Dyonic black hole counting in supersymmetric and non-supersymmetric backgrounds.

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This thesis is dedicated to the beautiful harmonies of life and fate that weave their colorful patterns in our universe, and to the intelligence that attempts to make sense of them all.

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Synopsis

String theory aims to achieve a complete understanding of quantum gravity. Einstein's theory of general relativity is a classical theory of gravity, and black holes are interesting solutions in this theory. It is possible to associate with black holes, thermodynamic properties like temperature and entropy, and to formulate the three laws of thermodynamics in terms of these quantities. A statistical description of the thermodynamics of black holes is an outstanding problem in gravity and needs to be necessarily addressed by any theory of quantum gravity. String theory has a set of duality symmetries which map theories in one region of their moduli space to the same or different theories in other regions of moduli space. These symmetries have proved to be extremely useful in understanding black holes since they can be used to map a non-perturbative description of a state in the theory to a description which is perturbatively accessible. This is a powerful tool especially when it comes to understanding the non-perturbative structure of string theory. To see this consider the T-duality group of string theory. A particle on a circle will have quantized momentum which is the charge of the U(1) translation symmetry along the compact direction. But since strings can wind on a circle, string theory compactified on a circle has a 2d self dual integral Lorentzian lattice consisting of momentum and winding. The $SO(1,1,\mathbf{Z})$ Lorentz group of this lattice is called the T-duality symmetry of the theory. This concept generalizes to bigger T-duality groups obtained by compactification on 6d compact manifolds to obtain string theories in 4 noncompact dimensions. T-duality is a symmetry that is realized perturbatively. There is an additional symmetry in string theory which takes a strongly coupled theory to a weakly coupled theory and exchanges fundamental strings with solitonic states. This is called S-duality and evidently is a symmetry that is not accessible in perturbation theory. The net symmetry group of string theory called U-duality contains both S and T dualities and is called the U-duality group of string theory. Hence using dualities we can make statements about non-perturbative aspects of string theory by doing a perturbatively accessible computation.

A very powerful technique to do the same is to compute the degeneracy of states in a theory and demand that this degeneracy be invariant under S-duality. This lays down strong constraints on the form of the degeneracy function and this function will be highly sensitive to underlying non-perturbative structures in the theory like lines of marginal stability across which the degeneracy jumps. Hence counting degeneracies becomes a outstanding problem in string theory and dualities provide a powerful tool to do the same.

Not surprisingly, therefore, string theory has had some spectacular successes along these directions. Strominger and Vafa [1] performed a microscopic computation of the Bekenstein Hawking entropy of a certain class of supersymmetric black holes. This entropy was derived in the gravity theory from the Einstein-Hilbert action. Subsequently, Wald presented a formula to compute entropy from a general gravitational action[1] and the resulting sub-leading corrections to S_{BH} have been successfully compared with those arising from an exact microscopic counting [2].

In this dissertation, we are going to deal with a similar program for four dimensional charged extremal black holes which arise as solutions of the $\mathcal{N} = 4$ theory obtained by compactifying Type II string theory on $K3 \times T^2$ or its heterotic dual on $T^4 \times T^2$, and an orbifold of the same theory. We encapsulate in the remaining part of this synopsis a summary description of the research covered in this dissertation. We first summarize the contents of [3]. Herein, we look at 1/4 BPS

[1] This included higher derivative corrections to the Einstein Hilbert action

supersymmetric dyonic configurations in the two theories that we consider in this paper. For, Type II string theory on $K3 \times T^2$, [4], proposed an exact counting formula for the degeneracy of these configurations. This formula was derived by Jatkar and Sen in [5] using a prescription called the Arithmetic lift. We offer an alternate derivation of the same using a procedure called the Borcherds lift. The 4d dyonic charges are first lifted to a 5d configuration of a D1-D5-P bound state moving in the Taub-NUT geometry of a KK monopole. The generating function of degeneracies is then a product of the degeneracy of 1/4 BPS states in the world volume theory of the D1-D5-P system, the KK-P bound state degeneracy and the degeneracy due to motion of the D1-D5-P in Taub-NUT. The first of these is the elliptic genus of the symmetric product of $K3$ and is obtained by the Borcherds lift while the remaining two are obtained by multiplying additional terms to preserve S-duality invariance of the final answer. The Borcherds lift prescription is then used for the orbifolded theory to get a counting formula for quarter BPS dyons in this theory.

As a sequel to this work, we summarize [6] which analyzes the degeneracy formula for the Type II theory on $K3 \times T^2$, which we derived before. An important consistency requirement of any degeneracy formula is that it be invariant under the U-duality symmetry of the theory. Now the degeneracy formula is already given in terms of the T-duality invariants. Now, under S-duality, both charges and moduli change. We showed that in different regions of moduli space the contour of integration used to extract degeneracies from the generating function needs to be chosen differently and these contours can not be deformed smoothly to each other because of the existence of poles in the function. These poles correspond to lines of marginal stability in moduli space across which the degeneracy jumps discontinuously. We also build on a picture of the dyonic configurations being represented as string webs in Type IIB string theory[7] to find a new discrete invariant I of the exact U-duality group of the theory. I is a function of the integral electric and magnetic charges of the theory and we showed that the degeneracy formula that was derived applied only to dyonic configurations with I=1.

An extension of the above work is performed in the contents of [8]. Here, we try to extend the class of dyons for which exact counting formulas can be derived beyond those counted by the above function. We focus on charge configurations which never form black holes but nevertheless exist in the $\mathcal{N} = 4$ field theory limit of the string theory. We concentrate on a special class of dyons in SU(N) gauge theory called Stern-Yi dyons. We start from our previous observation that these dyons have degeneracy jumps across lines of marginal stability we analyze these configurations near a line of marginal stability where they decay. Near these lines the 1/4 BPS dyon splits into a constituent 1/2 BPS and another 1/2 or 1/4 BPS dyon which are far separated from each other. Then the interaction between these two centers can be ignored and the net degeneracy comes from the product of the degeneracies of the individual centers and the electromagnetic field angular momentum associated with their bound state. This heuristic picture was used to compute the entropy of the Stern-Yi dyons and the results are found to be in agreement with previously known exact results.

We now summarize the contents of [9]. Here, we shift focus and move on to understanding the entropy of non-supersymmetric dyonic configurations. In this case even getting a microscopic understanding of black hole entropy is a formidable challenge. For a certain class of extremal black holes, namely those which admit a description as BTZ black holes in AdS_3, the black hole can then be viewed as a state in the boundary CFT_2. If the charges are in a certain specific ratio, the leading order entropy can then be computed using the Cardy formula . We use the exact symmetries of string theory to answer the question of whether other charge configurations can be brought to regions in charge and moduli space which allow a microscopic description of their entropy. We managed to demonstrate that a large class of dyonic configurations could be brought to the required regime in charge space. Hence if it was possible to use the residual symmetries of string theory to go to the relevant region of moduli space, we could have a microscopic derivation of the black hole entropy.

