Black holes and the finite temperature gauge theory

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Contents

Chapter 1

Introduction

Black holes are one of the most well known but enigmatic predictions of the classical general relativity. As their name suggests nothing can escape from a black hole, not even the radiation. However in 1972, Stephen Hawking demonstrated a surprising result that blackholes radiate semi-classically[1] and blackholes have finite temperature. It has also been argued that blackholes have finite entropy proportional to the area of the event horizon, which could be arbitrarily large. On the other hand the no-hair theorem, which is valid in the range of classical general relativity, states that a static blackhole horizon has almost no structure and contains no information other than the mass and the conserved charges of the black hole. This implies very little or no entropy for the horizon and presents an immediate conundrum. It is believed that the discrepancies should sort out within the frame work of a quantum theory of gravity and it remains a challenge for any successful theory of quantum gravity to understand the black hole entropy from a microscopic viewpoint. String theory, which has been proposed as a perturbatively finite theory of quantum gravity, is shown to give the correct entropy for a certain class of supersymmetric extremal black holes [2]. However the case of more natural non-supersymmetric black holes are poorly understood. String theories generally lacks a non-perturbative definition and it is difficult to formulate the non-perturbative questions like black hole entropy within the framework of string theory.

It has long been suspected that large N gauge theories leads to a non-perturbative formulation of string theory in certain back grounds. But the first concrete proposal was made by Maldacena [3] who proposed a duality between large N, $\mathcal{N} = 4$ SYM theory and IIB string theory in asymptotic

$AdS_5 \times S^5$ back ground. In a certain limit IIB theory reproduces supergravity and this provides a non-perturbative way to formulate the properties of the black hole in the context of gauge theory. In this thesis we will take this approach and study thermal gauge theory to understand the properties of the black holes. Let us start by briefly reviewing the AdS/CFT duality.

1.0.1 AdS/CFT

IIB string theory on $AdS_5 \times S^5$ background is equivalent to $\mathcal{N} = 4$, $SU(N)$ super Yang-Mills theory defined on the boundary of AdS_5 space that is $S^3 \times R$. The parameters of the two theory are related by

$$\frac{R^4}{l_s^4} = g_{YM}^2 N, \qquad \frac{G}{R^8} = N^{-2}, \qquad G = g_s^2 l_s^8 = l_p^8, \qquad g_s = g_{YM}^2$$

where R is the curvature radius of AdS_5, G is the Newton gravitational constant in AdS_5, l_s, l_p are string length and planck length respectively, g_s is the string coupling. g_{YM}^2 is the gauge theory coupling. The t'Hooft $1/N$ expansion in Yang-Mills theory corresponds to the g_s expansion in AdS_5, and a small t'Hooft coupling $g_{YM}^2 N$ implies a strongly coupled world-sheet. The metric of $AdS_5 \times S^5$ space in the global co-ordinate is given by.

$$ds^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_3^2 + R^2 d\Omega_5^2, \tag{1.1}$$

The AdS/CFT correspondence implies that the thermal phases of string theory can be studied by studying those of the dual gauge theory([4],[5],[9]). In this thesis we will primarily take the approach of studying gauge theories to learn facts about the gravity theory. In the next sections we review the phase structure of finite temperature supergravity in AdS_5 and the dual gauge theory interpretation.

Before moving further, we like to mention that the general correspondence between gauge theory and gravity is not confined to the particular case of the AdS/CFT correspondence that we discussed in the previous paragraph. String theories defined in AdS space have conformal gauge theory duals. However gauge/gravity duality has also been extended to the case of confining gauge theories ([5],[6],[7],[8]). These metrics can be written as,

$$ds^2 = W^2(u) dx_\mu^2 + ds_{int}^2 \tag{1.2}$$

where μ indices are in the non-compact direction, and ds^2_{int} is the metric on an internal manifold, one of whose coordinates is u, and whose constant u slices are compact. The variable u has the range $u_0 < u < \infty$ and the function $W(u)$ increases monotonically from a positive nonzero value at u_0 to infinity at $u = \infty$. The geometry is smooth and without boundaries everywhere, including at the IR wall $u = u_0$. This is possible because a k-cycle of the internal manifold shrinks to zero size at $u = u_0$, so that, locally, $u - u_0$ may simply be thought of as the radial coordinate of an R^{k+1} component of the geometry. In the third part of our work we will consider localized black hole solutions in these geometries.

Phase structure of Supergravity in AdS_5

The canonical ensemble for quantum gravity in AdS can be defined as a path integral over the metric and all other fields asymptotic to AdS with time direction periodically identified with a period $\beta = 1/T$. At semi-classical level, i.e. $R^2/l_p^2 \gg 1$, such a path integral is dominated by configurations near the saddle points, i.e. classical solutions to the Einstein equations. If we assume spherical symmetry and zero charge, there are three possible critical points, which are thermal AdS_5 (Euclidean AdS with time direction periodically identified), a big (Schwarzschild) black hole (BBH) and a small black hole (SBH). Among them thermal AdS and BBH are locally stable, while SBH has a negative mode and it is unstable. The thermal AdS background has topology $S^1 \times R^4$, while SBH and BBH have topology $R^2 \times S^3$, all of them have a common boundary $S^1 \times S^3$. The Euclidean time circle in a black hole background is contractible and hence the winding numbers are not conserved. In contrast the time circle in thermal AdS is noncontractible and the winding number is conserved.

The classical action for thermal AdS is $I_1 = 0$. This is standard in string theory: with a noncontractible time circle, there is no genus zero contribution to the free energy. A Schwarzschild black hole solution exists in AdS only for a Hawking temperature greater than

$$T_0 = \frac{\sqrt{2}}{\pi R}, \quad \beta_0 = \frac{1}{T_0} = \frac{\pi R}{\sqrt{2}} \tag{1.3}$$

The eucledian metric of the black hole solution is,

$$ds^2 = (1 + \frac{r^2}{R^2} - \frac{m^2}{r^2})d\tau^2 + \frac{dr^2}{1 + \frac{r^2}{R^2} - \frac{m^2}{r^2}} + r^2 d\Omega_3^2 + R^2 d\Omega_5^2, \tag{1.4}$$

For $T > T_0$, there are two possible black holes, whose horizon sizes are given by

$$\frac{r_\pm}{R} = \frac{1}{\sqrt{2}} \left[\frac{\beta_0}{\beta} \pm \sqrt{\frac{\beta_0^2}{\beta^2} - 1} \right] \tag{1.5}$$

The corresponding classical Euclidean action is given by

$$I_\pm = \frac{R^3}{G} 2\pi V_3 \left(\frac{r_\pm}{R} \right)^3 \frac{1 - \left(\frac{r_\pm}{R}\right)^2}{1 + 2\left(\frac{r_\pm}{R}\right)^2} \, , \tag{1.6}$$

where G is the five-dimensional Newton's constant. We will denote I_+, I_- the classical actions for large and small black hole respectively. The specific heat of the large black hole is positive and thus it is thermodynamically stable (i.e. it can reach locally stable thermal equilibrium with thermal radiation). The small black hole has a negative specific heat. The action I_- of the small black hole is always greater than the action of thermal AdS and of the big black hole. At temperature $T_1 = \frac{3}{2\pi R} > T_0$ the action for the big black hole is $I_+ = 0$. When $T_0 < T < T_1$, $I_+ > 0$, and the saddle corresponding to thermal AdS dominates. When $T > T_1$, $I_+ < 0$, the big black hole (BBH) dominates. There is a change of dominance at T_1. This is the Hawking-Page transition. In the classical limit $G \to 0$, this is a sharp first order transition. We expect that at finite G the transition should be smoothed out. This we will see explicitly in the gauge theory description.

When $T_0 < T < T_1$, the big black hole phase is metastable, since it has a higher free energy than that of that of thermal AdS. But string perturbation theory around it is well defined until T_0 is reached Hagedorn temperature(T_H) where we expect the perturbation theory to break down. Similarly, when $T > T_1$, thermal AdS becomes metastable. For a large AdS with $R \gg l_s$ (l_s is the string length) the perturbation theory around thermal AdS breaks down at a much higher Hagedorn temperature $T_H \sim \frac{1}{l_s}$. In the Hawking-Page discussion, there also exists a temperature T_2 beyond which the thermal graviton gas in AdS will collapse into a big black hole. For a weakly coupled string theory in $AdS_5 \times S_5$, T_2 is of order $\frac{1}{(Rl_p^4)^{\frac{1}{5}}}$ and is much higher than the Hagedorn temperature $T_H \sim \frac{1}{l_s}$ for thermal AdS.

Gauge theory analysis

The lagrangian of $\mathcal{N}=4$ SYM theory on $S^3\times R$ is given by,

$$L=\frac{1}{g^2}Tr\left[F^2+(D\phi)^2+\overline{\chi}\ \not{D}\chi+\sum_{IJ}[\phi^I,\phi^J]^2+\overline{\chi}\Gamma^I\phi^I\chi+m^2\phi^I\phi^I\right] \quad (1.7)$$

The field content of the theory is six scalar fields ϕ^I, gauge fields A^μ and four fermion fields χ^α. The theory has an internal $SO(6)$ R-symmetry. The theory is thought to be conformally invariant and the mass of the scalar field comes from the conformal coupling of the scalar fields with the background curvature. As discussed by Witten [5], there is one to one mapping between the thermal phases of IIB string theory and that of $\mathcal{N}=4$ SYM theory on $S^3\times R$. It has been argued that the black hole phase in the supergravity side is maped to the de-confined phase in the gauge theory and AdS_5 corresponds to the confined phase. The above mentioned identification is motivated by $O(N^2)$ free energy of the black hole phase.

The phases of large N guage theory can be studied using the unitary matrix model([10, 11, 12, 13]). The unitary matrix is the finite temperature Polyakov loop which does not depend on the points of S^3. This fortunate circumstance is due to the fact that in the Hamiltonian formulation, $\mathcal{N}=4$ SYM theory at a given time slice, is defined on the compact space S^3 and the $SO(6)$ scalars are massive because of their coupling to the curvature of S^3. These facts imply that, in principal, one can integrate out almost all the fields and obtain an effective theory of the zero mode of the gauge potential A_0. Using this method a detailed correspondence of the critical points of the gauge theory effective lagrangian and the critical points of supergravity can be constructed at the leading order of the $1/N$ expansion [20].

1.0.2 Plan of the thesis

Here we describe each chapter of the thesis.

R-charged AdS_5 black holes and large N unitary matrix models

In the second chapter we have generalized the above discussed correspondence to the case of R-charged AdS-black holes [15]. R-charged AdS black holes are known to have a rich phase structure in the canonical and grand canonical ensemble. Following the discussion of the previous chapter we have integrated out all the fluctuations to a get an effective action of the gauge theory in terms of the unitary matrix model. In the canonical ensemble we have introduced a fixed charge constraint in the thermal gauge theory. This is shown to contribute an additional logarithmic term $\log(TrUTrU^{\dagger})$ involving the order parameter, to the gauge theory effective action. It should be noted that the perturbative correction to the gauge theory effective action only gives rise to polynomial terms. This logarithmic term is crucial for matching with supergravity. We analyze the implications of this term in the large N limit and successfully compare with the various supergravity properties like the existence of only blackhole solutions in the canonical ensemble and also the existence of a point of cusp-catastrophe in the phase diagram. We also discussed the effect of inclusion of fermions and shown that the main conclusions are not modified.

Blackhole/String Transition in AdS_5 and Critical Unitary Matrix Models

In the third chapter we discuss the blackhole-string transition of the small Schwarzschild blackhole of $AdS_5 \times S^5$ using the AdS/CFT correspondence at finite temperature [16]. When the horizon of this blackhole approaches the string scale l_s, we expect the supergravity (geometric) description to break down and be replaced by a description in terms of degrees of freedom more appropriate at this scale. Presently we have no idea how to discuss this crossover in the bulk IIB string theory. Hence we will discuss this transition and its smoothening in the framework of a general finite temperature effective action of the dual $SU(N)$ gauge theory on $S^3 \times S^1$. The finite temperature gauge theory effective action, at weak *and* strong coupling, can be expressed entirely in terms of constant Polyakov lines which are $SU(N)$ matrices. In showing this we have taken into account that there are no Nambu-Goldstone modes associated with the fact that the 10 dimensional blackhole solution sits at a point in S^5. We show that the phase of the gauge theory in which the eigenvalue spectrum has a gap corresponds to supergravity saddle points in the bulk theory. We identify the third order $N = \infty$ phase transition with the blackhole-string transition. This singularity can be resolved using a double scaling limit in the transition region where the large N expansion is organized in terms of powers of $N^{-2/3}$. The $N = \infty$ transition now becomes a smooth crossover in terms of a renormalized string coupling constant, reflecting the physics of large but finite N. Multiply wound Polyakov lines condense in the crossover region. We also discuss the implications of our results for the resolution of the singularity of the Lorentzian section of the small Schwarzschild blackhole.

Plasma balls / kinks as solitons of large N confining gauge theories

In the fourth chapter we discuss finite regions of the de-confining phase of a confining gauge theory (plasma balls/kinks) as solitons of the large N, long wavelength, effective Lagrangian of the thermal gauge theory expressed in terms of suitable order parameters [17]. We consider a class of confining gauge theories whose effective Lagrangian turns out to be a generic 1 dim. unitary matrix model. The dynamics of this matrix model can be studied by an exact mapping to a non-relativistic many fermion problem on a circle. We present an approximate solution to the equations of motion which corresponds to the motion (in Euclidean time) of the Fermi surface interpolating between the phase where the fermions are uniformly distributed on the circle (confinement phase) and the phase where the fermion distribution has a gap on the circle (de-confinement phase). We later self-consistently verify that the approximation is a good one. We discuss some properties and implications of the solution including the surface tension which turns out to be positive.

Shock formation and the breakdown of collective field theory

In our investigations we realized that it is imperative to use the $2 + 1$ dimensional phase space formulation of the classical Fermi fluid theory. The collective field formalism, which is a hydro-dynamical description in $1 + 1$ dimensions inevitably leads to shock formation and singularities. It is not clear whether a finite energy density soliton solution can be obtained within collective field theory. The shocks are spurious singularities due to the collective field description which correspond to the folds on the Fermi surface, which we have argued is inevitable.

Epilogue

Here we have discussed the significance, general conclusions and possible future directions of our work.

Chapter 2

R-charged AdS_5 black holes and large N unitary matrix models

The AdS/CFT correspondence implies that the phases of string theory can be studied by studying those of the dual gauge theory. In the case of type IIB string theory in $AdS^5 \times S^5$ the phases can be studied using the large N limit of a unitary matrix model([10, 11, 12, 13])[1]. The unitary matrix is the finite temperature Polyakov loop which does not depend on the points of S^3. This fortunate circumstance is due to the fact that in the Hamiltonian formulation, $\mathcal{N} = 4$ SYM theory at a given time slice, is defined on the compact space S^3 and the $SO(6)$ scalars are massive because of their coupling to the curvature of S^3. These facts imply that, in principal, one can integrate out almost all the fields and obtain an effective theory of the zero mode of the gauge potential A_0.

Using this method a detailed correspondence of the critical points of the gauge theory effective lagrangian and the critical points of supergravity (discussed by Hawking and Page[14]) can be constructed at the leading order of the $1/N$ expansion. These are AdS_5 and the small and big black holes (as [20] we refer to these as SSB and BBH.) It turns out that in the gauge theory these critical points are in the gaped phase, where the density of eigenvalues vanishes in a finite arc of the circle around which the eigenvalues are distributed. The closing of the gap, corresponds to the Gross-Witten-Wadia(GWW) phase transition([21, 22, 23]). In a window around this transition, the supergravity description of string theory is likely to smoothly

[1]Phases of large N gauge theory is also discussed in [18, 19]

cross over into a description in terms of heavy string modes.

In this chapter we extend the discussion of the correspondence between R-charged AdS_5 blackholes([27, 28, 25, 26, 29]), and the effective unitary matrix model([21, 22, 23, 24]). R-charged black holes are known to have a rich phase structure in the canonical and grand canonical ensemble. In the canonical ensemble the fixed charge constraint, contributes an additional logarithmic term $\log(TrUTrU^{\dagger})$ involving the order parameter, to the gauge theory effective action. This term is crucial for matching with supergravity. We analyze the implications of this term in the large N limit and compare with the various supergravity properties like the existence of only blackhole solutions in the canonical ensemble and also the existence of a point of cusp-catastrophe in the phase diagram.

The plan of this chapter is as follows. In section 2.1 we give a brief review of charged AdS_5 blackholes. In section 2.2 and section 2.3 we discuss the effective action of the gauge theory at zero and small coupling, in the fixed charged sector. At zero coupling there is exactly one saddle point and the value of $TrUTrU^{-1}$ at the saddle point is always non-zero. For a small positive coupling there are two stable and one unstable saddle points, all with a non-zero value of $TrUTrU^{-1}$. They merge at the GWW point. In section (2.4) we discuss the model effective action at strong coupling. Here too, there are three saddle points,two stable(I,III) and one unstable(II). In the region $\rho > \frac{1}{2}$, I and III can be identified with a stable small blackhole and stable big blackhole respectively. Saddle point II is identified with the small unstable black hole. The merging of saddle points leads to critical phenomenon whose exponents can be calculated and shown to agree with supergravity. This is discussed in section 2.5. We have also calculated the $o(1)$ part of the partition function near the critical point.

2.1 R-charged blackholes in AdS_5 and critical phenomena

The R-charged AdS_5 black hole and relevant phase structure were discussed by A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers ([26],[25]). Here we review their result. The Einstein–Maxwell–anti–deSitter

(EMadS$_{n+1}$) action may be written as

$$I = -\frac{1}{16\pi G}\int_M d^{n+1}x\sqrt{-g}\left[\widetilde{R} - F^2 + \frac{n(n-1)}{R^2}\right] \ , \tag{2.1}$$

where $\widetilde{R}$ is the Ricci scalar and R is the characteristic length scale of AdS. The metric of the Reissner–Nordström–anti–deSitter (RNadS) solution is given in static coordinates by

$$ds^2 = -(1-\frac{m}{r^{n-2}}+\frac{q^2}{r^{2n-4}}+\frac{r^2}{R^2})dt^2+\frac{dr^2}{1-\frac{m}{r^{n-2}}+\frac{q^2}{r^{2n-4}}+\frac{r^2}{R^2}}+r^2d\Omega_{n-1}^2 \ , \tag{2.2}$$

The parameter q is proportional to the charge

$$Q = \sqrt{2(n-1)(n-2)}\left(\frac{\omega_{n-1}}{8\pi G}\right)q \tag{2.3}$$

and m is proportional to the ADM mass M of the blackhole.

$$M = \frac{(n-1)\omega_{n-1}}{16\pi G}m \tag{2.4}$$

ω_{n-1} is the volume of the unit $(n-1)$–sphere, and the gauge potential is given by

$$A_0 = \left(-\frac{1}{\sqrt{\frac{2(n-2)}{n-1}}}\frac{q}{r^{n-2}} + \Phi\right) \tag{2.5}$$

where Φ is the electrostatic potential difference between the black hole horizon and infinity.

For n=4 the solution (2.2) can be considered as a rotating black hole in $AdS_5 \times S^5$ with angular momentum in the internal space S^5. The symmetry group of S^5 is $SO(6)$ and the black hole we are discussing has equal $U(1)$ charges for all the three commuting $U(1)$ subgroups of $SO(6)$, the R-symmetry group of the $\mathcal{N} = 4$ SYM. Hence we are dealing with a system which has the same chemical potential μ for all three $U(1)$ charges in the grand canonical ensemble or equivalently three fixed equal U(1) charges in the canonical ensemble.

2.1.1 Equation of state

In order to discuss the thermodynamics, we consider the Euclidean continuation ($t{\rightarrow}i\tau$) of the solution, and identify the imaginary time period β with the inverse temperature. Using the formula for the period, $\beta{=}4\pi/V'(r_+)$ (for a review see [32]), we get

$$\beta = \frac{4\pi l^2 r_+^{2n-3}}{n r_+^{2n-2} + (n-2) l^2 r_+^{2n-4} - (n-2) q^2 l^2} \; . \tag{2.6}$$

This may be rewritten in terms of the potential as:

$$\beta = \frac{4\pi l^2 r_+}{(n-2) l^2 (1 - c^2 \Phi^2) + n r_+^2} \; . \tag{2.7}$$

The condition for euclidean regularity used to derive (2.6) is equivalent to the condition that the black hole is in thermodynamical equilibrium. The equation (2.6) may therefore be written as an equation of state $T{=}T(\Phi, Q)$. From this equation of state we see that for fixed Φ we get two branches, one for each sign, when the discriminant under the square root is positive[26]. For fixed Q, $T(\Phi)$ has three branches for $Q{<}Q_{crit}$ (Let us call them I, II, III) and one for $Q{>}Q_{crit}$. The critical charge is determined at the "point of inflection" by $(\partial Q/\partial\Phi)_T = (\partial^2 Q/\partial\Phi^2)_T = 0$.

The qualitative features of $\beta(r_+)$ for varying q are shown in Fig 2.1. There

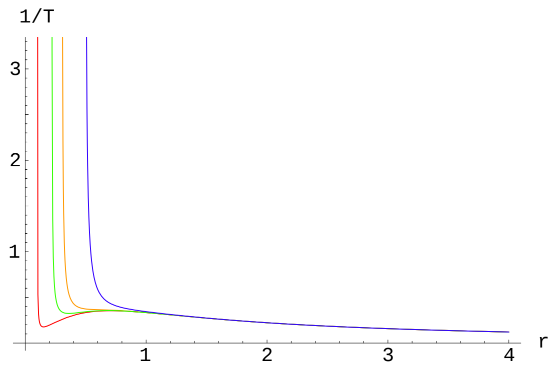

Figure 2.1: Plot of $\beta(r_+)$ for q increasing from the left. The third graph from the left is for q_{crit}

is a critical charge, $q_{\rm crit}$, below which there are three solutions for r_+ for a range of values of T, corresponding to small (I), unstable(II), and large (III) black holes. For fixed $q < q_{crit}$ only branch I is available at low temperatures. At $T = T_{02}(q)$, there is a nucleation of two new solutions, II the unstable small black hole solution, and III the stable big black hole solution. As the temperature is increased further, the black hole II approaches black hole I and at $T = T_{02}(q)$ the two solutions merge.

As q is increased further, T_{02} increases, whereas T_{01} decreases. At $q = q_{crit}$ we have $T_{crit} = T_{01} = T_{02}$. At q_{crit} and T_{crit} all three solutions merge. In the language of catastrophe theory this is a cusp catastrophe. As we increase q beyond q_{crit} there will be just one solution for all temperatures T.

We wish to take note of some properties of the phase diagram.

1. Thermal AdS is not a solution and all three branches of the solution represent black holes.

2. There exists a critical point where the three solutions of the system merge. It is a point of fold catastrophe.

2.1.2 Critical Phenomena

The critical point $(Q_{cit}$,$T_{crit})$ may be approached from various directions in the parameter space. If we set $T = T_{crit}$, then the equation determining $(r - r_{crit})$ takes the form (r_{crit} is the radius of the critical black hole)

$$(r - r_{crit})^3 = \mathcal{C}(Q - Q_{crit}) \tag{2.8}$$

$\mathcal{C}$ is a numerical constant. The critical exponent here is $\frac{1}{3}$, since

$$(r - r_{crit}) \propto (Q - Q_{crit})^{\frac{1}{3}} \tag{2.9}$$

As discussed in [26] (Fig16), the critical point may also be approached through the coexistence line in the parameter space. The coexistence line is the line with the property

$$S_I = S_{III}$$

for the parametric range $q < q_{crit}$. It is the line where the Hawking-Page (first order) transition from the small black hole to the big black hole takes place. As we approach the critical point through this line, we have the relation

$$(r_I - r_{II}) \propto (T - T_{crit})^{\frac{1}{2}} \tag{2.10}$$

In the following we will present an understanding of these properties. Before we do the matching with supergravity we would like to present a discussion of the gauge theory in the limit when $\lambda = g^2 N = 0$ and also when $\lambda << 1$. In these cases we of course can not compare with supergravity which requires $\lambda >> 1$.

2.2 Free YM theory

2.2.1 Effective action with chemical potential

In this section we briefly review (see [13],[12]) the effective action for a free $SU(N)$ Yang-Mills theory (with adjoint matter) on a compact manifold $\Im$ in the large N limit. The basic idea is to integrate out all fields in the theory except for the zero mode of the Polyakov line. The partition function is then reduced to a single unitary matrix integral.

Expanding all fields in the gauge theory in terms of harmonics on $\Im$, the theory reduces to a zero dimensional problem of free $N \times N$ Hermitian matrices

$$\mathcal{L} = \frac{1}{2} \sum_a Tr \left[(D_t M_n)^2 - \omega_n^2 M_n^2 \right] \tag{2.11}$$

The sum in (2.11))is over all field types and their Kaluza-Klein modes on $\Im$. ω_n is the frequency of each mode. The covariant derivative in (2.11) is

$$D_t M_n = \partial_t M_n - i[A_0, M_n] \tag{2.12}$$

A_0 comes from the zero mode (i.e. the mode independent of coordinates on $\Im$) of the time component of the gauge field and is not dynamical. The partition function of the theory at finite temperature can be written as a unitary matrix integral by integrating out all fields in (2.11) except for A_0. Hence we have

$$Z \ = \int DU \ \prod_n \left(\det_{adj} \left(1 - \epsilon_n e^{-\beta \omega_n} U \right) \right)^{-\epsilon_n} \tag{2.13}$$

where $U = \exp(i\beta A_0)$ is a $U(N)$. $\det_{adj}$ denotes the determinant in the adjoint representation and $\epsilon_n = 1$ (-1) for bosonic (fermionic) M_n. The above equation can be expressed as

$$Z = \int DU \exp(\sum_{n=1}^{\infty} \frac{z_n(\beta)}{n}) TrU^n TrU^{-n}) \tag{2.14}$$

where

$$z_n(\beta) = z_B(n\beta) + (-1)^{n+1} z_F(n\beta). \tag{2.15}$$

Here $z_B(\beta), z_F(\beta)$ are the single particle partition functions of the bosonic and fermionic sectors respectively (see [12] for the explicit formulas for $z(\beta)$ in various gauge theories).

If we introduce a chemical potential $\mu = (\log m)$, the formula for the partition function changes to

$$Z[\beta, \mu] = \int DU e^{(\sum_{n=1}^{\infty} \frac{z_n(\beta,\mu)}{n} TrU^n TrU^{-n})} \tag{2.16}$$

where

$$z_n(\beta, \mu) = z_B(n\beta, n\mu) + (-1)^{n+1} z_F(n\beta, n\mu) \tag{2.17}$$

$z_B(\beta, \mu), z_F(\beta, \mu)$ are now the single particle partition functions with chemical potential μ. Hence

$$z_B(\beta, \mu) = \sum_{bosons} \exp(-E_i\beta + Q_i\mu) \tag{2.18}$$

$$z_F(\beta, \mu) = \sum_{fermions} \exp(-E_i\beta + Q_i\mu) \tag{2.19}$$

Q_i is the charge of the state whose energy is E_i (see [12]).

As we have already mentioned, we wish to describe a system which has the same chemical potential μ for all three $U(1)$ charges of the R-symmetry group $SO(6)$ in grand canonical ensemble. Equivalently, we can work with fixed and equal values of the $U(1)$ charges in the canonical ensemble.

Let us for simplicity confine ourselves to the bosonic sector of the $\mathcal{N} = 4$ SYM theory. The gauge fields have no R-charge. The six scalars ϕ_i $(i = 1, ..., 6)$ can be grouped in pairs of two. We define

$$\phi^+ = \phi^1 + i\phi^2, \qquad \phi^- = \phi^1 - i\phi^2 \tag{2.20}$$

(We can similarly define complex fields for the other two pairs.) $\phi^\pm$ have charge ± 1 for each of the three commuting $U(1)s$ of $SO(6)$. Hence, if we consider the single particle partition function for these fields, it will be

$$z[x, \mu] = (exp(+\mu) + exp(-\mu))z[x, 0]/2 = \cosh(\mu)z[x, 0] \tag{2.21}$$

where $z[x, 0]$ is the single particle partition function without any chemical potential.

