylinder from the origin). The leading order solution, called the *thin ring*, (correct up to $\mathcal{O}(\epsilon)$) was obtained analytically in [6].

This method can be used to construct rings and other exotic plasma configurations in higher dimensions. For example in one more dimension (i.e. 4+1 dimensions) we can have a cylinder solution with topology $B^3 \times \mathbb{R}^1$ (B^3 being the ball type solution in $3+1$ dimension) then we can bend this cylinder to form a ring. In this way we would be able to obtain ring type solutions from the plasmaball configuration in one lower dimension. Further besides the ring we can also construct other configurations by a similar method. Again for concreteness let us consider the example of $4+1$ dimension where besides the $B^3 \times \mathbb{R}^1$ topology we can also have a topology $B^2 \times \mathbb{R}^2$ constructed out of the B^2 solution in $2+1$ dimensions (two dimensions lower). Now we can bend the $B^2 \times \mathbb{R}^2$ configuration along two directions in the $\mathbb{R}^2$ to form $B^2 \times S^1 \times S^1$. Another way to think of it is that in $3+1$ dimensions the $B^2 \times S^1$ topology existed. So (just like the construction of the cylinder) in $4+1$ dimension we have a topology $B^2 \times S^1 \times \mathbb{R}^1$. Now we can bend this cylinder along the $\mathbb{R}^1$ to form the topology $B^2 \times S^1 \times S^1$. Thus in d space-time dimensions we can use this method to study a topology

$$B^{(d-1-n)} \times T^n = B^{(d-1-n)} \times \underbrace{S^1 \times S^1 \ldots\ldots S^1}_{\text{n times}}. \tag{4.2.43}$$

Note that there can be solutions with other topologies in d dimensions which we will not be able to capture by our method, therefore we shall look at these topologies only. Here n cannot be arbitrary. Naively we would expect $n \leq d-3$ so that the ball is at least a B^2 [1]. However there is a stronger bound on n. We recall the fact that for the ring (or more exotic objects as above) to exist we must have non-zero angular momentum in the plane of the S^1(s). However in d dimensions we can independently turn on angular momentum only along $\frac{d-1}{2}$ ($\frac{d-2}{2}$) directions for odd (even) d. This is because the group of spatial rotations in d dimensions is $SO(d-1)$ which has rank $\frac{d-1}{2}$ ($\frac{d-2}{2}$) for odd (even) d. Thus n should not be more than the number of Cartans of the (spatial) rotation group in a particular dimension. Hence n should satisfy

$$\begin{aligned} &n = 0, \quad \text{for } d = 3. \\ &n \leq \frac{d-1}{2}, \quad \text{for odd } d \text{ greater than 3.} \\ &n \leq \frac{d-2}{2}, \quad \text{for even d.} \end{aligned} \tag{4.2.44}$$

[1] If the ball is a B^1 then we would be considering hollow solutions; however we are unable to capture the description of those solutions by our method. In addition, such hollow solutions might not exist in all dimension. In $2+1$ dimensions the existence of such hollow solutions (the annular ring) was reported in [80]; while there is strong evidence (see [6, 80]) to suggest that such hollow solutions do not exist in $3+1$ dimensions.

Here note that $d = 3$ is a special case because for this dimension $n \leq d-3$ is a stronger bound than $n \leq \frac{d-1}{2}$. Also note that a new topology of the plasma configuration is obtained for every odd d (≥ 5). Thus in an even d we only have the solutions that existed in $d-1$ dimensions.

Here we shall analytically show (in an approximation analogous to the thin ring approximation) that these topologies exist as plasma configurations which are solutions of the relativistic Navier-Stokes equations expressed in the form (4.1.3). The most suitable coordinate system for studying the general topology (4.2.43) is the one in which we choose $\{r_a, \phi_a\}$ with $a = 1, \ldots n$ as the coordinates on the n planes containing the n S^1s and in the rest of the space (which exists exists when $d > 2n+1$) we choose spherical polar coordinates $\{r, \theta_1, \ldots, \theta_{(d-2n-2)}\}$. The coordinates ϕ_a and θ_j are angular coordinates; the coordinates $\{\theta_j\}$ with $j = 1, \ldots, (d-2n-2)$ may be taken to be the coordinates on a unit sphere in a $d-2n-1$ dimensional space. The angles θ_j will be absent if $d = 2n+2$. The range of all the radial coordinates $\{r, r_a\}$ are as usual $[0, \infty)$ while the angles $\phi_a \in [0, 2\pi)$. Besides these spatial coordinate we denote the time coordinate by t.

The metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{a=1}^{n} r_a^2 \mathrm{d}\phi_a^2 + \sum_{a=1}^{n} \mathrm{d}r_a^2 + \mathrm{d}r^2 + r^{(d-2n-2)} \mathrm{d}\Omega^2_{(d-2n-2)}(\theta_1, \theta_2, \ldots, \theta_{d-2n-2}) \quad (4.2.45)$$

where $\mathrm{d}\Omega^2_{(d-2n-2)}$ is the round metric on a unit sphere in $d-2n-1$ dimensions. In such a space we consider the fluid surface to be given by

$$f \equiv h(r_1, r_2, \ldots, r_n) - r = 0. \quad (4.2.46)$$

For $n = \frac{d-1}{2}$ we have a special case because then there is no r coordinate. In this case we consider $f \equiv h(r_2, \ldots, r_n) - r_1 = 0$ as the equation of the surface.

The velocity is given by $u^\mu = \gamma(1, \omega_1, \ldots, \omega_n, 0, 0, \ldots, 0)$, with $\gamma = (1 - \sum_{a=1}^{n} \omega_a^2 r_a^2)^{\frac{1}{2}}$ being the normalization factor. Note that here we are not considering any angular velocity in the θ_i directions. The topologies that we explore could also be spinning along the θ_i directions. However, if that were the case the zeroth order solution would not be simply a round ball times a plane and we would lose analytic control. It might be possible to turn on infinitesimal angular velocities in these directions, but we will not consider that here.

The equation for h(r_a) that follows from (4.1.37) is given by

$$\frac{T^{d+1}}{(1 - \sum_{a=1}^{n} \omega_a^2 r_a^2)^{(\frac{d+1}{2})}} - 1 = \frac{1}{(1 + \sum_{a=1}^{n} (\partial_{r_a} h)(\partial_{r_a} h))^{\frac{3}{2}}} \left(\left(\frac{m}{h} - \sum_{a=1}^{n} \frac{\partial_{r_a} h}{r_a} - \sum_{a=1}^{n} \partial_{r_a} \partial_{r_a} h \right) \left(1 + \sum_{a=1}^{n} (\partial_{r_a} h)(\partial_{r_a} h) \right) + \sum_{a,b=1}^{n} (\partial_{r_a} h)(\partial_{r_b} h)(\partial_{r_a} \partial_{r_b} h) \right). \quad (4.2.47)$$

where $m = d - 2n - 2$. In order to obtain (4.2.47) from (4.1.37) we set the typical length scale of the problem ξ' to 1 by suitable choice of units and we are measuring temperature in units of $\mathcal{T}_c$, as described in §4.1.4. For $n = \frac{d-1}{2}$ the sum starts running from 2 (instead of 1) in the above equation. Now we separately consider the space of $\{r, r_a\}$ (only the first 'quadrant' of this space is physical because $\{r, r_a\} \in [0, \infty)$). We now shift the origin to a new point $\overrightarrow{\mathcal{L}}$ such that it is given by

$$\overrightarrow{\mathcal{L}} = (0, \mathcal{L}P_1, \mathcal{L}P_2, \ldots, \mathcal{L}P_n) \tag{4.2.48}$$

P_as are projectors along various r_a directions, so that $\sum P_a^2 = 1$ and $\mathcal{L}$ is the magnitude of the vector $\overrightarrow{\mathcal{L}}$. Again in the special case of $n = \frac{d-1}{2}$ we take $\overrightarrow{\mathcal{L}} = (\mathcal{L}P_1, \mathcal{L}P_2, \ldots, \mathcal{L}P_n)$. Let $\{x_a\}$ be the new shifted coordinates such that

$$r_a = \mathcal{L}P_a + x_a. \tag{4.2.49}$$

As is apparent the coordinate r (if it exists) remains unchanged by this coordinate change.

Now we perform the following scaling

$$\omega_a = \epsilon w_a; \quad \mathcal{L} = \frac{\ell_0}{\epsilon}; \quad h = y(\{x_a\}). \tag{4.2.50}$$

Then (4.2.47) at leading order in ϵ (which is ϵ^0) reduces to the equation

$$\begin{aligned}\frac{T^{d+1}}{(1-\sum_a (w_a \ell_0 P_a)^2)^{\left(\frac{d+1}{2}\right)}} - 1 =& \frac{1}{(1+\sum_{a=1}^n \partial_{x_a} y \partial_{x_a} y)^{\frac{3}{2}}} \left(\left(\frac{m}{y} - \sum_{a=1}^n \partial_{x_a}\partial_{x_a} y \right) \right. \\ & \left. \left(1 + \sum_{a=1}^n \partial_{x_a} y \partial_{x_a} y \right) + \sum_{a,b=1}^n (\partial_{x_a} y)(\partial_{x_b} y)(\partial_{x_s}\partial_{x_b} y) \right).\end{aligned} \tag{4.2.51}$$

Now (4.2.51) is satisfied by the function

$$y(\{x_a\}) = \left(R^2 - \sum_{a=1}^n x_a^2 \right)^{\frac{1}{2}}, \tag{4.2.52}$$

provided the following equation is satisfied by the parameters

$$T^{d+1} = \left(\frac{(d-n-2)+R}{R} \right) \left(1 - \sum_a (\ell_0 P_a w_a)^2 \right)^{\left(\frac{d+1}{2}\right)} \tag{4.2.53}$$

Also the equation (4.2.47) at $O(\epsilon)$ yields

$$\frac{(d+1)T^{d+1}\sum_a (w_a^2 P_a \ell_0 x_a)}{(1-\sum_a (w_a P_a \ell_0)^2)^{\frac{d+3}{2}}} + \frac{1}{(1+\sum_{a=1}^n \partial_{x_a} y \partial_{x_a} y)^{\frac{1}{2}}} \sum_a \frac{\partial_{x_a} y}{\ell_0 P_a} = 0. \tag{4.2.54}$$

In the above equation if we substitute (4.2.52) and set the coefficients of x_a to zero then we get (after using (4.2.53))

$$w_a^2 = \frac{1}{(\ell_0 P_a)^2 \left(((d-n-2)+R)(d+1)+n\right)}. \tag{4.2.55}$$

Note that here we have $n+1$ equations for $n+2$ parameters R, ℓ_0 and n P_as. However there is one constraint among the P_as (namely $\sum P_a^2 = 1$) which gives us the correct number of equations for the parameters to be determined.

Again for the special case of $n = \frac{d-1}{2}$ the chief results (4.2.52), (4.2.53) and (4.2.55) remain unchanged. However we have $h = \mathcal{L}P_1 + y(\{x_i\})$. As a result the 0th and 1st order equations, (4.2.51) and (4.2.54), are changed respectively to

$$\frac{T^{d+1}}{(1-\sum_a(w_a\ell_0P_a)^2)^{\left(\frac{d+1}{2}\right)}} - 1 = \frac{1}{(1+\sum_a \partial_{x_a}y\partial_{x_a}y)^{\frac{3}{2}}}\left(\left(-\sum_a \partial_{x_a}\partial_{x_a}y\right)\left(1+\sum_a \partial_{x_a}y\partial_{x_a}y\right) + \sum_{a,b}(\partial_{x_a}y)(\partial_{x_b}y)(\partial_{x_a}\partial_{x_b}y)\right), \tag{4.2.56}$$

and

$$\frac{(d+1)T^{d+1}\sum_a(w_a^2P_a\ell_0x_a)}{(1-\sum_a(w_aP_a\ell_0)^2)^{\frac{d+3}{2}}} + \frac{1}{(1+\sum_a \partial_{x_a}y\partial_{x_a}y)^{\frac{1}{2}}}\left(\frac{1}{\ell_0P_1} + \sum_a \frac{\partial_{x_a}y}{\ell_0P_{(a+1)}}\right) = 0. \tag{4.2.57}$$

As mentioned before here the sums run from 2 to n. Also in this case besides equating the coefficients of x_as to zero in (4.2.57) we also have set the coefficient of $(R^2 - \sum_a x_a^2)^{\frac{1}{2}}$ to zero to obtain (4.2.55).

In (4.2.55) we find a very curious fact about the speed of the class of solutions that we analyze here. The velocities $w_aP_a\ell_0$ reach a maximum when $R \to 0$ and the maximum value is given by

$$w_a^{max}P_a\ell_0 = \frac{1}{\sqrt{(d-n-2)(d+1)+n}}. \tag{4.2.58}$$

This value is consistent with the maximum speed for the ring in $3+1$ dimensions quoted in [6]. Also note that for the ring ($n=1$) and large d this maximum value goes as $\frac{1}{d}$; this is unlike (although consistent with) the behavior of the ring in asymptotically flat space where this goes as $\frac{1}{\sqrt{d}}$ (see [83, 89]). However, this limiting value occurs when $R \to 0$, where our approximation of the surface as having no thickness breaks down. Nevertheless, for large space time dimension the behavior of black holes in asymptotically flat spaces and that in asymptotically AdS spaces are expected to be similar (see [88]). In the light of this fact we may conclude that our fluid approximation is unable to capture this phenomenon correctly unless there actually exist a better bound in flat space (which would go as $\frac{1}{d}$ for large d).

In the coordinates that we have used in this section, the first derivative of our solution $\left(\partial_b y(x_a) = \frac{-x_b}{(R^2-\sum x_a^2)^{\frac{1}{2}}}\right)$ is singular near $\sum x_a^2 = R^2$. In fact (from our analysis so far) it is unclear whether a consistent perturbation theory can be performed in the ϵ parameter about the solution (4.2.52). In the later sections we shall move to better coordinates and exhibit the existence of a well controlled perturbation theory about (4.2.52). In certain special cases we shall explicitly compute the first correction to (4.2.52) which occurs at $O(\epsilon^2)$.

Summary

Here we have constructed a class of fluid configurations to the d dimensional Navier-Stokes equation in the generalized thin ring limit. To leading order in the parameter ϵ these fluid configurations are given by,

$$B^{(d-1-n)} \times T^n = B^{(d-1-n)} \times \underbrace{S^1 \times S^1 \ldots\ldots S^1}_{\text{n times}},$$

where $n \leq [\frac{d-1}{2}]$. These configurations are parameterized by the radius of the ball (R) and the radii of the various S^1s ($\ell_0 P_a$). In the generalized thin ring limit locally these configurations are like filled cylinders with the topology $B^{(d-1-n)} \times \mathbb{R}^n$. Then we can bend the different directions in $\mathbb{R}^n$ into S^1s in a controlled way with a perturbation expansion in ϵ. Now the intrinsic fluid parameters (namely the temperature (T) and the angular velocities (ω_a)) are related to the parameters of the fluid configuration (R and $\ell_0 P_a$) by the force balance conditions. The pressure along the radial direction of the ball is balanced by the surface tension. This condition yields

$$T^{d+1} = \left(\frac{(d-n-2)+R}{R}\right)\left(1 - \sum_a (\ell_0 P_a w_a)^2\right)^{\left(\frac{d+1}{2}\right)}$$

On the other hand the pressure along the radial direction of the S^1s is balanced by the centrifugal force. In order to obtain this force balance we require these configurations to be rotating (at least) in the planes in which the S^1s lie. For the sake of simplicity we have turned off angular velocity along any other direction. This force balance determines the angular velocities to be

$$w_a^2 = \frac{1}{(\ell_0 P_a)^2 \left(((d-n-2)+R)(d+1)+n\right)}.$$

These fluid configurations are dual to the horizon topology $S^{(d-n)} \times T^n$. Thus by exploiting the AdS/CFT correspondence we indirectly confirm the existence of such exotic horizon topologies. It is possible to generate a perturbation expansion (in the ϵ parameter) about these solutions as we shall demonstrate in some explicit examples in the later sections. Also the thermodynamic properties of the fluid configurations directly map to that of these exotic black objects. This provides us with a opportunity to study the thermodynamics of these black objects without performing a direct gravity calculation.

4.3 Rings

In this section we shall analyze the topology $B^{d-2} \times S^1$. This topology is the special case of (4.2.43) (for $n = 1$). We shall study this ring type solutions in $3+1$ and $4+1$ dimensions in detail. We use regular coordinates to set up a well controlled perturbation expansion in the parameter ϵ to compute corrections to the *thin ring*. The thin ring solution in $3+1$ dimensions has been well studied (including the thermodynamic properties) in [6]. Here we present the first correction to that solution demonstrating the fact that it is possible to construct such rings as a series in the ϵ parameter. This method can in principle be generalised to higher dimensions (we present the results in $4+1$ dimensions in §4.3.2). All these solutions are consistent with the general case discussed in §4.2 to zeroth order in ϵ.