List of Publications

1. Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift.
   Atish Dabholkar, Suresh Nampuri (Tata Inst.), TIFR-TH-06-07, Mar 2006.
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2. Comments on the spectrum of CHL dyons.
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3. On The Stability of Non-Supersymmetric Attractors in String Theory.
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4. Duality Symmetry and the Cardy Limit.
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   e-Print: arXiv:0711.4671 [hep-th]

5. Degeneracy of Decadent Dyons.
   Atish Dabholkar (Tata Inst. and Paris, LPTHE), K. Narayan (Chennai Math. Inst.), Suresh Nampuri (Tata Inst.) . TIFR-TH-06-07, Feb 2008. 15pp.
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Chapter 1

Introduction

A complete understanding of a quantum theory gravity involves understanding physics in the realm of length scales of the order of 10^{-33}cm (Planck length). The curvature of spacetime at these scales is correspondingly high and these are referred to as singularities in the classical description of spacetime given by general relativity. At a singularity all unitary evolution of wavefunctions breaks down simply because the conditions at the singularity are not well-defined. One of the main aims of string theory, as a purported theory of quantum gravity, is to resolve classical singularities and provide a consistent description of the quantum mechanical Hilbert space of states obtained by quantizing both gravity as well as the other forces in nature. A consistent theory of strings has both open and closed strings, which move about in spacetime. The induced metric on the string worldsheet and the pull-back of the background 2 form fluxes gives rise to a 1+1 CFT on the string worldsheet whose quantization yields a Hilbert space of spacetime states. These states can be thought of as different vibrational harmonics of the string and each harmonic corresponds to a distinct representation of the Poincare algebra i.e. a distinct particle with unique mass and quantum numbers. To ensure the absence of a tachyonic state we consider a supersymmetric worldsheet theory which is consistent only in 10 dimensions. In ordinary field theories, a KK-reduction or compactification on a circle converts particle momentum along that circle into the charge of the compact U(1) group generating translations along the circle. A string wrapped on a circle in addition to this charge also has a winding number associated with the circle. These charges are integral and fill out a lattice called the Narain lattice. The symmetry group of this lattice is called the T-duality group. In addition to the fundamental string, there are also solitonic objects called D-branes and NS5 branes in the theory and a strong weak coupling S-duality exchanges the fundamental strings with the soliton states. One of the best laboratories to test conjectures and results of string theory is in the vicinity of a singularity (either naked or covered by an event horizon-a black hole), where the classical geometry undergoes stringy corrections. These can be either as tree level corrections suppressed by powers of the square of the string length α' or as loop corrections suppressed by the string coupling constant g_s.

In string theory, 4d black holes are realized as states of strings or solitons wrapped on non-trivial cycles in a 6d Ricci flat compact manifold. The black holes carry charges corresponding to the momenta and winding number of strings or the number of branes wrapped on various cycles in the manifold. The intersection number of the cycles in the manifold define a symplectic structure in the charge lattice and the S-and T dualities can then be used to generate new solutions. The biggest symmetry of the compactification includes both S and T dualities and is called the U duality group. Depending on whether the bound state of strings and branes breaks the background supersymmetry or not, we get SUSY and non-SUSY black holes. If N be the order of the charges then the effective coupling constant for the world volume field theory living in the common world volume of these objects is g_sN and a counting of the degeneracy of the state in this theory carrying the same charges and quantum numbers as the black hole under consideration should give the entropy of the black hole. For small g_sN we can think of this system as a bound state of branes and strings while for large coupling the gravitational description becomes valid and this system can be viewed as a black hole solution of the low energy supergravity. It turns out that a degeneracy counting of the total number of bosonic and fermionic states in a given charge sector is not protected in string theory under a renormalization flow from small to large coupling. What are actually protected are indices which count solutions that retain full or partial (BPS) supersymmetry. These indices remain unchanged as one moves through the moduli space of compactification except for discrete jumps. Hence we can obtain exact counting formulas for supersymmetric black holes which are also extremal. Thus we have a way of achieving a statistical description of a certain class of black holes in string theory and there by understanding the laws of black hole thermodynamics from first principles. Also an exact counting formula should be invariant under strong weak coupling dualities and so must encode information about non-perturbative structures in string theory. Yet another technique to explore black holes is to use the AdS/CFT paradigm which conjectures an equivalence of the partition function of strings in spacetimes that are asymptotically AdS and a CFT living on the boundary of the AdS space. So a black hole in the bulk can be viewed as a state in the dual CFT and CFT counting formulas can be used to obtain atleast the leading order entropy for non-SUSY black holes.

In the following chapters, we use the Borcherds lift to derive the elliptic genus of the symmetric product of K3 and finally to construct the conjectured exact counting formula for 1/4 BPS dyonic black holes. We further explore the question of whether in the non-SUSY case, all extremal charged configurations can be brought by U-duality to a point in charge space where they are amenable to a approximation to the full CFT partition function called the Cardy formula. Each chapter in this dissertation is based on a research paper that I have worked on, and is titled by the name of the paper. All conclusions of various chapters are grouped together in a separate chapter at the end as are the appendices of all chapters.

1.1 Background

The work summarized in this thesis deals with microscopic counting of dyonic configurations in $\mathcal{N} = 4$ 4d string theories. We deal with both supersymmetric as well as non-supersymmetric configurations in these theories. In the supersymmetric case we derive exact counting formulas and examine their properties while in the non-supersymmetric case we use the Cardy formula extensively. So in this section, some basic facts of the theory under consideration as well as the Cardy formula are set out and notation and terminology is set up which will be used extensively in the remainder of this article. The compactification of Type IIA theory on $K3 \times T^2$ preserves 16 supersymmetries. It is dual to Heterotic theory on T^6[10]. The resulting four dimensional theory has 28 gauge fields. In the IIA description these arise as follows. One gauge field comes from the RR 1-form gauge potential, C_1; 23 gauge fields from the KK reduction of the RR 3-form gauge potential, C_3, on the 22 non-trivial 2-cycles of $K3$ and on the T^2; and 4 gauge fields from the KK reduction of the metric and the 2-form NS field, B_2 on the 1-cycles of the T^2. The duality group is $O(6,22,\mathbb{Z}) \times SL(2,\mathbb{Z})$. $O(6,22,\mathbb{Z})$ is the T-duality group of the Heterotic theory, and $SL(2,\mathbb{Z})$ is the S-duality symmetry of the 4 dimensional Heterotic theory.