2.2.2 Canonical Ensemble

We will now discuss the free gauge theory partition function for a canonical ensemble with constant charge Q_0, by introducing a delta function $\delta(\widehat{Q}-Q_0)$ in the functional integration of the gauge theory. $\widehat{Q} = Q[\phi]$ is the corresponding functional for the charge which we want to keep fixed. In our case $\widehat{Q}$ is just the functional for R-charge in gauge theory.

The fixed charge partition function is defined by

$$\begin{aligned} Z(\beta, Q_0) &= \int DX e^{\int_0^\beta S[X]} \delta(\widehat{Q} - Q_0) \\ &= \int DX e^{\int_0^\beta S[X]} \int \exp(i\mu\widehat{Q})\exp(-i\mu Q_0) d\mu \\ &= \int d\mu \exp(-i\mu Q_0) (\int DX e^{\int_0^\beta S[X]} e^{i\mu\widehat{Q}}) \\ &= \int d\mu \exp(-i\mu Q_0) \int DU \exp(\Sigma z_n[\beta, i\mu] TrU^n TrU^{-n}) \end{aligned} \tag{2.22}$$

where $z_n[\beta, i\mu] = z_n^V[\beta, 0] + cos(\mu) z_n^S[\beta, 0] + \cos(\frac{\mu}{2}) z_n^F[\beta, 0]$.

We can now make the approximation [2] that $|z_n[x, i\mu]|$ for $n > 1$ is negligible in comparison to $|z_1[x, i\mu]|$. Neglecting the contribution from the $n > 1$ modes we arrive at a model which contains only $TrUTrU^{-1}$. Using the specific formula for $z[x, \mu]$, of the bosonic [3] sector ,

$$\begin{aligned} Z(\beta, Q_0) &= \int d\mu \exp(-i\mu Q_0) \int DU \exp((a + c\cos(\mu)) TrUTrU^{-1}) \\ &= \int DU \exp(aTrUTrU^{-1}) \int d\mu \exp(-i\mu Q_0) \exp(c\cos(\mu) TrUTrU^{-1}) \\ &= \int DU \exp(aTrUTrU^{-1}) I_{Q_0}(cTrUTrU^{-1}) \end{aligned} \tag{2.23}$$

Here $a(\beta) = z_n^V[\beta, 0]$,$c(\beta) = z_n^S[\beta, 0]$ and for convenience we did not show the explicit β dependence in the equations. $I_n(x)$ is the Bessel function.

[2] This approximation can be thought of as a low temperature approximation. This is because at low temperatures $z_n^S[\beta, 0]$ approaches zero as $e^{-\beta n}$. Hence the higher z_n are suppressed relative to z_1. It is also true that for all temperatures, $z_n < z_1$ and for very high temperatures we have $z_n \sim \frac{z_1}{n}$. As an example, the total contribution for all other z_n, even near hagedorn transition in free $\mathcal{N} = 4$ SYM theory, is only about 7% of z_1[12].

Unitary matrix models involving $TrU^n, n > 1$ has been discussed in [33, 34].

[3] Effect of the fermions is discussed in appendix 2.7.

Hence we end up with a matrix model with an effective potential

$$S_{eff} = a(TrUTrU^{-1}) + \log[I_{Q_0}(cTrUTrU^{-1})] \tag{2.24}$$

where $a > 0, c > 0$. We define $\rho^2 = TrUTrU^{-1}/N^2$ to get

$$S_{eff}(\rho) = N^2(a\rho^2 + (1/N^2)\log[I_{Q_0}(cN^2\rho^2)]) \tag{2.25}$$

It may seem that the logarithmic term is suppressed by a factor of $1/N^2$ and hence negligible for large N. But this need not be the case because, in the semicalssical large N limit, we must deal with a system with a charge of order N^2. Hence we define $Q_0 = N^2 q$ ($q \sim o(1)$). Using the asymptotic expansion of $I_n(nx)$ for large n, the effective action [4] becomes

$$S_{eff}(\rho) = N^2(a\rho^2 + q(\sqrt{(1+\frac{c^2}{q^2}\rho^4)} + \log(\frac{\frac{c}{q}\rho^2}{1+\sqrt{1+\frac{c^2}{q^2}\rho^4}}))) + O(1) \tag{2.26}$$

2.2.3 Phase Structure

To understand the phase diagram of this model at large N, we have to locate the saddle points of (2.26) after including the relevant contribution from the path integral measure depending on whether $\rho < \frac{1}{2}$ or $\rho > \frac{1}{2}$.[5]

Differentiating $S_{eff}(\rho)$ we get

$$\begin{aligned}
\frac{\partial}{\partial\rho^2} S_{eff}(\rho) &= a + \frac{\partial}{\partial\rho^2}(\frac{1}{N^2}\log[I_Q(Q\frac{cN^2\rho^2}{Q})] \\
&= a + b\frac{I'_Q(Q\frac{c\rho^2}{q})}{I_Q(Q\frac{c\rho^2}{q})} \\
&= a + \frac{q}{\rho^2}(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + O(1/Q)
\end{aligned} \tag{2.27}$$

[4]It should be noted that when Q=0 we should use the asymptotic expansion of $I_0(x)$ for large x. Then we get the expected answer $S_{eff} = (a+c)TrUTrU^{-1}$ which is same as a model without any constraint on charge.

[5]The term in the right hand side of equation (2.28) originates from the path integral measure over an unitary matrix.(see [35],Appendix of [20]).

Hence the equations to solve are

$$
\begin{aligned}
a\rho + \frac{q}{\rho}(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} &= \rho, \rho < \frac{1}{2} \\
a\rho + \frac{q}{\rho}(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} &= \frac{1}{4(1-\rho)}, \rho > \frac{1}{2}
\end{aligned} \tag{2.28}
$$

The left side in (2.28) can be written as

$$
a\rho + \frac{q}{\rho}(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} = a\rho + c\rho + \frac{q}{\rho}\frac{1}{(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} \tag{2.29}
$$

So fixing the charge gives rise to a term of type $\frac{q}{\rho}\frac{1}{(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2}$ which has some important properties.

1. For all values of $q > 0, c > 0$, this term is a decreasing positive function of ρ and it diverges as $\rho \to 0$.

2. For all values of $c > 0$ it is a monotonically increasing function of q.

We can now discuss the solution of this model at $N = \infty$. Let us assume that we are discussing the phase where $a(T) + c(T) < 1$. This condition is valid for low temperatures since $a(T), c(T) \to 0$ as $T \to 0$. It should also be recalled that without any charge fixing the hagedorn transition occurs when $a(T) + c(T) = 1$. Unlike the situation with no charge, here we have a function, on the left hand side of (2.28), which diverges as $\rho \to 0$. Hence $\rho = 0$ can not be a solution. We get only one solution at a finite value of ρ which we will describe in the next paragraph.

Equation (2.28) is solved in the region $\rho < \frac{1}{2}$ with solution

$$
\rho^4 = \frac{q^2}{(1-a)^2 - c^2} = \frac{q^2}{(1-a-c)(1-a+c)} \tag{2.30}
$$

The self consistency condition for a solution in the region $\rho < \frac{1}{2}$ is

$$
\rho^4 = \frac{q^2}{(1-a)^2 - c^2} < \frac{1}{16} \tag{2.31}
$$

At low temperatures the condition is naturally satisfied for a small enough value of q. If we gradually increase the temperature (i.e. the value of $a(T)$ and $c(T)$) while keeping the value of the q fixed, then the value of ρ at this saddle point will increase. At some temperature $T_3(q)$, ρ will become equal to $\frac{1}{2}$. Since the measure part (i.e. right hand side of (2.28)) has a third order discontinuity at $\rho = \frac{1}{2}$, we will get a third order phase transition at the temperature T_3. From (2.31) we have the following condition at T_3

$$\frac{q^2}{(1-a)^2 - c^2} = \frac{1}{16} \tag{2.32}$$

If the temperature is increased beyond T_3 then we have to solve (2.28) in the region $\rho > \frac{1}{2}$.

If we increase q, then the value of ρ at the saddle point for a fixed temperature will increase. At some q_3 we get a third order phase transition satisfying

$$16 q_3(T)^2 = (a-1)^2 - c^2 \tag{2.33}$$

Since the minimum value of $a(T)$ and $c(T)$ is zero, the maximum possible value of $q_3^2(T)$ is $q_{crit}^2 = \frac{1}{16}$. If we increase the q beyond q_{crit}, the saddle point will always be confined in the parameter range $\rho > \frac{1}{2}$. Consequently as we increase the temperature from zero we will not get a third order phase transition for $q > q_{crit} = \frac{1}{4}$.

This free model, at zero gauge coupling ($l_s >> R$ in bulk), has some similarities with AdS_5 black holes in a fixed charge ensemble. However unlike the three black hole branches in AdS_5, we get only one branch in the free theory. But most importantly the solution always has a nonzero value of ρ.

It should be recalled that before the Hagedorn transition, a free gauge theory with zero charge has the solution $\rho = 0$ [10, 12].

Some properties of the free theory will be important when we analyze the situation for the weakly coupled gauge theory. Just above the temperature $T_3(q)$, the difference of the two sides of (2.28) can be expanded in the region $\rho > \frac{1}{2}$. Defining $\rho = \frac{1}{2} + x$,$(x > 0)$ the difference is

$$-\epsilon(q)x - C_1 x^2 \tag{2.34}$$

Here $\epsilon(q) > 0$ and $\epsilon(q) \to 0$ as $q \to 0$. It is important to note that $C_1 > 2$ because the measure function (i.e. right hand side of (2.28) has a third order discontinuity at the point $\rho = \frac{1}{2}$. We will also discuss the significance of this in what follows.

2.3 Small coupling model

We will now discuss the problem with a small non-zero gauge coupling $\lambda = g_{YM}^2 N$. By AdS/CFT correspondence it corresponds to a finite string length in AdS. It has been shown in [20] that by considering a phenomenological model of type

$$S[TrUTrU^{-1}] = a(\lambda, T)(TrUTrU^{-1}) + \frac{b(\lambda, T)}{N^2}(TrUTrU^{-1})^2, b > 0 \quad (2.35)$$

we can map out the possible phase diagram of type IIB string theory in AdS_5.[6] Even though the model in (2.35) can be derived from a weak coupling analysis of the gauge theory, it can be thought of as a phenomenological model describing supergravity in AdS_5.

We are motivated to discuss the fixed charge ensemble in the same spirit. Let us add a small interaction term to (2.25). The effective action is then given by

$$S_{eff}(\rho) = N^2(a\rho^2 + b\rho^4 + q(\sqrt{(1 + \frac{c^2}{q^2}\rho^4)} \quad (2.36)$$
$$+ \log(\frac{\frac{c}{q}\rho^2}{1 + \sqrt{1 + \frac{c^2}{q^2}\rho^4}}))) + O(1)$$

Here b is proportional to λ and is also a function of charge. Depending on the theory considered, the sign of b can be either positive or negative. It has been shown in [36] that b is positive for a pure YM theory. In the following discussion we will assume that this is the case in order to motivate a similarity with the supergravity picture. The equations determining the saddle points of (2.37) , including the contribution from the path integral measure, are

$$\begin{aligned}(a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} &= \rho^2, \rho < \frac{1}{2} \\ (a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} &= \frac{\rho}{4(1 - \rho)}, \rho > \frac{1}{2} \quad (2.37)\end{aligned}$$

[6] In fact in [20] an arbitrary convex function is considered, and shown to map out the phase diagram of IIB theory in AdS_5. The simplified model (2.35) leads to similar qualitative result.

In what follows it will be useful to introduce a function

$$\begin{aligned} M(\rho) &= \rho^2, \rho < \frac{1}{2} \\ &= \frac{\rho}{4(1-\rho)}, \rho > \frac{1}{2} \end{aligned} \tag{2.38}$$

$M(\rho)$ is an increasing convex function and is the right hand side of (2.37). It is also useful to introduce $D(x) = S'_{eff}(x) - M(x)$. Eqn (2.37) is then equivalent to $D(\rho) = 0$.

It has been shown in [20] that for an interacting model with zero charge, we get nucleation of black holes along the curve given by $a = \frac{1-2w}{(1-w)^2(1+w)}$ and $b = \frac{2w}{(1-w)^2(1+w)^3}$, $1 > w > 0$. Here we want to analyze a similar type of phenomenon. Let us consider the different cases.

T is varying and q is small

Let us start with a value of charge which is small (i.e. $q \ll 1$) and let us increase the temperature from zero. At low temperatures all the parameters a, c will be small (for small T these parameters have a dependence like $e^{-c\beta}$, c is a constant). Hence we get just one solution for $\rho < \frac{1}{2}$ which we call I. There is no solution for $\rho > \frac{1}{2}$ (see the topmost curve of Fig 2.2) because the left hand side of (2.37) is less than the right hand side.

The situation is quite similar to supergravity where for small charge and low enough temperatures we get a stable small black hole solution. [7]

The function $M(\rho)$ (i.e. right hand side of (2.37)) is a convex increasing function. Hence we will generate new solutions of (2.37) in the region $\rho > \frac{1}{2}$ as we increase temperature (i.e. $a(T), c(T)$ as discussed in appendix 2.8) keeping q fixed (Fig 2.2, Fig 2.3). The new solutions will always come in pairs (Fig 2.2). Let us call the solution nucleation temperature as $T_{01}(q)$. At $T = T_{01}$ we have

$$\begin{aligned} D(\rho) = S'_{eff}(\rho) - M(\rho) &= 0 \\ D'(\rho) = 0 & \end{aligned} \tag{2.39}$$

As we increase the temperature further (i.e. $a(T), c(T)$) the function on the left hand side of (2.37) will also increase. Hence the solutions will start

[7] We should keep in mind that $\rho < \frac{1}{2}$ is not the supergravity regime. The solutions of gauge theory effective action there should be represented as excited string states.

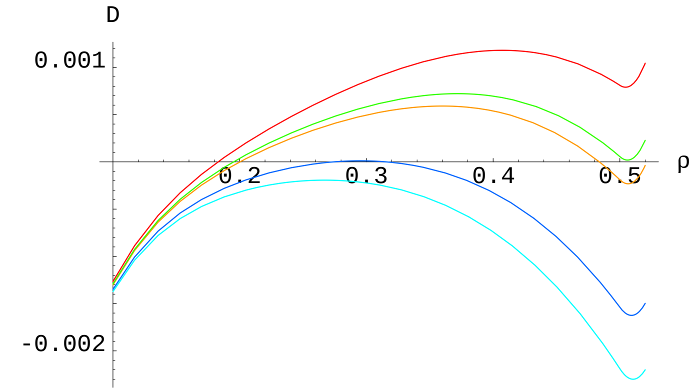

Figure 2.2: Plots of $D(\rho)$ with increasing $a = c$ from the above and with fixed b and $q < q_{crit}$

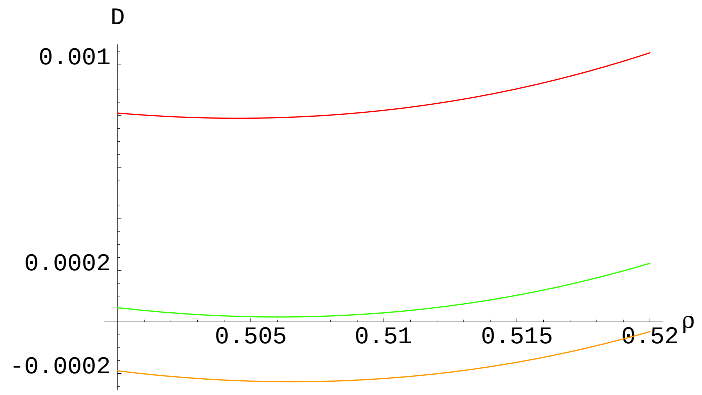

Figure 2.3: Top three graphs of Fig 2 showing emergence of two saddle points in the region $\rho > \frac{1}{2}$

to separate. Let us call them II and III. Here $\rho_{III} > \rho_{II}$. Also, III is a stable saddle point whereas II is an unstable one. They are similar to stable big and unstable small black holes in supergravity. As the temperature is increased beyond T_{01} the value ρ_{II} decreases whereas ρ_{III} increases. At some temperature $T_H(q)$, we will have $S(III) < S(I)$ and consequently we expect a first order phase transition. At a temperature T_{HP}, the dominant saddle point of the system changes from I to III. As the temperature increases the saddle point II goes through a third order transition when $\rho(II)$ crosses the

$\rho = \frac{1}{2}$ point. Call this temperature T_3 which is determined by the following relation between the parameters

$$(a + \frac{b}{2}) + 4q(1 + \frac{c^2}{16q^2})^{\frac{1}{2}} = 1 \qquad (2.40)$$

Increasing the temperature further, the saddle point II approaches the saddle point I and they merge at a temperature T_{02} ($4th$ graph from above in Fig 2.2). In the language of catastrophe theory this is a fold catastrophe. For $T > T_{02}$, the only saddle point is III. This then is the thermal history as we increase the temperature for a small q.

In summary at low temperatures, we have only one saddle point I and then two new saddle point II, III are created at T_{01}. As the temperature increases the saddle points I, II merge at a temperature T_{02}. Beyond that we have only one saddle point III. In the next paragraph we will discuss what happens when we increase the value of the q.

Varying q:

Let us discuss how the various temperatures discussed above change as we increase the value of q.

1. The first is $T_{01}(q)$, the nucleation temperature for saddle points II and III. T_{01} will decrease as we increase the value of q. This is so because all three terms in the right hand side of (2.37) are positive and increasing functions of ρ and the left hand side is a positive convex function. Consequently the saddle point value of ρ at T_{01} will also decrease.

2. The temperature T_{02} at which the stable saddle point I and unstable saddle point II merge and the value of ρ at T_{02}, will increase as we increase q. The reason is that all the coefficients in (2.37) are positive and hence increasing the value of q increases the function on the left hand side.

As we increase the value of q further, T_{01} and T_{02} will become equal for some value of $q = q_{crit}$. In the language of catastrophe theory this is a cusp catastrophe. Corresponding to q_{crit} there will be a $T_{crit} = T_{01} = T_{02}$ and a value of ρ_{crit} (saddle point ρ at q_{crit} , T_{crit}). As we increase the q beyond

q_{crit}, we do not get any new saddle point and consequently there is only one saddle point for all temperatures. We will discuss the physics near this phase transition in detail in what follows.

Value of ρ_{crit} :

Let us determine the value of ρ_{crit}. As already discussed at the end of the previous chapter, in (2.34) C is always a finite quantity. Hence, (2.37) cannot have three solutions in the region $\rho < \frac{1}{2}$ for small b. Therefore the saddle point I is always in the region $\rho < \frac{1}{2}$. Whereas the saddle point III will be in the region $\rho > \frac{1}{2}$. Hence the only place where these three saddle points can meet is $\rho_{crit} = \frac{1}{2}$ which is also the point of the third order phase transition.

Physical Interpretation

Before proceeding further we will briefly discuss the bulk interpretation of saddle points of weakly coupled gauge theory. Weak coupling in gauge theory means $l_s \gg R_{AdS_5}$. Hence the supergravity picture is not valid in the bulk. However at large N, the string coupling (i.e. $\frac{1}{N}$) will be small and we may conclude that the saddle points discussed above can be described by exact (in all orders in l_s) conformal field theories. These CFTs are characterized by the values of q and ρ at the saddle points.

We would like to end this section by emphasizing that the coincidence of the three saddle points at $\rho = \frac{1}{2}$, is a property of the weak coupling ($\lambda << 1$) limit of the gauge theory. In what follows we will see that this fact is not necessarily true at strong coupling. We will show that there the coincidence happens in the gaped phase where $\rho > \frac{1}{2}$.

2.4 Effective action and phase diagram at strong coupling

In this section we will discuss the effective action and the phase diagram in the strongly coupled gauge theory which is dual to the supergravity (discussed in section 2.1) regime of IIB string theory.

2.4.1 Finite temperature effective actions in the gauge theory

Let us first summarize the situation in the zero charge sector. The propagator of adjoint fields in the free gauge theory, on a compact manifold S^3, coupled to a space independent A_0 is given by (see [37])

$$G^{ij}_{U\ kl}(x,t,y,0) = \sum_{n=-\infty}^{n=\infty} (U^n)^i_j G_0(x,t+n\beta,y,0)(U^{-n})^k_j \tag{2.41}$$

where $G_0(x,t,y,0)$ is the zero temperature Green's function and U is the constant Polykov line. We know that at any temperature [8] $G_0(x,t+n\beta,y,0) > G_0(x,t,y,0)$ and also at low temperatures

$$G_0(y,t+n\beta,x,0) \sim e^{-n\beta} \tag{2.42}$$

Using the above Green's function one can develop the large N diagrammatic to arrive at an effective action involving Z_N invariant terms built out of products of trU^n. In fact one can imagine integrating out all the modes trU^n for $n > 1$ in favor of $trUtrU^{-1}$. In this way one gets an effective action of the form

$$S_{eff} = \sum_{n=1}^{n=\infty} a_n(\beta,\lambda)(\frac{TrUTrU}{N^2})^n \tag{2.43}$$

As one increases the coupling constant we would expect that the form of the effective action would remain the same except that the dependence of the parameters on temperature and the 'thooft coupling would change.

2.4.2 Non-zero charge sector

If we include the fixed charge constraint, as in (2.23), then we get the following expression for the fixed charge path integral

$$Z_q = \int d\mu e^{i\mu Q} \int DU e^{N^2(\sum_{n=0}^{n=\infty} S_n(\rho,\lambda,\beta)\cos(n\mu))} \tag{2.44}$$

[8] Temperature here is measured in units of $\frac{1}{R_{S^3}}$.

For large N we can do the μ integral by the saddle point method. The saddle point of μ is on the imaginary axis. Hence we set $\mu = im$, to get the saddle point equation

$$q = \sum_{n=1}^{n=\infty} nS_n(\rho, \lambda, \beta)\sinh(nm)), q = Q/N^2 \tag{2.45}$$

At small values of ρ, $S_n(\rho)$ goes as ρ^n [9] and hence in the $\rho \to 0$ limit we can approximate the equation (2.45) as

$$q \approx \mathcal{C}\rho\sinh(m) \tag{2.47}$$

where $\mathcal{C}$ is a constant independent of ρ. The solution is

$$m \approx q\log(\rho) + \mathcal{C}, \rho \to 0 \tag{2.48}$$

Substituting m in (2.44) we get a logarithmic term for ρ. We conclude that the logarithmic term is a general feature and it implies among other things that $TrU = 0$ is never a solution in the non-zero charge sector.

2.4.3 Model effective action at strong coupling

Following our previous discussion, we will include the generic logarithmic term in the effective potential for a fixed non-zero charge, and proceed to analyze the saddle point structure following [20].

Our proposal for the gauge theory effective action is

$$S_q = S(a(T), b(T), ..., \rho) + q\log(\rho) \tag{2.49}$$

[9] Introduction of a chemical potential changes the formula (2.41) as

$$G^{\mu}{}_{U}{}^{ij}_{kl}(x,t,y,0) = e^{\frac{i\mu t}{\beta}} \sum_{n=-\infty}^{n=\infty} (U^n)^i_j G^{\mu}_0(x, t+n\beta, y, 0)(U^{-n})^k_j$$

where

$$G^{\mu}_0(x, t+n\beta, y, 0) = e^{in\mu} G_0(x, t+n\beta, y, 0) \tag{2.46}$$

Hence in each order in perturbation theory the terms containing $\cos(n\mu)$ also get multiplied by $(TrUTrU^{-1})^m, m > n$ or the higher operators like TrU^n which can be integrated out to give again a term like $(TrUTrU^{-1})^m$.

and the saddle point equations are

$$\begin{aligned} \rho F(a(T), b(T), c(T), ...\rho^2) + q = \rho^2, \rho < \frac{1}{2} \\ \rho F(a(T), b(T), c(T), ...\rho^2) + q = \frac{\rho}{4(1-\rho)}, \rho > \frac{1}{2} \end{aligned} \tag{2.50}$$

Where $F(\rho) = S'_{eff}(\rho)$. We assume that

1. $F(x,T)$ is a monotonically increasing function of x and F(0,T)=0
2. Value of $F(x,T)$ increases for fixed x as we increase the temperature and F(x,0)=0.

   These global properties of $F(x)$ reproduce the phase diagram of supergravity.

Analysis of solution structure

Let us consider the function $D(T,\rho) = \rho F(\rho,T) - M(\rho)$ (Where $M(\rho)$ is the contribution from measure appearing at the right hand side of (2.50)). At $T = 0$, $F(\rho,T)$ is zero. Hence $D(T,\rho)$ is a monotonically decreasing function of ρ at $T = 0$.

We know that at $T = T_{01}$ a pair of two new saddle points appear at $\rho_{01} > \frac{1}{2}$. Hence at $T = T_{01}$ we have $D(T_{01},\rho_{01}) = 0$ and $D'(T_{01},\rho_{01}) = 0$. $D(T_{01},\rho)$ has a zero for $\rho = 0$ and it is a decreasing function in the neighborhood of $\rho = 0$. It again increases and become zero at ρ_{01} and then the function again decreases as $\rho \to 1$. This implies that the function has a local maximum and local minimum.

In summary:

1. $D(0,\rho)$ is a monotonically decreasing function of ρ.
2. $D(T_{01},\rho)$ has a maximum and minimum.

There is a temperature T_{crit} at which the local maximum and local minimum appear (Fig 2.4.3). Let us call this temperature T_{crit}. At T_{crit} the curve $D(T_{crit},\rho)$ will have a point of inflection at $\rho = \rho_{crit}$, say. Let the value of $D(T_{crit},\rho_{crit}) = q_{crit}$.

Increasing the value of q from zero we need to solve the equation $D(T,\rho) = q$. We will get a solution for a non-zero value of ρ. Denote

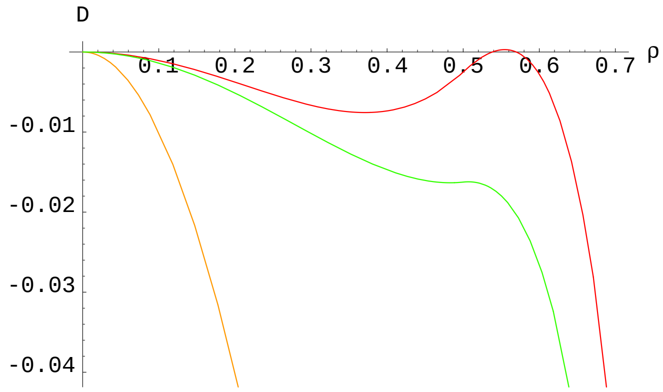

Figure 2.4: Plots of $D(T, \rho)$ for $T = 0$, $T = T_{crit}$,$T = T_{01}$ from below.

this solution as I. As the temperature increases, two new solutions appear at $T = T_{01}$. Call the stable solution as III, and the unstable solution II. As the temperature is further increased to T_{02}, the unstable solution II and the stable solution I merge. For $T > T_{02}$, the only solution is III.

As q approaches q_{crit} from below the two temperatures T_{01} and T_{02} approach each other. At $q = q_{crit}$, we have $T_{01} = T_{02} = T_{crit}$. If we increase q beyond q_{crit} only one solution appears for all temperature. These facts are consistent with supergravity solutions(section 2.1).

With a sufficiently sharp rising function $F(T, \rho)$ in (2.50) we can obtain this critical point in the region $\rho > \frac{1}{2}$.[10]. As the function $D(\rho, T)$ is smooth in the region $\rho > \frac{1}{2}$ the second derivative of the function $D(T, \rho)$ will vanish at the inflection point, and we will get a third order phase transition. We can calculate the partition function in suitable double scaling limit near the critical point. This is discussed in section 2.5.

A specific example

We will now illustrate the above phenomenon in a simple model defined by

$$F(\rho) = a\rho + b\rho^3 \tag{2.51}$$

where $a, b > 0$.