4.3.1 Rings in 3+1 dimensions

As pointed out previously in §4.2 the coordinates that we used for our general discussion (which are those that are used in [6, 80]) are not suitable for the perturbation theory. Therefore, for the construction of a regular well controlled perturbation theory, we move to some regular coordinates.

Here we use the coordinates $\{t, \rho, \theta, \phi\}$, t being the time coordinate, which are related to those used previously by

$$\begin{aligned} r &= \mathcal{L} + \rho\cos\theta, \\ z &= \rho\sin\theta. \end{aligned} \tag{4.3.59}$$

The metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + (\mathcal{L} + \rho\cos\theta)^2\mathrm{d}\phi^2. \tag{4.3.60}$$

Here $\mathcal{L}$ is a number which we shall determine in terms of the parameters of our system order by order in perturbation theory. Physically $\mathcal{L}$ is the distance of the center of the cylinder (our 0th order solution) from the origin. We simply redefine this center to be our origin in the above metric.

These coordinates are described in fig.4.2.

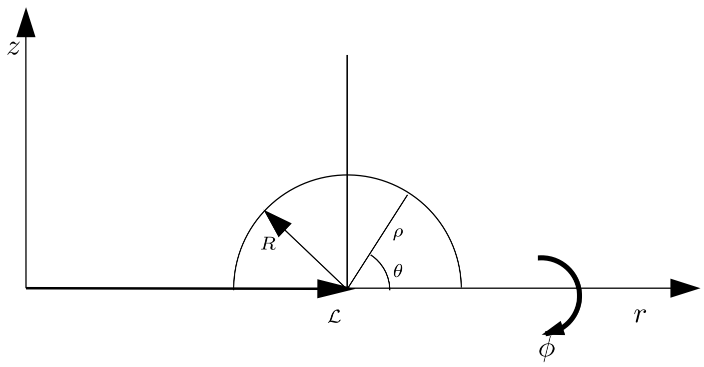


Figure 4.2: Cross section of the 3+1 dimensional ring. The curved arrow labelled ϕ indicates a direction that has been suppressed.

The velocity vector is given by $u^\mu = \gamma(1, 0, 0, \omega)$, where again $\gamma = \left(1 - \omega^2(\mathcal{L} + \rho\cos(\theta))^2\right)^{-\frac{1}{2}}$ is the normalization constant. In this case we consider the fluid surface to be given by

$$f \equiv g(\theta) - \rho = 0.$$

Then (4.1.37) reduces to the following differential equation for the function $g(\theta)$

$$\begin{aligned}
&\frac{T^5}{\left(1-\omega^2(\mathcal{L}+\cos(\theta)g(\theta))^2\right)^{5/2}} - 1 - \frac{1}{(\mathcal{L}+\cos(\theta)g(\theta))\left(g(\theta)^2+g'(\theta)^2\right)^{3/2}}\left(\sin(\theta)g'(\theta)^3\right.\\
&+(2\mathcal{L}+3\cos(\theta)g(\theta))g'(\theta)^2 + g(\theta)^2\sin(\theta)g'(\theta) + g(\theta)\left(g(\theta)(\mathcal{L}+2\cos(\theta)g(\theta))\right.\\
&\left.\left.-(\mathcal{L}+\cos(\theta)g(\theta))g''(\theta)\right)\right) = 0.
\end{aligned} \tag{4.3.61}$$

Now we perform the following scaling:

$$\begin{aligned}
\omega &= \epsilon\, w \\
g(\theta) &= R + \epsilon g_1(\theta) + \epsilon^2 g_2(\theta) + \epsilon^3 g_3(\theta) \\
\mathcal{L} &= \frac{1}{\epsilon}\ell_0 + \ell_1 + \epsilon\ell_2 + \epsilon^2\ell_3
\end{aligned} \tag{4.3.62}$$

where ϵ is the small parameter with which we wish to perform the perturbation. Here g_1, g_2, g_3 are functions to be determined and $\ell_0, \ell_1, \ell_2, \ell_3$ are to be expressed in terms of the fluid parameters. Also note that to zeroth order $g(\theta) = R$, which is the same solution that has been obtained in [6]. Also the solution in [6] (as in the general discussion above) was true up to first order in ϵ. This implies that the first order correction to $g(\theta)$ (i.e. $g_1(\theta)$) should vanish, as we shall shortly show.

To first order in ϵ (4.3.61) reduces to

$$-1 - \frac{1}{R} + \frac{T^5}{(1-w^2\ell_0^2)^{\frac{5}{2}}} = 0. \tag{4.3.63}$$

To higher order in ϵ we obtain differential equations for g_1, g_2, g_3 etc. These differential equations are of the general form

$$g_i(\theta) + g_i''(\theta) = \mathcal{S}_j(\theta). \tag{4.3.64}$$

where $\mathcal{S}(\theta)$ is the source which is determined at a particular order once the the complete solution up to one lower order is completely known. Also note that the homogeneous part of the equation is the same at all orders.

The equation that we obtain at first order is

$$g_1(\theta) + g_1''(\theta) = -R^2\left(\frac{5T^5\ell_0\ell_1 w^2}{\left(1-\ell_0^2w^2\right)^{7/2}} + \left(\frac{5T^5\ell_0 R w^2}{\left(1-\ell_0^2w^2\right)^{7/2}} - \frac{1}{\ell_0}\right)\cos(\theta)\right) \tag{4.3.65}$$

Solving the above equation we obtain

$$\begin{aligned}
g_1(\theta) =& \mathcal{C}_1\cos(\theta) + \mathcal{C}_2\sin\theta \\
&+ \frac{R^2\left(\left(\left(1-\ell_0^2w^2\right)^{7/2} - 5T^5\ell_0^2 R w^2\right)(\cos(\theta)+2\theta\sin(\theta)) - 20T^5\ell_0^2\ell_1 w^2\right)}{4\ell_0\left(1-\ell_0^2w^2\right)^{7/2}}
\end{aligned} \tag{4.3.66}$$

where the $\mathcal{C}_1$ and $\mathcal{C}_1$ are the integration constants to be determined by the boundary conditions. We must also remember that the constants R, ℓ_o, ℓ_1 are also to be determined in terms of the other fluid parameters, viz. K and w.

Now from physical considerations we shall demand that the surface should be a closed surface. This results in the boundary condition

$$g_1'(0) = 0; \quad g_1'(\pi) = 0. \tag{4.3.67}$$

The first condition demands $\mathcal{C}_2 = 0$, while the second condition yields

$$\frac{5T^5\ell_0^2 R w^2}{\left(1-\ell_0^2 w^2\right)^{7/2}} - 1 = 0. \tag{4.3.68}$$

Here the relations (4.3.63) and (4.3.68) may be used to express R and ℓ_0 in terms of the fluid parameters T and w. However for performing the calculations it is more convenient to do the reverse. We then have

$$\begin{aligned} T^5 &= \left(1+\frac{1}{R}\right)\left(\frac{5(1+R)}{6+5R}\right)^{\frac{5}{2}} \\ w &= \frac{1}{\ell_0\sqrt{6+5R}}. \end{aligned} \tag{4.3.69}$$

Plugging back these relations into (4.3.66) we find $g(\theta)$ up to order ϵ, which is given by

$$g(\theta) = R + \epsilon\left(\mathcal{C}_1\cos(\theta) - \frac{\ell_1 R}{\ell_0}\right) + O(\epsilon^2), \tag{4.3.70}$$

where $\mathcal{C}_1$ and ℓ_1 are still to be determined.

Now once we start including corrections we should consider a redundancy in description of the ring which we have to remove by proper gauge fixing. This pertains to the fact that we haven't defined $\mathcal{L}$ properly yet. Vaguely, it is r coordinate of the center of the ring. This becomes ill-defined once we take into consideration the corrections to the thin ring, which was a circle in the ρ-θ plane to zeroth order. This is taken care by the following gauge-fixing condition

$$\int_0^\pi g(\theta)\cos(\theta) = 0. \tag{4.3.71}$$

In words, this condition states that the average r coordinate of the surface in the r-z plane is $\mathcal{L}$. This condition implies $\mathcal{C}_1 = 0$. The constant ℓ_1 will only be determined at next order in epsilon just as ℓ_0 was determined at $\mathcal{O}(\epsilon)$. Therefore we now proceed to the calculation of the second order corrections to ϵ.

The equation for $g_2(\theta)$ is given by the coefficient of ϵ^2 in (4.3.61). After plugging in the solution for $g_1(\theta)$ we obtain

$$g_2(\theta) + g_2''(\theta) = -\frac{R}{10\ell_0^2(R+1)} \left((2-5R)\ell_1^2 + 10\ell_0\ell_2(R+1)\right. \\ \left. + R\cos(\theta)(2\ell_1(5R+12) + R(15R+22)\cos(\theta))\right). \tag{4.3.72}$$

The solution to the above equation is given by

$$\begin{aligned} g_2(\theta) =& \mathcal{C}_3\cos(\theta) + \mathcal{C}_4\sin\theta \\ &+ \frac{R}{60\ell_0^2(R+1)}\left(R(-3\ell_1(5R+19)\cos(\theta) + R(15R+22)\cos(2\theta)\right. \\ &\left. -6\ell_1(5R+12)\theta\sin(\theta)) - 3\left((4-10R)\ell_1^2 + 20\ell_0\ell_2(R+1) + R^2(15R+22)\right)\right) \end{aligned} \tag{4.3.73}$$

where again $\mathcal{C}_3$ and $\mathcal{C}_4$ are integration constants to be determined. The boundary conditions (i.e. $g_2'(0) = 0 = g_2'(\pi)$) again imply $\mathcal{C}_4 = 0$ and the condition

$$\frac{\ell_1\pi R^2(5R+12)}{10\ell_0^2(R+1)} = 0. \tag{4.3.74}$$

The above condition implies $\ell_1 = 0$. At this point the ϵ order solution is completely determined; and we find that corrections to $g(\theta)$ and $\mathcal{L}$ all vanish. However we go ahead further to compute the $\mathcal{O}(\epsilon^2)$ correction completely as it will provide us with the leading order correction. Using the fact $\ell_1 = 0$ and $\mathcal{C}_4 = 0$ we find $g(\theta)$ to be

$$\begin{aligned} g(\theta) =& R + \mathcal{C}_3\cos(\theta)\epsilon^2 \\ &+ \frac{\left(+R\left(R^2(15R+22)\cos(2\theta) - 3\left((15R+22)R^2 + 20\ell_0\ell_2(R+1)\right)\right)\right)}{60\ell_0^2(R+1)}\epsilon^2 + O(\epsilon)^3. \end{aligned} \tag{4.3.75}$$

Now the condition (4.3.71) again implies $\mathcal{C}_3 = 0$. Again in order to determine ℓ_2 we have to perform one higher order calculation. We can then determine ℓ_2 by imposing the boundary conditions on $g_3(\theta)$. Here we intend to present only the leading order corrections and therefore do not specify the details of the third order calculation. However the value of ℓ_2 that we obtain is given by

$$\ell_2 = \frac{R^2\left(225R^2 + 380R + 92\right)}{40\ell_0\left(5R^2 + 17R + 12\right)}. \tag{4.3.76}$$

Note that the denominator in the above expression never vanishes for positive values of R.

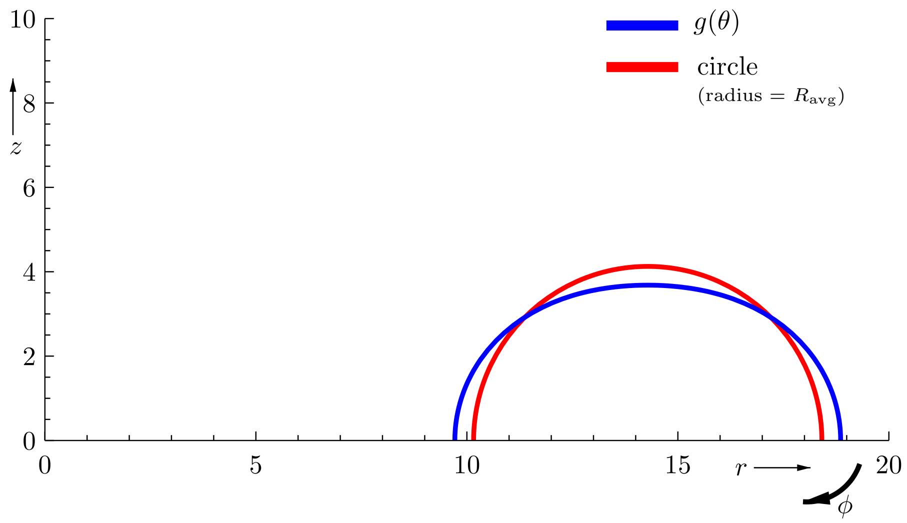


Figure 4.3: A plot of the corrected solution (4.3.77) for the 3+1 dimensional ring with $\frac{\ell_0}{\epsilon} = 14$ and $R = 7$.

In summary we can write

$$\begin{aligned} g(\theta) &= R + \frac{R^3(2(5R+12)(15R+22)\cos(2\theta) - 15(3R(25R+64)+124))\epsilon^2}{120\ell_0^2(R+1)(5R+12)} + O\left(\epsilon^3\right) \\ \mathcal{L} &= \frac{1}{\epsilon}\ell_0 + \epsilon\frac{R^2\left(225R^2+380R+92\right)}{40\ell_0\left(5R^2+17R+12\right)} + O(\epsilon^2) \end{aligned} \tag{4.3.77}$$

As mentioned earlier R and ℓ_0 are related to the fluid parameters K and w through the inverse of the relations (4.3.69). We can carry forward to arbitrary order in ϵ in a perfectly well controlled fashion.

Finally we would like to mention that it is possible to obtain corrections to the radius R from (4.3.77). However, just like the center, the notion of the radius has to be redefined for the corrected solution. This we may do by defining an average corrected radius,

$$R_{\text{avg}} = \int_0^{2\pi} g(\theta) d\theta. \tag{4.3.78}$$

Using (4.3.77) we can compute R_{avg} to be

$$R_{\text{avg}} = R - \frac{\left(R^3(3R(25R+64)+124)\right)\epsilon^2}{8\left(\ell_0^2(R+1)(5R+12)\right)} + \mathcal{O}\left(\epsilon^3\right). \tag{4.3.79}$$

We present a plot of this corrected solution in fig.4.3.

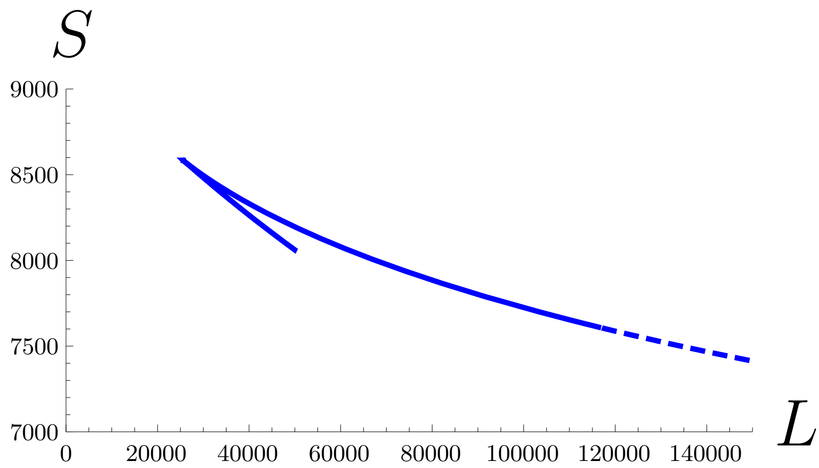


Figure 4.4: Plot of entropy, S, vs. angular momentum, L, at fixed energy, $E = 10,000$, for the ring in 3+1 dimensions. The dotted portion of the curve represents the region where $R < 1$ and hence lie outside the surface tension approximation.