A general state carries electric and magnetic charges with respect to these gauge fields. The electric charges, $\vec{Q}_e$, and the magnetic charges, $\vec{Q}_m$, take values in a lattice, $\Gamma^{6,22}$, which is even, self-dual and of signature, $(6,22)$. The lattice is invariant under the group, $O(6,22,\mathbb{Z})$. The electric and magnetic charges, $\vec{Q}_e, \vec{Q}_m$, transform as vectors of $O(6,22,\mathbb{Z})$. And together, $(\vec{Q}_e, \vec{Q}_m)$, transform as a doublet of $SL(2,\mathbb{Z})$. In a particular basis, $\{e_i\}$ of $\Gamma^{6,22}$, the matrix of inner products,

$$\eta_{ij} \equiv (e_i, e_j), \tag{1.1}$$

takes the form,

$$\eta = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{E}_8 \oplus \mathcal{E}_8 \oplus \mathcal{H} \oplus \mathcal{H}. \tag{1.2}$$

Here $\mathcal{H}$, is given by,

$$\mathcal{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.3}$$

and $\mathcal{E}_8$ is the Cartan matrix of E_8.

In this basis, the electric charge vector has components,

$$\vec{Q}_e = (q_0, -p^1, q_i, n_1, NS_1, n_2, NS_2). \tag{1.4}$$

Here, q_0 is the $D0$-brane charge; p^1 is the charge due to $D4$-branes wrapping $K3$; $q_i, i = 2, \cdots 23$ are the charges due to D2-branes wrapping the 22 2-cycles of $K3$ which we denote as C_i; n_1, n_2 are the momenta along the two 1-cycles of T^2 and NS_1, NS_2 are the charges due to NS_5 branes wrapping $K3 \times S^1$ where S^1 is one of the two 1-cycles of T^2.

And the magnetic charge vector has components,

$$\vec{Q}_m = (q_1, p^0, p^i, w_1, KK_1, w_2, KK_2). \tag{1.5}$$

Here, q_1 is the charge due to $D2$-branes wrapping T^2; p^0 is the $D6$-brane charge; $p^i, i = 2 \cdots 23$, are the charges due to $D4$-branes wrapping the cycle $\tilde{C}_i \times T^2$, where $\tilde{C}_i$ is the 2-cycle on K3 dual to C_i; w_1, w_2 are charges due to the winding modes of the fundamental string along the two 1-cycles of T^2; and KK_1, KK_2 are the KK-monopole charges that arise along the two 1-cycles of the T^2.

Three bilinears in the charges can be defined,

$$\begin{aligned} \vec{Q}_e^2 &\equiv (\vec{Q}_e, \vec{Q}_e) \\ \vec{Q}_m^2 &\equiv (\vec{Q}_m, \vec{Q}_m) \\ \vec{Q}_e \cdot \vec{Q}_m &\equiv (\vec{Q}_e, \vec{Q}_m). \end{aligned} \tag{1.6}$$

These are invariant under $O(6, 22, \mathbb{Z})$.

An invariant under the full duality group is,

$$I = (\vec{Q}_e)^2 (\vec{Q}_m)^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2. \tag{1.7}$$

It is quartic in the charges. For a big supersymmetric black hole, I is positive, and the entropy of the black hole[11] is,

$$S = \pi \sqrt{\vec{Q}_e^2 \vec{Q}_m^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2}. \tag{1.8}$$

In contrast, for a big non-supersymmetric extremal black hole, I is negative and the entropy is,

$$S = \pi \sqrt{(\vec{Q}_e \cdot \vec{Q}_m)^2 - \vec{Q}_e^2 \vec{Q}_m^2}. \tag{1.9}$$

We now turn to discussing the Cardy limit. Consider a Black hole carrying $D0 - D4$ brane charge. In our notation the non-zero charges are, $q_0, p^1, p^i, i =$

$2, \cdots 23$. This solution can be lifted to M-theory, and the near horizon geometry in M-theory is given by a BTZ black hole in $AdS_3 \times S^2$. The AdS_3 space-time admits a dual description in terms of a 1+1 dim. CFT living on its boundary. The central charge, C, of the CFT can be calculated from the bulk, it is determined by the curvature of the AdS_3 spacetime. For large charges we get,

$$C = 3|p^1 d_{ij} p^i p^j|, \tag{1.10}$$

where d_{ij} is the matrix η_{ij}, eq.(1.1), restricted to the 22 dimensional subspace of charges given by $D4$-branes wrapping two-cycles of $K3$ and T^2. This corresponds to the second, third and fourth factor of $\mathcal{H}$ and the two $\mathcal{E}_8$'s in eq.(3.27). In the Cardy limit the condition,

$$|q_0| \gg C, \tag{1.11}$$

is satisfied. The well known Cardy formula is

$$S = 2\pi \sqrt{\frac{C|q_0|}{6}}. \tag{1.12}$$

For a generic charge configuration, the central charge is

$$C = 3|p^1 \vec{Q}_m^2|, \tag{1.13}$$

and the Cardy limit is

$$I \gg 6(p^1)^2(\vec{Q}_m^2)^2. \tag{1.14}$$

Chapter 2

Spectrum of Dyons and Black Holes in CHL orbifolds using Borcherds Lift.

In this chapter, the degeneracies of supersymmetric quarter BPS dyons in four dimensions and of spinning black holes in five dimensions in a CHL compactification are computed exactly using Borcherds lift. The Hodge anomaly in the construction has a physical interpretation as the contribution of a single M-theory Kaluza-Klein 6-brane in the 4d-5d lift. Using factorization, it is shown that the resulting formula has a natural interpretation as a two-loop partition function of left-moving heterotic string, consistent with the heuristic picture of dyons in the M-theory lift of string webs.