[10] Which describes supergravity in the bulk[20].

We will determine the parameter ranges of a, b for which all the three saddle points of (2.51) are in the range $\rho > \frac{1}{2}$.

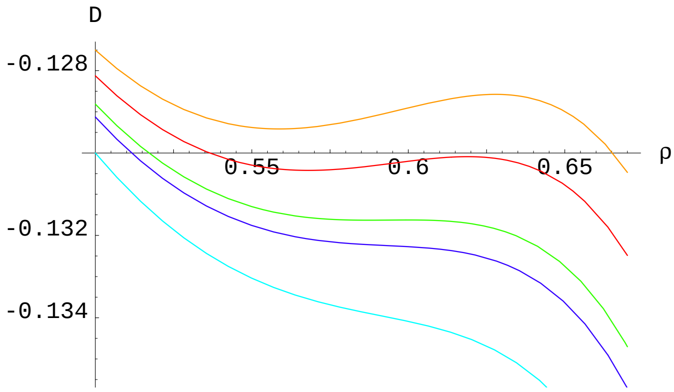

Figure 2.5: Plots of $D(T, \rho)$ with with fixed a and increasing b from the top, showing a critical transition in the region $\rho > \frac{1}{2}$ (from the top, the $3rd$ graph has a point of inflection)

At $\rho = \frac{1}{2}$ we have the constraints

$$\partial_\rho(\rho F(\rho) - \frac{\rho}{4(1-\rho)}) < 0$$
$$\partial_\rho^2(\rho F(\rho) - \frac{\rho}{4(1-\rho)} > 0 \tag{2.52}$$

Putting the value of $\rho = \frac{1}{2}$ in the above inequality we get the following constraints on the parameters $a + b < 1$ and $a + 3b > 2$. Simplifying we have $b > \frac{1}{2}$ and $a < \frac{1}{2}$.

As the coupling become stronger, we expect that b is not necessarily small and will be of $o(1)$ or greater. All the saddle points of (2.50) are then naturally shifted to the region $\rho > \frac{1}{2}$. Here, as was discussed in [20], we can expect to match the solutions of the gauge theory with those of supergravity. The stable saddle point I corresponds to the stable black hole branch I of supergravity. And unstable saddle point II is matched with the unstable black hole branch II. The stable saddle point III is matched with the big stable black hole in supergravity. With this identification the thermal history and critical behavior of the gauge theory, discussed earlier in this chapter, match with the thermal history and critical behavior of supergravity (discussed in section 2.1 and [26]).

2.5 Universal neighborhood of critical point and the critical exponents

Let us consider the effective action $S_{tot}(\rho, T, q)$ which includes the contribution from the path integral measure over an unitary matrix. The derivative of S_{tot} with respect to ρ, say $G(\rho, T, q)$, gives the saddle point equations (2.50). We have already discussed that by a suitable choice of parameters the critical point appears in the region $\rho > \frac{1}{2}$. This critical point is a third order critical point because three saddle points of the system merges here. Hence the first and second derivatives of $G_{tot}(\rho, T, q)$ with respect to ρ vanish at $\rho = \rho_{crit}, q = q_{crit}, T = T_{crit}$. Expanding $G(\rho, T, q)$ around the critical point , we get

$$G(\rho,T,q) = (\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta T)\partial_T G + (\delta q)\partial_q G + (\delta\rho)(\delta T)\partial_\rho\partial_T G + (\delta q)(\delta T)\partial_\rho\partial_q G \tag{2.53}$$

Let us fix $T = T_{crit}$ or $\delta T = 0$. Then the equation (2.53) has one solution. In order to know how the saddle point value of ρ approaches ρ_{crit} ($\delta\rho \to 0$) as $\delta q \to 0$, we equate the leading part of (2.53) to zero.

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta q)\partial_q G = 0 \tag{2.54}$$

Hence $\delta\rho \propto \delta q^{\frac{1}{3}}$ and we get the same universal exponent $\frac{1}{3}$ as in supergravity [26].

2.5.1 Partition function near the critical point

Near the critical point we can write the S_{tot} as

$$S_{tot} = S_{tot}(\rho_{crit}, T_{crit}, q_{crit}) + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + (\delta q)\partial_q S + (\delta q)(\delta\rho)\partial_\rho\partial_q S + O(\delta\rho^5) \tag{2.55}$$

If we define a double scaling limit $N^{\frac{1}{2}}\rho = x, N^{\frac{3}{2}}q = z$, we can write the $o(1)$ part of the partition function, after suitable rescaling of the variables, as

$$Z_2 \propto \int dx e^{-(x^4 - zx)} \tag{2.56}$$

This can be calculated in a power series

$$Z_2 \propto \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \Gamma(\frac{n}{2} + \frac{1}{4}) \tag{2.57}$$

2.5.2 Approaching the critical point through a line of first order transitions

Another type of double scaling limit is possible in this problem. We can set[11]

$$(\delta T)\partial_T G + (\delta q)\partial_q G = 0 \tag{2.58}$$

by choosing a suitable relation between δT and δq. Using (2.58) in (2.53) we get

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta\rho)((\delta T)\partial_\rho\partial_T G + (\delta q)\partial_\rho\partial_q G) = 0 \tag{2.59}$$

with the solutions

$$\begin{aligned} \delta\rho &= 0 \\ \delta\rho &\propto \pm(\delta T)^{\frac{1}{2}} \end{aligned} \tag{2.60}$$

We can expand S_{eff} as

$$S_{eff} \approx S_{crit} + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + C_1(\delta T)(\delta\rho)^2 + OT \tag{2.61}$$

where OT are terms independent of $\delta\rho$. Defining a suitable double scaling limit. $N^{\frac{3}{2}}\delta T = z, N^{\frac{1}{2}}\delta\rho = x$ and a suitable rescaling of the parameters we can evaluate the $o(1)$ factors in the partition function as

$$Z_2 \propto \int dx e^{-(x^4+2zx^2)} \propto \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} \Gamma(\frac{n}{2} + \frac{1}{4}) \propto \sqrt{z} e^{\frac{z^2}{2}} K_{\frac{1}{4}}(\frac{z^2}{2}) \tag{2.62}$$

where $z < 0$.

[11] It is same as following the HP(first order) transition line in parameter space.

2.6 Conclusion

In this chapter we have studied the logarithmic matrix model generated by fixing the R-charge in the gauge theory partition function. In the free gauge theory it has been shown that there is no solution with $\rho = 0$ (AdS type solution). We then studied the effect of adding an interaction term in our model and discussed the generic nature of the logarithmic term even at arbitrary value of the coupling. We identified the supergravity saddle points and their critical behavior which was discussed in ([26]).

Our main aim was to give another example of the utility of unitary matrix methods in providing a non-perturbative dual description of blakholes in AdS and to understand the relation between matrix models and string theory in general. It would be interesting to consider an effective unitary matrix model to describe phases of Kerr-Ads black holes.

2.7 Appendix: Inclusion of Fermions

Including the contributions from the fermions of $\mathcal{N} = 4$ SYM theory change (2.23) to

$$Z(\beta, Q_0) = \int DU \int d\mu \exp(N^2(a + c\cos(\mu) + d\cos(\frac{\mu}{2}))\rho^2 - i\mu Q_0) \quad (2.63)$$

Where $d(\beta)$ is the single particle partition function for the fermions.

At large N , the integral in (2.63) could be evaluated by the saddle point method. The equations determining the saddle points of $\mu = im$ and ρ are

$$(c\,\sinh(m) + \frac{d}{2}\sinh(\frac{m}{2})) = \frac{q}{\rho^2} \quad (2.64)$$

and

$$\rho(a + \cosh(m) + d\cosh(\frac{m}{2})) = \rho \quad (2.65)$$

We would like to see weather there is a solution with $\rho = 0$. As the right hand side of (2.64) becomes large in the limit $\rho \to 0$, we can self consistently approximate $\cosh(m)$ and $\sinh(m)$ as e^m and we get

$$m \approx \log\frac{q}{c\rho^2} \quad (2.66)$$

Hence a logarithmic potential for ρ is once again generated. One can also confirm this by putting (2.66) in (2.65).

2.8 Appendix: Positivity of the coefficient of the quadratic term in the effective action

Let us consider the partition function of YM theory on a compact manifold written as an integral over the effective action of $\rho = TrUTrU^{-1}$.

$$Z(\beta) = \int DU e^{N^2(S_{eff}(\rho))} \tag{2.67}$$

$$= \int d\rho e^{N^2(S_{eff}(\rho) - S_M(\rho))} \tag{2.68}$$

Where $S_M(\rho)$ is the contribution from the measure part[12] of path integral and

$$S_{eff}(\rho) = a(\beta)\rho^2 + \sum_{n=4} a_n(\beta)\rho^n \tag{2.69}$$

i.e. a polynomial in ρ. As $\beta \to \infty$ we have $S_{eff}(\rho) \to 0$. Contribution from the measure part $S_M(0)$ has only one minimum at $\rho = 0$. Hence at low temperature the system will have a saddle point at $\rho = 0$. Expanding ρ around this saddle point as $\rho = 0 + \frac{\delta\rho}{N}$ we get

$$Z(\beta) = \int_{-\infty}^{\infty} d(\delta\rho) e^{-(1-a(\beta))(\delta\rho)^2 + o(\frac{1}{N^2})} \tag{2.70}$$

Like any thermal partition function, (2.68) or (2.70) is a decreasing function of the β. Hence $a(\beta)$ should also be a decreasing function of β. Since $a(\infty) = 0$, for any finite β, $a(\beta)$ is a positive decreasing function. Hagedorn transition happens when $a(\beta_H) = 1$, but whether $a(\beta)$ will reach 1 or not depends on the model.

[12] See discussions before (2.27).

Chapter 3

Blackhole/String Transition in AdS_5 and Critical Unitary Matrix Models

The problem of the fate of small Schwarzchild blackholes is important to understand, in a quantum theory of gravity. In a unitary theory this problem is the same as the formation of a small blackhole. An understanding of this phenomenon has bearing on the problem of spacelike singularities in quantum gravity and also (to some extent) on the information puzzle in blackhole physics. It would also teach us something about non-perturbative string physics.

In the past Susskind [38], Horowitz and Polchinski (SHP) [39] and others [40, 41, 42] have discussed this, in the framework of string theory, as a blackhole-string transition or more appropriately a crossover. Their proposal is that this crossover is parametrically smooth and it simply amounts to a change of description of the same quantum state in terms of degrees of freedom appropriate to the strength of the string coupling. The entropy and mass of the state change at most by $o(1)$. By matching the entropy formulas for blackholes and perturbative string states, they arrived at a crude estimate of the small but non-zero string coupling at the crossover. The SHP description is difficult to make more precise because a formulation of string theory in the crossover regime is not yet explicitly known.

There are many studies on the blackhole–string transition and the nature of the blackhole singularity in the case of two and three–dimensional blackholes[48, 49, 50, 47, 51, 52]. Small extremal supersymmetric blackholes have been discussed in string theory with enormous success [43, 44, 45, 46]. In particular the α' corrections to the entropy of the supersymmetric string sized blackholes has been matched to the microscopic counting.

In this work we discuss the blackhole–string crossover for the small 10 dimensional Schwarzschild blackhole in the framework of the AdS/CFT correspondence. In [20], building on the work of [10, 11, 36, 13, 53, 18], a simplified model for the thermal history of small and big blackholes in AdS_5 (which were originally discussed by Hawking and Page [14]) was discussed in detail . In particular, the large N Gross-Witten-Wadia (GWW) transition [21, 22, 23] was identified with the SHP transition for the small AdS_5 blackhole.

However it turns out that the small blackhole in AdS_5, which is uniformly spread over S^5, has a Gregory-Laflamme instability. When the horizon radius $r_h \sim 0.5R$ [54] the $l = 1$ perturbation is unstable. The final configuration this instability leads to, as r_h decreases and the horizon becomes less and less uniform over S^5, is most likely to be the 10 dim Schwarzschild blackhole. This small 10 dim Schwarzschild black hole does not have any further instability of Gregory-Laflamme type. This blackhole also happens to be a solution with asymptotic $AdS_5 \times S^5$ geometry for $l_s \ll r_h \ll R$(3.16).

When the horizon of this blackhole approaches the string scale l_s, we expect the supergravity (geometric) description to break down and be replaced by a description in terms of degrees of freedom more appropriate at this scale. Presently we have no idea how to discuss this crossover in the bulk IIB string theory. Hence we will discuss this transition and its smoothening in the framework of a general finite temperature effective action of the dual $SU(N)$ gauge theory on $S^3 \times S^1$. In fact it is fair to say that in the crossover region we are really using the gauge theory as a definition of the non-perturbative string theory.

At large but finite N, since S^3 is compact, the partition function and all correlation functions are smooth functions of the temperature and other chemical potentials. There is no phase transition. However in order to make a connection with a theory of gravity, which has infinite number of degrees of freedom, we have to take the $N \to \infty$ limit and study the saddle point expansion in powers of $\frac{1}{N}$. It is this procedure that leads to non-analytic behavior. It turns out that by taking into account exact results in the $\frac{1}{N}$ expansion it is possible to resolve this singularity and recover a smooth crossover in a suitable double scaling limit.

In the specific problem at hand, it turns out that in the transition region the large N expansion is organized in powers of $N^{-2/3}$. In the bulk theory, assuming AdS/CFT, this would naively mean a string coupling expansion in powers of $g_s^{2/3}$. However in a double scaling limit, a renormalized string coupling $\widetilde{g} = N^{\frac{2}{3}}(\beta_c - \beta)$ once again organizes the coupling constant expansion in integral powers. The free energy and correlators are smooth functions of $\widetilde{g}$.

The use of the AdS/CFT correspondence for studying the blackhole-string crossover requires that there is a description of small Schwarzschild blackholes as solutions of type IIB string theory in $AdS_5 \times S^5$. Fortunately, Horowitz and Hubeny [55] have studied this problem with a positive conclusion. This result enables us to use the boundary gauge theory to address the crossover of the small Schwarzschild blackhole into a state described in terms of 'stringy' degrees of freedom. Even so the gauge theory is very hard to deal with as we have to solve it in the $\frac{1}{N}$ expansion for large but finite values of the 'tHooft coupling λ.

However there is a window of opportunity to do some precise calculations because it can be shown that the effective action of the gauge theory at finite temperature can be expressed entirely in terms of the Polyakov loop which does not depend on points on S^3. This is a single $N \times N$ unitary matrix, albeit with a complicated interaction among the winding modes $\mathrm{tr}U^n$. This circumstance, that the order parameter U in the gauge theory is a constant on S^3, matches well on the supergravity side with the fact that all the zero angular momentum blackhole solutions are also invariant under the $SO(4)$ symmetry of S^3. The blackhole may be localized in S^5, but it does not depend on the co-ordinates of S^3. The coefficients of the effective action depend upon the temperature, the 't Hooft coupling λ and the vevs of the scalar fields. Since the 10-dimensional blackhole sits at a point in S^5, one may be concerned about the spontaneous breaking of $SO(6)$ R-symmetry and corresponding Nambu-Goldstone modes. We will conclude, using a supergravity analysis, that the symmetry is not spontaneously broken. Instead we have to introduce collective coordinates for treating the zero modes associated with this symmetry.

The general unitary matrix model can be analyzed due to a technical progress we have made in discussing the general multi-trace unitary matrix model. We prove an identity that enables us to express and study critical properties of a general multi-trace unitary matrix model in terms of the critical properties of a general single trace matrix model.

As is well known, the single trace unitary matrix model at $N = \infty$ has a third order GWW transition, which occurs when the density of eigenvalues of the unitary matrix develop a gap on the unit circle. The vanishing of the density at a point on the circle leads to a relation among the coupling constants of the matrix model which defines a surface in the space of couplings (parameters of the effective action). The behavior of the matrix model in the neighborhood of this surface (call it critical surface) is characterized by universal properties which are entirely determined by the way the gap in the eigenvalue density opens: $\rho(\theta) \sim (\pi - \theta)^{2m}$, where m is a positive integer. In our problem, there is only one tunable parameter, namely the temperature. Hence we will mainly focus only on the lowest $m = 1$ critical point and present the relevant operator that opens the gap. We also discuss the possible relevance of higher order multi-critical points.

Using the properties of the $\frac{1}{N}$ expansion near and away from the critical surface, we will argue that the small blackhole (or for that matter any saddle point of supergravity around which a well defined closed string perturbation expansion exists) corresponds to the phase of the matrix model where the density of eigenvalues on the unit circle has a gap. The small blackhole therefore corresponds to the gapped phase of the unitary matrix model.

We make a reasonable physical assumption based on the proposal of SHP, that the thermal history of the unstable saddle point corresponding to the small blackhole, eventually intersects the critical surface at a critical temperature T_c, which is $o(1/l_s)$. T_c is smaller than the Hagedorn temperature. Once the thermal history crosses the critical surface it would eventually meet the $AdS_5 \times S^5$ critical point corresponding to a uniform eigenvalue distribution. (Such a history was already discussed in the context of a simplified model in [20].) It is natural to identify the crossover across the critical surface in the gauge theory as the bulk blackhole-string crossover.

At the crossover, the $o(1)$ part of the gauge theory partition function (which depends on the renormalized string coupling) can be exactly calculated in a double scaling limit. This is a universal result in a sense that it does not depend on the location of the critical point on the critical surface but depends only on deviations which are normal to the critical surface. If we parametrize this by t, the the free energy $-F(t)$ solves the differential equation $\frac{\partial^2 F}{\partial t^2} = -f^2(t)$ where $f(t)$ satisfies the Painlevé II equation. The exact analytic form of $F(t)$ is not known, but $F(t)$ is a smooth function in the domain $(-\infty, \infty)$[1]. All the operators $\rho_k = \frac{TrU^k}{N}$ condense in the crossover region. In fact $\left\langle N^{\frac{2}{3}}(\rho_k - \rho_k^{ug}) \right\rangle = C_k \frac{d}{dt} F$, where $C_k = \frac{(-1)^k}{k}$ and ρ_k^{ug} represents the expectation value of ρ_k in the ungapped phase.

The smooth crossover of the Euclidean blackhole possibly has implications for the resolution of the singularity of the Lorentzian blackhole, because within the AdS/CFT correspondence we should be able to address all physical questions of the bulk theory in the corresponding gauge theory. In particular we should be able to address phenomena both outside and inside the blackhole horizon. The plan of this chapter is as follows. Section 3.1 discusses the string-blackhole transition. Section 3.2 discusses the small 10-dimensional blackhole in $AdS_5 \times S^5$. Section 3.3 discusses the finite temperature gauge theory and the effective action in terms of the unitary matrix model. Section 3.4 presents the multi-trace partition function as the calculable integral transform of the single trace unitary matrix model. Section 3.5 discusses critical behavior in the unitary matrix model. Section 3.6 discusses the saddle point equations of the matrix model. Section 3.7 discusses the double scaled partition function. Section 3.8 discusses the introduction of chemical potentials and higher critical points. Section 3.9 discusses the applications of the critical matrix model to the small 10-dimensional blackhole. Final section discusses the Lorentzian blackhole. We also include appendices explaining and extending some of the results.

3.1 Blackhole-string transition

In this section we review the blackhole-string crossover. Consider the 10-dim Schwarzschild blackhole. As long as its horizon radius $r_h \gg l_s$ (l_s is the string length), the supergravity description is valid and we can trust the lowest order effective action in l_s. When $r_h \sim l_s$, this description breaks down and one learns to derive an effective action valid to all orders in l_s or devises other methods to deal with the problem. Let us assume that the all orders in l_s description is available, then presumably the geometrical description is still valid in principle, and one can indeed discuss the notion of a string size horizon with radius $r_h \sim l_s$ [43, 44, 46]. It is reasonable to expect that in such a description the qualitative fact that the mass decreases

[1]This universal formula also appeared in the discussion of the simplified model in [20]

with the horizon radius and increasing temperature, is still valid. These facts are obviously valid to lowest order in l_s, because $r_h = 2G_N M$ and $T_h = (G_N M)^{1/7}$. Here G_N is Newton's coupling and M is the mass of the blackhole. For definitiveness let us fix the mass and the entropy of the blackhole. Then the r_h and T_h vary with the gravitational coupling G_N. Now since $g_s^2 = G_N l_s^{-8}$, we can say that r_h and T_h vary with g_s and hence a crossover at $r_h \sim l_s$ happens at a specific value of the string coupling.

When $r_h \lesssim l_s$ the above description of the state has to be replaced by a description in terms of microscopic degrees of freedom relevant to the scale l_s. Even in this description it is reasonable to assume that the temperature of the state varies as we change the string coupling. The assumption of Susskind-Horowitz-Polchinski is that the mass of the state would change by at most o(1) in the string coupling.

From the above discussion it is clear that the blackhole-string crossover occurs in a regime where the curvature of the blackhole is $o(1)$ in string units, so as to render the supergravity description invalid. It is also clear that besides l_s related effects, the string coupling is non-zero and its effects have to be taken into account. Presently our understanding of string theory is not good enough for us to make a precise and quantitative discussion of the crossover. Hence we will discuss the problem using the AdS/CFT correspondence. In order to do this we need to be able to embed the small blackhole in $AdS_5 \times S^5$. This has been discussed by Horowitz and Hubeny [55]. We briefly review their work in the next section.

3.2 Embedding the 10-dimensional Schwarzschild blackhole in $AdS_5 \times S^5$

It is not difficult to argue that the small 10-dim Schwarzschild blackhole exists as a solution of Einstein's equation in $AdS_5 \times S^5$. A small patch of the $AdS_5 \times S^5$ space is locally identical to 10 dim Euclidean space. Since the horizon radius of this blackhole $r_h \ll R$, we can have a solution which is locally identical to a 10 dim Schwarzschild blackhole in flat space-time. We would also require that the solution for large 10 dimensional radial distances asymptotes to $AdS^5 \times S^5$. This solution is not explicitly known, but can be numerically constructed given the boundary conditions on the radial functions. The more non-trivial issue is concerning the fact that the type

IIB theory also has a 5-form. In the absence of the blackhole this form is the volume form of S^5 and carries N units of flux. It turns out that in the presence of the small blackhole, a consistent solution to the equations of motion, is such that there is no energy flux into the blackhole. Hence the small blackhole remains small. In the above analysis one neglects the back reaction on the metric due to the fact that the blackhole is small and the curvature near its horizon is large.

The solution is conveniently represented if we use a 10 dimensional radial coordinate system (fixed by the area of S^8) in $AdS_5 \times S^5$. One splits S^8 into S^3 and S^4, corresponding to the rotational $SO(4)$ symmetry of AdS_5 and the remaining (unbroken) $SO(5)$ symmetry of S^5. This is achieved by using the following coordinate transformation in (3.16)

$$\begin{aligned} r &= \rho \sin\theta \\ \chi &= \rho\cos\theta \end{aligned} \tag{3.1}$$

In these coordinates, a flat patch within AdS is achieved in the limit $r, \xi \ll R$, where R is the radius of AdS_5. The metric takes the form

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\Omega_4^2) \tag{3.2}$$

(The angular term in parenthesis is equivalent to $d\Omega_8^2$). Similarly the 5-form field strength takes the form

$$\begin{aligned} F = -\rho^3 \sin^4\theta dt \wedge d\rho \wedge d\Omega_3 - \rho^4 \sin^3\theta\cos\theta dt \wedge d\theta \wedge d\Omega_3 + \\ r^4\cos^5\theta d\rho\wedge d\Omega_4 - r^5\sin\theta\cos^4(\theta) d\theta\wedge d\Omega_4 \end{aligned} \tag{3.3}$$

In this metric the Schwarzschild solution is given by

$$ds^2 = -f(\rho)dt^2 + f^{-1}(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\Omega_4^2) \tag{3.4}$$

$$\begin{aligned} F = g_1(\rho,\theta)[-\rho^3\sin^4\theta dt\wedge d\rho\wedge d\Omega_3 - r^5\sin\theta\cos^4\theta d\theta\wedge d\Omega_4] \\ +g_2(\rho,\theta)[\rho^4\sin^3\theta\cos\theta dt\wedge d\theta\wedge d\Omega_3 + r^4\cos^5\theta d\rho\wedge d\Omega_4] \end{aligned} \tag{3.5}$$

where near the blackhole horizon $f = 1 - \frac{r_h^7}{r^7}$. As $r \to \infty$, the functions $f(r), g_1(r,\theta), g_2(r,\theta)$ approach their corresponding values in $AdS_5 \times S^5$ geometry. The explicit solution for these functions are not known but their form can be determined by numerically integrating a set of coupled linear differential equations. These solutions have the desired property that imply that the small blackhole remains small.

3.3 Finite temperature gauge theory, order parameter and effective action

We first present a general discussion of the order parameter of $SU(N)$ YM theory on the compact manifold S^3. We consider the theory in the canonical ensemble, i.e. the Euclidean time direction is periodically identified with a period of $\beta = \frac{1}{T}$. It was pointed out in [10, 12] that the Yang-Mills theory partition function on S^3 at a temperature T can be reduced to an integral over a unitary $SU(N)$ matrix U, which is the zero mode of Polyakov loop on the euclidean time circle. Their analysis was done in the limit when the 'tHooft coupling $\lambda \to 0$.

$$Z(\lambda, T) = \int dU\, e^{S(U)} \tag{3.6}$$

with

$$U = P\mathrm{exp}\Big(i\int_0^\beta A_0 d\tau\Big) \tag{3.7}$$

where $A_0(\tau)$ is the zero mode of the time component of the gauge field on S^3. This follows from the fact that apart from A_0 all modes of the gauge theory on S^3 are massive. We will discuss the validity of the above expression in both strong and weak (λ) coupling regimes. Hence we can use U as an order parameter. Gauge invariance requires that the effective action of U be expressed in terms of products of $\mathrm{tr}U^n$, with n an integer, since these are the only gauge invariant quantities that can be constructed from A_0 alone. $S_{\rm eff}(U)$ also has a Z_N symmetry under

$$U \to e^{\frac{2\pi i}{N}} U \tag{3.8}$$

This is due to the global gauge transformations which are periodic in the Euclidean time direction up to Z_N factors. Z_N invariance puts further restriction on the form of the effective action and a generic term in $S(U, U^\dagger)$ will have the form

$$\mathrm{tr}\, U^{n_1} \mathrm{tr}\, U^{n_2} \cdots \mathrm{tr} U^{n_k}, \qquad n_1 + \cdots n_k = 0 \ (\mathrm{mod}\ N), \qquad k > 1 \tag{3.9}$$

In the large N limit we can work with $U(N)$ rather than $SU(N)$, and in that case Z_N is replaced by $U(1)$.

We can expand S_{eff} in terms of a complete set of such operators. The first few terms are

$$S(U, U^\dagger) = a \text{tr}\, U \text{tr}\, U^{-1} + \frac{b}{N} \text{tr}\, U^2 \text{tr}\, U^{-1} \text{tr}\, U^{-1} + \frac{c}{N^2} \text{tr}\, U^3 \text{tr}\, U^{-1} \text{tr}\, U^{-1} \text{tr}\, U^{-1} + \cdots \tag{3.10}$$

More generally we will write the effective action (3.10) in a form which will be convenient for later discussion,

$$S(U, U^\dagger) = \sum_{i=1}^{p} a_i \text{tr}\, U^i \text{tr}\, U^{\dagger i} + \sum_{\vec{k}, \vec{k}'} \alpha_{\vec{k}, \vec{k}'} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{k}'}(U^\dagger), \tag{3.11}$$

where $\vec{k}$, $\vec{k}'$ are arbitrary vectors of nonnegative entries, and

$$\Upsilon_{\vec{k}}(U) = \prod_j \left(\text{tr}\, U^j\right)^{k_j}. \tag{3.12}$$

It is useful to define

$$\ell(\vec{k}) = \sum_j j k_j, \quad |\vec{k}| = \sum_j k_j. \tag{3.13}$$

The above parametrization of the general action is slightly redundant, since the second summand in (3.11) is already the most general gauge-invariant action for U, $U^\dagger$, but writing it this way will be very useful. Reality of the action (3.11) requires $\alpha_{\vec{k}\vec{k}'} = \alpha^*_{\vec{k}'\vec{k}}$. In fact, using the explicit perturbative rules to compute $S(U, U^\dagger)$ in (3.11), one can show that the $\alpha_{\vec{k}\vec{k}'}$ are real, therefore

$$\alpha_{\vec{k}\vec{k}'} = \alpha_{\vec{k}'\vec{k}}. \tag{3.14}$$

On the other hand, invariance of $S(U, U^\dagger)$ under $U \to e^{i\theta} U$ requires that

$$\ell(\vec{k}) = \ell(\vec{k}'). \tag{3.15}$$

We now present evidence at both weak and strong λ that the above effective action is correct.