The thermodynamic properties of the solution can be computed from (4.1.38) as follows

$$T \ln \mathcal{Z}_{\text{gc}} = 2\pi \left[\int_0^\pi \mathrm{d}\theta \int_0^{g(\theta)} \mathrm{d}\rho\, \rho(\mathcal{L} + \rho\cos\theta)(K\gamma^5 - 1) \right. \\ \left. - \int_0^\pi \mathrm{d}\theta \sqrt{g(\theta)^2 + g'(\theta)^2}(\mathcal{L} + g(\theta)\cos\theta) \right] \tag{4.3.80}$$

We find

$$T \ln \mathcal{Z}_{\text{gc}} = -\frac{2\pi^2 \ell_0 R}{\epsilon} + \frac{2\pi^2 R^3(15R+22)}{40\ell_0(R+1)}\,\epsilon + \mathcal{O}\left(\epsilon^2\right), \tag{4.3.81}$$

The other thermodynamic properties can be found by differentiating this (4.1.33):

$$\begin{aligned} E &= \frac{2\pi^2\ell_0 R(5R+7)}{\epsilon} - \frac{2\pi^2 R^3(R(5R(105R+353)+2018)+792)}{8(\ell_0(R+1)(5R+12))}\,\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ S &= \frac{2\pi^2\sqrt{5(5R+6)}\ell_0 R^{6/5}(R+1)^{3/10}}{\epsilon} \\ &\qquad - \frac{2\pi^2 R^{16/5}\sqrt{5R+6}(R(5R(105R+353)+2018)+792)}{8\sqrt{5}\ell_0(R+1)^{17/10}(5R+12)}\,\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ L &= \frac{2\pi^2\ell_0^2 R\sqrt{5R+6}}{\epsilon^2} + \frac{2\pi^2 R^3\sqrt{5R+6}(15R+22)}{40(R+1)} + \mathcal{O}\left(\epsilon^1\right). \end{aligned} \tag{4.3.82}$$

Note that to leading order these expressions for energy, entropy and angular momentum match with that presented in [6] after performing the necessary variable transformations.

We present a plot of the thermodynamic properties of this ring in fig.4.4

4.3.2 Rings in 4+1 dimensions

We now analyze the ring in one higher dimension i.e. $4+1$ dimension. The construction is exactly parallel to that in $3+1$ dimensions and therefore we skip most of the details and specify only the result.

The coordinates that we use here are $\{t, \rho, \theta, \phi_1, \phi_2\}$, which are related to the old coordinates by

$$\begin{aligned} r_1 &= \mathcal{L} + \rho \cos\theta, \\ r_2 &= \rho \sin\theta, \end{aligned} \tag{4.3.83}$$

with the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + (\mathcal{L} + \rho\cos\theta)^2 \mathrm{d}\phi_1^2 + (\rho \sin\theta)^2 \mathrm{d}\phi_2^2. \tag{4.3.84}$$

These coordinates are described in fig.4.5.

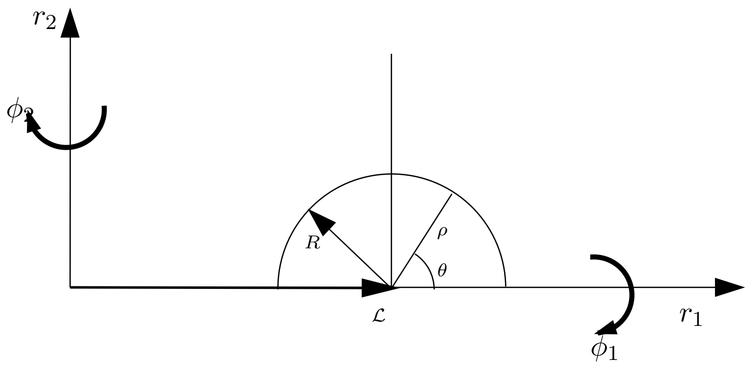


Figure 4.5: Cross section of the 4+1 dimensional ring. The curved arrows labelled ϕ indicate a direction that has been suppressed.

The velocity is given by $u^\mu = \gamma(1,0,0,\omega,0)$ where the normalization is given by $\gamma = (1 - \omega^2(\mathcal{L} + \rho\cos(\theta))^2)^{-\frac{1}{2}}$. We take the fluid surface to be $f \equiv g(\theta) - \rho = 0$. Then the equation (4.1.37) reduces to

$$\begin{aligned} &\frac{T^6}{\left(1-\omega^2(\mathcal{L}+\cos(\theta)g(\theta))^2\right)^3} - 1 - \frac{\left(g(\theta)+g''(\theta)\right)g'(\theta)^2}{g(\theta)^4\left(\frac{g'(\theta)^2}{g(\theta)^2}+1\right)^{3/2}} \\ &+ \frac{1}{(\mathcal{L}+\cos(\theta)g(\theta))\left(g(\theta)^4\left(\frac{g'(\theta)^2}{g(\theta)^2}+1\right)^{3/2}\right)}\left(g(\theta)^2+g'(\theta)^2\right)\left(g(\theta)(2\mathcal{L}+3\cos(\theta)g(\theta)) -\right. \\ &\left. - \csc(\theta)(\mathcal{L}\cos(\theta)+\cos(2\theta)g(\theta))g'(\theta) - (\mathcal{L}+\cos(\theta)g(\theta))g''(\theta)\right) = 0, \end{aligned} \tag{4.3.85}$$

Now we plug in the scaling with ϵ as in the previously discussed cases:

$$\begin{aligned} \omega &= \epsilon\ w \\ g(\theta) &= R + \epsilon g_1(\theta) + \epsilon^2 g_2(\theta) + \dots \\ \mathcal{L} &= \frac{1}{\epsilon}\ell_0 + \ell_1 + \epsilon \ell_2 + \dots \end{aligned} \tag{4.3.86}$$

Then, performing an analysis exactly the same as the one performed in §4.3.1, we obtain

$$\begin{aligned} g(\theta) &= R + \frac{R^3\left(5\left(81R^2+432R+572\right)\cos(2\theta) - 3\left(243R^2+1040R+1108\right)\right)}{240\ell_0^2\left(9R^2+44R+52\right)}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \\ \mathcal{L} &= \frac{1}{\epsilon}\ell_0 + \frac{R^2\left(81R^2+240R+116\right)}{30\ell_0\left(9R^2+44R+52\right)}\epsilon + \mathcal{O}\left(\epsilon^2\right), \end{aligned} \tag{4.3.87}$$

with R and ℓ_0 being expressed in terms of the fluid parameters T and w (implicitly) by the following relations

$$\begin{aligned} T^6 &= \frac{216(R+2)^4}{R(6R+13)^3}, \\ w &= \frac{1}{\ell_0\sqrt{6R+13}}. \end{aligned} \tag{4.3.88}$$

Even in this case we note that all the $\mathcal{O}(\epsilon)$ corrections vanish.

Finally the average radius of the curve in (4.3.87) (R_{avg} as defined in (4.3.78)) is given in this case by

$$R_{\text{avg}} = R - \frac{\left(R^3(243R+554)\right)\epsilon^2}{80\left(\ell_0^2(9R+26)\right)} + \mathcal{O}\left(\epsilon^3\right). \tag{4.3.89}$$

We present a plot of this corrected solution in fig.4.6.

The thermodynamic properties of the solution can be computed from (4.1.38) as follows

$$\begin{aligned} T\ln\mathcal{Z}_{\text{gc}} = (2\pi)^2 \Bigg[&\int_0^\pi \mathrm{d}\theta \int_0^{g(\theta)} \mathrm{d}\rho\,\rho^2 \sin\theta(\mathcal{L}+\rho\cos\theta)(K\gamma^6-1) \\ &- \int_0^\pi \mathrm{d}\theta \sqrt{g(\theta)^2+g'(\theta)^2}(\mathcal{L}+g(\theta)\cos\theta)g(\theta)\sin\theta \Bigg] \end{aligned} \tag{4.3.90}$$

We find

$$\begin{aligned} T\ln\mathcal{Z}_{\text{gc}} =& \frac{4\pi^2\ell_0 R^2}{9\epsilon}\left(-6R + \sqrt[3]{R+2}\sqrt{36R+78}\sqrt[6]{R} - 18\right) \\ &+ \frac{2\pi^2 R^4}{135\ell_0(R+2)^{5/3}(9R+26)}\Big(-3216\sqrt{36R+78}R^{7/6} - 243\sqrt{36R+78}R^{19/6} \\ &-1524\sqrt{36R+78}R^{13/6} + 1458(R+2)^{2/3}R^3 + 9630(R+2)^{2/3}R^2 \\ &+21888(R+2)^{2/3}R - 2288\sqrt{36R+78}\sqrt[6]{R} + 17160(R+2)^{2/3}\Big)\epsilon \\ &+ \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.3.91}$$

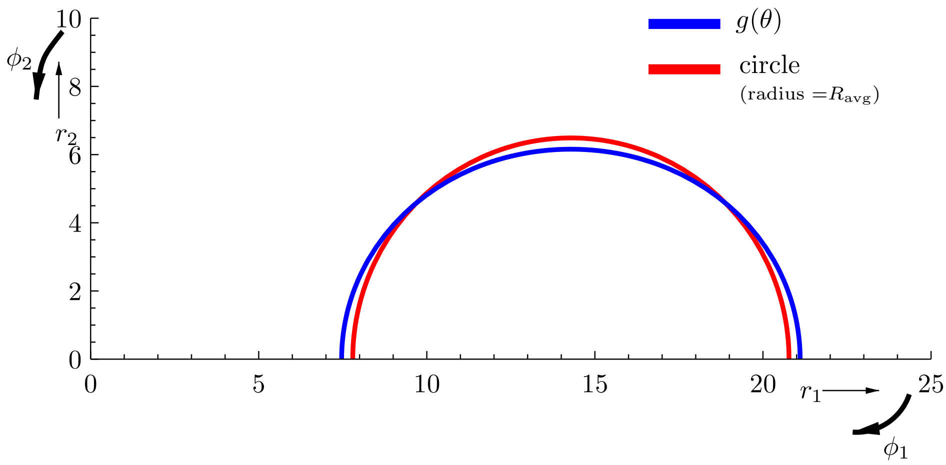


Figure 4.6: A plot of the corrected solution (4.3.87) for the 4+1 dimensional ring with $\frac{\ell_0}{\epsilon} = 14$ and $R = 7$.

For expressions of the energy, angular momentum and entropy refer to Appendix C.2. We present a plot of the thermodynamic properties of this ring in fig.4.7

4.4 'Torus' in 4+1 dimension

Here we analyze the solution with the topology $B^2 \times S^1 \times S^1$ which for the lack of terminology we refer to as the 'torus'. Although the perturbation theory for the torus is almost exactly parallel to that for the ring, however there are certain differences as far as imposing the boundary condition is concerned. We consider spatial part of the 4+1 dimensional space to consist of 2 independent planes. We turn on two independent angular velocities (ω_1, ω_2) along a direction orthogonal to these planes. The two S^1s of the topology $B^2 \times S^1 \times S^1$ lie on these two planes. Initially we put polar coordinates r_1, ϕ_1 and r_2, ϕ_2 on these two planes. Further in the (r_1, r_2) plane we shift to coordinates ρ, θ after a shift in the origin by the vector $\overrightarrow{\mathcal{L}} = (\mathcal{L}\cos(\chi), \mathcal{L}\sin(\chi))$ (expressed in the (r_1, r_2) coordinates). The various coordinates have been represented in fig.4.8. Thus finally we work with the coordinates $\{t, \rho, \theta, \phi_1, \phi_2\}$, which are related to the old coordinates by

$$\begin{aligned} r_1 &= \mathcal{L}\cos\chi + \rho\cos\theta, \\ r_2 &= \mathcal{L}\sin\chi + \rho\sin\theta, \end{aligned} \tag{4.4.92}$$

with the metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2 + (\mathcal{L}\cos\chi + \rho\cos\theta)^2 d\phi_1^2 + (\mathcal{L}\sin\chi + \rho\sin\theta)^2 d\phi_2^2 \tag{4.4.93}$$

Here the velocity is given by $u^\mu = \gamma(1, 0, 0, \omega_1, \omega_2)$ where again the normalization is given by $\gamma = (1 - (\mathcal{L}\cos\chi + \rho\cos\theta)^2\omega_1^2 - (\mathcal{L}\sin\chi + \rho\sin\theta)^2\omega_2^2)^{-\frac{1}{2}}$. We consider the fluid surface

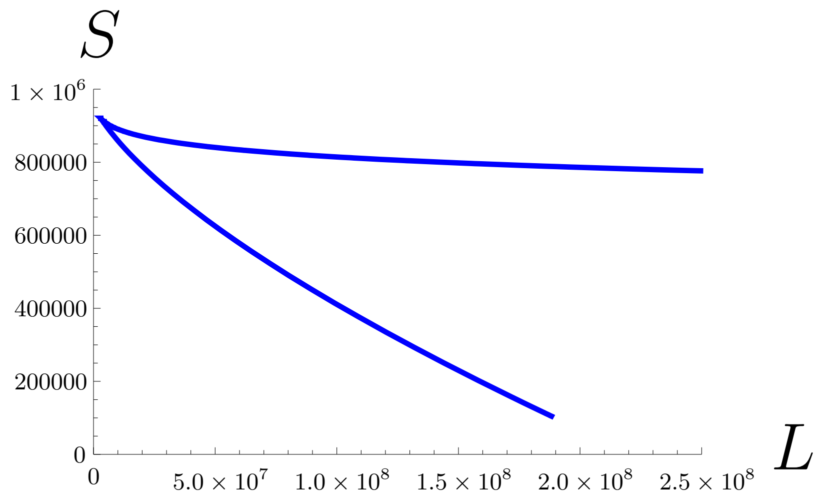


Figure 4.7: Plot of entropy, S, vs. angular momentum, L, at fixed energy, $E = 1,000,000$, for the ring in 4+1 dimensions.

to be given by $f \equiv g(\theta) - \rho = 0$.

Now the differential equation satisfied by $g(\theta)$ (which is (4.1.37) for the present case) is given by

$$
\begin{aligned}
&\frac{T^6}{\left(1-\omega_1^2(\mathcal{L}\cos\chi + g(\theta)\cos\theta)^2 - \omega_2^2(\mathcal{L}\sin\chi + g(\theta)\sin\theta)^2\right)^3} - 1 - \frac{(g(\theta)+g''(\theta))\,g'(\theta)^2}{g(\theta)^4\left(\frac{g'(\theta)^2}{g(\theta)^2}+1\right)^{3/2}} \\
&+ \frac{\left(g(\theta)^2+g'(\theta)^2\right)}{g(\theta)^2\left(\frac{g'(\theta)^2}{g(\theta)^2}+1\right)^{3/2}}\left(\frac{\cos(\theta)g'(\theta)-g(\theta)\sin(\theta)}{\sin(\theta)g(\theta)^2+\mathcal{L}\sin(\chi)g(\theta)} - \frac{\cos(\theta)g(\theta)+\sin(\theta)g'(\theta)}{\cos(\theta)g(\theta)^2+\mathcal{L}\cos(\chi)g(\theta)}\right. \\
&\left. + \frac{g''(\theta)-g(\theta)}{g(\theta)^2}\right) = 0,
\end{aligned}
\tag{4.4.94}
$$

Now we again consider the following scaling:

$$
\begin{aligned}
\omega_1 &= \epsilon\ w_1 \\
\omega_1 &= \epsilon\ w_2 \\
g(\theta) &= R + \epsilon\ g_1(\theta) + \epsilon^2\ g_2(\theta) + \dots \\
\mathcal{L} &= \frac{1}{\epsilon}\ell_0 + \ell_1 + \epsilon\ \ell_2 + \dots \\
\chi &= \chi_0 + \epsilon\ \chi_1 + \epsilon^2\ \chi_2 + \dots
\end{aligned}
\tag{4.4.95}
$$

We shall determine the unknown functions in a similar way as we did for the ring. However there is a crucial difference between the two. Firstly here we have one more parameter (since we have two angular velocities instead of one). Secondly the boundary condition that we have to impose on $g(\theta)$ is different from the previous case because here we are dealing with a different closed surface. Although physically it is the same criterion – the fact that we

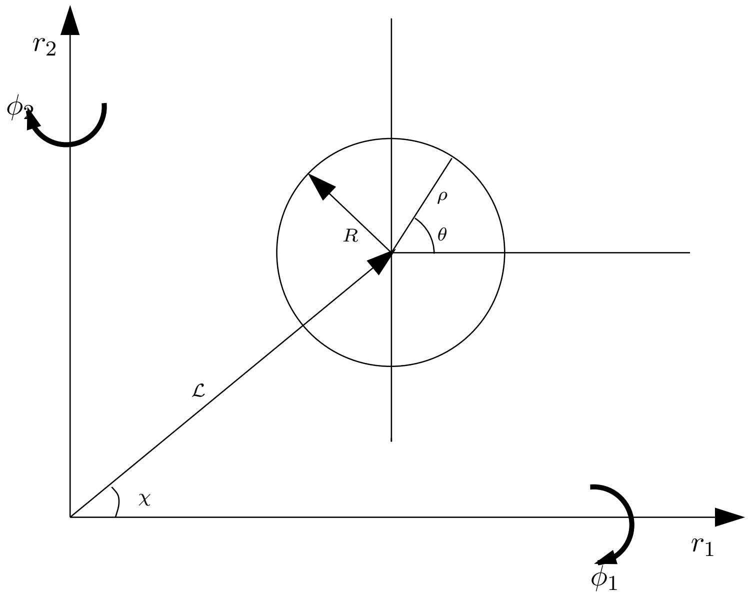


Figure 4.8: Cross section of the 4+1 dimensional torus. The curved arrows labelled ϕ indicate a direction that has been suppressed.

should have a closed surface, the mathematical formulation of the statement is different as we shall now describe.