2.1 Siegel Modular Forms of Level N

Let us recall some relevant facts about Siegel modular forms. Let Ω be the period matrix of a genus two Riemann surface. It is given by a (2×2) symmetric matrix with complex entries

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix} \tag{2.1}$$

satisfying

$$\mathrm{Im}\,(\rho) > 0, \quad \mathrm{Im}\,(\sigma) > 0, \quad \mathrm{Im}\,(\rho)\,\mathrm{Im}\,(\sigma) > \mathrm{Im}\,(\nu)^2, \tag{2.2}$$

and parameterizes the 'Siegel upper half plane' in the space of (ρ, ν, σ). There is a natural symplectic action on the period matrix by the group $Sp(2, \mathbb{Z})$ as follows. We write an element g of $Sp(2, \mathbf{Z})$ as a (4×4) matrix in a block-diagonal form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.3}$$

where A, B, C, D are all (2×2) matrices with integer entries. They satisfy

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I\,, \tag{2.4}$$

so that $g^t J g = J$ where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the symplectic form. The action of g on the period matrix is then given by

$$\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}. \tag{2.5}$$

The $Sp(2, \mathbb{Z})$ group is generated by the following three types of (4×4) matrices with integer entries

$$\begin{aligned}
g_1(a,b,c,d) &\equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad ad - bc = 1, \\
g_2 &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
g_3(\lambda, \mu) &\equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{2.6}$$

We are interested in a subgroup by $G_1(N)$ of $Sp(2, \mathbb{Z})$ generated by the matrices in (2.6) with the additional restriction

$$c = 0 \text{ mod } N, \quad a, d = 1 \text{ mod } N. \tag{2.7}$$

Note that with the restriction (2.7), the elements $g_1(a, b, c, d)$ generate the congruence subgroup $\Gamma_1(N)$ of $SL(2, \mathbb{Z})$ which is the reason for choosing the name $G_1(N)$ for the subgroup of $Sp(2, \mathbb{Z})$ in this case. From the definition of $G_1(N)$ it follows that if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_1(N), \tag{2.8}$$

then

$$C = \mathbf{0} \text{ mod } N, \quad \det A = 1 \text{ mod } N, \quad \det D = 1 \text{ mod } N \ . \tag{2.9}$$

One can similarly define $G_0(N)$ corresponding to $\Gamma_0(N)$ by relaxing the condition $a, d = 1$ mod N in (2.7).

We are interested in a modular form $\Phi_k(\Omega)$ which transforms as

$$\Phi_k[(A\Omega + B)(C\Omega + D)^{-1}] = \{\det (C\Omega + D)\}^k \Phi_k(\Omega), \tag{2.10}$$

for matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belonging to $G_1(N)$. We will actually construct modular forms of the bigger group $G_0(N)$. Such a modular form is called a Siegel modular form of level N and weight k. From the definition (3.6) it is clear that a product of two Siegel modular forms Φ_{k_1} and Φ_{k_2} gives a Siegel modular form $\Phi_{k_1+k_2}$. The space of modular forms is therefore a ring, graded by the integer k. The graded ring of Siegel Modular forms for $N = 1, 2, 3, 4$ is determined in a number of papers in the mathematics literature [12, 13, 14, 15, 16]. The special cases of our interest for the pairs (N, k) listed in the introduction were constructed explicitly in [5].

In the theory of Siegel modular forms, the weak Jacobi forms of genus one play a fundamental role. A weak Jacobi form $\phi_{k,m}(\tau, z)$ of $\Gamma_0(N)$ transforms under modular transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

as

$$\phi_{k,m}(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k \exp\,[\frac{2\pi imcz^2}{c\tau+d}]\,\phi_{k,m}(\tau,z). \tag{2.11}$$

and under lattice shifts as

$$\phi_{k,m}(\tau, z+\lambda\tau+\mu) = \exp\left[-2\pi im(\lambda^2\tau+2\lambda z)\right]\phi_{k,m}(\tau,z)\,, \quad \lambda,\mu\in\mathbb{Z}\,. \tag{2.12}$$

Furthermore, it has a Fourier expansion

$$\phi_{k,m}(\tau,z) = \sum_{n\geq 0,\, r\in\mathbb{Z}} c(4nm-r^2)q^n y^r. \tag{2.13}$$

The significance of weak Jacobi forms in this context stems from the fact that, with the transformation properties (2.11) and (2.12), the combination $\phi_{k,m}(\rho,\nu)\cdot\exp(2\pi im\sigma)$ transforms with weight k under the group elements $g_1(a,b,c,d)$ and $g_3(\lambda,\mu)$ in (2.6). Then, with some additional ingredients using the property (2.13), one can also ensure the required transformation properties under g_2 to obtain a Siegel modular form.

There are two methods for constructing a Siegel modular form starting with a weak Jacobi form which we summarize below.

- Additive Lift

  This procedure generalizes the Maaß-Saito-Kurokawa lift explained in detail for example in [17]. We refer to it as the 'additive' lift because it naturally gives the sum representation of the modular form in terms of its Fourier expansion. The starting 'seed' for the additive lift is in general a weak Jacobi form $\phi_{k,1}(\rho,\nu)$ of weight k and index 1. Let us denote the operation of additive lift by the symbol $\mathcal{A}[.]$. If a given weak Jacobi form $\phi_{k,1}$ results in a Siegel modular form Φ_k after the additive lift, then we can write

$$\Phi_k(\Omega) = \mathcal{A}[\phi_{k,1}(\rho,\nu)]. \tag{2.14}$$

  In the cases of our interest for the pairs (N,k) above, this procedure was used in [5] to obtain the modular forms Φ_k listed there. The seed in these cases can be expressed in terms of the unique cusp forms $f_k(\rho)$ of $\Gamma_1(N)$ of weight $(k+2)$,

$$f_k(\rho) = \eta^{k+2}(\rho)\eta^{k+2}(N\rho), \tag{2.15}$$

where $\eta(\rho)$ is the Dedekind eta function. The seed for the additive lift is then given by

$$\phi_{k,1}(\rho,\nu) = f_k(\rho)\,\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)}, \tag{2.16}$$

where $\theta_1(\rho,\nu)$ is the usual Jacobi theta function.