3.3.1 Perturbative analysis

In perturbation theory one can integrate out all fields, except the zero-mode A_0 of the time component of a gauge field, to get an effective action of U [12]. All fields other than this mode are massive in a free YM theory on S^3. The scalar fields get their mass due to the curvature coupling. We can expand all other fields on S^3, and due to the finite radius of S^3 all the harmonics are massive. Hence at small coupling (small λ) one may integrate out all the fields and derive an effective action for U. In [36] the perturbative (up to three loop order) effective action was calculated and it has the form (3.10).

3.3.2 Strong coupling analysis

The above discussion is perturbative and there is no guarantee that the scalar fields remain massive in the expansion of the theory around $\lambda = \infty$. We will now show, using the AdS/CFT correspondence, that even at strong coupling (large λ), all the excitations of $\mathcal{N} = 4$ SYM theory on S^3 are massive [5]. For illustration we consider the wave equation of a scalar field $\phi(r,t)$ in a general blackhole background which is asymptotically $AdS_5 \times S^5$.

The $AdS_5 \times S^5$ metric is given by

$$ds^2 = (1 + \frac{r^2}{R^2})d\tau^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_3^2 + R^2 d\Omega_5^2 \tag{3.16}$$

Let us consider the situation when the asymptotic solution depends on the co-ordinates of S^5 and S^3. Since S^5 and S^3 are compact spaces, their laplacians have a discrete spectrum. We focus on the radial part and consider a finite energy solution of energy E, $\phi(r,\theta_3,\theta_5,\tau) = f(r,\theta_3,\theta_5)\exp(E\tau)$. The wave equation in the asymptotic metric (3.16) is given by

$$\begin{aligned}(3+5r^2)f'(r,\theta_3,\theta_5) + r(1+r^2)f''(r,\theta_3,\theta_5) + (\frac{r}{1+r^2}E^2 + \frac{1}{r}\Delta^2_{\Omega 3} + \\ r\Delta^2_{\Omega_5})f(r,\theta_3,\theta_5) &= 0 \\ (3+5r^2)f'(r) + r(1+r^2)f''(r) + (\frac{r}{1+r^2}E^2 - \frac{1}{r}L^2_{\Omega 3} - rM^2_{\Omega_5})f(r) &= 0\end{aligned}$$

where $'$ is the partial derivative with respect to r and $L_{\Omega 3}$ is the contribution from S^3 harmonics and $M^2_{\Omega 5}$ is the contribution from S^5 harmonic.

For $f(r) \sim r^\alpha$, as $r \to \infty$, equation (3.17) reduces to

$$5r^{\alpha+2}((\alpha(\alpha-1)+5\alpha) - M^2_{\Omega 5}) = 0 \tag{3.17}$$

In the last equation we have neglected the term $E^2 r^\alpha$ and the S^3 harmonics part, as it is suppressed by a factor of order $\frac{1}{r}$. Hence $\alpha_1 = -2 + \sqrt{4 + M^2}$ or $\alpha_2 = -2 - \sqrt{4 + M^2}$ are two solutions of (3.17). Consequently , $f(r) \sim r^{\alpha_2}$ is the only solutions which is normalizable.

Let us now analyze the situation near the blackhole horizon which, in the euclidean continuation, acts like the origin of polar co-ordinates. Hence, we have the boundary condition,

$$\frac{df}{dr} = 0 \tag{3.18}$$

Near the origin, the scalar field laplacian in the blackhole back ground will have two solutions for a given E. One of them diverges at the horizon and other maintains the condition (3.18). For a generic E, a well-behaved solution in general approaches a non-renormalizable solution as $r \to \infty$. As in quantum mechanical problems, a normalizable solution exists only for those values of E for which, the solution that behaves correctly at the lower endpoint also vanishes for $r \to \infty$. This eigenvalue condition determines a discrete value of E. Hence the mass gap in SYM theory on S^3 persists at the strong coupling. The basic physical reason for the discrete spectrum is that the asymptotic $AdS_5 \times S^5$ geometry gives rise to an infinitely rising potential for large r.

In order to make an estimate of the mass gap we note that the blackhole metric depends on the combination GM, where $G \sim \frac{1}{N^2}$ is Newton's coupling and $M \sim N^2$ is the mass of the blackhole. Further using standard formulas of blackhole thermodynamics it is possible to express GM entirely in terms of the temperature of the blackhole, which sets the scale of the mass gap.

We also expect the single negative eigenvalue in the spectrum of the euclidean Schwarzschild solution in asymptotically flat space-time to persist in the present case. Next we discuss the zero modes.

$SO(6)$ non-invariance of the 10-dimensional blackhole

As discussed in the introduction, our main interest is the study of the 10 dimensional small blackhole to string transition in $AdS_5 \times S^5$. The metric of the small 10 dimensional blackhole in $AdS_5 \times S^5$ is not symmetric under the $SO(6)$ transformations of S^5. Hence the corresponding saddle point in the gauge theory would transform under the $SO(6)$ R-symmetry group and a natural question is whether the $SO(6)$ symmetry is spontaneously broken in the dual gauge theory with associated massless Nambu-Goldstone modes. If this were true, then we would have to include additional degrees of freedom in the effective action (3.10).

Fortunately even though the small 10 dimensional blackhole sits at a point in S^5 the massless modes associated with motions about this point correspond to normalizable solutions of the small fluctuations equation. Let us discuss this point in more detail.

We have already discussed in the section 3 that the small 10 dim blackhole is invariant under an "unbroken" $SO(5)$ subgroup of $SO(6)$. The remaining broken generators of $SO(6)$ rotate the blackhole in S^5. The blackhole is labeled by its mass (equivalently temperature) and its position in S^5, which we denote by the co-ordinates θ_5. $SO(6)$ rotations can rotate the blackhole to any point in S^5. The action of the initial and final blackhole is the same, because we get the final solution just by a co-ordinate rotation of the initial solution. As there is an orbit of blackhole solutions with the same action, it is expected that there is a zero mode in the spectrum of the small oscillations operator around the blackhole.

Let us clarify this point in more detail. Consider a blackhole metric $(g^0_{\mu\nu}(\theta_5))$ as a function of θ_5. As we mentioned before, an infinitesimal rotation in S^5 creates a new black solution which is given by $g^1_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$. As both the matrices $g^0_{\mu\nu}$ and $g^1_{\mu\nu}$ solve the equations of motion, their difference $\delta g_{\mu\nu}$ will be a zero mode. The existence of such a zero mode does not necessarily signal the onset of spontaneous symmetry breaking. The important point is whether the zero mode is normalizable or not. We will show that $\delta g_{\mu\nu}$ is a *normalizable* zero mode.

We make the assumption that the asymptotic geometry of an uncharged blackhole solution is determined by its mass. Hence the asymptotic geometry of the blackhole is given by that of a small AdS_5 blackhole [14] with corrections $f_{\mu\nu}$,

$$ds^2 = (1 + \frac{r^2}{R^2} - \frac{m}{r^2})dt^2 + \frac{dr^2}{(1 + \frac{r^2}{R^2} - \frac{m}{r^2})} + r^2 d\theta_3^2 + R^2 d\theta_5^2 + f_{\mu\nu} dx^\mu dx^\nu \quad (3.19)$$

where $f_{\mu\nu} \sim \frac{1}{r^3}$ as $r \to \infty$. Hence the difference of $g^0(\mu,\nu)$ and $g^1(\mu,\nu)$ can be written as

$$\delta g(\mu,\nu) = f^1_{\mu,\nu} - f^0_{\mu,\nu} \quad (3.20)$$

where f^0 and f^1 denotes the f's corresponding to g^0 and g^1. Now $f_{\mu\nu} \sim \frac{1}{r^3}$

implies $\delta g_{\mu\nu} \sim \frac{1}{r^3}$. Hence $\delta g_{\mu\nu}$ is square integrable [2],

$$\int d^4x \delta g_{\mu\nu}^2 \propto \int dr r^3 \frac{1}{r^6} \propto \int dr \frac{1}{r^3} \tag{3.21}$$

Since the symmetry is not spontaneously broken, we should consider the full orbit of the classical field under SO(6) (or its coset) using the method of collective coordinates [62]. Hence we have the situation in which the degrees of freedom correspond to two sets of zero modes: those corresponding to A_0 and those corresponding to $SO(6)$ symmetry. In the method of collective coordinates we make the following change of variables in the gauge theory path integral.

For simplicity of presentation we denote the fields of the gauge theory that transform under $SO(6)$ by $\phi(x)$ and consider

$$\phi(x) = \phi_0(x)^{[\Omega^5]} + \eta(x) \tag{3.22}$$

$$\tag{3.23}$$

and the gauge condition,

$$(\eta, \phi_0^{[\Omega^5]}) = 0 \tag{3.24}$$

where $\phi_0(x)^{[\Omega^5]}$ is the orbit under $SO(6)$ of the classical configuration $\phi_0(x)$. The path integral measure now becomes

$$D\phi(x) = d\Omega^5 D\eta(x) \delta(\eta, \phi_0^{[\Omega^5]}) \Delta \tag{3.25}$$

where Δ is the Faddev-Popov determinant. Then by standard means we can see that the zero mode is eliminated by the delta-function and the collective coordinate (compact group measure) factors out of the path integral and the remaining action is a functional of the classical field $\phi_0(x)$. Integrating out the fluctuations η, we will obtain an effective action entirely in terms of the unitary matrix U. The coefficients of the effective action will now depend on the vevs of the scalar fields.

[2] This argument seems to be independent of α' corrections as the asymptotic geometry is always weakly curved for any black hole situated in a asymptotic AdS space with $l_s << R$.

3.3.3 Comments on the effective theory

It should be mentioned that the effective action (3.10) is constructed only from the zero mode of A_0 on a compact manifold. Hence this effective action will not be able to describe physical situations which depend on the co-ordinates of the compact manifold S^3. However on the supergravity side all the zero angular momentum blackhole solutions are invariant under the $SO(4)$ symmetry of S^3. The blackhole may be localized in S^5, but it does not depend on the co-ordinates of S^3. This fortunate circumstance enables us to use (3.10) as a reliable effective action to describe some aspects of the string theory in $AdS_5 \times S^5$.

The saddle points of (3.10) corresponding to the $\mathcal{N} = 4$ SYM theory are in one to one correspondence with the bulk supergravity (more precisely IIB string theory) saddle points. For example, the $AdS_5 \times S^5$ geometry corresponds to a saddle point such that $\langle \mathrm{tr}\, U^n \rangle = 0 \ \forall n \neq 0$. Hence the eigenvalue density function is a uniform function on the circle. Now, depending on the co-efficients in (3.10) the saddle point $\langle \mathrm{tr}\, U^n \rangle$ can have a non-uniform gaped or ungapped eigenvalue density profile. Changing the values of the coefficients, by varying the temperature, may open or close the gap and lead to non-analytic behavior in the temperature dependence of the free energy at $N = \infty$. We will interpret this phenomenon as the string-blackhole transition. As we shall see this non-analytic behavior can be smoothened out by a double scaling technique in the vicinity of the phase transition.

3.4 Exact integral transform for the partition function

We start with the most general effective action given in the equation (3.11). The partition function is given by

$$Z = \int [d\,U] e^{S(U,U^\dagger)}. \tag{3.26}$$

We will assume in the following that $a_i > 0$ in (3.11). This amounts to the assumption that $\rho_i = \langle \frac{1}{N} \mathrm{tr}\, U^i \rangle = 0$ is always a saddle point of the effective action. It corresponds to the $AdS_5 \times S^5$ saddle point of IIB string theory. In [15] it was shown that, at sufficiently low temperatures, $a_1 > 0$.

We now use the standard Gaussian trick to write,

$$\begin{aligned}
&\exp\Big\{\textstyle\sum_{i=1}^{p} a_i \mathrm{tr}\, U^i \mathrm{tr}\, U^{\dagger i}\Big\} \\
&= \left(\tfrac{N^2}{2\pi}\right)^p \textstyle\int \prod_{i=1}^{p} \frac{dg_i\, d\overline{g}_i}{a_i} \exp\Big\{-N^2 \sum_{i=1}^{p} \frac{g_i \overline{g}_i}{a_i} + N \sum_{i=1}^{p} (g_i \mathrm{tr}\, U^i + \overline{g}_i \mathrm{tr}\, U^{\dagger i})\Big\}
\end{aligned} \tag{3.27}$$

Using this trick a second time we have,

$$\begin{aligned}
&\exp(-N^2 \textstyle\sum_{j=1}^{p} \frac{g_j \overline{g}_j}{a_j}) \\
&= \left(\tfrac{N^2}{\pi}\right)^p \textstyle\int \prod_{j=1}^{p} a_j d\mu_j\, d\overline{\mu}_j \exp\Big\{-N^2 \sum_{j=1}^{p} a_j \mu_j \overline{\mu}_j + \mathrm{i} N^2 \sum_j (\mu_j \overline{g}_j + \overline{\mu}_j g_j)\Big\}
\end{aligned} \tag{3.28}$$

In order to deal with an arbitrary polynomial P of $\mathrm{tr}\, U^i, \mathrm{tr}\, U^{\dagger i}$, we use the following identity in (3.27),

$$\begin{aligned}
&\exp\Big\{P(\mathrm{tr}\, U^i, \mathrm{tr}\, U^{\dagger i}) + \sum_{i=1}^{p} a_i \mathrm{Tr}\, U^i \mathrm{Tr}\, U^{\dagger i}\Big\} \\
= \;& \left(\frac{N^2}{2\pi}\right)^p \int \prod_{i=1}^{p} \frac{dg_i\, d\overline{g}_i}{a_i} \exp\Big\{-N^2 \sum_{i=1}^{p} \frac{g_i \overline{g}_i}{a_i}\Big\} \exp\Big\{P(\frac{\partial}{N\partial g_i}, \frac{\partial}{N\partial \overline{g}_i})\Big\} \\
& \cdot \exp\Big\{N \sum_{i=1}^{p} (g_i \mathrm{Tr}\, U^i + \overline{g}_i \mathrm{Tr}\, U^{\dagger i})\Big\} \\
= \;& \left(\frac{N^2}{2\pi}\right)^p \int \prod_{i=1}^{p} \frac{dg_i\, d\overline{g}_i}{a_i} \exp\Big\{N \sum_{i=1}^{p} (g_i \mathrm{Tr}\, U^i + \overline{g}_i \mathrm{Tr}\, U^{\dagger i})\Big\} \\
& \cdot \exp\Big\{P(-\frac{\partial}{N\partial g_i}, -\frac{\partial}{N\partial \overline{g}_i})\Big\} \exp\Big\{-N^2 \sum_{i=1}^{p} \frac{g_i \overline{g}_i}{a_i}\Big\}
\end{aligned} \tag{3.29}$$

In the last line we have integrated by parts. Then we use (3.28) to write

$$
\begin{aligned}
& \exp\Big\{P(-\frac{\partial}{N\partial g_i},-\frac{\partial}{N\partial \overline{g}_i})\Big\}\exp\Big\{-N^2\sum_{i=1}^{p}\frac{g_i\overline{g}_i}{a_i}\Big\} \qquad (3.30)\\
= & \left(\frac{N^2}{\pi}\right)^p \exp\Big\{P(-\frac{\partial}{N\partial g_i},-\frac{\partial}{N\partial \overline{g}_i})\Big\}\\
& \cdot\int\prod_{j=1}^{p} a_j d\mu_j\, d\overline{\mu}_j \exp\Big\{-N^2\sum_{j=1}^{p}a_j\mu_j\overline{\mu}_j+\mathrm{i}N^2\sum_j(\mu_j\overline{g}_j+\overline{\mu}_j g_j)\Big\}\\
= & \left(\frac{N^2}{\pi}\right)^p \int\prod_{j=1}^{p} a_j d\mu_j\, d\overline{\mu}_j \exp\Big\{-N^2\sum_{j=1}^{p}a_j\mu_j\overline{\mu}_j+\mathrm{i}N^2\sum_j(\mu_j\overline{g}_j+\overline{\mu}_j g_j+P(N\mu_j,N\overline{\mu}_j))\Big\}
\end{aligned}
$$

Since the effective action (3.11) is a polynomial in $\mathrm{tr}\, U^i$, $\mathrm{tr}\, U^{\dagger i}$, we can use the procedure discussed above to write the partition function (3.26) as

$$
Z=\left(\frac{N^4}{2\pi^2}\right)^p\int\prod_{i=1}^{p} dg_i\, d\overline{g}_i\, d\mu_i\, d\overline{\mu}_i \exp(N^2 S_{\text{eff}}) \qquad (3.31)
$$

where

$$
\begin{aligned}
S_{\text{eff}} &= -\sum_{j=1}^{p}a_j\mu_j\overline{\mu}_j+\mathrm{i}\sum_j(\mu_j\overline{g}_j+\overline{\mu}_j g_j)+ \qquad (3.32)\\
&\quad \sum_{\vec{k},\vec{k}'}\alpha_{\vec{k},\vec{k}'}(-\mathrm{i})^{|\vec{k}|+|\vec{k}'|}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu)+F(g_k,\overline{g}_k).
\end{aligned}
$$

In the above formula we have introduced the definition

$$
\Upsilon_{\vec{k}}(\mu)=\prod_j \mu_j^{k_j}. \qquad (3.33)
$$

and the free energy $F(g_k,\overline{g}_k)$ is defined by

$$
\exp(N^2F(g_k,\overline{g}_k)) = \int[dU]\exp\Big\{N\sum_{i\geq 1}(g_i\mathrm{tr}\, U^i+\overline{g}_i\mathrm{tr}\, U^{\dagger i})\Big\}, \qquad (3.34)
$$

It is important to note that given the effective action $S(U,U^\dagger)$ of the gauge theory, S_{eff} can be exactly calculated.

One notes that $F(g_i, \overline{g}_i)$ depends only on the $p-1$ phases, since one of the phases of the g_i can be absorbed by a rotation of U in the unitary integral in (3.34). The full integrand (3.31) can be shown to be independent of one phase of g_i by a redefinition of the auxiliary variables $\mu_j, \overline{\mu}_j$.

The significance of (3.31) is that the partition function (3.26) can be expressed as an exact integral transformation of the linear matrix model (3.34). The phase structure and the critical behavior of the linear matrix model is well understood, and hence we can study these to learn about the critical behavior and the phase structure of (3.26). In the next section we will discuss the phase structure of (3.34).

3.5 Critical behavior in matrix model

The eigenvalues of an unitary matrix U are the complex numbers $e^{i\theta_i}$.[3] In the large N limit, we can consider an eigenvalue density $\rho(\theta)$ defined on the unit circle by,

$$\rho(\theta) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i) = \frac{1}{2\pi}\sum_{n}\exp(in\theta)\frac{1}{N}\mathrm{tr}\,U^{-n} \tag{3.35}$$

The density function is non-negative and normalized,

$$\int \rho(\theta)\mathrm{d}\theta = 1 \tag{3.36}$$

$$\rho(\theta) \geq 0 \tag{3.37}$$

It is well known that in the limit of $N \to \infty$, $\rho(\theta)$ can develop gaps, i.e. it can be non-zero only in bounded intervals. For example, in the case of a single gap when $\rho(\theta)$ is non-zero only in the interval $(-\frac{\theta_0}{2}, \frac{\theta_0}{2})$, it is given by the classical formula

$$\rho(\theta) = f(\theta)\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}} \tag{3.38}$$

A well known example of a $\rho(\theta)$ which does not have a gap is

[3]Phase structure of a generic unitary matrix model has been discussed in [34].

$$\rho(\theta) = \frac{1}{2\pi}(1 + a\cos(\theta)), \quad a < 1 \tag{3.39}$$

At $a = 1$, $\rho(\pi) = 0$, and a gap will begin to open. For $a > 1$ the functional form of $\rho(\theta)$ is as given by (3.38).

The matrix model under investigation has a complicated effective action. The saddle point distribution of the eigenvalues of the matrix U may or may not have a gap, depending on the values of parameters g_k in (3.34). In the large N expansion, the functional dependence of $F(g_k, \overline{g}_k)$ on $g_k, \overline{g}_k$ depends on the phase, and we quote from the known results [35, 56, 58, 66],

$$\begin{aligned}
N^2 F(g_k, \overline{g}_k) &= N^2 \sum_k k g_k \overline{g}_k + e^{-2Nf(g_k, \overline{g}_k)} \sum_{n=1}^{n=\infty} \frac{1}{N^n} F_n^{(1)}, \text{ungapped} \quad (3.40)\\
N^2 F(g_k, \overline{g}_k) &= N^2 \sum_k k g_k \overline{g}_k + \sum_{n=0}^{n=\infty} N^{-\frac{2}{3}n} F_n^{(2)}, \quad g - g_c \sim o(N^{-\frac{2}{3}})\\
N^2 F(g_k, \overline{g}_k) &= N^2 G(g_k, \overline{g}_k) + \sum_{n=1}^{n=\infty} \frac{G^{(n)}}{N^2}, \quad \text{gapped}
\end{aligned}$$

In the above, we have assumed for simplicity that the eigenvalue distribution has only one gap. (In principle we can not exclude the possibility of a multi gap solution. But in this chapter, since we are interested in the critical phenomena that results when the gap opens (or closes) we will concentrate on the single gap solution.) Near the boundary of phases, the functions $F_n(g)$ and $G_n(g)$ diverge. It is well known that in the leading order N, $F(g_k, \overline{g}_k)$ has a third order discontinuity at the phase boundary. This non-analytic behavior is responsible for the large N GWW type transition. In the $o(N^{-\frac{2}{3}})$ scaling region near the phase boundary (the middle expansion in (3.40)) this non-analytic behavior can be smoothened by the method of double scaling. This smoothening is important for our calculation of the double scaled partition function near the critical surface.

In (3.40) $f(g_k, \overline{g}_k)$, $F_n^{(1)}$, $F_n^{(2)}$ and $G^n(g_k, \overline{g}_k)$ are calculable functions using standard techniques of orthogonal polynomials. As an example, $G(g_k, \overline{g}_k)$ can be expressed as,

$$G(g_k, \overline{g}_k) = \frac{1}{N} \log h_0 + \int_0^1 d\xi (1 - \xi) \log\, f_0(\xi) \tag{3.41}$$

where $f_0(\xi)$ and h_0 are determined in terms of $g_k, \overline{g}_k$ by a recursion relation of orthogonal polynomials for the unitary matrix model. It should be noted that in the ungapped phase all perturbative $(\frac{1}{N^2})$ corrections to the leading free energy vanishes. This follows from the fact that in the character expansion (strong coupling expansion) the ungapped free energy becomes an exact result. We also note that at $g_k = 0 = \overline{g}_k$, $f = 0$ and the non-pertubative term is absent.

3.5.1 Gap opening critical operator at m=1 critical point

We now derive the form of the critical operator that opens the gap and corresponds to the scaling region of width $o(N^{-\frac{2}{3}})$.

From (3.40) we can easily find the density of eigenvalues in the ungapped phase.

$$\begin{aligned} \rho(\theta) &= \frac{1}{2\pi}(1 + \sum_{k\neq 0}(kg_k \exp(ik\theta) + k\overline{g}_k \exp(-ik\theta)) \qquad (3.42) \\ \text{and} \quad \rho_{\mathrm{k}} &= k\overline{g}_k \end{aligned}$$

For a set of real g_k, the lagrangian (3.34) is invariant under $U \to U^\dagger$. We will assume that the gap opens at $\theta = \pi$ according to $\rho(\pi - \theta) \sim (\pi - \theta)^2$, which characterizes the first critical point[4]. At the boundary of the gapped-ungapped phase (critical surface) we have $\rho(\pi) = 0$. In terms of the critical fourier components ρ_k^c, it is the equation of a plane with normal vector $\widetilde{D}_k = (-1)^k$

$$\sum_{k=-\infty}^{\infty} (-1)^k (\rho_k^c + \overline{\rho}_k^c) = -1 \tag{3.43}$$

Now since $\rho_k^c = kg_k^c$ (up to non-perturbative corrections), we get the equation of a plane

$$\sum_{k=-\infty}^{\infty} (-1)^k k(g_k^c + \overline{g}_k^c) = -1 \tag{3.44}$$

[4]In general the mth critical point is characterized by $\rho(\pi - \theta) \sim (\pi - \theta)^{2m}$.

where g_k^c are the values of g_k at the critical plane. Since the metric induced in the space of g_k from the space of ρ_k is $G_{k,k'} = k^2\delta_{k,k'}$, the vector that defines this plane is

$$C_k = \frac{(-1)^k}{k} \tag{3.45}$$

We mention that the exact values of g_k^c where the thermal history of the small blackhole intersects the critical surface are not known to us as we do not know the coefficients of the effective lagrangian. However this information, which depends on the details of dynamics, does not influence the critical behavior. The information where the small blackhole crosses the critical surface is given by the saddle point equations (3.50), which are in turn determined by the $o(N^2)$ part of the partition function.

Below we will show that the critical behavior is determined by the departure from the critical surface and not on where the thermal history intersects it, and conclude that the $o(1)$ part of the doubled scaled partition function is always determined in terms of the solution of the Painlevé II equation.

If we go slightly away from the critical surface by setting $g_k = g_k^c + \delta g_k$ and $\overline{g}_k = g_k^c + \delta\overline{g}_k$, then the gap opens provided $\rho(\pi) < 0$ [5]. This condition is easily ensured by the choice $\delta g_k + \delta\overline{g}_k = tN^{-\frac{2}{3}}C_k$, $t < 0$, which is normal to the critical plane (3.44).

The operator that corresponds to $\rho(\pi) = 0$ at the first critical point is

$$\widehat{O} \;=\; \sum_{k=1}^{\infty}(g_k^c \mathrm{tr}\, U^k + \overline{g}_k \mathrm{tr}\, U^{\dagger k}) \tag{3.46}$$

The gap at $\theta = \pi$ opens if we add a perturbation that leads to a small negative value for the ungapped solution of $\rho(\pi)$. Such a perturbation is necessarily in the direction of the vector C_k, because a perturbation that lies in the critical plane does not contribute to the opening of the gap. Hence we will set $(g_k - g_k^c) = N^{-\frac{2}{3}}\widetilde{t}_k$. As we shall explain in Appendix A, $\widetilde{t}_k = tC_k$, where $t = \widetilde{C} \cdot \widetilde{t}$ is an arbitrary parameter and $\widetilde{C}$ is the unit vector corresponding to C. Therefore the *relevant* gap opening perturbation to be added to the action is

$$\widehat{O}_t \;=\; N^{-\frac{2}{3}}t\sum_{k=1}^{\infty} C_k(\mathrm{tr}\, U^k + \mathrm{tr}\, U^{\dagger k}) \tag{3.47}$$

[5] To calculate $\rho(\theta)$ we have used the ungapped solution in (3.40)

The factor $N^{-\frac{2}{3}}$ is indicative that the perturbation is relevant and has exponent $-\frac{2}{3}$. N acts like an infrared cutoff .

In the double scaling limit, near the critical surface, $F_0^{(2)}$ in (3.40) is a function of the parameter t (see Appendix A). It is known that $F_0^{(2)}(t)$ (from now on we will call it $F(t)$) satisfies the following differential equation,

$$\frac{\partial^2 F}{\partial t^2} = -f^2(t) \tag{3.48}$$

where $f(t)$ satisfies the Painleve II equation,

$$\frac{1}{2}\frac{\partial^2 f}{\partial t^2} = tf + f^3 \tag{3.49}$$

The exact analytic form of $F(t)$ is not known, but $F(t)$ is a smooth function in the domain $(-\infty, \infty)$. Smoothness of $F(t)$ guarantees the smoothening of large N transition in the double scaling limit.