Instead of the boundary condition (4.3.67) we should use the condition

$$g(0) = g(2\pi); \quad g'(0) = g'(2\pi). \tag{4.4.96}$$

which is the statement that we should have a closed curve in the ρ-θ plane and that the curve must close in a regular fashion such that the derivatives on either side of the point of closing (which we take to be $\theta = 0$) are equal. As the differential equation is second order and periodic, this ensures that all higher derivatives are continuous at $\theta = 0$ and the solution is fully periodic.

Besides the boundary conditions we will also have to fix the ambiguity regarding the center of the torus just as we did for the ring. However unlike the ring the center here does not lie on the r_2 axis. Therefore in order to fix the center we will have to use two conditions namely

$$\begin{aligned} \int_0^{2\pi} g(\theta)\cos(\theta) = 0, \\ \int_0^{2\pi} g(\theta)\sin(\theta) = 0. \end{aligned} \tag{4.4.97}$$

In words, these conditions states that the average r_1 coordinate of the surface in the r_1-r_2 plane is $\mathcal{L}\cos\chi$ and the average r_2 coordinate is $\mathcal{L}\sin\chi$. Then proceeding in the same way as the ring (after including the above modifications) we find the following result:

$$
\begin{aligned}
g(\theta) &= R - \frac{1}{72\left(\ell_0^2(R+1)\right)}\left(R^3\left(9\left(27R^2+74R+51\right)+36(R+1)\cos(2\theta-2\chi_0)\right.\right.\\
&\quad \left.\left.+4(9R+17)\cos(2(\theta+\chi_0))\right)\csc^2(2\chi_0)\right)\epsilon^2 + O\left(\epsilon^3\right)\\
\mathcal{L} &= \frac{\ell_0}{\epsilon} + \frac{R^2\left(81R^2+150R+8\cos(4\chi_0)+57\right)\csc^2(2\chi_0)}{48\ell_0(R+1)}\epsilon + O(\epsilon^2)\\
\chi &= \chi_0 - \frac{R^2\cot(2\chi_0)}{3\ell_0^2(R+1)}\epsilon^2 .
\end{aligned} \tag{4.4.98}
$$

Note that just as for the ring all the order ϵ corrections vanish. Here R, ℓ_0 and χ_0 are again given implicitly in terms of the fluid parameters T, w_1 and w_2 by the following relations,

$$
\begin{aligned}
T^6 &= \frac{27(R+1)^4}{R(3R+4)^3}\,,\\
w_1 &= \frac{1}{\ell_0\cos(\chi_0)}\sqrt{\frac{1}{8+6R}}\,,\\
w_2 &= \frac{1}{\ell_0\sin(\chi_0)}\sqrt{\frac{1}{8+6R}}\,.
\end{aligned} \tag{4.4.99}
$$

Note that this entirely matches the general results (4.2.53) and (4.2.55) for $d=5$ and $n=2$.

Again in this case, the average radius of the closed curve (4.4.98) (with R_{avg} as defined in (4.3.78)) is given by,

$$
R_{\text{avg}} = R - \frac{\epsilon^2\left(R^3\left(27R^2+74R+51\right)\csc^2(2\chi_0)\right)}{8\left(\ell_0^2(R+1)\right)} + \mathcal{O}\left(\epsilon^3\right). \tag{4.4.100}
$$

We present a plot of this corrected solution in fig.4.9.

We expect our construction of the torus solution to break down when $\ell_0 \sim R$. Which is reflected in the fact that $\frac{g_2(\theta)}{R}$, $\frac{\ell_2}{\ell_o}$ and $\frac{\chi_2}{\chi_0}$ are all proportional to $\left(\frac{R}{\ell_0}\right)^2$.

The thermodynamic properties of the solution can be computed from (4.1.38) as follows

$$
\begin{aligned}
T\ln\mathcal{Z}_{\text{gc}} = (2\pi)^2 &\left[\int_0^{2\pi}\mathrm{d}\theta\int_0^{g(\theta)}\mathrm{d}\rho\,\rho(\mathcal{L}\cos\chi+\rho\cos\theta)(\mathcal{L}\sin\chi+\rho\sin\theta)(K\gamma^6-1)\right.\\
&\left.-\int_0^{2\pi}\mathrm{d}\theta\sqrt{g(\theta)^2+g'(\theta)^2}(\mathcal{L}\cos\chi+g(\theta)\cos\theta)(\mathcal{L}\sin\chi+g(\theta)\sin\theta)\right]
\end{aligned} \tag{4.4.101}
$$

We find

$$
\begin{aligned}
T\ln\mathcal{Z}_{\text{gc}} &= \frac{2\pi^3\ell_0^2R\sin(2\chi_0)}{3\epsilon^2}\left(-3R+\sqrt[3]{R+1}\sqrt{9R+12}\sqrt[6]{R}-6\right)\\
&+\frac{\pi^3R^3\csc(2\chi_0)}{36(R+1)^{5/3}}(3(R+1)^{2/3}(R(3R(27R+98)+401)+208)\\
&-\sqrt[6]{R}(3R+4)\sqrt{9R+12}(R(27R+62)+39))+\mathcal{O}\left(\epsilon^1\right),
\end{aligned} \tag{4.4.102}
$$

Again the expressions for the energy, angular momentum and entropy are given in Appendix

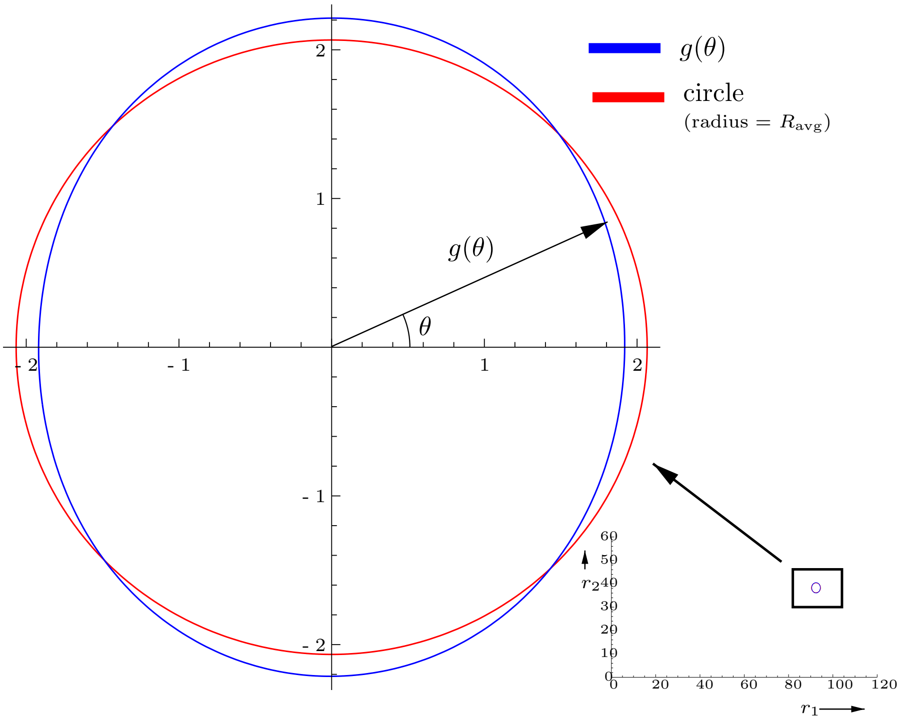


Figure 4.9: A plot of the corrected solution (4.4.98) for the 4+1 dimensional torus with $\frac{\ell_0}{\epsilon} = 100$, $\chi_0 = \frac{\pi}{8}$ and $R = 10$.

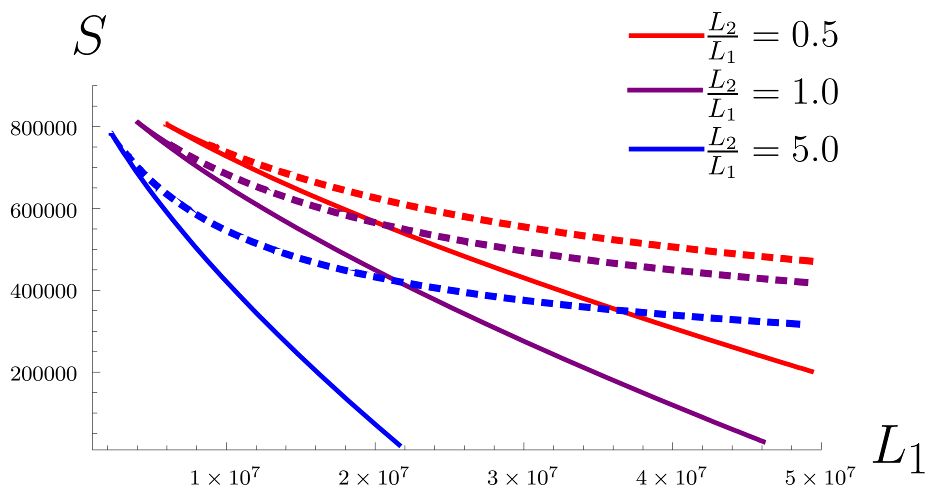


Figure 4.10: Plot of entropy, S, vs. angular momentum, L_1, at fixed energy, $E = 1,000,000$, and different values of $\frac{L_2}{L_1}$ for the torus in 4+1 dimensions. Here again the dotted portion of the curve represents the region $R < 1$ which is outside the validity of the surface tension approximation.

C.2. We present a plot of the thermodynamic properties of this torus in fig.4.10

4.5 Numerical Results

We performed a through numerical analysis of the possible fluid configurations in 3+1 dimensions (which corresponded to Scherk-Schwarz compactified AdS_6). Upto numerical acuracies we found that the only configurations that are allowed in this dimension is a ball, a pinched ball and a torus. We then went on to plot the regions of their existance in the energy-angular momentum plane (see fig.4.11)

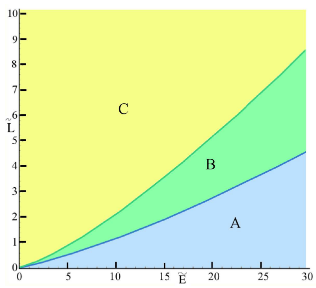


Figure 4.11: Here we present the $\tilde{E}$ $\tilde{L}$ plane showing regions where the various solutions viz. the ordinary ball, the pinched ball and the ring exists. In region A we have only a single ball solution. In region B we have one ball (either ordinary or pinched) and two ring solutions. In region C we have a single (thin) ring solution.

we see that the $\tilde{E}\tilde{L}$ plane is divided into three distinct regions, A, B, C in fig:4.12. In region A we have only a single ball solution. In region B we have one ball and two ring solutions.

In region C we have a single (thin) ring solution [2].

In the region B of fig:4.12, the solution of maximum entropy will dominate the thermodynamics of the system. In order to see how this goes we once again fix to a given value of energy and plot the entropy versus angular momentum of ring solutions fig:4.12(see right inset), where we superpose this plot with that of the ball solution to obtain the entropy versus angular momentum plot of all solutions in fig:4.12 at constant energy. The shape of this final plot is schematically depicted in fig:4.13(b) We see from this plot that the thick ring solution is always entropically subdominant compared to the ball and the thin ring. Thus the ball dominates at smaller angular momenta, while the thin ring is entropically dominant at larger angular momenta.

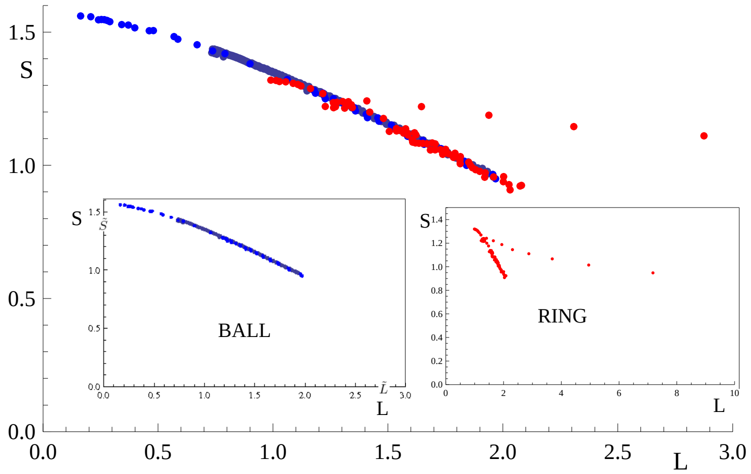


Figure 4.12: Plot showing $\tilde{S}$(y-axis) vs $\tilde{L}$(x-axis) for all the solutions together.

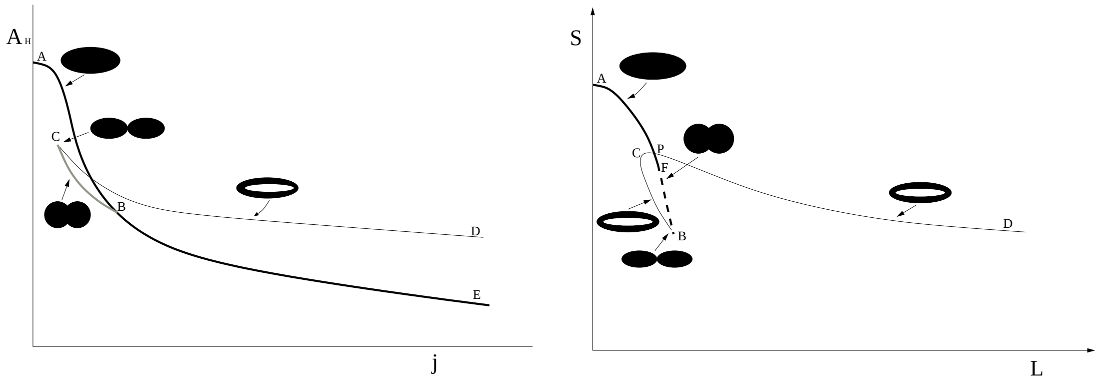
4.5.1 Comparison with the results in flat space

Above we have used the effective fluid dynamical description of the dynamics of the deconfined phase of the field theory dual to gravity on AdS_6 compactified on a Scherk-Schwarz circle to investigate the structure of large rotating black holes and black rings in this gravitational background. In this section we qualitatively compare our results with known results and conjectures about the structure of black holes and black rings in flat six dimensional space.

While rotating Myers Perry black holes have been analytically constructed in flat space six dimensional space, black ring solutions have not yet been constructed. Nonetheless Emparan and collaborators [83] have presented physically motivated conjectures for the structure of these solutions. In this section we will compare the moduli space of solutions obtained in this paper with that conjectured in [83]. We will find some similarities but also other differences.

In fig:4.13(a) we present the relevant part of the conjectured phase diagram (the area of the horizon vs the angular momentum) in asymptotically flat six dimensional space. Here

[2] A portion of this thin ring solution are captubred by our perturbation theory.



(a) (b)

Figure 4.13: In (a) the area of the horizon has been plotted against angular momentum for black hole topologies in six dimensional asymptotically flat space as in [83] (we present only that part of the phase curve that is relevant for comparison with our result). In (b) we summarize our result qualitatively.

the dark line from A to E through B represents the well known Myers-Perry black holes which is a class of rotating black holes with a spherical (S^4-ordinary ball) horizon topology. The thin line from C to D represents the black ring (with $S^3 \otimes S^1$ topology) and the grey thick line from B to C is the conjectured smooth interpolation from the ordinary ball to the ring type through the 'pinched ball' solutions, with the point C being the point of extreme pinching. It is important to note however that the authors in [83] make it clear that this part of the diagram is a guess.