- Multiplicative Lift

  This procedure is in a sense a logarithmic version of the Maaß-Saito-Kurokawa lift. We call it 'multiplicative' because it naturally results in the Borcherds product representation of the modular form. The starting 'seed' for this lift is a weak Jacobi form $\phi_{0,1}^k$ of weight zero and index one and the superscript k is added to denote the fact after multiplicative lift it gives a weight k form Φ_k. Let us denote the operation of multiplicative lift by the symbol $\mathcal{M}[.]$. If a given weak Jacobi form $\phi_{0,1}^k$ results in a Siegel modular form Φ_k after the multiplicative lift, then we can write

$$\Phi_k(\Omega) = \mathcal{M}[\phi_{0,1}^k(\rho,\nu)]. \tag{2.17}$$

Given the specific Siegel modular forms $\Phi_k(\Omega)$ obtained by additively lifting the seeds $\phi_{k,1}$ in (2.16) for the pairs $(N,k) = (1,10),(2,6),(3,4),(7,1)$ as in [5], we would like to know if the same Siegel forms can be obtained as multiplicative lifts of some weak Jacobi forms $\phi_{0,1}^k$. Such a relation between the additive and the multiplicative lift is very interesting mathematically for if it exists, it gives a Borcherds product representation of a given modular form. However, to our knowledge, at present there are no general theorems relating the two. Fortunately, as we describe next, in the examples of interest to us, it seems possible to determine the seed for the multiplicative lift from the seed for the additive lift quite easily and explicitly. Finding such a multiplicative seed to start with is a nontrivial step and is not guaranteed to work in general. But if one succeeds in finding the multiplicative seed $\phi_{0,1}^{k}$ given a Φ_k obtained from the additive seeds $\phi_{k,1}$ in (2.16) then one can write

$$\Phi_k(\Omega) = \mathcal{A}[f_k(\rho)\,\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)}] = \mathcal{M}[\phi_{0,1}^{k}(\rho,\nu)]. \tag{2.18}$$

2.2 Multiplicative Lift

We now outline the general procedure for constructing modular forms $\Phi_k(\Omega)$ as a Borcherds product [18] by a multiplicative lift following closely the treatment in [14, 15, 16]

For the special pair $(1, 10)$, which results in the Igusa cusp form Φ_{10}, the product representation was obtained by Gritsenko and Nikulin [19, 20]. The starting seed for this lift is a weak Jacobi form $\phi_{0,1}^{10}$ of weight zero and index one

$$\phi_{0,1}^{10} = 8[\frac{\theta_2(\rho,\nu)^2}{\theta_2(\rho)^2} + \frac{\theta_3(\rho,\nu)^2}{\theta_3(\rho)^2} + \frac{\theta_4(\rho,\nu)^2}{\theta_4(\rho)^2}], \tag{2.19}$$

where $\theta_i(\rho,\nu)$ are the usual Jacobi theta functions. We therefore have in this case the desired result

$$\Phi_{10}(\Omega) = \mathcal{A}(\phi_{10,1}) = \mathcal{M}(\phi_{0,1}^{10}). \tag{2.20}$$

This weak Jacobi form happens to also equal the elliptic genus of **K3**. As a result, the multiplicative lift is closely related to the elliptic genus of the symmetric product of **K3** [21] which counts the bound states of the D1-D5-P system in five dimensions. This coincidence, which at first sight is purely accidental, turns out to have a deeper significance based on the 4d-5d lift [22].

We would now like find a similar product representation for the remaining pairs of (N, k) using the multiplicative lift so that we can then try to find a similar physical interpretation using 4d-5d lift. We first describe the general procedure of the multiplicative lift for the group $G_0(N)$ and then specialize to the illustrative case $(2, 6)$ of our interest, to obtain the product representation of Φ_6 using these methods.

As we have defined in 2.1, the group $G_0(N)$ consists of matrices with integer entries of the block-diagonal form

$$\{\begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in Sp(2,\mathbb{Z})\} \tag{2.21}$$

which contains the subgroup $\Gamma_0(N)$. A basic ingredient in the construction of Siegel modular forms is the Hecke operator T_t of $\Gamma_0(N)$ where t is an integer. The main property of our interest is that acting on a weak Jacobi form $\phi_{k,m}$ of weight k and index m, the Hecke operator T_t generates a weak Jacobi form $\phi_{k,mt} = T_t(\phi_{k,m})$ of weight k and index mt. Thus, on a modular form $\phi_{k,1}$, the Hecke operator T_t acts like a raising operator that raises the index by $(t-1)$ units. One subtlety that needs to be taken into account in the case of $\Gamma_0(N)$ that does not arise for $SL(2,\mathbb{Z})$ is the fact that $\Gamma_0(N)$ has multiple cusps in its fundamental domain whereas $SL(2,\mathbb{Z})$ has a unique cusp at $i\infty$. As a result, the Hecke operators that appear in the construction in this case are a little more involved as we review in 6.1 in the appendix. .

Let us now explain the basic idea behind the lift. Given a seed weak Jacobi form $\phi^k_{0,1}(\rho,\nu)$ for the multiplicative lift, we define

$$(L\phi^k_{0,1})(\rho,\nu,\sigma) = \sum_{t=1}^{\infty} T_t(\phi^k_{0,1})(\rho,\nu)\exp(2\pi i\sigma t). \tag{2.22}$$

Now, $T_t(\phi^k_{0,1})$ is a weak Jacobi form of weight 0 and index t. It then follows as explained in 2.1, with the transformation properties (2.11) and (2.12), the combination $T_t(\phi^k_{0,m})(\rho,\nu)\cdot\exp(2\pi it\sigma)$ is invariant under the group elements $g_1(a,b,c,d)$ and $g_3(\lambda,\mu)$ in (2.6). Thus, each term in the sum in (2.22) and therefor $L\phi$ is also invariant under these two elements.

If $L\phi$ were invariant also under the exchange of p and q then it would be invariant under the element g_2 defined in (2.6) and one would obtain a Siegel modular form of weight zero. This is almost true. To see this, we note that $\exp(L\phi^k_{0,1})$ can be written as an infinite product using the explicit representation of Hecke operators given in the appendix :

$$\prod_{\substack{l,m,n\in\mathbb{Z}\\ m>0}} (1-(q^n y^l p^m)^{n_s})^{h_s n_s^{-1} c_{s,l}(4mn-l^2)}, \tag{2.23}$$

where $q \equiv \exp(2\pi i\rho), y \equiv \exp(2\pi i\nu), p \equiv \exp(2\pi i\sigma)$ (6.15). In the product presentation (2.23), the coefficients $c_{s,l}(4mn - l^2)$ are manifestly invariant under the exchange of m and n. The product, however, is not quite symmetric because the range of the products in (2.23) is not quite symmetric: m is strictly positive whereas n can be zero. This asymmetry can be remedied by multiplying the product (2.23) by an appropriate function as in [16, 23]. The required function can essentially be determined by inspection to render the final product symmetric in p and q. Following this procedure one then obtains a Siegel modular form as the multiplicative lift of the weak Jacobi form $\phi^k_{0,1}(\rho, \nu)$,

$$\Phi_k(\Omega) = \mathcal{M}[\phi^k_{0,1}] = q^a y^b p^c \prod_{(n,l,m)>0} (1 - (q^n y^l p^m)^{n_s})^{h_s n_s^{-1} c_{s,l}(4mn - l^2)}, \tag{2.24}$$

for some integer b and positive integers a, c. Here the notation $(n, l, m) > 0$ means that if (i) $m > 0, n, l \in \mathbb{Z}$, or (ii) $m = 0, n > 0, l \in \mathbb{Z}$, or (iii) $m = n = 0, l < 0$.