In the gapped phase of the matrix model, $F(g_k, \overline{g}_k)$ has a standard expansion in integer powers of $\frac{1}{N^2}$, which becomes divergent as one approaches the critical surface. In the double scaling region (3.40) $(g - g_c) \sim \mathcal{O}(N^{-\frac{2}{3}})$, and the the perturbation series (3.40) is organized in an expansion in powers of $N^{-\frac{2}{3}}$. The reason for the origin of such an expansion is not clear from the viewpoint of the bulk string theory. However, it is indeed possible to organize the perturbation series, in the scaling region, in terms of integral powers of a renormalized coupling constant. We will come back to this point later. In the ungapped phase the occurrence of $o(e^{-N})$ terms is also interesting. Here too we lack a clear bulk understanding of the non-perturbative terms which naturally remind us of the D-branes.

3.6 Saddle point equations at large N

In this section we will use the results of the previous section to write down the large N saddle point equations for the multi-trace matrix model (3.31). We treat μ_j and $\overline{\mu}_j$ as independent complex variables. This is natural as the saddle point of the theory may occur at complex values of the variable, though at the end we will find that for real $\alpha_{\vec{k},\vec{k}'}$ in (3.11) we have saddle points in imaginary μ_i and real g_i. From (3.11) we deduce the saddle point at large N by including the leading $o(N^2)$ contribution of $F(g_k, \overline{g}_k)$ to the free energy. The equations for saddle points are given by

$$\begin{aligned}
\frac{\partial S_{\rm eff}}{\partial g_j} &= i\overline{\mu}_j + \frac{1}{2j}\overline{g}_j = 0, \\
\frac{\partial S_{\rm eff}}{\partial \overline{g}_j} &= i\mu_j + \frac{1}{2j}g_j = 0, \\
\frac{\partial S_{\rm eff}}{\partial \mu_j} &= -a_j\overline{\mu}_j + i\overline{g}_j + \sum_{\vec{k},\vec{k}'}\alpha_{\vec{k},\vec{k}'}(-\mathrm{i})^{|\vec{k}|+|\vec{k}'|}\frac{k'_j}{\mu_j}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu) = 0 \\
\frac{\partial S_{\rm eff}}{\partial \overline{\mu}_j} &= -a_j\mu_j + ig_j + \sum_{\vec{k},\vec{k}'}\alpha_{\vec{k},\vec{k}'}(-\mathrm{i})^{|\vec{k}|+|\vec{k}'|}\frac{k_j}{\overline{\mu}_j}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu) = 0
\end{aligned} \tag{3.50}$$

These equations correspond to the ungapped phase. Equations similar to equation (3.50) can also be written using $F(g_k, \overline{g}_k)$ in the gapped phase.

By the AdS/CFT correspondence the solutions to (3.50) are dual to supergravity/string theory solutions, like $AdS_5 \times S^5$ and various $AdS_5 \times S^5$ blackholes. The number and types of saddle points and their thermal histories depends on the dynamics of the gauge theory (i.e. on the numerical values of the parameter a_j and $\alpha_{\vec{k},\vec{k}'}$, which in turn are complicated functions of λ and β). These issues have been discussed in the frame work of simpler models in [20], where the first order confinement/deconfinement transition and its relation with the Hawking-Page type transition in the bulk has also been discussed. Here we will not address these issues, but focus on the phenomenon when an *unstable saddle point* of (3.50) crosses the critical surface (3.44).

By solving the eqn.(3.50) we can write g_j in terms of μ_j and the coefficients $a_j(\beta), \alpha_{\vec{k},\vec{k}'}(\beta)$. Using the critical values of g_j (3.44), we get the relation between $a_j(\beta), \alpha_{\vec{k},\vec{k}'}(\beta)$ at the critical surface,

$$g_j^c(ja_j - 1) + \frac{\widehat{g}_j^c}{j} + \sum_{\vec{k},\vec{k}'} 2^{2-|\vec{k}|-|\vec{k}'|}(-1)^{|\vec{k}|+|\vec{k}'|}\alpha_{\vec{k},\vec{k}'}\frac{k_j}{g_j^c}\Upsilon_{\vec{k}+\vec{k}'}(g_j^c) = 0, \quad j = 1, \cdots, p. \tag{3.51}$$

Whether the above relation is achieved for some values of the coefficients $a_j(\beta), \alpha_{\vec{k},\vec{k}'}(\beta)$ is a difficult question which again needs a detailed understanding of the gauge theory dynamics. The coefficients $a_j(\beta), \alpha_{\vec{k},\vec{k}'}(\beta)$

have been perturbatively calculated in [36] and it can be shown that at some specific $\beta < \beta_{\mathrm{HG}}$ [6] the condition (3.51) is satisfied.

We would like to mention that there is no fine tuning associated with the relation (3.44) or (3.51) being satisfied. This is because we have one tunable parameter, the temperature, and one relation (3.44) to satisfy. Hence one may hope that in the most general situation the relation (3.44) will be satisfied. In the next section we will discuss the doubled scaled partition function near the critical point.

In a later section we will use the AdS/CFT correspondence to argue that in the strongly coupled gauge theory, a 10 dimensional "small blackhole" saddle point reaches the critical surface (3.51). The interpretation of this phenomenon in the bulk string theory, as a blackhole to excited string transition will also be discussed.

3.7 Double scaled partition function at crossover

We will assume that the matrix model (3.34) has a saddle point which makes a gapped to ungapped transition as we change the parameters of the theory($\alpha^c_{\vec{k},\vec{k}'}$,$a_j$) by tuning the temperature β^{-1}. We will also assume that, this saddle point has one unstable direction which corresponds to opening the gap as we lower the temperature. These assumptions are motivated by the fact that the small (euclidean) Schwarzchild blackhole crosses the critical surface and merges with the $AdS_5 \times S^5$ and that it is an unstable saddle point of the bulk theory. To calculate the doubled scaled partition function near this transition point, we basically follow the method used in [20]. We expand the effective action (3.34) around the $1st$ critical point, and we simultaneously expand the original couplings a_j, g_j, $\overline{g}_j$ and $\alpha_{\vec{k},\vec{k}'}$ around their critical values a^c_j, β^c_j, $g^c_j = 0$, and $\alpha^c_{\vec{k},\vec{k}'}$. For clarity we define

$$P(\mu,\overline{\mu},\alpha) = \sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'}(-\mathrm{i})^{|\vec{k}|+|\vec{k}'|}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu) \tag{3.52}$$

We also introduce the column vectors,

[6] β_{HG}^{-1} is the temperature of Hagedorn transition.

$$\mu = \begin{pmatrix} \mu_j \\ \overline{\mu}_j \end{pmatrix}, \quad A = \begin{pmatrix} a_j \\ \alpha_{\vec{k},\vec{k}'} \end{pmatrix}, \quad g = \begin{pmatrix} g_j \\ \overline{g}_j \end{pmatrix} \tag{3.53}$$

and expand the above mentioned vector variables

$$\begin{aligned} g - g^c &= N^{-\frac{2}{3}}\widetilde{t} \\ \mu - \mu^c &= N^{-\frac{4}{3}} n \\ A - A^c &= \widetilde{g} N^{-\frac{2}{3}} \alpha \end{aligned} \tag{3.54}$$

where $\widetilde{g} = N^{\frac{2}{3}}(\beta - \beta_c)$ and $\alpha = \frac{\partial A}{\partial \beta}|_{\beta=\beta_c}$. The expansion of the co-efficients a_j and $\alpha^c_{\vec{k},\vec{k}'}$ are proportional to the deviation of the tuning parameter β from its critical value, i.e. $\widetilde{g} = N^{\frac{2}{3}}(\beta_c - \beta)$.

The expanded action takes the following form,

$$N^2 S_{eff} = -\frac{1}{2} N^{-\frac{2}{3}} n^t \mathcal{L}\, n + n^t(\mathcal{J} t - \widetilde{g}\mathcal{H}\alpha) + F(C \cdot \widetilde{t}) + O(N^{-\frac{4}{3}}) \tag{3.55}$$

In the above we have, following the discussion in Appendix A, used the fact that the $o(1)$ function F depends on the scaled variable through the combination $t = C \cdot \widetilde{t}$. Recall that C is the constant vector normal to the critical plane and the matrices $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ are given by

$$\begin{aligned} \mathcal{L} &= \begin{pmatrix} -\frac{\partial^2 P}{\partial \mu_j \partial \mu_k} & a_j^{(c)} \delta_{jk} - \frac{\partial^2 P}{\partial \mu_j \partial \overline{\mu}_k} \\ a_j^{(c)} \delta_{jk} - \frac{\partial^2 P}{\partial \mu_j \partial \overline{\mu}_k} & -\frac{\partial^2 P}{\partial \overline{\mu}_j \partial \overline{\mu}_k} \end{pmatrix}, \\ \mathcal{H} &= \begin{pmatrix} -\overline{\mu}_j \delta_{jk} & \frac{\partial^2 P}{\partial \mu_j \partial \alpha_{\vec{k},\vec{k}'}} \\ -\mu_j \delta_{jk} & \frac{\partial^2 P}{\partial \overline{\mu}_j \partial \alpha_{\vec{k},\vec{k}'}} \end{pmatrix}, \\ \mathcal{J} &= \frac{1}{2} \begin{pmatrix} \mathrm{i}\mathcal{F} & \mathcal{F} \\ \mathrm{i}\mathcal{F} & -\mathcal{F} \end{pmatrix}, \end{aligned} \tag{3.56}$$

In the above we have introduced the diagonal matrix

$$\mathcal{F}_{jk} = \frac{1}{j} \delta_{jk}, \quad j, k = 1, \cdots, p. \tag{3.57}$$

All quantities appearing in the matrices are calculated at the first critical point. Here $o(N^2)$ part of the action does not depend on $n, \widetilde{t}$ and hence we do not show this part of the action explicitly.

We now do the Gaussian integration over n_k in the functional integral

$$Z \sim \int d\widetilde{t}(\det(N^{-\frac{2}{3}}\mathcal{L}))^{-\frac{1}{2}}\exp\left\{\frac{1}{2}N^{\frac{2}{3}}(\widetilde{t}-\widetilde{g}\mathcal{C}\alpha)^t\mathcal{M}(\widetilde{t}-\widetilde{g}\mathcal{C}\alpha)+F(C\cdot\widetilde{t})+O(N^{-\frac{2}{3}})\right\}, \tag{3.58}$$

The matrices appearing here can be easily obtained,

$$\mathcal{D} = \frac{1}{2}\begin{pmatrix}\mathcal{F} & 0\\ 0 & \mathcal{F}\end{pmatrix}, \quad \mathcal{M} = \mathcal{J}^t\mathcal{L}^{-1}\mathcal{J} + \mathcal{D}, \quad \mathcal{C} = -\mathcal{M}^{-1}\mathcal{J}^t\mathcal{L}^{-1}\mathcal{H}. \tag{3.59}$$

Notice that the Hessian associated with S_{eff} is given by

$$H = \begin{pmatrix}-\mathcal{L} & \mathcal{J}\\ \mathcal{J} & \mathcal{D}\end{pmatrix}. \tag{3.60}$$

In order to discuss the further evaluation of the integral (3.58), we must take into account the fact that we are evaluating the integral near an unstable saddle point. That the saddle point has precisely one unstable direction is motivated by the fact that in the bulk theory the euclidean 10-dimensional blackhole has one negative eigenvalue. This statement strictly speaking should apply to the saddle point in the gapped phase. However since the GWW phase transition is third order an unstable saddle point in the gapped phase should continue to be unstable at the crossover.

In order to render the gaussian integral (3.58) along the unstable direction well defined, we should make an analytic continuation. Once this is done we can easily see that as $N \to \infty$ the integral in (3.58) is localized at

$$\widetilde{t} = \widetilde{g}\mathcal{C}\alpha \tag{3.61}$$

This follows from a matrix generalization of the gaussian representation of the delta function.

Putting the above expression in (3.58) we get the final result,

$$Z \sim i(\det(H))^{-\frac{1}{2}}\exp F(\widetilde{g}C\cdot\mathcal{C}\alpha), \tag{3.62}$$

where $C\cdot\mathcal{C}\alpha$ is a constant independent of $\widetilde{g}$. We have assumed that the Hessian H does not have a zero mode, but the one negative eigenvalue accounts for the i in front of (3.62).

The $o(1)$ part of the partition function, (3.62) is universal in the sense that the appearance of the function $F(\widetilde{g}\times constant)$, does not depend on the exact values of the parameters of the theory. In the double scaling limit the partition function becomes a function of a single scaling variable $\widetilde{g}$. Exact values of the couplings and the $o(N^2)$ part of the partition function determine where the thermal history crosses the critical surface (3.44). However the form of the function F and the double scaling limit of (3.55) are independent of the exact values of g_k^c. They only depend on the fact that one is moving away perpendicular to the critical surface. This is the reason why in [20] we obtained exactly the same equation when $g_1^c \neq 0$ but all other $g_k^c = 0$.

3.7.1 Condensation of winding modes at the crossover

We will now discuss the condensation of the winding Polyakov lines in the crossover region. Specifically we will discuss the expectation value of the critical operator (3.46). In the leading order in large N we have already seen in (3.43), that $\rho_k^c = kg_k^c$. In order to calculate subleading corrections it can be easily seen that all the ρ_k's condenses in the scaling region,

$$\left\langle N^{\frac{2}{3}}(\rho_k - \rho_k^{ug}) \right\rangle = C_k \frac{dF}{dt} \tag{3.63}$$

where $\rho_k^{ug} = kg_k$. This smoothness of the expectation value of the ρ_k's follows from the smooth nature of $F(t)$. The exact form of $F(t)$ is not known but it is known that it is a smooth function with the following asymptotic expansion.

$$\begin{aligned} F(t) &= \frac{t^3}{6} - \frac{1}{8}\log(-t) - \frac{3}{128t^3} + \frac{63}{1024t^6} + \cdots, \quad -t \gg 1 \qquad (3.64) \\ F(t) &= \frac{1}{2\pi} e^{-\frac{4\sqrt{2}}{3}t^{\frac{3}{2}}} \left(-\frac{1}{8\sqrt{2}t^{\frac{3}{2}}} + \frac{35}{384t^3} - \frac{3745}{18432\sqrt{2}t^{\frac{9}{2}}} + \cdots\right), \quad t \gg 1 \end{aligned}$$

The derivative of $F(t)$ diverges as $t \to -\infty$ and goes to zero as $t \to \infty$. This behavior tallies with the condensation of winding mode in one phase (the gapped phase) and the non-condensation of winding modes in the ungapped phase. The condensation of the winding modes also indicates that the $U(1)$ symmetry (which is the Z_N symmetry of the $SU(N)$ gauge theory in the large N limit) is broken at the crossover, but restored in the limit $t \to \infty$.

3.8 Higher critical points and the introduction of chemical potentials

Besides the first critical point, single trace unitary matrix models can have higher critical points. The mth critical point is characterized by,

$$\rho_m(\theta) \sim (\theta - \pi)^{2m}, \quad \theta \to \pi, \tag{3.65}$$

and hence it is specified by the following relations,

$$\rho^{(2n)}(\pi) = 0, \quad 0 \leq n < m \tag{3.66}$$

Writing the above in terms of g_k's we get,

$$\sum_{k=-\infty}^{\infty} (-1)^k k^{2n-1} (g_k^c + \overline{g}_k^c) = 0, \quad 0 \leq n < m \tag{3.67}$$

A particular choice for the density of eigenvalues with this behavior is

$$\rho_m(\theta) = c_m \left(2 \cos \frac{\theta}{2}\right)^{2m}, \tag{3.68}$$

where

$$c_m = \frac{2^{2m}}{2\pi} \frac{(m!)^2}{(2m)!}. \tag{3.69}$$

By expanding in Fourier modes, one finds

$$\rho_m(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{m} \frac{(m!)^2}{(m-k)!(m+k)!} \cos k\theta\right) \tag{3.70}$$

Using the relation between the density of eigenvalues in the ungapped phase and the matrix model potential one recovers the critical potential of Periwal and Shevitz.

As the plane (3.67) is determined by more than two equations, a generic curve in the space of couplings g_k's will not necessarily intersect the plane. Hence by tuning one parameter, the history of a saddle point may not reach the higher critical points. But one may consider a situation where along with temperature, some additional chemical potentials are also turned on

[59]. Using these chemical potentials (like say the R-charge) we may be able to reach higher multicritical points.

In appendix (3.12), we have considered a more general effective action which includes general source terms in addition to (3.11),

$$\widetilde{S}(U,U^\dagger) = S(U,U^\dagger) + N\sum_{k\geq 1}(b_k \mathrm{tr}\, U^k + \bar{b}_k \mathrm{tr}\, U^{\dagger k}). \tag{3.71}$$

Using the above action, we have calculated the doubled scale partition function near higher critical points. Similar to our result in (3.62), the $o(1)$ part of the doubled scaled partition function becomes a universal function determined by the mKdV hierarchy. It should be mentioned that the calculation is performed near the m–th multicritical point characterized by,

$$g_n = 0 \quad , n > m \tag{3.72}$$

According to the comments at the end of section 3.7 the final form of the doubled scaled partition function(3.119) and the double scaling limit (3.106) is universal and independent of the particular choice of (3.72).

3.9 Applications to the small 10-dimensional blackhole

We now apply what we have learned about the matrix model (gauge theory) GWW transition and its smoothening in the critical region to the blackhole-string transition in the bulk theory. The first step is to identify the matrix model phase in which the blackhole or for that matter the supergravity saddle points occur. We will argue that they belong to the gapped phase of the matrix model. This inference is related to the way perturbation theory in $\frac{1}{N}$ is organized in the gapped, and ungapped phase as discussed in (3.40). Note that it is only in the gapped phase, that the $\frac{1}{N}$ expansion is organized in powers of $\frac{1}{N^2}$, exactly in the way perturbation theory is organized around classical supergravity solutions in closed string theory. Hence at the strong gauge theory coupling($\lambda \gg 1$), it is natural to identify the small 10 dimensional blackhole with a saddle point of the equations of motion like (3.50) but obtained by using $F(g_k, \overline{g}_k)$ corresponding to the gapped phase. [7]

[7] A saddle point of the weakly coupled gauge theory may also exist in the gapped phase. With a change in the temperature the saddle point can transit through the critical

One can associate a temperature with this saddle point which would satisfy $l_s^{-1} \gg T \gg R^{-1}$.

As the temperature increases towards l_s^{-1}, one traces out a curve (thermal history) in the space of the parameters $a_i, \alpha_{k,k'}$ of the effective theory. One can also say that a thermal history is traced in the space of $\rho_i = \langle \frac{1}{N} \mathrm{tr}\, U^i \rangle$, which depends on the parameters of the effective theory. We will now make the reasonable assumption that the thermal history, at a temperature $T_c \sim l_s^{-1}$, intersects the critical surface (3.43) (equivalently the plane (3.44) and then as the temperature increases further it reaches the point $\rho_i = \langle \frac{1}{N} \mathrm{tr}\, U^i \rangle = 0$, which corresponds to $AdS_5 \times S^5$. Once the thermal history crosses the critical surface, the gauge theory saddle points are controlled by the free energy of the ungapped phase in (3.40). The saddle points of eqns. (3.50) which were obtained using this free energy do not correspond to supergravity backgrounds, because the temperature, on crossing the critical surface is very high $T \gtrsim l_s^{-1}$. Besides this the free energy in the gapped phase has unconventional exponential factors (except at $g_k = 0$ which corresponds to $AdS_5 \times S^5$). It is likely that these saddle points define in the correspondence, exact conformal field theories/non-critical string theories in the bulk. Neglecting the exponential corrections $exp(-N)$, it seems reasonable, by inspecting the saddle point equations, that in this phase the spectrum would be qualitatively similar to that around $\rho_i = 0$. Since this corresponds to $AdS_5 \times S^5$, we expect the fluctuations to resemble a string spectrum.

As we saw in the previous section, our techniques are good enough only to compute the $o(1)$ part of the partition function in the vicinity of the critical surface which depends on the renormalized coupling. The exact solution of the free energy (in the single trace model) in the transition region in (3.40) enabled us to define a double scaling limit in which the non-analyticity of the partition function could be smoothened out, by a redefinition of the string coupling constant according to $\widetilde{g} = N^{\frac{2}{3}}(\beta_c - \beta)$. This smooth crossover corresponds to the blackhole crossing over to a state of strings corresponding to the ungapped phase.

We have also computed the vev of the scaling operator and hence at the crossover the winding modes $\rho_i = \langle \frac{1}{N} \mathrm{tr}\, U^i \rangle$ condense (3.63). They also have

surface. Using the results of [20], it is easy to see that this is precisely what happens for the perturbative gauge theory discussed in [36]. We note that in the corresponding bulk picture since $l_s >> R_{AdS}$, the supergravity approximation is not valid. It would be interesting to understand the bulk interpretation in this case.

a smooth parametric dependence across the transition. This phenomenon in the bulk theory may have the interpretation of smooth topology change of a blackhole spacetime to a spacetime without any blackhole and only with a gas of excited string states. However in the crossover region a geometric spacetime interpretation is unlikely. We may be dealing with the exact description of a non-critical string in 5-dims. in which only the zero mode along the S^3 directions is taken into account. This interpretation is inspired by the fact that the free energy $F(t)$ also describes the non-critical type 0B theory as was already discussed in [57, 20].

3.10 Implications for the Lorentzian blackhole

All our discussion has been in the context of the euclidean time, both in the bulk and the boundary theory. Since the boundary theory is governed by a well defined positive Hamiltonian the analytic continuation from euclidean to lorentzian signature is well understood and simple. Hence the partition function gives a way of computing the density of states at a particular energy using the formula,

$$Z(\beta) = \int_0^\infty dE \rho(E) e^{-\beta E} \tag{3.73}$$

where $\rho(E) = \mathrm{tr}\,\delta(H - E)$ is the density of states at energy E. Since the partition function, in an appropriate scaling limit, is a smooth function of the renormalized coupling constant $\widetilde{g}$, at the crossover between the gapped and ungapped phase, (3.73) implies that $\rho(E)$ inherits the same property. Since $\rho(E)$ is as well a quantity that has meaning when the signature of time is Lorentzian, it would imply that the blackhole-string crossover in the Lorentzian signature is also smooth. This is an interesting conclusion especially because we do not know the AdS/CFT correspondence for the small Lorentzian blackhole. The Lorentzian section of the blackhole has a horizon and singularity. Since the gauge theory should also describe this configuration, a smooth density of states in the cross over would imply that the blackhole singularity was resolved in the gauge theory.

We believe in this conclusion but an understanding of this can only be possible if we have an explicit model in the gauge theory of the small

Lorentzian blackhole. Work in this direction is in progress drawing lessons from [5, 8, 60, 61, 63, 64].

This program was originally motivated by an attempt to understand and resolve the information puzzle in blackhole physics. In the AdS/CFT correspondence we know that the $SU(N)$ gauge theory is defined by a hermitian hamiltonian defined on $S^3 \times R$. The $N \to \infty$ limit and the $\lambda \to \infty$ limits make contact with semi-classical gravity limit of the type IIB string theory in the bulk. In this limit, one can represent the quantum gravity theory path integral as an integral which splits into a sum over distinct topologies. In particular in the euclidean framework the path integral splits as a sum of contributions from histories with and without a blackhole. However this representation arises by a naive consideration of the large N limit. We know that as long as N is finite the notion of summing over distinct topologies does not exist. A careful understanding of the double scaling limit has indeed made it possible to treat finite N effects in a saddle point expansion around large N and smoothened the GWW transition. Since we have identified this gauge theory phenomenon with a smooth blackhole-string crossover, we conclude that topology change is indeed possible in the bulk string theory.

In light of our results we are not convinced about Hawking's proposed solution to the information puzzle [65] which uses the notion of representing the quantum gravity path integral as a sum over all topologies. At large but finite N (or equivalently at small but finite string coupling) this notion is not neccessarily valid.

3.11 Appendix A: Discrete recursion relations, $m = 1$ critical point and Painleve II

In this appendix we discuss the appearance of the m=1 critical point in the discrete recursion relations in the presence of general couplings g_k, where k is a positive integer. The main point can be explicitly illustrated in the case of two couplings g_1 and g_2, and the generalization to more general potentials is straightforward. We briefly review how we find scaling regions in matrix models and how double scaling limits are implemented. We follow closely the work of Periwal-Shevitz [56]. The action we consider is:

$$g_1\Big(\operatorname{tr} U + \operatorname{tr} U^\dagger\Big) + g_2\Big(\operatorname{tr} U^2 + \operatorname{tr} U^{\dagger 2}\Big) = \mu_1 V_1 + \mu_2 V_2, \tag{3.74}$$

where $V_{1,2}$ are the first critical potentials found in [56]:

$$\begin{aligned} V_1 &= \tfrac{1}{2}\Big(\operatorname{tr} U + \operatorname{tr} U^\dagger\Big), \\ V_2 &= \tfrac{4}{3}\Big(\operatorname{tr} U + \operatorname{tr} U^\dagger\Big) + \tfrac{1}{12}\Big(\operatorname{tr} U^2 + \operatorname{tr} U^{\dagger 2}\Big). \end{aligned} \tag{3.75}$$

and

$$\mu_1 = 2g_1 - 16g_2, \qquad \mu_2 = 12g_2. \tag{3.76}$$

For those interested in the details we have modified the critical potentials by making the transformation $g_k \to (-1)^k g_k$, $U \to -U$. This is a symmetry of the action that guarantees that the gap opens at $\theta = \pi$. In the original paper [56] the gap opens at $\theta = 0$. Obviously the gap can open anywhere on the circle, but we simply have to be consistent once a convention is chosen. The Periwal-Shevitz's [56] equation with two couplings g_1 and g_2, in our convention takes the following form,

$$\begin{aligned} -R_n\frac{n+1}{N} &= (1-R_n^2)[-(R_{n+1}+R_{n-1})g_1 - 2g_2(R_{n-1}R_{n-1}^2 + \\ & R_{n-1}^2R_n + 2R_{n-1}R_nR_{n+1} + R_nR_{n+1}^2 - R_{n+2} - R_{n-2} + R_{n+1}^2R_{n+2})] \end{aligned} \tag{3.77}$$

We will show that this equation besides the m=2 fixed point also has the m=1 fixed point. The latter is well known to be described by Painleve II equation with just one coupling. (The derivation of Painleve II from the one coupling case has been discussed in the original paper [56]).

As usual to find scaling regions we first solve the planar theory. However we have to solve it for any n, in other words, in the planar case R_n becomes a function $R(\xi)$, where $\xi = n/N$ which completely determines the planar limit of the theory. The equation that determines $R(\xi)$ is obtaining by ignoring in 3.78 the above the shifts in the $R's$. This yields the planar string equation:

$$R\xi = (1-R^2)(2(g_1 - 2g_2)R + 12g_2R^3) \tag{3.78}$$

If we take the scaling region to be close to the endpoint of the ξ interval, i.e. 1, we introduce the scaling variable:

$$\xi = 1 - a^2 t \tag{3.79}$$

as is standard in matrix models, and a is a small "lattice" parameter that is necessary to study the scaling region. Since in these theories the critical value of $R = 0$, we have to write the function R in terms of some scaling funcion with appropriate exponents:

$$R = a^{\gamma} f(t) \tag{3.80}$$

Since we want to consider only the first critical point $m = 1$, this implies that $\gamma = 1$ and the scaling behavior of R is

$$R = af(t) \tag{3.81}$$

Substituting in the planar string equation we obtain:

$$af(1 - a^2 t) = (1 - a^2 f^2)(2(g_1 - 2g_2)af + 12g_2 a^3 f^3) \tag{3.82}$$

The terms of order a determine the criticality condition, which as expected is the gap opening condition

$$g_1 - 2g_2 = \frac{1}{2} \tag{3.83}$$

The terms of order a^3 now provide the planar string equation that determines the functional form of f as a function of t to leading order in $1/N$:

$$-a^3 tf(t) = -a^3 f^3(2(g_1 - 2g_2) - 12g_2) \tag{3.84}$$

all other terms are irrelevant to this order, and what this equation does is to determine $f(t)$, and also it provides the first term in the expansion of the P-II equation in powers of fractional powers of t. The condition 3.83 determines the first critical point of the theory, $m = 1$, which implies that near $\xi = 1$ equation 3.78 has a second order zero in R. If we require that the zero is of order 4 (after dividing by a common R on both sides) we obtain the conditions for the $m = 2$ critical point governed by the scaling action V_2 above. Since in our problem we have a single control paremeter, i.e. the temperature, we focus on the $m = 1$ condition 3.83 and study next the double scaling limit. To make contact with the arguments of section VI we will study this limit for generic coupling g_1, g_2, this way we include also the perturbations of a given model on the "critical surface" 3.83 by the gap opening operator (VI.44).