In fig:4.13(b) we present qualitatively the phase diagram (Entropy vs angular momentum at constant energy) obtained by us for asymptotically AdS_6 spaces. Here the dark line from A to F represents the rotating black hole with spherical topology which is the analog of Myers-Perry black holes in asymptotically flat space. The two phase diagrams have several points of difference. Firstly, at any given energy there exists a Myers-Perry black holes at every value of angular momentum no matter how large. In contrast, at any given energy the ball like fluid solutions determined in this paper exist up to only a finite value of angular momentum. Also in fig:4.13(b) the segment F to B represents the pinched ball configuration. Note that the ordinary ball smoothly continues into the pinched ball at F. This is unlike the flat space predictions where there is a kink at the point where the ordinary ball continues into a pinched ball (see point B in fig:4.13(a)).

This important difference feeds into the next point of distinction between fig:4.13(a) and fig:4.13(b). The moduli space of balls ends, at large angular momenta, as an extreme pinched ball. This ball smoothly turns into a thick ring giving rise to the segment BC in fig:4.13(b) with point B representing the extreme pinched ring. The natural continuation of fig:4.13(b) to flat space would be to push the point B to infinity as a consequence of which the Myers Perry black holes would 'pinch off' only at infinite angular momentum. This would result in a phase diagram with three solutions at all angular momenta larger than a critical value, with the thick ring always entropically subdominant, and approaching a pinch at infinite angular momenta. In such a scenario the fact that the thin ring is entropically dominant compared to the Myers Perry black holes is similar to that in fig:4.13(a). However it would still be qualitatively different from fig:4.13(a). This perhaps suggest that despite similarities there are considerable difference between asymptotically AdS and flat spaces. Such differences between AdS and flat space in five dimensions regarding existence of Saturn type solutions have also been reported in [90]. Alternatively the continuation of fig:4.13(b) to flat space could be such that the ordinary ball solution is continued from the point F to infinite angular momentum such that the thin ring at large angular momentum is always entropically dominant compared to the ordinary ball. This diagram would be more close to fig:4.13(a). Both these diagrams would have a special point where the solution with ball topology would split up into two lines. It would clearly be interesting to have a better understanding of this point.

Chapter 5

Conclusions

In this thesis we have explored the deep connection between black holes in AdS spaces and the hydrodynamic limit of Yang-Mills theories. This fluid gravity map constitutes an area of active recent research. One of the primary reasons for interest in fluids with a dual gravity description is the fact that these are fluids with a microscopic theory that is strongly coupled (see chapter 1) and therefore it is extremely difficult to get analytical control on the transport properties of these fluids. It is nevertheless possible to use the dual gravitational solutions for studying the hydrodynamics of these strongly coupled theories. Such recent studies has proved to provide us with new insights into fluid dynamics (for instance, the discovery of a lower bound on the sheer viscosity to entropy density ratio). In this thesis, we undertake a detailed study of this fluid gravity correspondence in a more general set up.

Firstly we generalized the fluid gravity map to include fluids with a global $U(1)$ charge, which may be anomalous. In this case we discovered a completely new transport phenomenon related to the vorticity of the fluid with the corresponding transport coefficient being proportional to the coefficient of the anomaly at first order in the derivative expansion. This transport coefficient was later argued to occur even for non-conformal fluids (which are not captured by our gravity analysis). All that was required for it to be non-zero was an anomaly of the global charge. Thus although discovered in the context of the gauge gravity duality, this transport coefficient has potential applications for real charged fluids if such a fluid suffers from an anomaly. It is also fascinating to realize that this transport coefficient is a macroscopic manifestation of a quantum phenomenon (anomalies) and therefore is very interesting in its own right.

We then went on to consider the case when this $U(1)$ symmetry of the charged fluid is spontaneously broken, i.e. superfluids. In this case, guided by our gravity calculations we were able to construct a theory of (parity even) superfluid hydrodynamics. Even in this case, we found the existence of a new transport coefficient which to our knowledge was not considered earlier in the superfluid literature (classic references on the subject like [71, 72] miss this term). This new term in the constitutive relations indicates the presence of a interesting transport phenomenon (not studied till date) which may even be observable in real superfluids like liquid helium. However the observation of such a phenomenon may be experimentally challenging as it is observable only for finite superfluid velocities and most superfluids are unstable beyond a particular superfluid velocity (which may be quite small for real superfluids).

Finally we used the fluid gravity map in a reverse fashion to study thermodynamic properties of a class of black objects in Scherk-Schwarz (SS) compactified AdS, with completely new horizon topologies. It will be fascinating to construct these solutions directly in gravity and verify our predictions. This investigation is primarily obstructed by the fact that the domain wall solution in the bulk separating the confined and deconfined phase in the boundary at the transition temperature is only known numerically. Further, the fluid configurations that we study have boundaries which play a crucial role in their dynamics. These boundaries should support local fluctuations which are expected to interact non-trivially with the bulk shear and density waves. A detailed study of these fluctuations which forms an important part of the dynamical perturbations of our static configurations may throw light on the stability properties of these objects.

The investigations referred to in this synopsis opens up several interesting questions that require future investigation. From the existence of soliton solutions in AdS reported in [84, 85], we may conclude that such non-trivial scalar field configurations would also exist for the SS compactified AdS [91]. It would be interesting to study the hydrodynamics dual to these solutions as it would be the first instance of a case where we expect to see non-trivial long wavelength phenomena even in the absence of a horizon. Also if these solutions exist then it is natural to wonder if they can be used to add scalar hair to our generalized black rings in SS compactified AdS.

Another very interesting avenue of research that we are guided into by our study here is as follows. In developing the theories of hydrodynamics both in the presence and absence of superfluidity we have found that the principle of local increase of entropy was extremely powerful. For example in the case of parity even superfluids considered here this principle cuts down the total number of allowed constitutive parameters from 50 (which are allowed by symmetry) to 21 (the Onsager's relations bringing it down further to 14). This throws open the question whether such a principle may be used to constrain (higher derivative) corrections to the theory of gravity. In fact this question may be addressed in the fluid gravity context which helps us identify non-trivial fluid configurations where there is no entropy production. We may try to see if small corrections to these configurations due to the addition of a higher derivative term in the bulk lagrangian renders the divergence of the entropy current negative. Then for consistency with second law of thermodynamics we should demand that such a higher derivative term is disallowed as a correction to Einstein general relativity. If this program can be realized then it would throw enormous light on the physics of gravity in the real world.

Appendix A

Charged conformal fluids: Weyl covariance and Source terms

A.1 Charged conformal fluids and Weyl covariance

Consider the hydrodynamic limit of a 3 + 1 dimensional CFT with one global conserved charge. The Weyl covariance of the CFT translates into the Weyl covariance of its hydrodynamics. In turn, this implies that the metric dual to fluid configurations of the CFT under consideration should also be invariant under boundary Weyl-transformations [9, 10, 14].

In this section, we use the manifestly Weyl-covariant formalism introduced in [14] to examine the constraints that Weyl-covariance imposes on the conformal hydrodynamics and its metric dual. We begin by introducing a Weyl-covariant derivative acting on a general tensor field $Q^{\mu\cdots}_{\nu\ldots}$ with weight w (by which we mean that the tensor field transforms as $Q^{\mu\cdots}_{\nu\ldots} = e^{-w\phi}\tilde{Q}^{\mu\cdots}_{\nu\ldots}$ under a Weyl transformation of the boundary metric $g_{\mu\nu} = e^{2\phi}g_{\mu\nu}$)

$$\begin{aligned}\mathcal{D}_\lambda\, Q^{\mu\cdots}_{\nu\ldots} \equiv \nabla_\lambda\, Q^{\mu\cdots}_{\nu\ldots} &+ w\ \mathcal{A}_\lambda Q^{\mu\cdots}_{\nu\ldots} \\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right] Q^{\alpha\cdots}_{\nu\ldots} + \ldots \\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right] Q^{\mu\cdots}_{\alpha\ldots} - \ldots\end{aligned} \tag{A.1.1}$$

where the Weyl-connection $\mathcal{A}_\mu$ is related to the fluid velocity u^μ via the relation

$$\mathcal{A}_\mu = u^\lambda\nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{3}u_\mu \tag{A.1.2}$$

We can now use this Weyl-covariant derivative to enumerate all the Weyl-covariant scalars, transverse vectors (i.e, vectors that are everywhere orthogonal to the fluid velocity field u^μ) and the transverse traceless tensors in the charged hydrodynamics that involve no more than second order derivatives. We will do this enumeration 'on-shell', i.e., we will enumerate those quantities which remain linearly independent even after the equations of motion are taken into account. Our discussion here will closely parallel the discussion in section 4.1 of [10] where a similar question was answered in the context of uncharged hydrodynamics coupled to a scalar with weight zero. However, we will use a slightly different basis of Weyl-covariant tensors which is more suited for purposes of this paper.

The basic fields in the charged hydrodynamics are the fluid velocity u^μ with weight unity, the fluid temperature T with with weight unity and the chemical potential μ with weight unity. This implies that an arbitrary function of μ/T is Weyl-invariant and hence one could always multiply a Weyl-covariant tensor by such a function to get another Weyl-covariant tensor. Hence, in the following list only linearly independent fields appear. To make contact with the conventional literature on hydrodynamics we will work with the charge density n (with weight 3) rather than the chemical potential μ.

At one derivative level, there are no Weyl invariant scalars or pseudo-scalars. The only Weyl invariant transverse vector is $n^{-1}P_\mu^\nu \mathcal{D}_\nu n$. Finally, the only Weyl-invariant transverse pseudo-vector l_μ and only one Weyl-invariant symmetric traceless transverse tensor $T\sigma_{\mu\nu}$.

At the two derivative level, there are five independent Weyl-invariant scalars[1]

$$T^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad T^{-2}\omega_{\mu\nu}\omega^{\mu\nu}, \quad T^{-2}\mathcal{R}, \quad T^{-2}n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n \quad \text{and} \quad T^{-2}n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\mathcal{D}_\nu n \tag{A.1.4}$$

one Weyl-invariant pseudo-scalar $T^{-2}n^{-1}l^\mu\mathcal{D}_\mu n$ and four independent Weyl-invariant transverse vectors

$$T^{-1}P_\mu^\nu\mathcal{D}_\lambda\sigma_\nu{}^\lambda, \qquad T^{-1}P_\mu^\nu\mathcal{D}_\lambda\omega_\nu{}^\lambda, \qquad T^{-1}n^{-1}\sigma_\mu{}^\lambda\mathcal{D}_\lambda n \qquad \text{and} \qquad T^{-1}n^{-1}\omega_\mu{}^\lambda\mathcal{D}_\lambda n \tag{A.1.5}$$

and one Weyl-invariant transverse pseudo-vector $T^{-1}\sigma_{\mu\nu}\, l^\nu$.

There are eight Weyl-invariant symmetric traceless transverse tensors -

$$\begin{gathered} u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}, \quad \omega_\mu{}^\lambda\sigma_{\lambda\nu}+\omega_\nu{}^\lambda\sigma_{\lambda\mu}, \quad \sigma_\mu{}^\lambda\sigma_{\lambda\nu}-\frac{P_{\mu\nu}}{3}\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad \omega_\mu{}^\lambda\omega_{\lambda\nu}+\frac{P_{\mu\nu}}{3}\,\omega_{\alpha\beta}\omega^{\alpha\beta}, \\ n^{-1}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha\mathcal{D}_\beta n, \quad n^{-2}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha n\,\mathcal{D}_\beta n, \quad C_{\mu\alpha\nu\beta}u^\alpha u^\beta \quad \text{and} \quad \frac{1}{4}\,\epsilon^{\alpha\beta\lambda}{}_\mu\,\epsilon^{\gamma\theta\sigma}{}_\nu C_{\alpha\beta\gamma\theta}\,u_\lambda u_\sigma. \end{gathered} \tag{A.1.6}$$

where we have introduced the projection tensor $\Pi_{\mu\nu}^{\alpha\beta}$ which projects out the transverse traceless symmetric part of second rank tensors

$$\Pi_{\mu\nu}^{\alpha\beta} \equiv \frac{1}{2}\left[P_\mu^\alpha P_\nu^\beta + P_\nu^\alpha P_\mu^\beta - \frac{2}{3}P^{\alpha\beta}P_{\mu\nu}\right]$$

and $C_{\mu\nu\alpha\beta}$ is the boundary Weyl curvature tensor. Further, there are four Weyl-invariant

[1] We shall follow the notations of [14] in the rest of this section(except for the curvature tensors which differ by a sign from the curvature tensors in [14]. In particular, we recall the following definitions

$$\begin{aligned} \mathcal{R} &= R + 6\nabla_\lambda\mathcal{A}^\lambda - 6\mathcal{A}_\lambda\mathcal{A}^\lambda\ ; \qquad & \mathcal{D}_\mu u_\nu &= \sigma_{\mu\nu}+\omega_{\mu\nu} \\ \mathcal{D}_\lambda\sigma^{\mu\lambda} &= \nabla_\lambda\sigma^{\mu\lambda} - 3\mathcal{A}_\lambda\sigma^{\mu\lambda}\ ; \qquad & \mathcal{D}_\lambda\omega^{\mu\lambda} &= \nabla_\lambda\omega^{\mu\lambda} - \mathcal{A}_\lambda\omega^{\mu\lambda} \end{aligned} \tag{A.1.3}$$

Note that in a flat space-time, R is zero but $\mathcal{R}$ is not.

symmetric traceless transverse pseudo-tensors

$$\mathcal{D}_{(\mu}l_{\nu)},\quad n^{-1}\Pi^{\alpha\beta}_{\mu\nu}l_\alpha\mathcal{D}_\beta n,\quad n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_\alpha\mathcal{D}_\beta n\quad\text{and}\quad \frac{1}{2}\epsilon_{\alpha\beta\lambda(\mu}C^{\alpha\beta}{}_{\nu)\sigma}u^\lambda u^\sigma. \tag{A.1.7}$$

We will now restrict ourselves to the case where the boundary metric is flat. In this case the last two tensors appearing in (A.1.6) and the last tensor appearing in (A.1.7) are identically zero whereas, contrary to what one might naively expect, the Weyl-covariantised Ricci scalar $\mathcal{R}$ would still be non-zero.

We will now relate the rest of the Weyl-covariant scalars, transverse vectors and symmetric, traceless transverse tensors listed above to the quantities appearing in the table 3.1.

There are six scalar/pseudo-scalar Weyl covariant combinations given by

$$\begin{aligned}
W_s^1 &\equiv \sigma_{\mu\nu}\sigma^{\mu\nu} = \mathrm{ST5}\\
W_s^2 &\equiv \omega_{\mu\nu}\omega^{\mu\nu} = \frac{1}{2}\mathrm{ST4}\\
W_s^3 &\equiv \mathcal{R} = 14\ \mathrm{ST1} + \frac{2}{3}\mathrm{ST3} - \mathrm{ST4} + 2\mathrm{ST5} - \frac{\mathrm{S3}}{m}\\
W_s^4 &\equiv n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n = \frac{1}{q}\left[\mathrm{QS2} - \frac{3q}{4m}\mathrm{S3} + 18\mathrm{qST1} + 5\mathrm{QS5}\right]\\
W_s^5 &\equiv n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\ \mathcal{D}_\nu n = \frac{1}{q^2}\left[\mathrm{QS4} + 6\mathrm{qQS5} + 9\mathrm{q}^2\mathrm{ST1}\right]\\
W_s^6 &\equiv l^\mu\mathcal{D}_\mu q = \mathrm{QS3} + 3\mathrm{qST2}.
\end{aligned} \tag{A.1.8}$$

and five vector/pseudo-vector Weyl covariant combinations given by

$$\begin{aligned}
(W_v)^1_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda\sigma_\nu{}^\lambda = \frac{5\mathrm{V4}}{9} + \frac{5\mathrm{V5}}{9} + \frac{5\mathrm{VT1}}{3} - \frac{5\mathrm{VT2}}{12} - \frac{11\mathrm{VT3}}{6}\\
(W_v)^2_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda\omega_\nu{}^\lambda = \frac{5\mathrm{V4}}{3} - \frac{\mathrm{V5}}{3} - \mathrm{VT1} - \frac{\mathrm{VT2}}{4} + \frac{\mathrm{VT3}}{2}\\
(W_v)^3_\mu &\equiv l^\lambda\sigma_{\mu\lambda} = \mathrm{VT5}\\
(W_v)^4_\mu &\equiv n^{-1}\sigma_\mu{}^\lambda\mathcal{D}_\lambda n = \frac{1}{q}\left[\mathrm{QV4} + 3\mathrm{qVT3}\right]\\
(W_v)^5_\mu &\equiv n^{-1}\omega_\mu{}^\lambda\mathcal{D}_\lambda n = \frac{1}{2q}\left[\mathrm{QV3} + 3\mathrm{qVT2}\right]
\end{aligned} \tag{A.1.9}$$

In the tensor sector, there are nine Weyl-covariant combinations

$$\begin{aligned}
WT^{(1)}_{\mu\nu} &= u^{\lambda}\mathcal{D}_{\lambda}\sigma_{\mu\nu} = TT1 + \frac{1}{3}TT4 + T3.\\
WT^{(2)}_{\mu\nu} &= -2\left(\omega^{\mu}{}_{\lambda}\sigma^{\lambda\nu} + \omega^{\nu}{}_{\lambda}\sigma^{\lambda\mu}\right) = TT7.\\
WT^{(3)}_{\mu\nu} &= \sigma^{\mu}{}_{\lambda}\sigma_{\lambda\nu} - \frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta} = TT6.\\
WT^{(4)}_{\mu\nu} &= 4\left(\omega^{\mu}{}_{\lambda}\omega_{\lambda\nu} + \frac{1}{3}P^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta}\right) = TT5.\\
WT^{(5)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_{\alpha}\mathcal{D}_{\beta}n\\
&= \frac{1}{q}\left[QT1 + 8QT4 + 15qTT1 + qTT4 + 3qT3 + 3qTT6 + \frac{3q}{4}TT5\right]\\
WT^{(6)}_{\mu\nu} &= n^{-2}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_{\alpha}n\mathcal{D}_{\beta}n = \frac{1}{q^2}\left[QT3 + 6qQT4 + 9q^2TT1\right]\\
WT^{(7)}_{\mu\nu} &= \mathcal{D}_{\mu}l_{\nu} + \mathcal{D}_{\nu}l_{\mu} = 4TT2 + 2T2 - TT3.\\
WT^{(8)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu}l_{\alpha}\mathcal{D}_{\beta}n = \frac{1}{q}\left[QT2 + 3\ q\ TT2\right].\\
WT^{(9)}_{\mu\nu} &= n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_{\alpha}\mathcal{D}_{\beta}n = \frac{1}{q}\left[QT5 - \frac{3}{2}\ q\ TT2 + \frac{3}{2}\ q\ TT3\right].
\end{aligned} \tag{A.1.10}$$

A.2 Source Terms in Scalar Sector: Second Order

There are three source terms in scalar sector at second order $S_k(r)$, $S_h(r)$ and $S_M(r)$. They are quite complicated functions. Here we provide the explicit form of these source terms in terms of weyl covariant quantities.