It is useful to write the final answer for $\Phi_k(\Omega)$ as follows

$$\Phi_k(\Omega) \quad = \quad p^c H(\rho, \nu) \exp[L\phi^k_{0,1}(\rho, \nu, \sigma)], \tag{2.25}$$

$$\begin{aligned} H(\rho, \nu) \quad &= \quad q^a y^b \prod_s \prod_{l,n\geq 1} (1 - (q^n y^l)^{n_s})(1 - (q^n y^l)^{n_s})^{n_s^{-1} h_s c_{s,l}(-l^2)} && (2.26) \\ &\times \quad \prod_{n=1}^{\infty} (1 - q^{n n_s})^{n_s^{-1} h_s c_{s,l}(0)} \prod_{l<0}^{\infty} (1 - y^{l n_s})^{n_s^{-1} h_s c_{s,l}(-l^2)}, && (2.27) \end{aligned}$$

in terms of the separate ingredients that go into the construction. This rewriting is more suggestive for the physical interpretation, as we discuss in the next section. Following Gritsenko [24], we refer to the function $H(\rho, \nu)$ as the 'Hodge Anomaly'. The construction thus far is general and applies to the construction of modular forms of weight k which may or may not be obtainable by an additive lift. In many cases however, as in the cases of our interest, it might be possible to obtain the same modular form by using the two different lifts. To see the relation between the two lifts in such a situation and to illustrate the significance of the Hodge anomaly for our purpose, we next specialize to the case $(2, 6)$. We show how to determine the multiplicative seed and the Borcherds product given the specific Φ_6 obtained from the additive lift.

2.3 Multiplicative Lift for Φ_6

We want to determine the seed $\phi^6_{0,1}$ whose multiplicative lift equals Φ_6 constructed from the additive lift of (2.16). From the p expansion of the additive representation of Φ_6 we conclude that the integer c in (2.24) and (2.25) equals one. Then we see from (2.25) that if Φ_6 is to be a weight six Siegel modular form, $H(\rho, \nu)$ must be a weak Jacobi form of weight six and index one. Such a weak Jacobi form is in fact unique and hence must equal the seed $\phi_{6,1}$ that we used for the additive lift. In summary, the Hodge anomaly is given by

$$\begin{aligned} H(\rho,\nu) &= \phi_{6,1}(\rho,\nu) = \eta^2(\rho)\eta^8(2\rho)\theta_1^2(\rho,\nu) && (2.28) \\ &= qy(1-y^{-1})^2\prod_{n=1}^{\infty}(1-q^{2n})^8(1-q^n)^4(1-q^ny)^2(1-q^ny^{-1})^2. && (2.29) \end{aligned}$$

Comparing this product representation with (2.26), we determine that

$$c_1(0) = 4, \quad c_1(-1) = 2; \quad c_2(0) = 8, \quad c_2(-1) = 0; \tag{2.30}$$

and moreover $c_1(n) = c_2(n) = 0, \quad \forall\, n < -1$. This information about the leading coefficients $c_s(n)$ obtained from the Hodge anomaly is sufficient to determine completely the multiplicative seed $\phi^6_{0,1}$. Let us assume the seed to be a weak Jacobi form[1]. Now, proposition (6.1) in [16] states that the space of weak Jacobi forms of even weight is generated as linear combinations of two weak forms $\phi_{-2,1}$ and $\phi_{0,1}$ which in turn are given in terms of elementary theta functions by

$$\phi_{-2,1}(\rho,\nu) = \frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)} \tag{2.31}$$

$$\phi_{0,1}(\rho,\nu) = 4\Big[\frac{\theta_2(\rho,\nu)^2}{\theta_2(\rho)^2} + \frac{\theta_3(\rho,\nu)^2}{\theta_3(\rho)^2} + \frac{\theta_4(\rho,\nu)^2}{\theta_4(\rho)^2}\Big] \tag{2.32}$$

The coefficients for this linear combination can take values in the ring $A(\Gamma(N))$ of holomorphic modular forms of $\Gamma(N)$. Basically, the coefficients have to be

[1] Strictly, it is enough that it is a 'very weak' Jacobi form as defined in [16] but from the physical interpretation that we give in the next section, we expect and hence assume it to be a weak Jacobi form to find a consistent solution.

chosen so as to get the correct weight. For our case, with $N = 2$, the relevant holomorphic modular form, is the one of weight two

$$\alpha(\rho) = \theta_3^4(2\rho) + \theta_2^4(2\rho). \tag{2.33}$$

By virtue of the above-mentioned proposition, and using the definitions in (2.31) and (2.33), we can then write our desired seed as the linear combination

$$\phi_{0,1}^6(\rho,\nu) = A\alpha(\rho)\phi_{-2,1}(\rho,\nu) + B\phi_{0,1}(\rho,\nu), \tag{2.34}$$

where A and B are constants. To determine the constants we investigate the behavior near the cusps. For $\Gamma_0(2)$, there are only two cusps, one at $i\infty$ and the other at 0 in the fundamental domain which we label by $s = 1, 2$ respectively. Then the various relevant quantities required in the final expression (2.24) are given in our case by

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_1 = 1, \quad z_1 = 0, \quad n_1 = 1 \tag{2.35}$$

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad h_2 = 2, \quad z_2 = 1, \quad n_2 = 2. \tag{2.36}$$

The q expansion for $\phi_{-2,1}$ and $\phi_{0,1}$ at the cusp $q = 0$ is given by

$$\phi_{-2,1} = (-2 + y + y^{-1}) + q(-12 + 8y + 8y^{-1} - 2y^2 - 2y^{-2}) + ... \tag{2.37}$$

$$\phi_{0,1} = (10 + y + y^{-1}) + \ .... \tag{2.38}$$

The Fourier expansion of $\alpha(\rho)$ at the cusps $i\infty$ and 0 is given by,

$$\alpha(\rho) \;\; = \;\; 1 + 24q + 24q^2 + \ldots \tag{2.39}$$

near infinity and by

$$\rho^{-2}\alpha(-1/\rho) \;\; = \;\; -\frac{1}{2}\alpha(\frac{\rho}{2}) \tag{2.40}$$

$$= \;\; -\frac{1}{2} + \ldots \tag{2.41}$$

near zero. Demanding that the leading terms in the Fourier expansion of the linear combination (2.34) match with those given by (2.30) determines the coefficients $A = 4/3$ and $B = 2/3$ in (2.34). The constraints are actually over-determined so the fact that a solution exists at all gives a check of the procedure. Our final answer for the multiplicative lift is then

$$\phi^6_{0,1}(\rho,\nu) = \frac{4}{3}\alpha(\rho)\phi_{-2,1}(\rho,\nu) + \frac{2}{3}\phi_{0,1}(\rho,\nu). \tag{2.42}$$

With this determination we can simply apply the formalism in the previous section to determine

$$\Phi_6(\Omega) = \mathcal{M}[\frac{4}{3}\alpha(\rho)\phi_{-2,1}(\rho,\nu) + \frac{2}{3}\phi_{0,1}(\rho,\nu)] \tag{2.43}$$

by using the formula (2.24).