So far the parameter a is just a small number, and for the time being it has no dependence on N. To get the N-dependence we do the double scaling limit, by expanding the full string equation, and see what is the relation beteween N and a that leads to a differential equation containing the string coupling constant, i.e. containing higher genus terms in the expansion and thus generating a string perturbation theory. Let us do it in general, but of course we have to keep track of the fact that we have already determined the scaling behavior of both ξ and $R(\xi)$, and we have to include it in 3.78:

$$\begin{aligned} af(\xi)(1-a^2t) &= \left(1-a^2\,f(\xi)^2\right)\left(2\,a\,g_1\,f(\xi)-4\,a\,g_2\,f(\xi)+12\,a^3\,g_2\,f(\xi)^3\right) \\ &+ \left(1-a^2\,f(\xi)^2\right)\Big(20\,a^3\,g_2\,f(\xi)\,f'(\xi)^2 \\ &+ a\,(g_1-8g_2)\,f''(\xi)+20\,a^3\,g_2\,f(\xi)^2\,f''(\xi)\Big)\,\frac{1}{N^2}+\ldots \end{aligned} \tag{3.85}$$

Now we are ready to get the relation between N, and a. In going from derivatives with respect to ξ to derivatives with respect to t, we obtain, including the factor of $1/N$ a term of the form $\frac{1}{Na^2}\frac{d}{dt}$ for each derivative. Since the first nontrivial terms with derivatives contains two of them, this means: $\frac{1}{(Na^2)^2}\frac{d^2}{dt^2}$.

The final result up to two derivatives (it is easy to show that higher ones are irrelevant) is:

$$\begin{aligned} -a^3tf(t) &= -(1-12g_2)a^3f(t)^3 \\ &+ (g_1-8g_2)a\frac{1}{(Na^2)^2}\frac{d^2f}{dt^2} \\ &+ 20\,g_2a^3\frac{1}{(Na^2)^2}(f\dot{f}^2+f^2\ddot{f}) \end{aligned} \tag{3.86}$$

where the dots are derivatives with respect to t. To get the double scaling limit, notice that we want that up to a numerical constant

$$a(\frac{1}{Na^2})^2 = g_{st}^2a^3 \tag{3.87}$$

Hence, up to $g_{\rm st}$ we obtain:

$$a \sim N^{-1/3} \tag{3.88}$$

Note that the terms in the third line of 3.86 will vanish like a^2 after we divide out by a^3 unless we force a strange scaling of g_2, but this is something we cannot do in the above procedure. The equation that survives is of course Painleve-II after some simple numerical rescalings. The computation has been carried out only for the two coupling case, but it is easy to generalize to a more general action. We have also included the case where we have a shift of the couplings of the model with respect to the critical surface. Of course the answer is the same, and the reason is that any of the terms $\mathrm{tr}(U^k + U^{-k})$ that appear in the gap opening operator have a component along the first scaling operator. For the two coupling theory this is the origin of the term $-12g_2$ in the f^3 piece and the term $-8g_2a$ in the term $\ddot{f}$. We get Painleve-II unless we do some unnatural fine tuning in the coupling g_2, a freedom we do not have at our disposal given that we have just one control parameter. Obviously, even if we consider more general potential, the same will happen with the gap opening operator. The operator identified with gap opening in the text should be more precisely be called the "bare" gap opening operator. After renormalization around any critical point, and in particular near the $m = 1$ it will be be dominated by the first scaling operator. We know also from [56] that the integrable hierarchy behind the unitary matrix model is the modified KdV (mKdV), and their flows can be identified with the expectation values of the scaling operators of the theory (including of course the irrelevant ones at the $m = 1$ critical surface.

One may wonder what happens with the expectation values of the ρ_n at the cross over region. This is however no problem, since we can renormalize these operators with more freedom than we have above, in fact, the way to argue that generically, at the initial conditions of the mKdV hierarchy that starts with Painleve-II; the continuum limit of the ρ_n get an expectation value is to use the renormalized Wilson loop operator of the matrix model, as it is done in [66]. The expansion of the Wilson loop $\langle w(t) \rangle$ has as coefficients, for each power of t^{n+1} precisely the expectation value of the corresponding σ_n which are the continuum limits of the ρ_n, and what follows from the double scaling limit of the loop equations is that to leading order those expectation values are not zero and are given by a power of f to leading planar order with corrections. This power of course is not zero, and hence it says that the corresponding derivative of the free energy with respect to the scaling parameter t_n that produces the expectation value of ρ_n is not zero even when we set $t_n = 0$ after taking the derivative.

3.12 Appendix B: Partition function near multicritical points

Here we will calculate the double scaled partition function near higher multicritical points. We start with eqn (3.71) and denote

$$Z = \int [dU] e^{\widetilde{S}(U,U^\dagger)}, \tag{3.89}$$

where $\widetilde{S}(U,U^\dagger)$ has the form (3.71). We will assume in the following that $a_i > 0$. We closely follow the discussion of section 3.7 and use the standard Gaussian trick discussed in section 3.4, to write

$$Z = \left(\frac{N^4}{2\pi^2}\right)^p \int \prod_{i=1}^{p} dg_i \, d\overline{g}_i \, d\mu_i \, d\overline{\mu}_i \exp(N^2 S_{\text{eff}}) \tag{3.90}$$

where

$$\begin{aligned} S_{\text{eff}} = &- \sum_{j=1}^{p} a_j \mu_j \overline{\mu}_j + \mathrm{i} \sum_j (\mu_j \overline{g}_j + \overline{\mu}_j g_j) + \sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'} (-\mathrm{i})^{|\vec{k}|+|\vec{k}'|} \Upsilon_{\vec{k}}(\overline{\mu}) \Upsilon_{\vec{k}'}(\mu) \\ &+ F(g_k + b_k, \overline{g}_k + \overline{b}_k). \end{aligned} \tag{3.91}$$

We now write g_k as

$$g_l = \frac{1}{2l}(\beta_l - i\gamma_l) \tag{3.92}$$

and we also write

$$b_k = \frac{1}{2k}(\widetilde{g}_k - i\widehat{\gamma}_k) \tag{3.93}$$

Performing change of the variables in the integral,

$$g_k \rightarrow g_k + b_k, \quad \overline{g}_k \rightarrow \overline{g}_k + \overline{b}_k. \tag{3.94}$$

we get,

$$\begin{aligned} S_{\text{eff}} = &\sum_{j=1}^{p} \left(-a_j \mu_j \overline{\mu}_j + \frac{\mathrm{i}}{2j} \Big((\beta_j - \widetilde{g}_j)(\mu_j + \overline{\mu}_j) + \mathrm{i}(\gamma_j - \widehat{\gamma}_j)(\mu_j - \overline{\mu}_j) \Big) \right) \\ &+ \sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'} (-\mathrm{i})^{|\vec{k}|+|\vec{k}'|} \Upsilon_{\vec{k}}(\overline{\mu}) \Upsilon_{\vec{k}'}(\mu) + F(\beta_k, \gamma_j). \end{aligned} \tag{3.95}$$

We will assume that we are analyzing the theory in the ungapped phase, in the proximity of the even multicritical point $m = 2k$. In this case we have,

$$N^2 F(\beta,\gamma) = N^2 F_{\rm ug}(\beta,\gamma) + N^2 F_{\rm scaling}(\beta,\gamma), \tag{3.96}$$

where $F_{\rm ug}(\beta,\gamma)$ is the planar free energy in the ungapped phase (3.40), and $F_{\rm scaling}(\beta,\gamma)$ satisfies

$$\lim_{N\to\infty} N^2 F_{\rm scaling}(\beta,\gamma) = F^{(m)}(t_l), \tag{3.97}$$

where $F^{(m)}(t_l)$ is the double-scaled free energy at the m-th multicritical point determined by the solution to the mKdV hierarchy [58].

To find the saddle point at large N we only have to consider the contribution of the free energy $F(\beta,\gamma)$ in the ungapped phase. The equations for the saddle point are given by,

$$\begin{aligned}
\frac{\partial S_{\rm eff}}{\partial \beta_j} =& \frac{\rm i}{2j}(\mu_j + \overline{\mu}_j) + \frac{1}{2j}\beta_j = 0,\\
\frac{\partial S_{\rm eff}}{\partial \gamma_j} =& -\frac{1}{2j}(\mu_j - \overline{\mu}_j) + \frac{1}{2j}\gamma_j = 0,\\
\frac{\partial S_{\rm eff}}{\partial \mu_j} =& -a_j\overline{\mu}_j + \frac{\rm i}{2j}(\beta_j - \widetilde{g}_j + {\rm i}\gamma_j - {\rm i}\widehat{\gamma}_j)\\
&+\sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'}(-{\rm i})^{|\vec{k}|+|\vec{k}'|}\frac{k'_j}{\mu_j}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu) = 0\\
\frac{\partial S_{\rm eff}}{\partial \overline{\mu}_j} =& -a_j\mu_j + \frac{\rm i}{2j}(\beta_j - \widetilde{g}_j - {\rm i}\gamma_j + {\rm i}\widehat{\gamma}_j)\\
&+\sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'}(-{\rm i})^{|\vec{k}|+|\vec{k}'|}\frac{k_j}{\overline{\mu}_j}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu) = 0
\end{aligned} \tag{3.98}$$

In the first two equations we have used that, in the ungapped phase,

$$\frac{\partial F_{\rm ug}}{\partial \beta_j} = \frac{1}{2j}\beta_j, \quad \frac{\partial F_{\rm ug}}{\partial \gamma_j} = \frac{1}{2j}\gamma_j, \tag{3.99}$$

We will assume that there is a solution to these equations corresponding to the m-multicritical even point of the model (3.91), which is characterized by

$$\gamma_j = 0, \quad \beta_j = \beta_j^{(m)}, \tag{3.100}$$

where the critical values of the couplings $\beta_j^{(m)}$ can be read from the particular solution (3.70). We find that this solution leads to the conditions

$$\mu_j^{(m)} = \overline{\mu}_j^{(m)} = \frac{\mathrm{i}}{2}\beta_j^{(m)}. \tag{3.101}$$

One finds the equations for the critical submanifolds in the original couplings, a_j, $\widetilde{g}_k$, and $\alpha_{\vec{k},\vec{k}'}$,

$$\beta_j^{(m)}(ja_j-1)+\frac{\widetilde{g}_j^c}{j}+\sum_{\vec{k},\vec{k}'} 2^{2-|\vec{k}|-|\vec{k}'|}(-1)^{|\vec{k}|+|\vec{k}'|}\alpha_{\vec{k},\vec{k}'}\frac{k_j}{\beta_j^{(m)}}\Upsilon_{\vec{k}+\vec{k}'}(\beta_j^{(m)}) = 0, \quad j=1,\cdots,p. \tag{3.102}$$

where $\widetilde{g}_j^c$ is the critical value of $\widetilde{g}_j$, and we have set $\widehat{\gamma}_j^c = 0$ for simplicity.

We now expand the effective action around the critical point, and we expand simultaneously the original couplings a_j, $\widetilde{g}_j$, $\widehat{\gamma}_j$ and $\alpha_{\vec{k},\vec{k}'}$ around a point a_j^c, $\widetilde{g}_j^c$, $\widehat{\gamma}_j^c = 0$, and $\alpha_{\vec{k},\vec{k}'}^c$ on the critical submanifold determined by (3.102). We denote

$$P(\mu,\overline{\mu},\alpha) = \sum_{\vec{k},\vec{k}'}\alpha_{\vec{k},\vec{k}'}(-\mathrm{i})^{|\vec{k}|+|\vec{k}'|}\Upsilon_{\vec{k}}(\overline{\mu})\Upsilon_{\vec{k}'}(\mu). \tag{3.103}$$

We introduce the column vectors of variables,

$$\begin{gathered}\xi(N)n = \begin{pmatrix}\mu_j - \mu_j^{(m)} \\ \overline{\mu}_j - \overline{\mu}_j^{(m)}\end{pmatrix}, \quad \alpha = \begin{pmatrix} a_j - a_j^c \\ \alpha_{\vec{k},\vec{k}'} - \alpha^c_{\vec{k},\vec{k}'}\end{pmatrix}, \\ g = \begin{pmatrix}\beta_j - \beta_j^{(m)} \\ \gamma_j\end{pmatrix}, \quad b = \begin{pmatrix}\widetilde{g}_j - \widetilde{g}_j^c \\ \widehat{\gamma}_j\end{pmatrix},\end{gathered} \tag{3.104}$$

where $\xi(N)$ is an appropriate scaling factor. When we expand the action in (3.91) around the m-th multicritical point, we obtain

$$\sum_l \Big(g_l \mathrm{tr}\, U^l + \overline{g}_l \mathrm{tr}\, U^{\dagger l}\Big) = V^{(m)} + \sum_n N^{\frac{n-2m}{2m+1}} t_n \widetilde{V}_n, \tag{3.105}$$

where $V^{(m)}$ is the critical potential associated to the m–th multicritical point, and $\widetilde{V}_n$ are scaling operators which can be explicitly written by using the results of [67]. In this way we find the relation between the variables g introduced in (3.104) and the scaling operators of the multicritical model,

$$g_a = \sum_{n\geq 0}\mathcal{G}_{an}N^{\frac{n-2m}{2m+1}}t_n, \tag{3.106}$$

where $\mathcal{G}$ is a matrix that can be explicitly determined from the expressions for the perturbations of the density of eigenvalues. The equation (3.106) determines the scaling properties of the g_a. Notice that we can use the freeedom to rotate U to get rid of one of the $2p$ parameters $g_i, \overline{g}_i$, so we will only have $2p-1$ times.

We now do a Gaussian integration over n. The relevant part of the action reads,

$$N^2\, S_{\rm eff} = -\frac{1}{2}N^2\xi(N)^2 n^t\,\mathcal{L}\,n + N^2\xi(N)n^t(\mathcal{J}g - \mathcal{J}b + \mathcal{H}\alpha) + \cdots\,, \quad (3.107)$$

where the matrices $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ are given by

$$\begin{aligned}
\mathcal{L} &= \begin{pmatrix} -\frac{\partial^2 P}{\partial\mu_j\partial\mu_k} & a_j^{(c)}\delta_{jk} - \frac{\partial^2 P}{\partial\mu_j\partial\overline{\mu}_k} \\ a_j^{(c)}\delta_{jk} - \frac{\partial^2 P}{\partial\mu_j\partial\overline{\mu}_k} & -\frac{\partial^2 P}{\partial\overline{\mu}_j\partial\overline{\mu}_k}\end{pmatrix}, \\
\mathcal{H} &= \begin{pmatrix} -\overline{\mu}_j\delta_{jk} & \frac{\partial^2 P}{\partial\mu_j\partial\alpha_{\vec{k},\vec{k}'}} \\ -\mu_j\delta_{jk} & \frac{\partial^2 P}{\partial\overline{\mu}_j\partial\alpha_{\vec{k},\vec{k}'}}\end{pmatrix}, \\
\mathcal{J} &= \frac{1}{2}\begin{pmatrix} \mathrm{i}\mathcal{F} & \mathcal{F} \\ \mathrm{i}\mathcal{F} & -\mathcal{F}\end{pmatrix},
\end{aligned} \quad (3.108)$$

and we have introduced the diagonal matrix

$$\mathcal{F}_{jk} = \frac{1}{j}\delta_{jk}, \quad j,k = 1,\cdots,p. \quad (3.109)$$

All quantities involved in these matrices are evaluated at the critical point. The Gaussian integration leads to

$$N^{2p}(\det(\mathcal{L}))^{-\frac{1}{2}}\exp\left\{\frac{1}{2}N^2(g-\mathcal{E}b-\mathcal{C}\alpha)^t\mathcal{M}(g-\mathcal{E}b-\mathcal{C}\alpha) + F^{(m)}(t_\ell) + \cdots\right\}, \quad (3.110)$$

where we have assumed that $\mathcal{L}$ does not have zero modes, and the fact that the Gaussian integration gives an overall factor N^{-2p} which combines with the overall N^{4p} in (3.90). Notice that the scaling $\xi(N)$ does not appear in this equation. The choice of $\xi(N)$ must be done in such a way that the rest of the terms involving n in the expansion of $N^2 S_{\rm eff}$ vanish in the limit $N\to\infty$. The matrices appearing here can be easily obtained from the above data.

Then, we have

$$\begin{aligned}
\mathcal{D} =& \frac{1}{2}\begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix}, \\
\mathcal{M} =& \mathcal{J}^t \mathcal{L}^{-1} \mathcal{J} + \mathcal{D}, \\
\mathcal{C} =& -\mathcal{M}^{-1} \mathcal{J}^t \mathcal{L}^{-1} \mathcal{H}, \\
\mathcal{E} =& \mathcal{M}^{-1} \mathcal{J}^t \mathcal{L}^{-1} \mathcal{J}.
\end{aligned} \tag{3.111}$$

Notice that the Hessian associated to S_{eff} is given by

$$H = \begin{pmatrix} -\mathcal{L} & \mathcal{J} \\ \mathcal{J} & \mathcal{D} \end{pmatrix}. \tag{3.112}$$

We now introduce scaling variables for the couplings g, α. The scaling of g is determined In this way we obtain for (3.110)

$$\exp\left\{ \frac{1}{2} \sum_{n,p} N^{\frac{2+n+p}{2m+1}} (t_n - t_n^0) \mathcal{A}_{np} (t_p - t_p^0) + F^{(m)}(t_\ell) + \cdots \right\}, \tag{3.113}$$

where

$$\begin{aligned}
\mathcal{A} =& \mathcal{G}^t \mathcal{M} \mathcal{G}, \\
t_n^0 =& N^{\frac{2m-n}{2m+1}} \sum_{\ell} \Big((\mathcal{G}^{-1}\mathcal{C})_{n\ell} \alpha_\ell + (\mathcal{G}^{-1}\mathcal{E})_{nj} b_j \Big).
\end{aligned} \tag{3.114}$$

As we see, the scaling of the original coupling constants packaged in α, b is determined by the scaling of the couplings in the m-th critical point.

In the limit $N \to \infty$, the integral localizes in

$$t_n = t_n^0. \tag{3.115}$$

To see this in detail, we use the following fact. Let B_ϵ be an $n \times n$ matrix whose entries go to $+\infty$ as $\epsilon \to 0$. Then, one has the following

$$\lim_{\epsilon \to 0} (\det(B_\epsilon))^{\frac{1}{2}} e^{-\frac{1}{2} x^t B_\epsilon x} = \pi^{\frac{n}{2}} \delta(x). \tag{3.116}$$

In our case we find that

$$\exp\left\{ \frac{1}{2} \sum_{n,p} N^{\frac{2+n+p}{2m+1}} (t_n - t_n^0) \mathcal{A}_{np} (t_p - t_p^0) \right\} \to N^{-\frac{\sum_{n\geq 0}(n+1)}{2m+1}} \frac{\pi^{p-\frac{1}{2}}}{\det(\mathcal{G})(\det(-\mathcal{M}))^{\frac{1}{2}}} \delta(t - t_0) \tag{3.117}$$

as $N \to \infty$. Remember that there are only $2p-1$ times involved. After changing variables in the integral from $g, \overline{g}$ to t, we inherit a Jacobian

$$N^{\frac{\sum_{n\geq 0}(n-2m)}{2m+1}} \det(\mathcal{G}). \tag{3.118}$$

Putting all these ingredients together, we finally obtain

$$Z \sim N(\det(H))^{-\frac{1}{2}} \exp F^{(m)}(t_n^0), \tag{3.119}$$

up to factors of π. We have assumed here that H has no zero modes. The factor of N comes from the fact that the quotient between the factors of N in (3.117) and (3.118) gives a power of N given simply by minus the number of times involved, which is $-2p+1$. This combines with the factor N^{2p} in (3.110) to give an overall factor of N. In the above derivation we have assumed that $\mathcal{M}$ (and therefore H has no zero eigenvalues).

We can also analyze the more general case in which $\mathcal{M}$ (which is a $p \times p$ matrix) has ℓ nonzero eigenvalues d_n, $n = 1, \cdots, \ell$, and $2p - \ell$ zero eigenvalues. Let R^{-1} be the orthogonal $2p \times 2p$ matrix that diagonalizes $\mathcal{M}$, i.e. $R^{-1t}\mathcal{M}R^{-1} = \mathrm{diag}(d_n, 0)$. Define now the following eigenvectors of $\mathcal{M}$

$$r = N^{\frac{2m}{2m+1}} Rg, \tag{3.120}$$

which in terms of the scaling operators means

$$r_n = \sum_q \mathcal{R}_{nq} t_q N^{\frac{q}{2m+1}}, \tag{3.121}$$

where $\mathcal{R} = R\mathcal{G}$. Then, the exponent in the Gaussian (3.110) becomes

$$\frac{1}{2} N^{\frac{2}{2m+1}} \sum_{n=1}^{\ell} d_n \Big(r_n - N^{\frac{2m}{2m+1}} c_n \Big)^2 + N^{\frac{2+2m}{2m+1}} \sum_{n=\ell+1}^{2p} r_n \zeta_n, \tag{3.122}$$

where

$$\begin{aligned} \zeta_n =& \sum_q R_{nq}^{-1t} \big(\mathcal{J}^t \mathcal{L}^{-1} \mathcal{H} \alpha - \mathcal{J}^t \mathcal{L}^{-1} \mathcal{J} b \big)_q, \quad n = \ell+1, \cdots, 2p \\ c_n =& -d_n^{-1} \sum_q R_{nq}^{-1t} \big(\mathcal{J}^t \mathcal{L}^{-1} \mathcal{H} \alpha - \mathcal{J}^t \mathcal{L}^{-1} \mathcal{J} b \big)_q, \quad n = 1, \cdots, \ell. \end{aligned} \tag{3.123}$$

As $N \to \infty$, the first term in (3.122) gives a delta function constraint of the form

$$\sum_{q\geq 0} \mathcal{R}_{nq} t_q N^{\frac{q}{2m+1}} = c_n, \quad n = 1, \cdots, \ell, \tag{3.124}$$

therefore there are only $2p - 1 - \ell$ independent times involved. From the behavior of the above equation as $N \to \infty$ it follows that we have to solve for the times with the higher scaling dimension in terms of the constants c_n. This in turn determines the scaling properties of c_n:

$$t_q = t_q^0 \equiv N^{\frac{2m-q}{2m+1}} \sum_{n=1}^{\ell} \mathcal{R}^{-1}_{qn} c_n, \quad q = 2p - 1 - \ell, \cdots, 2p - 2, \tag{3.125}$$

where we have inverted the $\ell \times \ell$ submatrix $\mathcal{R}_{qn}$, $q, n = 2p - 1 - \ell, p - 2$. This fixes the values of ℓ times in the free energy as functions of the scaled parameters c_n, $n = 1, \cdots, \ell$. The other times lead to a integral transform. To see this, let us define

$$\bar{t}_q = N^{\frac{2m+2+q}{2m+1}} \sum_{n=\ell+1}^{2p} \mathcal{R}_{nq} \zeta_n. \tag{3.126}$$

This equation determines the scaling of ζ_n. Notice that the scaling properties induced on c_n and ζ_n are very different. Up to overall factors, we end up with the integral

$$\int \prod_{n=0}^{2p-2} dt_n \prod_{q=2p-1-\ell}^{2p-2} \delta(t_q - t_q^0) \mathrm{exp}\left\{ \sum_{q=0}^{2p-2} t_q \bar{t}_q + F^{(m)}(t_q) \right\} =$$
$$e^{\sum_{q=2p-1-\ell}^{2p-2} t_q^0 \bar{t}_q} \int \prod_{n=0}^{2p-2-\ell} dt_n \mathrm{exp}\left\{ \sum_{q=0}^{2p-2-\ell} t_q \bar{t}_q + F^{(m)}(t_0, \cdots, t_{2p-2-\ell}, t^0_{2p-1-\ell}, \cdots, t^0_{2p-2}) \right\}. \tag{3.127}$$

For Hermitian matrix models, a similar result was obtained in [68]. Notice that the integral transform will change the critical exponents of the model, as noted in [68].

To illustrate our formalism we can look on to the example of free YM theories at finite temperature [12, 10]

$$S(U, U^\dagger) = \sum_{j=1}^{\infty} a_j \mathrm{tr}\, U^j \,\mathrm{tr}\, U^{\dagger j}, \tag{3.128}$$

where

$$a_j = \frac{1}{j}(z_B(x^j) + (-1)^{j+1} z_F(x^j)). \tag{3.129}$$

The equation for the critical surface reduces to

$$\beta_j^{(m)}(ja_j - 1) + \frac{\widetilde{g}_j^c}{j} = 0, \tag{3.130}$$

and by tuning the value of $\widetilde{g}_j^c$ we can reach any critical point. Notice that, if we do not include the b_k terms in the original action, only the first critical point $m = 1$ can be realized in the model. In that case, one has

$$a_1(T) = 1, \tag{3.131}$$

which defines the Hagedorn temperature $T = T_H$. Also, if we do not include the source terms involving b_k, we can turn on only a single scaling operator in the theory and we recover the $m = 1$ model. When one includes the $b_k, \bar{b}_k$ couplings one can also recover all the evolution times of the double-scaled matrix model.

Chapter 4

Plasma balls / kinks as solitons of large N confining gauge theories

In this work we would like to focus on confining gauge theories which have two length scales: the confinement scale Λ and the temperature $T = \beta^{-1}$. The relevant order parameters in such theories have a spatial variation on the scale of Λ^{-1}. The gravity duals of these theories have blackhole solutions which are localized on the boundary. It has been argued in [76, 8], that their holographic dual corresponds, in the large N limit, to a localized region of the de-confinement phase. This object has been called the plasma ball in [8], and it has a mass and a lifetime of $o(N^2)$. A qualitative gauge theory discussion in [8] uses a balancing of positive surface tension and negative pressure inside the plasma ball, to argue for its existence.

There is no doubt that it is important to study the plasma ball and its dynamics. Besides its utility for the physics of gauge theories at finite temperature, it is one more concrete laboratory for testing and studying various conundrums presented by blackholes [32]. The fact that the blackhole dual is localized on the boundary provides a greater handle on studying the horizon and what lies behind it.

Before we begin to make headway into an understanding of these problems, we need to have a dynamical handle on the plasma ball in the gauge theory. This is a standard hard strong coupling problem. Here by strong coupling we mean large t' Hooft coupling $\lambda = Ng^2_{YM}$. One natural strategy is to use numerical techniques. However a direct numerical approach is also difficult without developing a formalism within which we can ask the right questions.

We present a partial answer to this question in this work. We will discuss the plasma ball as a *large N soliton* which can be discussed in terms of various order parameters which distinguish between the confinement or deconfinement phases of the gauge theory. In order to do a concrete calculation we will focus on a concrete model that was discussed in [8] in which an interpolating solution was found between two bulk solutions of type IIB string theory: the AdS soliton [71] and a blackbrane. Both solutions are asymptotically $R^2 \times S^1_\tau \times S^1_\theta$, where S^1_τ is the thermal circle of radius β and S^1_θ is a Scherk-Schwarz spatial circle of radius 2π. The corresponding gauge theory is a Scherk-Schwarz compactification of $\mathcal{N} = 4$ $SU(N)$ gauge theory, on $R^2 \times S^1_\tau \times S^1_\theta$. The relevant and natural order parameters of this gauge theory are the holonomies of the gauge field around $S^1_\tau \times S^1_\theta$. In fact for technical reasons we will compactify R^2 to a Scherk-Schwarz cylinder, so that the Euclidean spacetime of the gauge theory is $R^1 \times S^1_\tau \times S^1_\theta \times S^1_\alpha$. The radius of S^1_α is chosen larger than that of the S^1_τ and S^1_θ [1].