The source term S_k is given by

$$S_C = \sum_{i=1}^{6} s_i^{(C)} W_s^i. \tag{A.2.11}$$

The Weyl covariant terms W_s^i are given in §A.1. The functions $s_i^{(k)}$s are given by,

$$\begin{aligned}
s_1^{(C)} &= \frac{r\left(4\left(m_0 - 3r^4\right)\left(r^2 + rR + R^2\right)F_2\left(\frac{r}{R}, \frac{m_0}{R^4}\right) + R\left(m_0(r+R) - 2R^3\left(r^2 + rR + R^2\right)\right)\right)}{3R(r+R)\left(-m_0 + r^4 + r^2R^2 + R^4\right)}\\
s_2^{(C)} &= \frac{1}{3m_0^2r^7}\Big(-m_0^3\left(r^4 + 2r^2R^2 + 36R^4\kappa^2\right) + 2m_0^2\left(18r^4R^4\kappa^2 + r^2R^6 + 36R^8\kappa^2\right)\\
&\quad -36m_0R^8\kappa^2\left(2r^4 + R^4\right) + 36r^4R^{12}\kappa^2\Big)\\
s_3^{(C)} &= \frac{r}{3}\\
s_4^{(C)} &= \frac{2r^2\left(m_0 - R^4\right)\left(rF_1^{(1,0)}\left(\frac{r}{R}, \frac{m_0}{R^4}\right) + 6RF_1\left(\frac{r}{R}, \frac{m_0}{R^4}\right)\right)}{R^6}
\end{aligned} \tag{A.2.12}$$

$$
\begin{aligned}
s_5^{(C)} &= -\frac{1}{2R^{16}\left(m_0-3R^4\right)}\Big(r^2\left(m_0-R^4\right)\Big(24R^4F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\left(r^3\left(m_0^2-4m_0R^4\right.\right.\right.\\
&+\ \left.3R^8\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+11r^2R\left(m_0^2-4m_0R^4+3R^8\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&+\ \left.6m_0R^7-4R^{11}\right)+r\left(r^2R^2\left(m_0^2\left(25r^2-13R^2\right)+m_0\left(-25r^6-75r^2R^4+52R^6\right)\right.\right.\\
&+\ \left.75r^6R^4-39R^{10}\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\left(4R^9\left(m_0-R^4\right)\right.\\
&-\ \left.\left.r^3\left(m_0^2\left(R^2-r^2\right)+m_0\left(r^6+3r^2R^4-4R^6\right)+3R^4\left(R^6-r^6\right)\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\\
&+\ 2RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\left(-5r^3\left(m_0^2\left(R^2-r^2\right)+m_0\left(r^6+3r^2R^4-4R^6\right)\right.\right.\\
&+\ \left.\left.3R^4\left(R^6-r^6\right)\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+26m_0R^9-22R^{13}\right)\\
&+\ \left.16m_0R^6\left(m_0-R^4\right)F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+96m_0R^7\left(m_0-R^4\right)F_1^{(0,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&+\ \left.\left.288rR^6\left(m_0-3R^4\right)\left(m_0-R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2\right)\right)\\
s_6^{(C)} &= \frac{2\sqrt{3}\kappa\left(m_0-r^4\right)\left(R^4-m_0\right)\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{m_0R^7}.
\end{aligned}
\tag{A.2.13}
$$

The source term S_h is given by

$$
S_h=\sum_{i=1}^{6}s_i^{(h)}W_s^i, \tag{A.2.14}
$$

where the functions $s_i^{(h)}$'s are given by

$$
\begin{aligned}
s_1^{(h)} &= \frac{1}{3R(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2}\left(2r\left(2\left(m_0\left(4r^3+8r^2R+6rR^2+3R^3\right)\right.\right.\right.\\
&\quad \left.\left.-3R^3\left(r^2+rR+R^2\right)^2\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)+r^2R\left(r^2+rR+R^2\right)^2\right)\Big),\\
s_2^{(h)} &= \frac{2}{3r^7}\left(r^4-\frac{36R^4\kappa^2\left(m_0-R^4\right)^2}{m_0^2}\right),\\
s_3^{(h)} &= 0,\\
s_4^{(h)} &= 0,\\
s_5^{(h)} &= \frac{r^7\left(R^4-m_0\right)}{R^{16}}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)^2\\
s_6^{(h)} &= \frac{4\sqrt{3}\kappa\left(R^4-m_0\right)}{m_0R^7}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right).
\end{aligned}
\tag{A.2.15}
$$

Finally the source term $S_M(r)$ is given by

$$
S_M(r)=\sum_{i=1}^{6}s_i^{(M)}W_s^i\ , \tag{A.2.16}
$$

with the functions $s_i^{(M)}$ being given by

$$
\begin{aligned}
s_1^{(M)} &= \frac{4r\sqrt{R^2\left(m_0-R^4\right)}\left(r^2+rR+R^2\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\\
s_2^{(M)} &= -\frac{2\sqrt{R^2\left(m_0-R^4\right)}\left(m_0^2r^4+12R^2\kappa^2\left(m_0-R^4\right)\left(2m_0r^2+3m_0R^2-3R^6\right)\right)}{m_0^2r^7}\\
s_3^{(M)} &= 0\\
s_4^{(M)} &= -\frac{r^5\sqrt{R^2\left(m_0-R^4\right)}\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{R^9}\\
s_5^{(M)} &= \frac{r^5\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{R^{17}\left(3R^4-m_0\right)}\left(r^2\left(-\left(6r\left(m_0-3R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^5\right)\right)F_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad -2\left(15r^2R\left(m_0-3R^4\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2+\left(3r\left(m_0-3R^4\right)\left(r^2F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\right.\right.\\
&\quad \left.\left.+35R^2F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+20R^7\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R\left(2m_0rRF_1^{(2,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad +r\left(39r\left(m_0-3R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+7R^5\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad \left.\left.\left.+10m_0R^2F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\right)\right)\\
s_6^{(M)} &= \frac{\sqrt{3}\kappa\left(R^4-m_0\right)}{m_0r^2R^7\sqrt{R^2\left(m_0-R^4\right)}}\left(-m_0r^4RF_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+r^4R^5F_1^{(3,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\\
&\quad +r^2R\left(20m_0r^2-17m_0R^2+17R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\quad \left.+r^3\left(4m_0r^2-7m_0R^2+7R^6\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+4R^9\right)
\end{aligned}
\tag{A.2.17}
$$

A.3 Source Terms in Vector Sector: Second Order

The source term in the vector sector at second order $S_E^{\text{vec}}(r)$ in (3.1.72) is given by

$$
(S_E^{\text{vec}})_i(r) = \sum_{l=1}^{5} r_l^{(E)}(W_v)_i^l \tag{A.3.18}
$$

where the Weyl covariant quantities W_v^i's are given in Appendix A.1 and the functions $s_i^{(E)}$ are given by

$$
\begin{aligned}
r_1^{(E)} &= \frac{r^2+rR+R^2}{3(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
r_2^{(E)} &= \frac{1}{3r^3},\\
r_3^{(E)} &= \frac{\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}\left(m_0(r+2R)+3r\left(r^2+rR+R^2\right)^2\right)}{\sqrt{3}m_0r^3(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2},
\end{aligned}
\tag{A.3.19}
$$

$$
\begin{aligned}
r_4^{(E)} =& \frac{\left(m_0-R^4\right)}{3R^6(r+R)^2\left(-m_0+r^4+r^2R^2+R^4\right)^2}\Bigg(-6r^2(r+R)\left(r^2+rR+R^2\right)\left(-m_0\right.\\
&\left.+r^4+r^2R^2+R^4\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)-6rR\left(3\left(r^2+rR+R^2\right)^2\left(r^3+2R^3\right)\right.\\
&\left.-m_0\left(7r^3+14r^2R+12rR^2+6R^3\right)\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&-\frac{R^8\left(m_0(2r+R)+3R\left(r^2+rR+R^2\right)^2\right)}{m_0-3R^4}\Bigg),\\
r_5^{(E)} =& \frac{\left(R^4-m_0\right)\left(r\left(9RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+6R^2F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)}{r^2R^7}.
\end{aligned}
\tag{A.3.20}
$$

The other source term in the vector sector at second order $S_M^{\text{vec}}(r)$ in (3.1.73) is given by

$$
(S_M^{\text{vec}})_i(r)=\sum_{l=1}^{5} r_l^{(M)}(W_v)_i^l, \tag{A.3.21}
$$

where the coefficient functions $r_i^{(M)}$ are given by

$$
\begin{aligned}
r_1^{(M)} =& 0,\\
r_2^{(M)} =& \frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}\left(m_0r^2+24R^2\kappa^2\left(R^4-m_0\right)\right)}{m_0r^5},\\
r_3^{(M)} =& \frac{6R\kappa\left(m_0-R^4\right)}{m_0r^5(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\Bigg(r^2R\left(r\left(r^2+rR+R^2\right)\left(3r^3+R^3\right)\right.\\
&\left.-m_0\left(3r^2+3rR+2R^2\right)\right)-8m_0(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)F_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Bigg),\\
r_4^{(M)} =& -\frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}}{R^6(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)}\Bigg(r^2(r+R)\left(-m_0+r^4\right.\\
&\left.+r^2R^2+R^4\right)\left(5RF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+rF_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\\
&+12rR\left(m_0-R^4\right)\left(r^2+rR+R^2\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^6\left(r^2+rR+R^2\right)\Bigg),\\
r_5^{(M)} =& \frac{2\sqrt{3}\sqrt{R^2\left(m_0-R^4\right)}}{m_0r^5R^6}\Bigg(6m_0r^3\left(m_0-R^4\right)\left(RF_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)-rF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\\
&+m_0r^2R^6+24R^8\kappa^2\left(R^4-m_0\right)\Bigg).
\end{aligned}
\tag{A.3.22}
$$

A.4 Source Terms in Tensor Sector: Second Order

In this appendix we provide the source of the dynamical equation (3.1.85). We report the result in terms of the parameters M and R and the variable ρ defined in (3.1.5). The source

$\mathbf{T}_{ij}(\rho)$ in (3.1.85) is given by

$$\mathbf{T}_{ij}(r) = \sum_{l=1}^{9} \tau_l(r)\ WT_{ij}^{(l)}, \tag{A.4.23}$$

where the weyl-covariant terms $WT_{ij}^{(l)}$ are defined in Appendix A.1 in equation (A.1.10). The coefficient of the weyl-covariant terms in the above source is given by

$$\begin{aligned}
\tau_1(r) =& \frac{3rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R} + \frac{m_0(r+R) - \left(r^2+rR+R^2\right)\left(3r^3+R^3\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
\tau_2(r) =& -\frac{1}{2R}\left(3rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right) - \frac{2r^3R\left(r^2+rR+R^2\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)} + R\right),\\
\tau_3(r) =& \frac{6rF_2\left(\frac{r}{R},\frac{m_0}{R^4}\right)}{R} + \frac{2\left(m_0(r+R) - 2r^3\left(r^2+rR+R^2\right)\right)}{(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)},\\
\tau_4(r) =& \frac{18R^4\kappa^2\left(m_0-R^4\right)^2\left(-m_0r^2+4m_0R^2+r^6-4R^6\right)}{m_0^2r^{10}} - \frac{m_0r^2+2m_0R^2+r^6-2R^6}{2r^6},\\
\tau_5(r) =& \frac{1}{R^6}\left(6r\left(R^4-m_0\right)\left(rF_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right) + 3RF_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)\right),\\
\tau_6(r) =& \frac{3r\left(m_0-R^4\right)}{2R^{16}\left(m_0-3R^4\right)}\Bigg(r\left(R^2F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\left(r^2\left(m_0-3R^4\right)\left(-25m_0r^2\right.\right.\right.\\
&\left.\left.\left.+37m_0R^2+25r^6-37R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+30m_0R^8-2R^{12}\right)\right.\\
&-r^4\left(3R^4-m_0\right)(r-R)(r+R)\left(-m_0+r^4+r^2R^2+R^4\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)^2\\
&+2rR\left(5r^2\left(3R^4-m_0\right)(R-r)(r+R)\left(-m_0+r^4\right.\right.\\
&\left.\left.+r^2R^2+R^4\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)+R^8\left(m_0+R^4\right)\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&\left.+16m_0R^6\left(m_0-R^4\right)F_1^{(1,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+48m_0R^7\left(m_0-R^4\right)F_1^{(0,1)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\\
&+24R^{11}\left(3m_0-2R^4\right)F_1\left(\frac{r}{R},\frac{m_0}{R^4}\right)\Bigg)\\
\tau_7(r) =& \frac{3\sqrt{3}\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{2m_0r^5},\\
\tau_8(r) =& \frac{3\sqrt{3}\kappa\left(R^2\left(m_0-R^4\right)\right)^{3/2}}{m_0r^5R^9}\left(2r^2\left(R\left(-m_0\left(5r^2+R^2\right)+5r^6+R^6\right)F_1^{(1,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right.\right.\\
&\left.\left.+r\left(m_0\left(R^2-r^2\right)+r^6-R^6\right)F_1^{(2,0)}\left(\frac{r}{R},\frac{m_0}{R^4}\right)\right)+R^9\right),\\
\tau_9(\rho) =& 0.
\end{aligned} \tag{A.4.24}$$

Appendix B

The linear independence of data for the parity even superfluid

B.1 The linear independence of first order terms

The vector sector

The equations of motion in the vector sector are

$$\begin{aligned} \tilde{P}_{\mu\beta}\partial_\nu T^{\nu\beta} &= \tilde{P}_{\mu\beta}F^{\beta\nu}J_\nu \\ \tilde{P}^{\mu\beta}u^\nu\left(\partial_\beta\xi_\nu - \partial_\nu\xi_\beta\right) &= \tilde{P}^{\mu\beta}E_\beta \\ \tilde{P}^{\mu\beta}\xi^\nu\left(\partial_\beta\xi_\nu - \partial_\nu\xi_\beta\right) &= \tilde{P}^{\mu\beta}F_{\beta\nu}\xi^\nu\,. \end{aligned} \tag{B.1.1}$$