2.4 Physical Interpretation of the Multiplicative Lift

Both $\exp(-L\phi)$ and the inverse of the Hodge anomaly $H^{-1}(\rho,\nu)$ that appear in the multiplicative lift in (2.2) have a natural physical interpretation using the 4d-5d lift, which we discuss in this section and also in terms of M-theory lift of string webs which we discuss in the next section.

Let us recall the basic idea behind the 4d-5d lift [22]. Consider Type-IIA compactified on a five-dimensional space $\mathbf{M_5}$ to five dimensions. Given a BPS black hole in Type-IIA string theory in five dimensions, we can obtain a black hole in four dimensions as follows. A five-dimensional black hole situated in an asymptotically flat space $\mathbb{R}^4$ can be embedded into an asymptotically Taub-NUT geometry of unit charge. Intuitively, this is possible because near the origin, the Taub-NUT geometry reduces to $\mathbb{R}^4$, so when the Taub-NUT radius is much larger than the black hole radius, the black hole does not see the difference between $\mathbb{R}^4$ and Taub-NUT. Asymptotically, however, the Taub-NUT geometry is $\mathbb{R}^3 \times \mathbf{S^1_{tn}}$. We can dimensionally reduce on the Taub-NUT circle to obtain a four-dimensional compactification. Now, Type-IIA is dual to M-theory compactified on the M-theory circle $\mathbf{S}^1_m$ so we can regard four-dimensional theory as an M-theory compactification on $\mathbf{M_5} \times \mathbf{S^1_{tn}} \times \mathbf{S^1_m}$. Now flipping the two circles, we can choose to regard the Taub-NUT circle $\mathbf{S^1_{tn}}$ as the new M-theory circle. This in turn is dual to a Type-IIA theory but in a different duality frame than the original one. In this duality frame, the Taub-NUT space is just the Kaluza-Klein 6-brane of M-theory dual to the D6-brane. Thus the Taub-NUT charge of the original Type-IIA frame is interpreted in as the D6 brane charge in the new Type-IIA frame and we obtain a BPS state in four dimensions with a D6-brane charge. Since we can go between the two descriptions by smoothly varying various moduli such as the Taub-NUT radius and choosing appropriate duality frames, the spectrum of BPS states is not expected to change. In this way, we relate the spectrum of four-dimensional BPS states with D6-brane charge to five-dimensional BPS states in Type-IIA.

With this physical picture in mind, we now interpret the term $\exp(-L\phi)$ in (2.25) as counting the degeneracies of the five dimensional BPS states that correspond to the four-dimensional BPS states after the 4d-5d lift. For example, in the familiar case $(1, 10)$ of toroidally compactified heterotic string, the dual Type-II theory is compactified on $\mathbf{K3} \times \mathbf{\tilde{S}^1} \times \mathbf{S^1}$. In the notation of the previous paragraph, we then have $\mathbf{M_5} = \mathbf{K3} \times \mathbf{\tilde{S}^1}$. The five-dimensional BPS state is described by the D1-D5-P system. Its degeneracies are counted by the elliptic genus of the symmetric product of $\mathbf{K_3}$. In this case, indeed $\exp(-L\phi)$ above gives nothing but the symmetric product elliptic genus evaluated in [21].

In our case $(2, 6)$, D-brane configuration in five dimensions corresponding to our dyonic state in four dimensions is obtained simply by implementing the CHL orbifolding action in the open string sector on the D1-D5-P system in five dimensions. The term $\exp(-L\phi)$ in (2.25) then has a natural interpretation as a symmetric product elliptic genus. Because of the shift in the orbifolding action, the resulting orbifold is a little unusual and the elliptic genus is weak Jacobi form not of $SL(2, \mathbb{Z})$ but of $\Gamma_0(2)$.

The Hodge anomaly plays a special role in the 4d-5d lift. It is naturally interpreted as the contribution of the bound states of momentum and the single

Taub-NUT 5-brane in the Type-IIB description. A KK5-brane of IIB wrapping $\mathbf{K_3} \times \mathbf{S^1}$ carrying momentum along the $\mathbf{S^1}$ is T-dual to an NS5-brane of IIA wrapping $\mathbf{K_3} \times \mathbf{S^1}$ carrying momentum which in turn is dual to the heterotic fundamental string wrapping the circle with momentum.[1]. These can be counted in perturbation theory [26, 27, 28, 29] in both cases $(1, 10)$ and $(2, 6)$. The $y(1 - y^{-1})$ term in the Hodge anomaly in (2.29) is more subtle and would require a more detailed analysis.

2.5 M-theory lift of String Webs

The appearance in the dyon counting formulae of objects related to a genus two Riemann surface such as the period matrix and the $G_0(N)$ subgroups of $Sp(2, \mathbb{Z})$ is quite surprising and demands a deeper physical explanation. We now offer such an explanation combining earlier work of [30] and [22] in the toroidal $(1, 10)$ case and generalizing it to CHL orbifolds.

To start with, let us reinterpret the Hodge anomaly following Kawai [30]. It can be written as

$$H(\rho, \nu) = \eta^8(\rho)\eta^8(2\rho)\frac{\theta_1^2(\rho, \nu)}{\eta^6(\rho)} = Z(\rho)K^2(\rho, \nu), \tag{2.44}$$

where $Z(\rho) \equiv \eta^8(\rho)\eta^8(2\rho)$ is the one-loop partition function of the left-moving chiral 24-dimensional bosonic string with the $\mathbb{Z}_2$ twist α of the CHL orbifold action, and $K(\rho, \tau)$ is the prime form on the torus. Let us also expand

$$\exp(-L\phi_{0,1}^6(\rho, \nu)) = \sum_{N=0}^{\infty} p^N \chi_N \tag{2.45}$$

[1] In [25], the Hodge anomaly for the $(1, 10)$ example is interpreted as a single 5-brane contribution. This, however, is not dual to the heterotic F1-P system and would not give the desired form of the Hodge anomaly. For the purposes of 4d-5d lift, it is essential to introduce Taub-NUT geometry which appears like KK5-brane in IIB. In the 5d elliptic genus the bound states of this KK5-brane and momentum are not accounted for. Therefore, the Hodge anomaly is naturally identified as this additional contribution that must be taken into account.