We discuss the effective action of the gauge theory in the long wavelength expansion defined by the confinement scale Λ. The effective action, in the axial gauge along the non-compact direction x, is a one dimensional model of three unitary matrices $U(x)$, $V(x)$ and $W(x)$ corresponding to the zero modes of the Wilson loops on $S^1_\tau \times S^1_\theta \times S^1_\alpha$. Using the fact that we are working with a confining gauge theory of adjoint fields which are all short ranged (of the order of Λ^{-1}) one can integrate out V(x) and W(x) to arrive at an effective action involving the single unitary matrix U(x), which has the general form

$$S = \Lambda^{-1} \int_{-\infty}^{\infty} dx f(U) \mathrm{tr}(\partial_x U \partial_x U^\dagger) + g(U) \tag{4.1}$$

where Λ^{-1} is the confinement scale, and $f(U)$ and $g(U)$ are gauge invariant functions of U. $f(U)$ and $g(U)$ contain the information that the gauge theory has a first order confinement/deconfinement phase transition.

It is possible to discuss soliton solutions of the general multi-trace model using the Hamiltonian formulation together with the method of dealing with multi-trace operators developed in [16]. However in order to exhibit a solution

[1]This additional compactification of the boundary does not disturb the bulk solution

we simplify the effective action even further and present the soliton (plasma kink) solution. It turns out to be the motion of the Fermi surface of the many fermion problem that is equivalent to the matrix model in the $SU(N)$ invariant sector. This solution interpolates between the confinement and deconfinement phases and has energy density peaked at the phase boundary.

In our investigations we realized that it is imperative to use the $2+1$ dimensional phase space formulation of the classical Fermi fluid theory. The collective field formalism, which is a hydrodynamical description in $1+1$ dimensions inevitably leads to shock formation and singularities. It is not clear whether a finite energy density soliton solution can be obtained within collective field theory. The shocks are spurious singularities due to the collective field description which correspond to the folds on the Fermi surface, which are inevitable.

The plan of the chapter is as follows. In section 4.1 we describe the two bulk geometries- the AdS soliton [71] and the black brane solution, for which an interpolating domain wall solution was constructed in [8]. In section 4.2, we present a qualitative discussion as to how one can arrive at an effective description of the thermal gauge theory in terms of the holonomy matrices around the various cycles of the boundary, starting from a four dimensional gauge theory compactified on Scherk-Schwarz circles. For technical reasons we will be working with a gauge theory compactified on two Scherk-Schwarz circles. One can have two dual effective descriptions, in terms of either the Polyakov line or the Wilson loop over the spatial cycle. We present the general class of such effective matrix models. In the following sections we will be working with a particular matrix model belonging to this class. This model can be discussed in terms of an exact fermionic description [78, 79, 80, 81, 82, 83]. We shall also discuss the collective field equations [84] and indicate that their solution develops shocks in finite time.

In section 4.3 we discuss the phase structure of the model. This model has two stable phases: the confined and deconfined phases, and it undergoes a first order phase transition at a particular temperature. Later in the section we construct the soliton (kink) solution which interpolates between the two phases at the phase transition temperature. We then discuss some of the properties of the solution, in particular the surface tension of the soliton is discussed. We also present the localised (in one dimension) soliton solution at temperatures greater than the phase transition temperature, which approaches the confined phase in the two ends, and discuss some of it's properties.

In appendix 4.5 we show that starting from the confined phase of the theory, where the density of eigenvalues of the Polyakov line is uniform, we reach the clumped eigenvalue distribution only asymptotically, and never in any finite time. In appendix 4.6 we discuss the relation of the shocks formed in the collective field theory description to the formation of folds in the Fermi description.

4.1 Plasma balls in the large N gauge theory and dual black holes

A plasma ball is a localized spherically symmetric bubble of the deconfining phase of a confining gauge theory. In [8] using the AdS/CFT correspondence, their existence was inferred by exhibiting a bulk solution that interpolates between the AdS soliton[71] and the black-brane solution. The AdS soliton (AdSS), is given by the metric,

$$ds^2 = L^2\alpha'(e^{+2u}(d\tau^2 + T_{2\pi}d\theta^2 + d\omega_i^2) + \frac{1}{T_{2\pi}(u)}du^2) \tag{4.2}$$

where,

$$T_{2\pi}(u) = 1 - (\frac{1}{2}(d+1)e^u)^{-(d+1)} \tag{4.3}$$

Here we will be working with $d \leq 3$. The coordinate θ is periodic with periodicity 2π, and τ is the angular coordinate along the thermal circle of the Euclidean theory, with periodicity $\tau \to \tau + \beta$, and the ω_i are the two non-compact coordinates, while u is the radial coordinate. The boundary topology is $R^2 \times S^1_\tau \times S^1_\theta$, where S^1_τ and S^1_θ are the thermal and spatial cycles respectively. From the expression for $T_{2\pi}$, one sees that the spatial circle shrinks to zero size at a finite value of u.

The black-brane (BB) geometry is given by the metric

$$ds^2 = L^2\alpha'(e^{+2u}(T_\beta d\tau^2 + d\theta^2 + d\omega_i^2) + \frac{1}{T_\beta(u)}du^2 \tag{4.4}$$

with $T_\beta(u) = 1 - (\frac{\beta}{4\pi}(d+1)e^u)^{-(d+1)}$. This metric continued to Lorentzian signature has a horizon. Notice that when $\beta = 2\pi$, the two metrics 4.2 and 4.4 are simply obtained from one other by interchanging the thermal circle with the spatial circle. Since geometrically there is no difference between the two, the free energy of the two configurations must be the same at this temperature. For $\beta < 2\pi$, the free energy of the BB geometry dominates the path integral while for $\beta > 2\pi$, the free energy of the AdSS geometry is dominant. In [8] a domain wall solution which interpolates between these two solutions was constructed. Clearly such a domain wall solution exists only for $\beta = 2\pi$ when the free energy of the two phases is equal. The domain wall is independent of one of the non-compact direction and in the other non-compact direction the BB and AdSS geometry are asymptotically reached at the two ends.

These solutions can be incorporated within the IIB string theory by compactifying on S^5, with the five-form RR flux turned on. This would then have a dual boundary description in terms of the Scherk-Schwartz compactification of the $\mathcal{N} = 4$ $SU(N)$ SYM theory on a spatial cycle, with thermal boundary condition on both the cycle S^1_τ and S^1_θ. The gauge theory lives on $R^2 \times S^1_\tau \times S^1_\theta$. At $\beta = 2\pi$ clearly the two circles are identical and can be interchanged.

The above discussion suggests that a ball of large but finite radius of the deconfined plasma can occur as a solution to the finite temperature effective action of the gauge theory, at a temperature slightly above T_c. At $T = T_c$ there exists a kink solution interpolating between the confined and deconfined phases.

4.2 Gauge theories on $R^2 \times S^1_\tau \times S^1_\theta$

From the AdS/CFT correspondence, these bulk geometries- the AdSS geometry and the BB geometry correspond in the thermal gauge theory to the confinement and deconfinement phases respectively [5]. These phases are characterised by the expectation value of the Polyakov line, which is the trace of the holonomy around the thermal circle,

$$U(w_1, w_2, \theta) = \mathcal{P}\exp(-\oint A_\tau d\tau) \tag{4.5}$$

w_i are the two non-compact coordinates and the θ is the angular coordinate along the spatial circle, while $\mathcal{P}$ denotes path ordering. In particular, the expectation value of tr U vanishes in the confined phase while in the deconfined phase it takes a non-zero value [2]. Similarly one can define the holonomy around the spatial cycle S^1_θ.

$$V(w_1, w_2, \tau) = \mathcal{P}\exp(-\oint A_\theta d\theta) \tag{4.6}$$

Since the role of the two circles are interchanged in the two bulk geometries, it follows from the AdS/CFT correspondence, that $\mathrm{tr}V = 0$ in the deconfined phase, and it is non-zero in the confined phase [3]. At $\beta = 2\pi$, because the two geometries are identical under the interchange of the thermal and spatial circles, the effective action in terms of V should be identical to the one in terms of U. Later in this section we will qualitatively argue as to how one can arrive at an effective action in terms of both U and V and then in terms of either U or V, starting from the four-dimensional gauge theory.

Since we will mainly be interested in the solution which interpolates between the confinement and deconfinement phases as a function of one of the non-compact direction, it should be possible to find the one dimensional kink solution in an effective one-dimensional unitary matrix model. In order to realize this in a gauge theory at large N, it turns out to be convenient to work with $R \times S^1_\tau \times S^1_\theta \times S^1_\alpha$, where the S^1_α is the spatial circle, obtained by compactifying a noncompact direction previously labelled by the coordinate w_2. We introduce the holonomy along the spatial cycle S^1_α

$$W(w_1, \tau, \theta) = \mathcal{P}\exp(-\oint A_\alpha d\alpha) \tag{4.7}$$

This would correspond to replacing one of the non-compact directions of the bulk geometry that we discussed earlier, with a circle without changing the solution. Henceforth we shall set the noncompact direction $w_1 \equiv x$.

4.2.1 Effective action in terms of the Polyakov lines and Wilson loops

The bosonic part of the action of the general gauge theory will be written in terms of the gauge degrees of freedom A_1, A_τ, A_θ, A_α as well as the scalar fields Φ_i which transform in the adjoint representation. Here A_1 corresponds

[2] This basically reflects the fact that a quark in the fundamental representation of SU(N) has infinite free energy in the confining phase and finite free energy in the deconfined phase

[3] This reflects a gluon condensate in the vacuum of the gauge theory [85].

to the gauge field in the non-compact direction and we can choose the axial gauge $A_1 = 0$. These fields are in general functions of $(x, \theta, \tau, \alpha)$. Since the Scherk-Schwarz compactification breaks supersymmetry, the fermions are massive and the scalar fields get mass at one loop from quantum corrections. They can therefore be integrated out from the quantum effective action. Fourier expanding the gauge fields in all the circles and integrating out all the higher KK modes around every circle, we get a effective theory in terms of the zero modes of the fields: $A^0_\tau(x)$, $A^0_\theta(x)$, $A^0_\alpha(x)$.

This effective theory in terms of the zero modes is gauge invariant, and therefore we should be able to write it down in terms of the zero modes of the the holonomy matrices U,V and W. From now on we will use the notation U, V, W, to denote the zero modes of the above holonomy matrices.

The effective action will be a function of all possible gauge invariant operators. The gauge invariant operators are constructed out of the Z_N invariant products of the polynomials of U, V and W and their covariant derivatives, D_xU, D_xV, D_xW, and are of the form $\Pi_i \mathrm{tr}(U^{l_i}V^{m_i}W^{p_i}(D_xU)^{n_i}...)$, where the exponents l_i, m_i, p_i, n_i,etc are integers, such that the sum of all the exponents $\sum_i l_i + m_i + p_i + n_i + ... = 0$. In the gauge $A_1 = 0$, the covariant derivatives are the same as the ordinary derivatives. At sufficiently long wavelengths we neglect the higher derivative terms which are suppressed by powers of the confining scale Λ^{-1}.

Depending on which of the holonomy matrices condense, there will be three phases in the gauge theory. In the BB phase $\mathrm{tr}U \neq 0$, $\mathrm{tr}V = 0$ and $\mathrm{tr}W = 0$. In the other two phases one of the spatial holonomy matrices V or W will get expectation values, while the expectation value for the other two vanish. However we are interested in an interpolating solution between the black brane and the AdS soliton, and not in the transitions involving all the three cycles. If we choose the radius $R(S^1_\alpha) > R(S^1_\tau)$, $R(S^1_\theta)$, at the temperature of interest, then from the supergravity solution it follows that the cycle S^1_α never shrinks and corresponds to $\langle W \rangle = 0$ in the gauge theory. We can therefore put $W = 0$ in the effective three matrix model to once again obtain a two matrix model. The action for this will in general be very complicated, with all terms that are allowed by gauge invariance. It will contain words of the type $\mathrm{tr}(U^{n_1}V^{n_2}U^{n_3}....)$, and also derivative terms. The general action in the long wavelength expansion will be of the form,

$$\begin{aligned} S_{eff} &= \Lambda^{-1}\int dx f_1(U,V)\mathrm{tr}|\partial_x U|^2 + f_2(U,V)\mathrm{tr}|\partial_x V|^2 + \\ & \quad f_3(U,V)\mathrm{tr}(\partial_x U\partial_x V^{\partial}) + f_4(U,V) + \mathrm{h.c} \end{aligned} \tag{4.8}$$

where, Λ^{-1} is the confinement length scale, f_i's are gauge invariant functions of arbitrary polynomials of U, V and β with appropriate factors of N. At $\beta = 1/2\pi$, when the size of the two cycles are equal, the effective action will be invariant under $U \Rightarrow V$. Integrating over either the U or the V we will get a single matrix model in terms of V or the U matrix.

Since the theory is confining and has a mass gap, we can integrate out the V matrix, without worrying about infrared divergences, and we will be left with a model, given by

$$S = \Lambda^{-1}\int dx f(U)\mathrm{tr}(\partial_x U\partial_x U^{\dagger}) + g(U) \tag{4.9}$$

where again Λ^{-1} is the confinement scale, and $f(U)$ and $g(U)$ are temperature dependent, gauge invariant functions. As in (4.8) we have neglected all the higher derivative terms in the action, which are suppressed by powers of the confining scale. Equivalently we could integrate out U to arrive at a matrix model of V.

In the sequel we will mainly study the soliton solution of the simplest of this class of models, given by [4].

$$S = \Lambda^{-1}\int dx N\mathrm{tr}(|\partial_x U|^2) + \xi|\mathrm{tr}U|^2 \tag{4.10}$$

Here we will assume that $\xi > 0$ which ensures the existence of a first order phase transition at some value of ξ. By rescaling $x \to \Lambda^{-1}x$, we can remove the explicit Λ dependence from the above action to get,

$$S = \int dx N\mathrm{tr}(|\partial_x U|^2) + \xi|\mathrm{tr}U|^2 \tag{4.11}$$

where x is now given in units of Λ^{-1}. Hence forth we will be using this form of the action[5].

[4] This model has previously appeared in the discussion of 1+1 dimensional gauge theories [86]

[5] Therefore all the quantities we calculate later in the text like the surface tension of the phase boundary of the soliton, for example, will be given in units of the confinement scale.

4.3 Analysis of the one dimensional matrix model

In this section we will analyze the phase diagram of the unitary matrix model given by the action of the form (4.10). The matrix model described by the action(4.10) can be discussed using two methods. One is to use the collective field theory techniques as was done by Jevicki and Sakita[84]. This is basically a collective field description in $1+1$ dimension. The Hamiltonian is written in terms of the density $\rho(\theta, x)$ and velocity $v(\theta, x) = \partial_\theta \Pi(\theta, x)$, where Π is the canonical conjugate of ρ. The $\rho(\theta, x)$ field is the eigenvalue density field constructed out of the matrix U,

$$\rho(\theta, x) = \sum_{n=-\infty}^{+\infty} \rho_n(x) e^{2i\pi n\theta} \tag{4.12}$$

where $\rho_n = \frac{1}{N}\mathrm{tr}(U^n)$. For example, from the matrix model described by equation (4.10), we get the following collective field Hamiltonian,

$$H_{cf} = \int d\theta \, (\frac{\rho v^2}{2} + \frac{\pi^2 \rho^3}{6}) - \xi |\rho_1|^2 \tag{4.13}$$

This Hamiltonian, gives rise to the following set of fluid dynamical equations,

$$\begin{aligned} \frac{\partial \rho(x,\theta)}{\partial x} + \frac{\partial}{\partial \theta}(\rho(x,\theta) v(x,\theta)) &= 0 \qquad (4.14) \\ \frac{\partial v(x,\theta)}{\partial x} + v(x,\theta)\frac{\partial v(x,\theta)}{\partial \theta} + \pi^2 \rho(x,\theta)\frac{\partial \rho(x,\theta)}{\partial \theta} &= -2\xi \rho_1(x) \sin\theta \end{aligned}$$

Here θ is a periodic variable defined in the range $[-\pi, \pi]$ and x is a variable defined in the range $(-\infty, +\infty)$. The collective field approach is only valid for solutions which are spatially uniform,(for which $v(x,\theta) = 0$ and $\frac{\partial}{\partial x}\rho(x,\theta) = 0$). The spatially non-uniform solutions generically develop shocks in finite time, after which the collective field equations are not valid. [6]

[6] As discussed in more detail in appendix 4.6, this phenomenon can be understood from the underlying fermionic theory. Infact if we change $x \to ix$ and $v \to -iv$ in equation (4.14), we get the inviscid Burgers equation with a source term. In [89], it has been shown, using the method of hodograph transformation, that the source free version of the Burgers equation develops shock in finite time.

A correct (and exact) way to analyze the model (4.10) is to rewrite the model as a theory of interacting fermions [81] with the Hamiltonian (where the 'x' direction is identified with the Euclidean time).

$$H = \int d\theta \psi^\dagger(\theta)\partial_\theta^2\psi(\theta) - \xi \left| \int d\theta e^{i\theta}\psi(\theta)\psi^\dagger(\theta)\right|^2 \tag{4.15}$$

In the large N limit the fermion system will be classical and one can use the phase space density, $\mathcal{U}(p,\theta,x)$ such that,

$$\int \frac{dp}{2\pi} d\theta\, \mathcal{U}(p,\theta,x) = 1 \tag{4.16}$$

If a phase space cell is occupied then $\mathcal{U}(p,\theta,x) = 1$ or else $\mathcal{U}(p,\theta,x) = 0$. Hence $\mathcal{U}(p,\theta,x)$ satisfies the relation[7]

$$\mathcal{U}(p,\theta,x)^2 = \mathcal{U}(p,\theta,x) \tag{4.17}$$

The Hamiltonian written in terms of the phase space density is,

$$\frac{H}{N^2} = \int dp d\theta \frac{p^2}{2}\mathcal{U}(p,\theta,x) - \xi \left| \int dp d\theta e^{i\theta}\mathcal{U}(p,\theta,x)\right|^2 \tag{4.18}$$

In terms of $\mathcal{U}(p,\theta,x)$, the density and velocity $\rho(\theta,x)$ and $v(\theta,x)$ are,

$$\rho(\theta,x) = \int \frac{dp}{2\pi}\mathcal{U}(p,\theta,x),\ v(\theta,x) = \frac{1}{\rho}\int \frac{dp}{2\pi} p\, \mathcal{U}(p,\theta,x) \tag{4.19}$$

In the appendix we will further discuss the relation between the phase space and collective field theory approach and we will interpret the shock formation as the formation of folds on the Fermi surface. Hence the shock singularities are artifacts of the collective field approach and are resolved by a more accurate treatment.

In the following sections we will analyze the solutions of the fermionic Hamiltonian (4.18). We will start by describing the spatially uniform solution (phases of the theory) and then describe the non-uniform interpolating solution (plasma kink).

[7] The relation (4.17) is true only at large N. At finite N, $\mathcal{U}$ satisfies the relation, $\mathcal{U} * \mathcal{U} \equiv \cos\frac{1}{2N}(\partial_\theta\partial_{p'} - \partial_p\partial_{\theta'})[\mathcal{U}(p,\theta)\mathcal{U}(p',\theta')]|_{p'=p,\theta'=\theta} = \mathcal{U}$ [83], which reduces to equation (4.17) at large N.

4.3.1 Spatially uniform solutions

Here we analyze those solutions where the density of the eigenvalues of the matrix U is uniform over the direction x. In this case the location of the Fermi level will also be constant in x. Classically ρ can always be chosen to be an even function of θ. Then the potential in the equation (4.18) becomes, $\xi(\int dp\frac{d\theta}{2\pi}\cos\theta\mathcal{U}(p,\theta,x))^2$. In the Hatree Fock approximation, the phase space evolution equation for a single particle is,

$$\begin{aligned} \dot{\theta} &= p \\ \dot{p} &= -2\xi\rho_1(x)\sin\theta \end{aligned} \tag{4.20}$$

Where $\rho_1(x) = \int dp\frac{d\theta}{2\pi}\cos\theta\mathcal{U}(p,\theta,x)$ and $\dot{\theta} \equiv \frac{d}{dx}\theta$, $\dot{p} \equiv \frac{d}{dx}p$. For a spatially uniform solution, ρ_1 is independent of x and we can integrate the above equations to get,

$$p^2 = 2(E + 2\xi\rho_1\cos\theta) \tag{4.21}$$

where E is the energy of the particle. Therefore for a particle on the Fermi level, we have,

$$\widehat{p}_{\pm} = \pm\sqrt{2(E_f + 2\xi\rho_1\cos\theta)} \tag{4.22}$$

where $\widehat{p}_{\pm}$ correspond to the upper and lower branches of the Fermi level. Consequently

$$\rho(\theta) = \frac{\sqrt{2}}{\pi}\sqrt{E_f + 2\xi\rho_1\cos\theta}. \tag{4.23}$$

One has to satisfy the normalization condition given in equation (4.16) and the self consistency condition for ρ_1, which effectively solves E_f in terms of ξ and ρ_1

$$\begin{aligned} \int d\theta\frac{\sqrt{2}}{\pi}\sqrt{E_f + 2\xi\rho_1\cos\theta} &= 1 \\ \int d\theta\frac{\sqrt{2}}{\pi}\cos\theta\sqrt{E_f + 2\xi\rho_1\cos\theta} &= \rho_1 \end{aligned} \tag{4.24}$$

Depending on whether $|\frac{E_f}{2\xi\rho_1}| < 1$ or $|\frac{E_f}{2\xi\rho_1}| \geq 1$, the integrals in equation(4.24) will be evaluated between the limits $[-\theta_0, \theta_0]$, with $\theta_0 < \pi$, or over the full range $[-\pi, +\pi]$. The former case corresponds to the gapped phase, as

$\rho(\theta) = 0$ outside$[-\theta_0, \theta_0]$.), while the latter case corresponds to the ungapped phase).

One can study the different static phases of the model, by solving the self-consistency and the normalization conditions given in equation (4.24) simultaneously. This is hard to do analytically, but can be studied numerically. However it would be useful to have an understanding of the various phases as extrema of the potential in terms of ρ. This potential can be obtained from the Hamiltonian given in equation (4.18), using equations (4.19, 4.22) to integrate over p. We then obtain,

$$\mathcal{H} = \int d\theta \frac{1}{2}\rho v^2 + V([\rho]) \tag{4.25}$$

where,

$$V([\rho]) = \int d\theta \frac{\pi^2\rho^3}{6} - \xi|\rho_1|^2 \tag{4.26}$$

The potential of the model is actually a function of the infinitely many Fourier modes of ρ. Note that the static phases are all of the form given by the equation (4.23). It is therefore useful to parametrize ρ by

$$\rho = \sqrt{\sum_{n=0}^{\infty} a_n \cos(n\theta)} \tag{4.27}$$

With this parametrization, the uniform phase solution is given by $a_0 = \frac{1}{2\pi}$, and all other $a_n = 0$, while the gapped phase corresponds to $a_n = 0$, for $n > 1$ and a_0, a_1 taking appropriate values. With this parametrization, the potential will be a function of the a_n. Since all the phases of the theory lie in the plane given by $a_{n>1} = 0$, it will be enough to restrict to this plane. We therefore parametrize $\widehat{p}_\pm$ by the following form.

$$\widehat{p}_\pm = \pm\sqrt{2(E + 2\xi C_1 \cos\theta)} \tag{4.28}$$

We determine E in terms of C_1 by the normalization condition (4.16). Then substituting this in the expression for the potential, the potential becomes a function of only one parameter C_1. Then we can numerically calculate the potential given by the equation(4.26) as a function of C_1 (see figure 4.1).

We now summarise the key points from our analysis of the phase structure of the model in consideration.

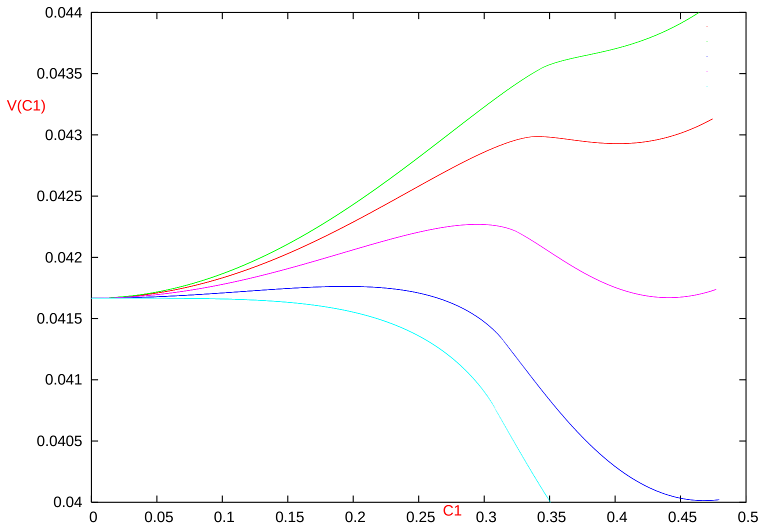

Figure 4.1: Plot of $V(C_1)$ with C_1 with $\xi = 0.22$, $\xi = 0.23$, $\xi = 0.237$, $\xi = 0.245$ and $\xi = 0.25$, with value of ξ increasing from the top curve to the bottom.

- At low enough values of ξ, there is a single phase where $\rho(\theta) = \frac{1}{2\pi}$ or $\widehat{p}_{\pm} = \pm\frac{1}{2}$. Here $C_1 = 0$. This is the uniform phase of the eigenvalue distribution.

- At $\xi = \xi_n = 0.227$ there is nucleation of two phases for which $\rho(\theta)$ is no more a constant. Both the phases have a gapped eigenvalue distribution. One phase is unstable (II) and the other is stable (III).

- The first order phase transition between the phase I and phase III occurs at $\xi = \xi_1 = 0.237$ and $C_1 = 0.4408$, $E = 0.1711$.

- The phase I becomes locally unstable at $\xi = \xi_2 = .25$

- At $\xi = \xi_3 = 0.23125$, and $C_1 = 0.3336$, phase II has a gapped to ungapped transition, this is the point of the third order GWW phase transition.

4.3.2 Spatially non-uniform solutions: plasma kinks

In the previous section we have analyzed the phase structure of our model. In particular we saw that at $\xi = 0.237$, the two stable phases (the confining and the deconfining phases) of the model have the same free energy. In this section we will first describe an interpolating domain wall type solution from the deconfined phase to the confining phase, at this value of ξ. Later in the section we will also construct a localised soliton solution which reaches the confined phase for large values of $|x|$.

The confining phase is described by a constant Fermi level which is given by the following equations in phase space,

$$\widehat{p}_{\pm} = \pm\frac{1}{2} \tag{4.29}$$

While in the deconfining phase, the Fermi levels were given by,

$$\widehat{p}_{\pm} = \pm\sqrt{2(E + 2\xi C_1 \cos\theta)} \tag{4.30}$$

Therefore we are looking for solutions in which the Fermi level evolves from (4.29) to (4.30). In terms of the geometry of the Fermi level it is a evolution from a band like to an ellipsoidal structure fig. 4.2.

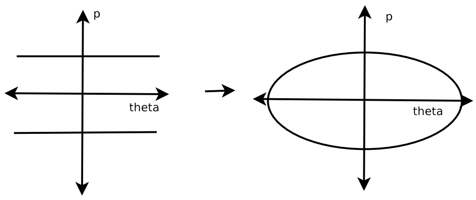

Figure 4.2: A schematic picture of the Fermi levels.