A basis of ten one derivative vectors (before using the equations of motion) was listed in table 2.3. It is given by

$$\begin{aligned} &\tilde{P}^{\mu\beta}(u\cdot\partial)u_\beta,\ \tilde{P}^{\mu\beta}(u\cdot\partial)\xi_\beta,\ \tilde{P}^{\mu\beta}(\xi\cdot\partial)u_\beta,\ \tilde{P}^{\mu\beta}(\xi\cdot\partial)\xi_\beta,\ \tilde{P}^{\mu\beta}\partial_\beta\left(\frac{\mu}{T}\right), \\ &\tilde{P}^{\mu\beta}\partial_\beta T,\ \tilde{P}^{\mu\beta}\partial_\beta\left(\frac{\xi}{T}\right),\ \tilde{P}^{\mu\beta}\xi^\nu\partial_\beta u_\nu,\ \tilde{P}^{\mu\beta}E_\beta,\ \tilde{P}^{\mu\beta}F_{\beta\nu}\xi^\nu\,. \end{aligned} \tag{B.1.2}$$

The quantities in (B.1.2) are not all on-shell inequivalent as they are constrained by the relations (B.1.1). In this subsection we will argue that it is consistent to choose the seven vectors listed in the third column of Table 2.3 as independent vector data. That is, we will show that it is possible to use the equations (B.1.1) to solve for $\tilde{P}^{\mu\beta}\partial_\beta T$, $\tilde{P}^{\mu\beta}\partial_\beta\left(\frac{\xi}{T}\right)$, $\tilde{P}^{\mu\beta}\xi^\nu\partial_\beta u_\nu$ in terms of

$$\begin{aligned} &\tilde{P}^{\mu\beta}(u\cdot\partial)u_\beta,\ \tilde{P}^{\mu\beta}(u\cdot\partial)\xi_\beta,\ \tilde{P}^{\mu\beta}(\xi\cdot\partial)u_\beta,\ \tilde{P}^{\mu\beta}(\xi\cdot\partial)\xi_\beta,\ \tilde{P}^{\mu\beta}\partial_\beta\left(\frac{\mu}{T}\right), \\ &\tilde{P}^{\mu\beta}E_\beta,\ \tilde{P}^{\mu\beta}F_{\beta\nu}\xi^\nu \end{aligned} \tag{B.1.3}$$

If we rewrite the equations of motion in (B.1.1) in terms of the quantities in (B.1.2) we find

$$
\begin{aligned}
\tilde{P}^{\mu\beta}\left((P+\rho)(u\cdot\partial)u_\beta+\partial_T P\partial_\beta T+\partial_{\frac{\mu}{T}}\partial_\beta\frac{\mu}{T}+\partial_{\frac{\xi}{T}}\partial_\beta\frac{\xi}{T}+f(\xi.\partial)\xi_\beta\right)=&\tilde{P}^{\mu\beta}\left(qE_\mu-fF_{\mu\nu}\xi^\nu\right)\\
\tilde{P}^{\mu\beta}\left((u\cdot\partial)\xi_\beta-T\partial_\beta\frac{\mu}{T}-\frac{\mu}{T}\partial_\beta T+\xi^\nu\partial_\beta u_\nu\right)=&\tilde{P}^{\mu\beta}E_\beta\\
\tilde{P}^{\mu\beta}\left((\xi.\partial)\xi_\beta+T\xi\partial_\beta\frac{\xi}{T}+\frac{\xi}{T}\partial_\beta T\right)=&\tilde{P}^{\mu\beta}F_{\beta\nu}\xi^\nu .
\end{aligned}
\tag{B.1.4}
$$

It is possible to use (B.1.4) to solve for the scalars listed in (B.1.3) if and only if the 3×3 matrix of the three vectors $\tilde{P}^{\mu\beta}\partial_\beta T,\ \tilde{P}^{\mu\beta}\partial_\beta\left(\frac{\xi}{T}\right),\ \tilde{P}^{\mu\beta}\xi^\nu\partial_\beta u_\nu$ in the three equations (B.1.4) has nonzero determinant. This 3×3 matrix is given by

$$
M_{(v)}=\begin{pmatrix}\partial_T P & \partial_{\xi/T}P & 0\\ -\mu/T & 0 & 1\\ \xi^2/T & T\xi & 0\end{pmatrix}
\tag{B.1.5}
$$

and its determinant is given by

$$
Det\left(M_{(v)}\right)=\xi\left(\frac{\xi}{T}\partial_{\frac{\xi}{T}}P-T\partial_T P\right).
\tag{B.1.6}
$$

It is nonzero for a generic functional form for $P(T,\mu,\xi)$. We conclude that the vectors (B.1.3) form a basis for onshell independent one derivative vectors.

The scalar sector

The equations of motion in the scalar sector are given by

$$
\begin{aligned}
\xi_\nu\partial_\mu T^{\mu\nu}&=qE\cdot\xi\\
u_\nu\partial_\mu T^{\mu\nu}&=fE\cdot\xi\\
\partial_\mu J^\mu&=cE\cdot B\\
u^\nu\xi^\mu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)&=E\cdot\xi\,.
\end{aligned}
\tag{B.1.7}
$$

A basis of 11 one derivative vectors (before using the equations of motion) was listed in Table 2.3. We denote them by $\{\mathcal{L}_j^{(a)},S_i^{(a)}\}$ for the first set of on shell independent scalars and $\{\mathcal{L}_j^{(b)},S_i^{(b)}\}$ for the second set. Here j runs from 1 to 4 and i runs from 1 to 7. We have used the notation in Table 2.4. The new quantities $\mathcal{L}_j^{(b)},\mathcal{L}_j^{(b)}$ are defined as follows

$$
\begin{aligned}
\mathcal{L}_1^{(a)}&=u\cdot\partial\Sigma_1, & \mathcal{L}_i^{(b)}&=\xi\cdot\partial\Sigma_1,\\
\mathcal{L}_2^{(a)}&=u\cdot\partial\Sigma_2, & \mathcal{L}_i^{(b)}&=\xi\cdot\partial\Sigma_2,\\
\mathcal{L}_3^{(a)}&=u\cdot\partial\Sigma_3, & \mathcal{L}_i^{(b)}&=\xi\cdot\partial\Sigma_3,\\
\mathcal{L}_4^{(a)}&=\xi^\mu u\cdot\partial u_\mu, & \mathcal{L}_i^{(b)}&=\xi^\mu\xi\cdot\partial u_\mu\,.
\end{aligned}
\tag{B.1.8}
$$

The quantities defined in (B.1.8) are the dependent data for the two choices of bases among the on-shell inequivalent quantities. These quantities are to be determined by the equation of motion (B.1.7) in terms of the dependent quantities S_i. Note that the sets $\{\mathcal{L}_i^{(a)}, S_i^{(a)}\}$ and $\{\mathcal{L}_i^{(b)}, S_i^{(b)}\}$ are different partitioning of the same set of quantities. The equation of motion in (B.1.7) expressed in terms of the quantities in (B.1.8) has the form

$$\sum_{i=1}^{7}(\mathbf{e}_i^{(a)})_p\ S_i^{(a)} + \sum_{j=1}^{4}(\ell_j^{(a)})_p\ \mathcal{L}_j^a = 0, \quad \sum_{i=1}^{7}(\mathbf{e}_i^{(b)})_p\ S_i^{(b)} + \sum_{j=1}^{4}(\ell_j^{(b)})_p\ \mathcal{L}_j^b = 0. \tag{B.1.9}$$

In the equations above the index p runs from 1 to 4 denoting the 4 equations in (B.1.7). Again note that both the equations in (B.1.9) refer to the same set of equations. We find it convenient to define the new set of quantities

$$A = -\frac{\chi^2}{T^2(\nu^2-\chi^2)}; \quad B = -\frac{1}{T^2(\mu^2-\xi^2)}; \quad C = -\frac{\nu}{T(\nu^2-\xi^2)}. \tag{B.1.10}$$

so that the projector

$$\tilde{P}^{\mu\nu} = \eta^{\mu\nu} + Au^\mu u^\nu + B\xi^\mu\xi^\nu + C\left(\xi^\mu u^\nu + u^\mu\xi^\nu\right). \tag{B.1.11}$$

The coefficients in (B.1.9) are given by

$$\begin{aligned}
&(\mathbf{e}_1^{(a)})_1 = -(P+\rho), && (\mathbf{e}_2^{(a)})_1 = f\nu,\\
&(\mathbf{e}_3^{(a)})_1 = B(P+\rho) + Cf\mu - f, && (\mathbf{e}_4^{(a)})_1 = B\xi Tf\mu + \mu\partial_\chi f,\\
&(\mathbf{e}_5^{(a)})_1 = \mu\partial_\nu f - CTf\mu + fT, && (\mathbf{e}_6^{(a)})_1 = \mu\partial_T f + 2f\nu,\\
&(\mathbf{e}_7^{(a)})_1 = -f &&\\
&(\mathbf{e}_1^{(a)})_2 = (P+\rho)\mu, && (\mathbf{e}_2^{(a)})_2 = -f\chi^2 T,\\
&(\mathbf{e}_3^{(a)})_1 = -(\mu B(\rho+P) + CTf\xi^2), && (\mathbf{e}_4^{(a)})_2 = \partial_\chi P - \xi^2\partial_\chi f - \xi fT - B\xi^3 Tf,\\
&(\mathbf{e}_5^{(a)})_2 = \partial_\nu P - \xi^2\partial_\nu f + CTf\xi^2, && (\mathbf{e}_6^{(a)})_2 = \partial_T P - \xi^2\partial_T f - 2f\chi^2 T,\\
&(\mathbf{e}_7^{(a)})_2 = -q, &&\\
&(\mathbf{e}_1^{(a)})_3 = q, && (\mathbf{e}_2^{(a)})_3 = \frac{f}{T}\\
&(\mathbf{e}_3^{(a)})_3 = -(Bq + Cf), && (\mathbf{e}_4^{(a)})_3 = -\partial_\chi f - B\xi fT\\
&(\mathbf{e}_5^{(a)})_3 = \partial_\nu f + CTf, && (\mathbf{e}_6^{(a)})_3 = -\partial_T f - \frac{f}{T}\\
&(\mathbf{e}_7^{(a)})_3 = 0, &&\\
&(\mathbf{e}_1^{(a)})_4 = 0, && (\mathbf{e}_2^{(a)})_4 = 0,\\
&(\mathbf{e}_3^{(a)})_4 = -1, && (\mathbf{e}_4^{(a)})_4 = 0\\
&(\mathbf{e}_5^{(a)})_4 = T, && (\mathbf{e}_5^{(a)})_6 = \nu,\\
&(\mathbf{e}_7^{(a)})_4 = -1. &&
\end{aligned} \tag{B.1.12}$$

and

$$
\begin{aligned}
&(\ell_1^{(a)})_1 = Cf\mu\xi T - \partial_\chi\rho, && (\ell_2^{(a)})_1 = -(Af\mu T + \partial_\nu\rho),\\
&(\ell_3^{(a)})_1 = -\partial_T\rho, && (\ell_4^{(a)})_1 = Af\mu + C(P+\rho),\\
&(\ell_1^{(a)})_2 = \mu\partial_\chi(P+\rho) - Cf\xi^3 T, && (\ell_2^{(a)})_2 = \mu\partial_\nu(P+\rho) + TA\xi^2 f,\\
&(\ell_3^{(a)})_2 = \mu\partial_T(P+\rho), && (\ell_1^{(a)})_2 = (P+\rho) - C\mu(P+\rho) - Af\xi^2\\
&(\ell_1^{(a)})_3 = \partial_\chi q - C\xi f T, && (\ell_2^{(a)})_3 = \partial_\nu q + ATf,\\
&(\ell_3^{(a)})_3 = \partial_T q, && (\ell_4^{(a)})_3 = -(Cq + Af)\\
&(\ell_1^{(a)})_4 = \xi T, && (\ell_2^{(a)})_4 = 0,\\
&(\ell_3^{(a)})_4 = \chi^2 T, && (\ell_4^{(a)})_4 = 0.
\end{aligned}
\tag{B.1.13}
$$

The other set of coefficients, with index (b), can be read from (B.1.12) and (B.1.13) using

$$
\begin{aligned}
(\ell_1^{(b)})_i =& (\mathbf{e}_4^{(a)})_i;\; (\ell_2^{(b)})_i = (\mathbf{e}_5^{(a)})_i;\; (\ell_3^{(b)})_i = (\mathbf{e}_6^{(a)})_i;\; (\ell_4^{(b)})_i = (\mathbf{e}_3^{(a)})_i;\\
(\mathbf{e}_4^{(b)})_i =& (\ell_1^{(a)})_i;\; (\mathbf{e}_5^{(b)})_i = (\ell_2^{(a)})_i;\; (\mathbf{e}_6^{(b)})_i = (\ell_3^{(a)})_i;\; (\mathbf{e}_3^{(b)})_i = (\ell_4^{(a)})_i;
\end{aligned}
\tag{B.1.14}
$$

We can express all the derivatives in (B.1.12) and (B.1.13) as derivatives of a single function, say, the pressure. Thermodynamic relations that enable us to do so are

$$
q = \frac{1}{T}\partial_{\mu/T}P\ ;\quad f = \frac{1}{T\xi}\partial_{\xi/T}P\ ;\quad \rho = -P + T\partial_T P - \frac{\xi}{T}\partial_{\frac{\xi}{T}}P. \tag{B.1.15}
$$

We make the following observations:

a) We can use the equations of motion (B.1.9) to solve for the 4 scalars $\xi^\mu u\cdot\nabla u_\mu$, $u\cdot\partial\Sigma_i$ $(i = 1,\ldots,3)$ in terms of the 7 independent scalars in the 3rd column of the first row of Table 2.3. This is possible if and only if the 4×4 matrix of coefficients of the four quantities in the first equation in (B.1.9) has nonzero determinant. This matrix is given by

$$
M_{ij}^{(a)} = (\ell_i^{(a)})_j; \tag{B.1.16}
$$

b) We can use the equations of motion (B.1.9) to solve for the quantities $\xi^\mu\xi\cdot\partial u_\mu$, $\xi\cdot\partial\Sigma_i$ in terms of the 7 independent scalars in the 3rd column of the second row of Table 2.3. This is possible if and only if the 4×4 matrix of coefficients of the 4 quantities in the second equation in (B.1.9) has nonzero determinant. This matrix is given

$$
M_{ij}^{(b)} = (\ell_i^{(b)})_j; \tag{B.1.17}
$$

The relations (B.1.15) allow us to express the matrices (B.1.16) and (B.1.17) in terms of the pressure. Using several reasonable equations of state we have used `Mathematica` to verify that the determinant of the matrices in (B.1.16) and (B.1.17) is generically non-zero.