In terms of ρ, the solution has the property,

$$\rho(\theta, x) \to \frac{1}{2\pi}, \quad x \to -\infty \tag{4.31}$$
$$\rho(\theta, x) \to \frac{\sqrt{2}}{\pi}\sqrt{E + 2\xi\rho_1 \cos\theta}, \quad x \to \infty$$

Now in general the Fermi level will be described by the vanishing of some implicit function $f(\theta, p, x) = 0$. In the static case,

$$f(p, \theta) \equiv (p_+ - \sqrt{2}\sqrt{E + 2\xi C_1 \cos\theta})(p_- + \sqrt{2}\sqrt{E + 2\xi C_1 \cos\theta}) = 0 \tag{4.32}$$

In the general case $f(p,\theta,x)$ is not of this simple form and may have more roots. This corresponds to the case where the upper and lower Fermi levels develop folds and become multi-valued in θ. [8]

As each point in the phase space satisfies the equation, $\dot{\theta} = p, \dot{p} = V'(\theta)$, one can derive the time evolution of the function f to be,

$$\partial_x f + p\partial_\theta f + V'(\theta)\partial_p f = 0 \tag{4.33}$$

It would be interesting to try and solve the above equations numerically as a boundary value problem. We have not been able to do this. Instead we take a variational approach to the problem, and make a simple but reasonably accurate ansatz for the Fermi level. We will now summarise the main steps of the analysis.

- We choose an ansatz for the Fermi level similar to the form in the static case,

  $$f(p,\theta,x) \equiv (p_+ - \pi\rho(\theta,x) + v(\theta,x))(p_- + \pi\rho(\theta,x) + v(\theta,x)) = 0 \tag{4.34}$$

  with ρ given by,

  $$\rho = \frac{\sqrt{2}}{\pi}\sqrt{E(x) + 2\xi C_1(x)\cos\theta} \tag{4.35}$$

  where the $E(x)$, $C_1(x)$ are functions of x. This would be a good approximation if the $E(x)$, $C_1(x)$ are slowly varying functions of x. What we are doing in effect is to approximate the actual solution by a two Fermi surface solution throughout the evolution of the system, always given by the two curves $p = \widehat{p}_\pm$. Therefore $\widehat{p}_\pm$ are of the form,

  $$\widehat{p}_\pm = \pm\sqrt{2}\sqrt{E(x) + 2\xi C_1(x)\cos\theta} + v(\theta,x) \tag{4.36}$$

  $E(x)$ is determined in terms of $C_1(x)$ by the condition (4.16) or equivalently

  $$\int d\theta \frac{\sqrt{2}}{\pi}\sqrt{E(x) + 2\xi C_1(x)\cos\theta} = 1$$

[8] In fact as is shown in appendix 4.6 the folds are inevitably formed no matter what Fermi level configuration one starts with.

We determine $v(\theta, x)$ by the continuity equation,

$$\frac{d}{dx}\int \mathcal{U}(p,\theta,x)dp\frac{d\theta}{2\pi} = 0 \tag{4.37}$$

The solution of the continuity equation is given by,

$$v(\theta, x) = \frac{1}{\rho(\theta, x)}\left(\frac{\partial}{\partial x}\int_0^{\theta} d\widetilde{\theta}\rho(\widetilde{\theta}, x)d\widetilde{\theta}\right) \tag{4.38}$$

- Next, substituting this form of $\rho(\theta, x)$ and $v(\theta, x)$ back into the Hamiltonian and performing the θ integral, we get,

$$\mathcal{H} = {C_1'}^2 K(C_1) - V(C_1) \tag{4.39}$$

where $C_1' = \frac{d}{dx}C_1(x)$. Hence the whole problem is reduced to a quantum mechanical problem of $C_1(x)$. The function $K(C_1)$ and $V(C_1)$ are determined numerically, and $K(C_1)$ is positive and non-zero. Along the propagation in x the quantity $\mathcal{H}$ is conserved. This conservation law is used to determine the relation,

$$\frac{d}{dx}C_1 = \sqrt{\frac{E + V(C_1)}{K(C_1)}} \tag{4.40}$$

- The above equation is integrated numerically to obtain $C_1(x)$ as a function of x. Knowing $C_1(x)$ enables us to determine the phase space density $\mathcal{U}(p, \theta, x)$. The plot of $C_1(x)$ as a function of x is shown in figure (4.3). It should be noted that the soliton rises slowly but approaches the other end relatively fast. This follows from the asymmetric nature of the potential.

- It is important to check for the self consistency of this ansatz. This can be done by substituting the $\rho_1(x)$ obtained from our ansatz into the single particle equations and see how they evolve in x under this ρ_1. One can then compute the $\rho_1(x)$ obtained from this exact evolution at each instance of x, which we denote by $\rho_1^a(x)$ and compare with $\rho_1(x)$ obtained from the ansatz. If $\rho_1(x)$ were an exact solution, then one would get $\rho_1^a(x) = \rho_1(x)$. This is checked numerically. We started with

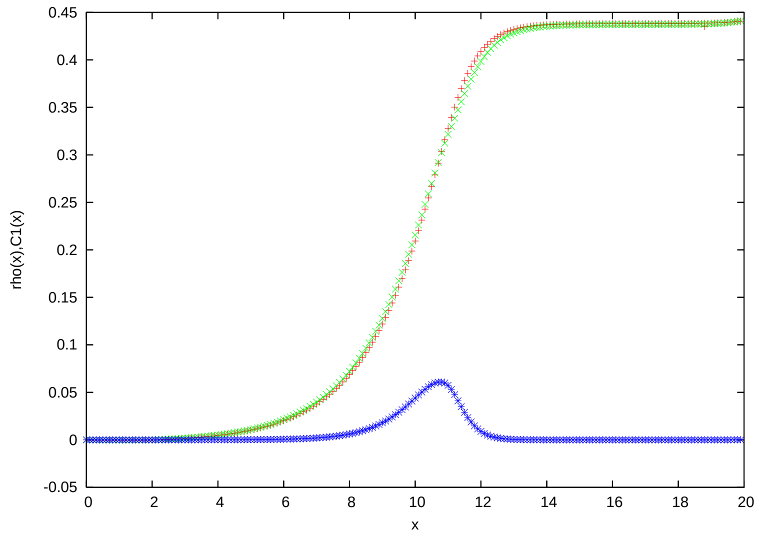

Figure 4.3: Plot of $C_1(x)$ (green), $\rho(x)$ (red) and free energy density (blue, not in scale)

50×50 particles uniformly distributed over the phase space region $p \in [-\frac{1}{2}, \frac{1}{2}], \theta \in [0, 2\pi]$. This gives us the band like Fermi level in figure 4.2. We study the evolution of the individual particles under the driving force $2\xi\rho_1(x)$ and calculate the $\rho_1^a(x)$ from the phase space distribution of the particles. We present the plots comparing the two values of $\rho_1(x)$, in figure 4.4.

- One may also look at the snapshots of the phase space particles. In figures 4.6 and 10 we have presented two snapshots taken at $x \approx 11.9$ and at $x \approx 11.6$. We see from the plots that the system is driven to the gapped phase configuration to a good accuracy. The phase space snapshot at the later value of x matches very well with the expected Fermi distribution in phase III at $\xi = 0.237$. This means that we are indeed reaching very near to the phase III.

  We also find that during the evolution of the Fermi sea, folds are formed on the Fermi level. As we discuss in the appendix 4.6 this is inevitable. However the area under the folds is a small fraction of the area of the full Fermi surface. This shows that our ansatz of a Fermi level with no folds, is self-consistent.

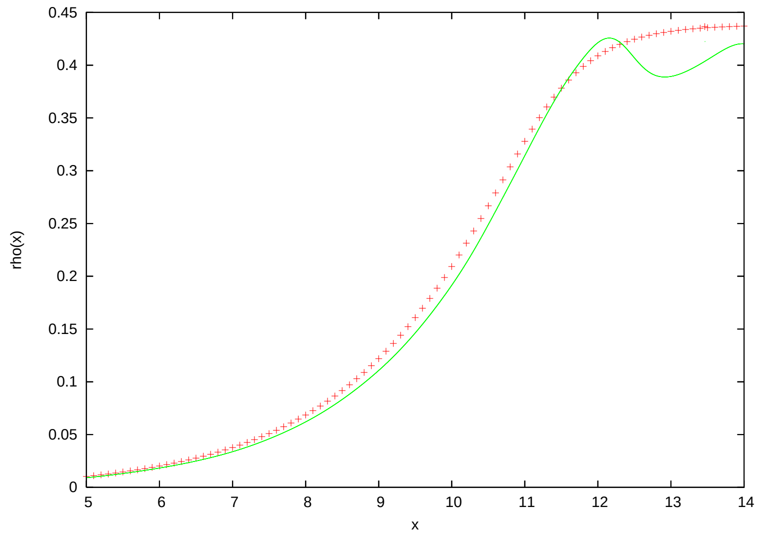

Figure 4.4: Plot of $\rho_1^a(x)$ (green) and $\rho_1(x)$ (red) with x

One also sees from the phase space plots that, as discussed in appendix 4.5, $\rho(0, x) \neq 0$ for all x.

- If we continue to plot the evolution of the phase space particles for long times, we will see that the value of $\rho_1^a(x)$ will start falling from it's value in the gapped phase, and the particles will disperse away from the ellipsoid as the system will move away from the gapped phase. This happens because even though the ρ_1 we obtain from our ansatz drives the system very near to the gapped phase starting from the uniform phase (as is evident from the phase space plots), it does not take it exactly to the gapped phase, since no matter how good the ansatz is it is not the exact solution[9]. If we continue to the evolve the system this error will start accumulating and the system will again disperse away from the gapped phase. This problem would not occur if we could do the exact numerical simulation for the soliton in the phase space as a boundary value problem with value of $\rho_1^a(x)$ fixed at both ends.

An important quantity that we can determine from our solution is the surface tension. The surface tension in general could either be positive or

[9]This is clear since in the correct solution folds are always formed no matter how small.

negative at the phase boundary. However, for lagrangians with positive kinetic terms, which is true in our case, the surface tension also turns out to be positive.

In one dimension surface tension is defined as the total free energy of the soliton, which in turn is the total action for the soliton. Hence the surface tension σ is, (see [87])

$$\sigma = 2 \int_{-\infty}^{+\infty} dx \, (V(C_1(x)) - V_{vacuum}) \tag{4.41}$$

This quantity at $\xi = \xi_1 = 0.237$ is numerically calculated to be, $\sigma = 0.0027$.

4.3.3 Localized soliton- plasma ball

In the previous section we constructed an interpolating kink solution for $\xi = \xi_1$. For ξ between ξ_1 and ξ_2, the two minima corresponding to phase I and phase III have different free energies (figure 4.1) and in particular, the minima corresponding to phase $I(C_1 = 0)$ is a false vacuum. In this case there exists a soliton solution which is localized in the x direction, and which goes to $C_1 = 0$ at both $x \to \pm\infty$ [88].

Such a solution has a simple interpretation in terms of a particle in real time moving in a potential $-V(x)$. From the conservation of the Hamiltonian (4.39), it is obvious that if we start from $C_1 = 0$ at $x = 0$, the solution never reaches phase III. It will bounces from a finite value of C_b and comes back to the phase I again, where C_b is determined by the relation $V(C_b) = V(0)$. In Fig 4.5 we present a schematic plot of $-V(C_1)$ and the bounce solution.

As before, one can construct such a solution numerically (see figure 4.6). This solution has a natural interpretation as a bubble of deconfined plasma within the confined phase. The plots shows two interesting trends. The first one is that the width and height of the soliton both increases as $\xi \to \xi_1 = 0.237$ from above. The second one is that as $\xi \to \xi_2$, the height of the soliton decreases, but the width of the soliton also increases. Hence width of the soliton comes to a minimum at some value of ξ between ξ_1 and ξ_2.

One can define the width w of the localized soliton as a measure of the spread in x over which the value of C_1 drops to a specified fraction C_* of it's maximum C_b. As $\xi \to \xi_1$ the localized soliton becomes the semi-infinite

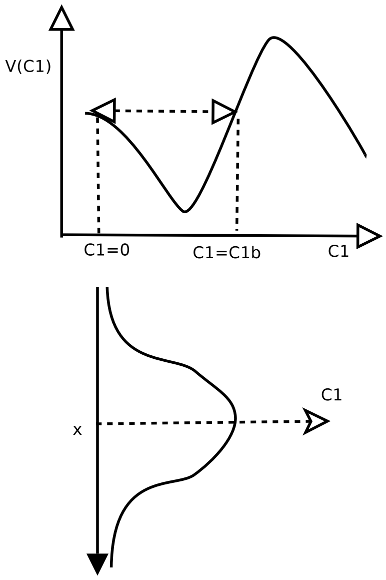

Figure 4.5: Plot of $V(C_1)$ showing the bounce solution below.

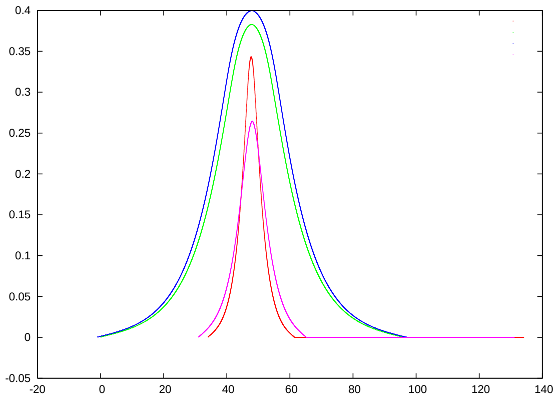

Figure 4.6: Plot of $C_1(x)$ as a function of x at $\xi = 0.245$ (violet), $\xi = 0.24$ (red), $\xi = 0.238$ (green) and $\xi = 0.2375$ (blue).

soliton discussed in the previous section and consequently the width of the soliton goes to infinity. It would be interesting to calculate the change in the width of the soliton with ξ as $\xi \to \xi_1$. In this limit, C_b almost reaches C_{III}. The equation of motion is given by,

$$2K(C_1)C_1^{''} + 2K'(C_1)C_1^{'2} = V'(C_1) \tag{4.42}$$

where $C_1' = \frac{d}{dx}C_1$, $K'(C_1) = \frac{d}{dC_1}K(C_1)$, and similarly $V'(C_1) = \frac{d}{dC_1}V(C_1)$. Expanding $K(C_1), V(C_1)$ around $C_1 = C_{III}$, and using the fact that $V'(C_{III}) = 0$, and C_1' will be small and negligible near $C_1 = C_{III}$ (because C_{III} is a turning point), we get from equation(4.42)

$$\frac{d^2}{dx^2}\delta C_1 = A(C_{III})\delta C_1 \tag{4.43}$$

where $\delta C_1 = C_{III} - C_1$ and $A = \frac{V''(C_{III})}{2K(C_{III})}$.

Using the boundary conditions, $\delta C_1(0) = (C_{III} - C_b)$ and $\frac{d}{dx}\delta C_1(0) = 0$, one can solve the above equation to obtain,

$$\delta C_1 = (C_{III} - C_b)(\cosh(\sqrt{A}x) \tag{4.44}$$

If we define $B = C_b - C_*$, then the width w is given by,

$$1 + \frac{B}{C_{III} - C_b} = \cosh(\sqrt{A}w) \tag{4.45}$$

Since $C_{III} - C_b \to 0$ as $\xi \to \xi_1$, it follows that in this limit the leading ξ dependence of $C_{III} - C_b$ will be of the form $C_{III} - C_b \sim (\xi - \xi_1)^a$, where a could be any real positive number. Putting this dependence back into the above equation, and solving in the $w \to \infty$ limit, we get,

$$w \propto -\log(\xi - \xi_1) \tag{4.46}$$

Hence we see that the width of the soliton diverges logarithmically with $\xi - \xi_1$.

4.4 Conclusion

In this chapter we have a presented a $o(N^2)$ soliton solution of a confining gauge theory which interpolates between the confining and deconfinement phases separated by a first order phase transition. The soliton is a solution of the large N, long wavelength effective action of the gauge theory expressed in terms of the thermal order parameter (Polyakov line). The general three dimensional effective Lagrangian would have to contain higher derivative terms to support a solitonsolution and this would make the problem technically very difficult. However, in the present work we have analyzed a simpler one dimensional example. We have presented a qualitative discussion on the possible connection of this model with a higher dimensional confining gauge theory which has a gravity dual. The soliton that we have found numerically is a finite region of the deconfinement phase (plasma kink/ball) with a positive surface tension at the phase boundary. The free energy density is also a smooth function every where in space.

Even though the soliton solution is obtained in a thermal gauge theory formulated in Euclidean spacetime it is reasonable to expect it to be a static solution in Lorentzian spacetime at finite temperature.[10] This fact can be inferred by observing that the bulk solution can be analytically continued from Euclidean to Lorentzian spacetime. Given these facts it is tempting to identify the phase boundary as dual to the horizon of the blackhole. A more precise understanding of this correspondence will enable us to explore the structure of blackholes, especially 'inside the horizon' and address very directly the persistent question of the blackhole singularity.

4.5 Appendix: Analysis of the clumping in the eigenvalue distribution in finite time

In this appendix we will prove that if we give a small perturbation around phase I, $\rho(\theta, x)$ never becomes 0 near the point $\theta = 0$, at any finite x. Let us solve the equations of motion for individual phase space points near $\theta = 0$. Near $\theta = 0$ we can make the approximation, $\sin\theta \sim \theta$. The equations of motion can be written as,

$$\begin{pmatrix} \dot{p} \\ \dot{\theta} \end{pmatrix} = M(x) \begin{pmatrix} p \\ \theta \end{pmatrix} \tag{4.47}$$

where,

$$M(x) = \begin{pmatrix} 0 & 2\xi\rho_1(x) \\ 1 & 0 \end{pmatrix} \tag{4.48}$$

[10] In this case the holonomy matrix $V(x)$ may be a more appropriate order parameter.

Here we start by approximating $\rho_1(x)$ with with a step function such that

$$\begin{aligned} \rho_1(x) &= \rho_1, \quad x > 0 && (4.49) \\ &= 0, \quad x < 0 && (4.50) \end{aligned}$$

The solution of the equation is given by the condition,

$$\exp(-Mx) \begin{pmatrix} p(x) \\ \theta(x) \end{pmatrix} = \begin{pmatrix} p(0) \\ \theta(0) \end{pmatrix} \tag{4.51}$$

If we look at the Fermi level given by, $\widehat{p}_{\pm}(0) = \pm p_0$, then at "time" x the position of the Fermi level will be,

$$\widehat{p}_{\pm}(x) = \frac{\pm p_0}{\cosh(\sqrt{2\xi x \rho_1})} \tag{4.52}$$

As $|\rho_1(x)| < 1$, $\widehat{p}_{\pm}(x)$ does not reach 0 at any finite time. Similar result seems to be true for a time dependent ρ_1. Consequently, eigenvalue density function $\rho(\theta) = \widehat{p}_+(\theta) - \widehat{p}_-(\theta)$ is always non-zero at the point $\theta = 0$. Hence any gap in the eigen value distribution can not open in finite time. However, the solution may asymptotically reach a gapped phase.

4.6 Appendix: Shock formation in the collective field equations and folds on the Fermi surface

In this section we will show that the collective field equations develop shocks in finite time which can be understood from the underlying phase space picture as the formation of folds on the Fermi surface. The collective field equations may be derived from a classical theory of fermions. Consider first the theory of free fermions. We are looking at the phase space description of this theory. The motion of individual phase space points are described by the equations,

$$\dot{\theta} = p, \dot{p} = 0 \tag{4.53}$$

From the above equation we can determine the equation of motion for a particle on the Fermi surface to be,

$$\partial_x \widehat{p} + p \partial_\theta \widehat{p} = 0 \tag{4.54}$$

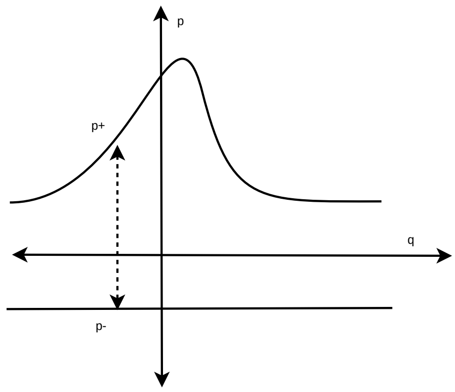

Figure 4.7: Fermi level

where $\widehat{p}$ denotes the value of p at any point on the Fermi surface.

Now if the profile of the Fermi surface is such that for each value of θ, there are exactly two points lying on the Fermi surface, one on the upper and lower Fermi level each (like in figure (4.7)), then we have

$$\partial_x \widehat{p}_\pm + \widehat{p}_\pm \partial_\theta \widehat{p}_\pm = 0 \tag{4.55}$$

where $\widehat{p}_\pm$ characterize the points on the upper and lower Fermi levels respectively. The source free version of the collective equations in (4.14) are simply linear combination of the above two equations (see [81]), governing the dynamics of $\widehat{p}_+ + \widehat{p}_-$ and $\widehat{p}_+ - \widehat{p}_-$, which are proportional to v and ρ respectively from (4.19).

This identification with the collective field equations is perfectly fine for a fluctuation of the form shown in the figure (4.7). However because of the equation of the motion, points of the curve which are higher, have greater velocity than the lower points, hence even if we start with a simple profile like that given in figure (4.7), the profile changes due to the unequal velocity of the various points lying on the Fermi level to a profile of the form given in figure (4.8). In figure(4.8), where the profile becomes multi-valued, the identification is not as before, since there are more than two values of p corresponding to the same value of x. For instance, if at a point the Fermi profile has a multi valuedness of the "order four", that is there are four values

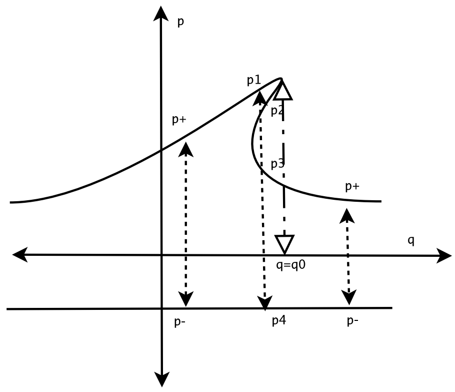

Figure 4.8: Fermi level

of $\widehat{p}$ corresponding to the same value of θ, then the equation for ρ becomes

$$2\pi\rho(\theta) = \int_{\widehat{p}_3}^{\widehat{p}_4} dp\, \mathcal{U}(p,\theta) + \int_{\widehat{p}_1}^{\widehat{p}_2} dp\, \mathcal{U}(p,\theta) \tag{4.56}$$

and similarly for the equation for ρv. One can easily see that one cannot derive the simple collective field equations in this case. Hence the collective field equations do not describe the dynamics of the Fermi surface at all times.

However we can still look at the the topmost value of p as $\widehat{p}_+$ and the lowest value of p as $\widehat{p}_-$. In that case the equations governing the dynamics of $p_+ + p_-$ and $\widehat{p}_+ - \widehat{p}_-$ are the same collective field equation throughout, but then we see clearly from figure (4.8). that the values of these variables jumps at $\theta = \theta_0$, and hence the θ derivative blows up at this point. This jump will correspond to the shock of the collective field equations. Note that the description in terms of the fermion phase space is always perfectly smooth since it is after all the theory of free fermions.

In our case we are dealing with a $1+1$ dimensional interacting Euclidean fermionic theory given by a Lagrangian of one fermionic field $\Psi(\theta)$

$$\begin{aligned} \mathcal{L} &= \int d\theta \Psi^\dagger \partial_x \Psi + |\partial_\theta \Psi|^2 \\ &+2\xi \int d\theta d\theta' \Psi^\dagger(\theta)\Psi(\theta)\cos(\theta-\theta')\Psi^\dagger(\theta')\Psi(\theta') \end{aligned} \tag{4.57}$$

These equations give rise to the equation of the form (4.14). The phase space arguments discussed here will continue to hold even in this case again leading to shock formation in finite time (see fig 4.10). But the theory viewed as a theory of fermions will still be valid.

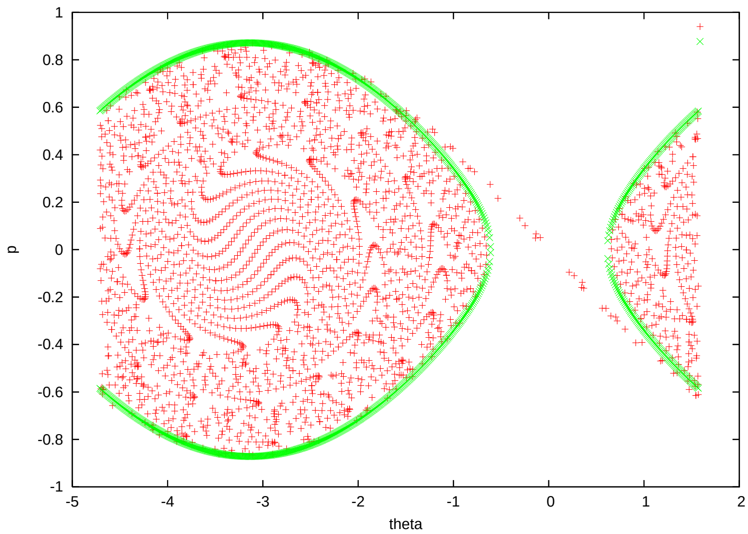

Figure 4.9: Phase space particles (red) at $x \approx 11.9$ showing the match with Fermi surface (green) in phase III.

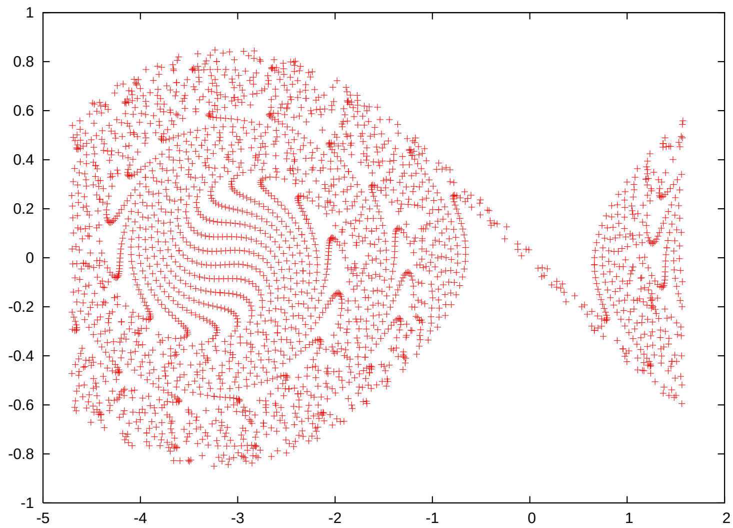

Figure 4.10: Phase space particles at $x \approx 11.6$ showing shocks at around $\theta \approx -0.8$ and $\theta \approx 0.8$.

Chapter 5

Epilogue

In our work we concentrated on finite temperature gauge theories and the AdS/CFT correspondence. Even if one starts with a super-symmetric gauge theory, the super-symmetry will be broken by finite temperature effects. It is difficult to make a systematic study of these problems using analytic or numerical methods. Our approach was to study gauge theories, using the Polyakov loop as an order parameter and arrive at an effective unitary matrix model. We have shown that analyzing these effective matrix models, one may learn interesting information about gauge theories and their gravity duals. With more thorough and improved analytical studies we believe that our type of approach will provide more information about gauge theories and the nature of the AdS/CFT correspondence. The exact value of the coefficients appearing in the effective unitary matrix models are difficult to calculate analytically and many important facts including the calculation of black hole entropy depend on them. Numerical techniques using monte-carlo simulations may be important in these situations [90]. In our studies we have entirely neglected the questions related to the real time dynamics of the finite temperature gauge theories. We used Polyakov loop as our order parameter and by construction it does not carry any information about the real time dynamics. Progress in fluid dynamical view point and understanding of the viscosity-entropy relation [91] is an important step to study the finite temperature , real time gauge theory. In our future work we would like to venture in this direction.

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