B.2 The linear independence of the second order terms

A list of second order scalar data, the second order equations of motion and a choice of second order independent scalar data can be found in Table 2.5. The second order scalar equations that follows from the first order vector equations are

$$\begin{aligned}\nabla_\alpha\left(\tilde{P}^{\alpha\mu}u^\nu\left(\partial_\mu\xi_\nu-\partial_\mu\xi_\nu\right)\right)&=\nabla_\alpha\left(\tilde{P}^{\alpha\mu}E_\mu\right),\\ \nabla_\mu\left(\tilde{P}^{\mu\nu}\nabla_\beta T^\beta_{\ \nu}\right)&=\nabla_\mu\left(\tilde{P}^{\mu\nu}F_{\nu\beta}J^\beta\right)\\ \nabla_\alpha\left(\tilde{P}^{\alpha\mu}\xi^\nu\left(\partial_\mu\xi_\nu-\partial_\mu\xi_\nu\right)\right)&=\nabla_\alpha\left(\tilde{P}^{\alpha\mu}F_{\mu\nu}\xi^\nu\right)\end{aligned}\tag{B.2.18}$$

The two derivative terms in these equations take the form

$$\tilde{P}^{\mu\nu}\left((P+\rho)u^\beta\nabla_\mu\nabla_\beta u_\nu+\partial_T\rho\nabla_\mu\partial_\nu\frac{\mu}{T}+\partial_{\xi/T}\nabla_\mu\partial_\nu\frac{\xi}{T}+f\xi^\beta\nabla_\mu\nabla_\beta\xi_\nu\right)=\dots\tag{B.2.19}$$

$$\tilde{P}^{\mu\nu}\left(u^\beta\nabla_\mu\nabla_\beta\xi_\nu-T\nabla_\mu\partial_\nu\frac{\mu}{T}-\frac{\mu}{T}\nabla_\mu\partial_\nu T+\xi^\beta\nabla_\mu\nabla_\nu u_\beta\right)=\dots\tag{B.2.20}$$

$$\tilde{P}^{\mu\nu}\left(\xi^\beta\nabla_\mu\nabla_\beta\xi_\nu+T\xi\nabla_\mu\partial_\nu\frac{\xi}{T}+\frac{\xi^2}{T}\nabla_\mu\partial_\nu T\right)=\dots.\tag{B.2.21}$$

The quantities $\tilde{P}^{\mu\nu}\nabla_\mu\partial_\nu T, \tilde{P}^{\mu\nu}\nabla_\mu\partial_\nu\frac{\xi}{T}, \tilde{P}^{\mu\nu}\xi^\beta\nabla_\mu\nabla_\nu u_\beta$ can be solved for using equations (B.2.19), (B.2.20), and (B.2.21). Note that these two derivative scalar quantities do not appear in any other equations of motion. We can then use the remaining 8 equations of motion to solve for the other 8 dependent data,

$$u^\mu u^\nu\nabla_\mu\partial_\nu\Sigma_i,\quad u^\mu u^\nu\xi^\beta\nabla_\mu\nabla_\nu u_\beta,\quad \xi^\mu\xi^\nu\xi^\beta\nabla_\mu\nabla_\nu u_\beta,\quad \xi^\mu\xi^\nu\nabla_\mu\partial_\nu\Sigma_i\tag{B.2.22}$$

where i runs from 1 to 3. The reaming 8 two derivative scalar equation of motion are

$$u^\beta\nabla_\beta\left(u_\mu\nabla_\nu T^{\mu\nu}\right)=u^\beta\nabla_\beta\left(-E_\mu J^\mu\right),\tag{B.2.23a}$$

$$u^\beta\nabla_\beta\left(\xi_\mu\nabla_\nu T^{\mu\nu}\right)=u^\beta\nabla_\beta\left(\xi^\mu F_{\mu\nu}J^\nu\right),\tag{B.2.23b}$$

$$u^\beta\nabla_\beta\left(\nabla_\mu J^\mu\right)=u^\beta\nabla_\beta\left(cE^\mu B_\mu\right),\tag{B.2.23c}$$

$$u^\beta\nabla_\beta\left(\xi^\mu u^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=u^\beta\nabla_\beta\left(\xi^\mu E_\mu\right)\tag{B.2.23d}$$

$$\xi^\beta\nabla_\beta\left(u_\mu\nabla_\nu T^{\mu\nu}\right)=\xi^\beta\nabla_\beta\left(-E_\mu J^\mu\right),\tag{B.2.23e}$$

$$\xi^\beta\nabla_\beta\left(\xi_\mu\nabla_\nu T^{\mu\nu}\right)=\xi^\beta\nabla_\beta\left(\xi^\mu F_{\mu\nu}J^\nu\right),\tag{B.2.23f}$$

$$\xi^\beta\nabla_\beta\left(\nabla_\mu J^\mu\right)=\xi^\beta\nabla_\beta\left(cE^\mu B_\mu\right),\tag{B.2.23g}$$

$$\xi^\beta\nabla_\beta\left(\xi^\mu u^\nu\left(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu\right)\right)=\xi^\beta\nabla_\beta\left(\xi^\mu E_\mu\right)\tag{B.2.23h}$$

The matrix of coefficients of the terms in (B.2.22) as they appear in the equation of motion (B.2.23) may be expressed as

$$N_{ij}=\begin{pmatrix}M^{(a)}_{ij}&0\\0&M^{(b)}_{ij}\end{pmatrix}\tag{B.2.24}$$

where the rows represent the ordered equations in (B.2.23) and the columns represent the ordered quantities in (B.2.22). It follows that

$$\mathrm{Det}[N_{ij}] = \mathrm{Det}[M_{ij}^{(a)}]\mathrm{Det}[M_{ij}^{(b)}]. \tag{B.2.25}$$

In the previous section we concluded that both $\mathrm{Det}[M_{ij}^{(b)}]$ and $\mathrm{Det}[M_{ij}^{(b)}]$ are generically non-zero. Therefore we can infer that $\mathrm{Det}[N_{ij}]$ is also generically non-zero.

In order to understand the structure of the matrix N we note that the first four equations in (B.2.23) are generated by the action of $u\cdot\partial$ on the first equation in (B.1.9). We then find that $u\cdot\partial$ acting on $S_i^{(a)}$ generates all the independent second order data as presented in Table 2.5. Likewise, the action of $u\cdot\partial$ on the $\mathcal{L}_i^{(a)}$ generates the four terms $u^\mu u^\nu \xi^\beta \nabla_\mu \nabla_\nu u_\beta$, $u^\mu u^\nu \nabla_\mu \partial_\nu \Sigma_i$ $(i = 1, \ldots, 3)$. In fact these dependent two derivative terms appear only in equations (B.2.23a), (B.2.23c), (B.2.23d), (B.2.23e) and is not there in the rest of the four equations in (B.2.23).

Similarly, we can think of the equations (B.2.23f), (B.2.23g), (B.2.23h), (B.2.23h) as being obtained by the action of $\xi\cdot\partial$ on the second equation in (B.1.9). Also here the terms $\xi^\mu\xi^\nu\xi^\beta\nabla_\mu\nabla_\nu u_\beta$, $\xi^\mu\xi^\nu\nabla_\mu\partial_\nu\Sigma_i$ (which constitutes the 4 remaining second order quantities which are determined by the equation of motion) are generated by $\xi\cdot\partial$ acting on the $\mathcal{L}_i^{(b)}$ terms. These dependent four second order quantities do not appear in the first four equations in (B.2.23). This structure justifies the block diagonal form of the coefficient matrix in (B.2.24).

Appendix C

Extrinsic Curvature and thermodynamics of the ring and torus in 4+1 dimensions

C.1 Extrinsic curvature

Suppose we have a timelike surface with unit normal vector n pointing toward us (spacelike surfaces will require some sign differences). The induced metric on the surface is

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \tag{C.1.1}$$

The extrinsic curvature is given by [**?**]

$$\Theta_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} = \nabla_\mu n_\nu. \tag{C.1.2}$$

We have to be a little careful with the last expression. It agrees with the first expression when projected tangent to the surface. The first expression has vanishing components normal to the surface. The normal components of the second expression depend on how we extend n off the surface.

The conventional choice for extending n is as follows: at each point on the surface, construct the geodesic that passes through that point tangent to n and parallel transport n along it. In other words

$$n^\mu \nabla_\mu n^\nu = 0. \tag{C.1.3}$$

This ensures that the second expression in (C.1.2) has vanishing components normal to the surface. The other normal component, $n^\nu \nabla_\mu n_\nu$, vanishes due to the normalisation of n.

For the surfaces given by $f(x) = 0$, considered in §4.1.3, the unit normal on the surface is given by

$$n_\mu = -\frac{\partial_\mu f}{\sqrt{\partial f \cdot \partial f}}. \tag{C.1.4}$$

However, if we used this vector away from the surface, it would not satisfy (C.1.3). We could still use either expression in (C.1.2) with this vector — we would just have to project the second one tangent to the surface. Alternatively, we can use

$$n_\mu = -\frac{\partial_\mu f}{(\partial f \cdot \partial f)^{1/2}} + \left[\frac{\partial^\nu f\, \nabla_\nu \partial_\mu f}{(\partial f \cdot \partial f)^{3/2}} - \frac{\partial_\mu f\, \partial^\lambda f\, \partial^\nu f\, \nabla_\lambda \partial_\nu f}{(\partial f \cdot \partial f)^{5/2}}\right] f + \mathcal{O}(f^2). \tag{C.1.5}$$

The $\mathcal{O}(f^2)$ terms don't contribute to (C.1.2) or (C.1.3) on the surface. The contribution of the $\mathcal{O}(f)$ terms on the surface to (C.1.2) are normal to the surface and ensure that n satisfies (C.1.3).

Either way, on the surface, we get

$$\Theta_{\mu\nu} = -\frac{\nabla_\mu \partial_\nu f}{(\partial f \cdot \partial f)^{1/2}} + \frac{\partial_\mu f\, \partial^\lambda f\, \nabla_\lambda \partial_\nu f + \partial_\nu f\, \partial^\lambda f\, \nabla_\lambda \partial_\mu f}{(\partial f \cdot \partial f)^{3/2}} - \frac{\partial_\mu f\, \partial_\nu f\, \partial^\lambda f\, \partial^\sigma f\, \nabla_\lambda \partial_\sigma f}{(\partial f \cdot \partial f)^{5/2}}. \tag{C.1.6}$$

As this is perpendicular to n, it doesn't matter if we contract its indices with the full metric $g_{\mu\nu}$ or the induced metric $h_{\mu\nu}$. We get

$$\Theta = \Theta^\mu_\mu = -\frac{\Box f}{(\partial f \cdot \partial f)^{1/2}} + \frac{\partial^\mu f\, \partial^\nu f\, \nabla_\mu \partial_\nu f}{(\partial f \cdot \partial f)^{3/2}}. \tag{C.1.7}$$

C.2 Energy, angular momentum, and entropy of the ring and torus in 4+1 dimensions

The energy, E, angular momentum, L, and entropy S for the $4+1$ dimensional ring are obtained from (4.3.91) by differentiation (4.1.33) and are given by

$$\begin{aligned}
E =& \frac{4\pi^2 \ell_0 R^2}{3\sqrt{6R+13}(9R+26)\epsilon}\Big(-2379\sqrt{6}\sqrt[3]{R+2}R^{7/6} - 180\sqrt{6}\sqrt[3]{R+2}R^{19/6} \\
&-1128\sqrt{6}\sqrt[3]{R+2}R^{13/6} + 180\sqrt{6R+13}R^3 + 1188\sqrt{6R+13}R^2 + 2732\sqrt{6R+13}R \\
&-1690\sqrt{6}\sqrt[3]{R+2}\sqrt[6]{R} + 2184\sqrt{6R+13}\Big) \\
&+ \frac{2\pi^2 R^4}{135\ell_0 (R+2)^{5/3}\sqrt{6R+13}(9R+26)^3}\Big(-2598156(R+2)^{2/3}\sqrt{6R+13}R^6 \\
&-34169688(R+2)^{2/3}\sqrt{6R+13}R^5 - 186252048(R+2)^{2/3}\sqrt{6R+13}R^4 \\
&-539837568(R+2)^{2/3}\sqrt{6R+13}R^3 - 879420672(R+2)^{2/3}\sqrt{6R+13}R^2 \\
&-765123840(R+2)^{2/3}\sqrt{6R+13}R \\
&+\sqrt{6}(6R+13)(3R(R(3R(3R(243R(66R+853)+1105784)+9364772) \\
&+44322304)+37098880)+38667200)\sqrt[6]{R} - 278403840(R+2)^{2/3}\sqrt{6R+13}\Big)\epsilon \\
&+ \mathcal{O}(\epsilon^2),
\end{aligned} \tag{C.2.8}$$

$$
\begin{aligned}
S =& \frac{4\pi^2\ell_0 R^{13/6}}{9\sqrt[3]{R+2}(9R+26)\epsilon}\Big(468(R+2)^{2/3}\sqrt{36R+78} \\
& -\sqrt[6]{R}\Big(-396(R+2)^{2/3}\sqrt{36R+78}R^{5/6}-90(R+2)^{2/3}\sqrt{36R+78}R^{11/6}+540R^3 \\
& +3276R^2+6591R+4394\Big)\Big)+\frac{\pi^2 R^{25/6}\sqrt{4R+\frac{26}{3}}}{135\ell_0(R+2)^{7/3}(9R+26)^3}\Big(111296640\sqrt{36R+78}R^{7/6} \\
& +433026\sqrt{36R+78}R^{37/6}+5596533\sqrt{36R+78}R^{31/6}+29856168\sqrt{36R+78}R^{25/6} \\
& +84282948\sqrt{36R+78}R^{19/6}+132966912\sqrt{36R+78}R^{13/6}-2598156(R+2)^{2/3}R^6 \\
& -34169688(R+2)^{2/3}R^5-186252048(R+2)^{2/3}R^4-539837568(R+2)^{2/3}R^3 \\
& -879420672(R+2)^{2/3}R^2-765123840(R+2)^{2/3}R+38667200\sqrt{36R+78}\sqrt[6]{R} \\
& 278403840(R+2)^{2/3}\Big)\,\epsilon+\mathcal{O}(\epsilon^2), \\
L =& -\frac{4}{9\epsilon^2}\Big(\pi^2\ell_0^2R^2\sqrt{6R+13}\Big(-6R+\sqrt[3]{R+2}\sqrt{36R+78}\sqrt[6]{R}-18\Big)\Big) \\
& +\frac{2\pi^2R^4\sqrt{6R+13}}{135(R+2)^{5/3}(9R+26)}\Big(-3216\sqrt{36R+78}R^{7/6}-243\sqrt{36R+78}R^{19/6} \\
& -1524\sqrt{36R+78}R^{13/6}+1458(R+2)^{2/3}R^3+9630(R+2)^{2/3}R^2 \\
& +21888(R+2)^{2/3}R-2288\sqrt{36R+78}\sqrt[6]{R}+17160(R+2)^{2/3}\Big)+\mathcal{O}(\epsilon^1),
\end{aligned}
\tag{C.2.9}
$$

These expressions for E, S and L are used for the plot in fig.4.7.

For the torus in 4+1 dimensions, the energy, entropy and angular momenta can be expressed in terms of the derivatives of $T\ln\mathcal{Z}_{\rm gc}$ as in (4.1.34) as well.

Using (4.4.102) we find

$$
\begin{aligned}
E =& \frac{\pi^3\ell_0^2R\sin(2\chi_0)}{3\epsilon^2}\Big(-R^{1/6}(1+R)^{1/3}\sqrt{12+9R}(20+3R(8+3R)) \\
& +9(12+R(18+R(11+3R))))+\frac{\pi^3R^3\csc(2\chi_0)}{72(R+1)^{5/3}}\Big(\sqrt[6]{R}(3R+4)\sqrt{9R+12}\,(R \\
& (9R(R(135R+554)+867)+5564)+1560) \\
& -3(R+1)^{2/3}(R(3R(3R(3R(135R+734)+4907)+16948)+30658)+7904)\Big) \\
& +\mathcal{O}\left(\epsilon^1\right),
\end{aligned}
\tag{C.2.10}
$$

$$
\begin{aligned}
S =& \frac{\pi^3\ell_0^2R^{7/6}\sin(2\chi_0)}{3\sqrt[3]{R+1}\epsilon^2}\Big(3(R+1)^{2/3}\sqrt{9R+12}(R(3R+8)+8) \\
& -\sqrt[6]{R}(3R+4)(3R(3R+8)+14)\Big)-\frac{\pi^3R^{19/6}(6R+8)\csc(2\chi_0)}{144\left(\sqrt{3}(R+1)^{7/3}\right)}\Big(9(R+1)^{2/3} \\
& \sqrt{3R+4}(R(3R(R(135R+554)+891)+2020)+624) \\
& -\sqrt{3}\sqrt[6]{R}(3R+4)(R(9R(R(135R+554)+861)+5440)+1482)\Big)+\mathcal{O}\left(\epsilon^1\right),
\end{aligned}
$$

$$
\begin{aligned}
L_1 = {} & \frac{4\sqrt{2}\pi^3\ell_0^3 R\sin(\chi_0)\cos^2(\chi_0)}{3\epsilon^3}\left(3(R+2)\sqrt{3R+4}-\sqrt{3}\sqrt[6]{R}\sqrt[3]{R+1}(3R+4)\right) \\
& + \frac{\pi^3\ell_0 R^3\sqrt{3R+4}\cos(2\chi_0)\csc(\chi_0)}{36\sqrt{2}(R+1)^{5/3}\epsilon}\Big(\sqrt[6]{R}(3R+4)\sqrt{9R+12}(R(27R+62)+39) \\
& -3(R+1)^{2/3}(R(3R(27R+98)+401)+208)\Big)+\mathcal{O}\left(\epsilon^0\right), \\
L_2 = {} & \frac{4\sqrt{2}\pi^3\ell_0^3 R\sin^2(\chi_0)\cos(\chi_0)}{3\epsilon^3}\left(3(R+2)\sqrt{3R+4}-\sqrt{3}\sqrt[6]{R}\sqrt[3]{R+1}(3R+4)\right) \\
& + \frac{\pi^3\ell_0 R^3\sqrt{3R+4}\cos(2\chi_0)\sec(\chi_0)}{36\sqrt{2}(R+1)^{5/3}\epsilon}\Big(3(R+1)^{2/3}(R(3R(27R+98)+401)+208) \\
& -\sqrt[6]{R}(3R+4)\sqrt{9R+12}(R(27R+62)+39)\Big)+\mathcal{O}\left(\epsilon^0\right).
\end{aligned}
\tag{C.2.11}
$$

We use these expressions for E, S, L_1 and L_2 to obtain the plots in fig.4.10.

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