We can then write from (2.25),

$$\begin{aligned}
\frac{1}{\Phi_6(\Omega)} &= \frac{1}{p}\frac{1}{H(\rho,\nu)}\exp(-L\phi^6_{0,1}(\rho,\nu)) && (2.46)\\
&= \sum_{N=0}^{\infty} p^{N-1}\frac{1}{K(\rho,\nu)^2}\chi_N && (2.47)\\
&\sim \frac{1}{p}\frac{1}{K(\rho,\nu)^2}\frac{1}{Z(\rho)} + \dots && (2.48)
\end{aligned}$$

In (2.48) above, we can identify $K^{-2}(\rho,\nu)$ as the on-shell (chiral) tachyon propagator, and $Z(\rho)$ as the one-loop left-moving partition function. If we denote by X the bosonic spacetime coordinate, then we have

$$< e^{ik\cdot X}(\nu)e^{-ik\cdot X}(0) >= K^{-2}(\rho,\nu), \tag{2.49}$$

where k is the momentum of an on-shell tachyon and the correlator is evaluated on a genus one Riemann surface with complex structure ρ. This is exactly the first term in a factorized expansion where the subleading terms at higher N denoted by $\dots$ in (2.48) come from contributions of string states at higher mass-level $N-1$. Summing over all states gives the genus two partition function.

This reinterpretation of $1/\Phi_6$ as the two-loop partition function of the chiral bosonic string explains at a mathematical level the appearance of genus two Riemann surface generalizing the results of Kawai to the $(2,6)$ case. Note that the partition function $Z(\rho)$ will be different in the two cases. In the (1, 10) case it equals $\eta^{-24}(\rho)$ and in the $(2,6)$ case it equals $\eta^{-8}(\rho)\eta^{-8}(2\rho)$. This precisely captures the effect of CHL orbifolding on the chiral left moving bosons of the heterotic string. To describe the $N=2$ orbifold action let us consider the $E_8\times E_8$ heterotic string. The orbifold twist α then simply flips the two E_8 factors. We can compute the partition function with a twist in the time direction $\mathrm{Tr}(\alpha q^H)$ where H is the left-moving bosonic Hamiltonian. Then, the eight light-cone bosons will contribute $\eta^{-8}(\rho)$ as usual to the trace, but the sixteen bosons of the internal $E_8\times E_8$ torus will contribute $\eta^{-8}(2\rho)$ instead of $\eta^{-16}(2\rho)$. The power changes from -16 to -8 because eight bosons of the first E_8 factor are mapped by α to the eight bosons of the second E_8. Thus only those states that have equal number of oscillators from the two E_8 factors contribute to the trace, thereby reducing effectively the number of oscillators to 8. The argument on the other hand is doubled to 2ρ because when equal number of oscillators from the two factors are present, the worldsheet energy is effectively doubled. The tachyon propagator in the two cases is unchanged because only light-cone bosons appear on shell which are not affected by the orbifolding.

This mathematical rewriting does not explain at a fundamental level why the chiral bosonic string has anything to do with dyon counting. This connection can be completed using with the heuristic picture suggested in [7].

Under string-string duality, the $SL(2,\mathbb{Z})$ S-duality group of the heterotic string gets mapped to the geometric $SL(2,\mathbb{Z})$ of the Type-IIB string [31, 32]. Thus, electric states correspond to branes wrapping the a cycle of the torus and magnetic states correspond to branes wrapping the b cycle of the torus. A general dyon with electric and magnetic charges (n_e, n_m) of a given $U(1)$ symmetry is then represented as a brane wrapping (n_e, n_m) cycle of the torus. If a state is charged under more than one $U(1)$ fields then one gets instead a (p, q) string web with different winding numbers along the a and the b cycles. The angles and lengths of the web are fixed by energetic considerations for a given charge assignment [33, 34]. For our purpose, we can consider D5 and NS5 branes wrapping the **K3** resulting in two different kinds of $(1, 0)$ and $(0, 1)$ strings. A dyon in a particular duality frame then looks like the string web made of these strings as in the first diagram. In the M-theory lift of this diagram, both D5 and NS5 branes correspond to M5 branes so the string in the web arises from M-theory brane wrapping **K3**. To count states, we require a partition function with Euclidean time. Adding the circle direction of time we can fatten the string web diagram which looks like a particle Feynman diagram into a genus-two Riemann surface representing a closed-string Feynman diagram. Now, **K3**-wrapped M5 brane is nothing but the heterotic string. Furthermore, since we are counting BPS states by an elliptic genus, the right-movers are in the ground state and we are left with the two-loop partition function of the bosonic string. This partition function is what we have constructed above using factorization.

Chapter 3

Comments on the Spectrum of CHL Dyons.

In this chapter based on [6], we address a number of puzzles relating to the proposed formulae for the degeneracies of dyons in orbifold compactifications of the heterotic string to four dimensions with $\mathcal{N} = 4$ supersymmetry. The partition function for these dyons is given in terms of Siegel modular forms associated with genus-two Riemann surfaces. We point out a subtlety in demonstrating S-duality invariance of the resulting degeneracies and give a prescription that makes the invariance manifest. We show, using M-theory lift of string webs, that the genus-two contribution captures the degeneracy only if a specific irreducibility criterion is satisfied by the charges. Otherwise, in general there can be additional contributions from higher genus Riemann surfaces. We analyze the negative discriminant states predicted by the formula. We show that even though there are no big black holes in supergravity corresponding to these states, there are multi-centered particle-like configurations with subleading entropy in agreement with the microscopic prediction and our prescription for S-duality invariance. The existence of the states is moduli dependent and we exhibit the curves of marginal stability and comment on its relation to S-duality invariance.

Let Ω be a (2×2) symmetric matrix with complex entries

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \tag{3.1}$$

satisfying

$$(\mathrm{Im}\rho) > 0, \quad (\mathrm{Im}\sigma) > 0, \quad (\mathrm{Im}\rho)(\mathrm{Im}\sigma) > (\mathrm{Im}v)^2 \tag{3.2}$$

which parameterizes the 'Siegel upper half plane' in the space of (ρ, v, σ). It can be thought of as the period matrix of a genus two Riemann surface. For a genus-two Riemann surface, there is a natural symplectic action of $Sp(2, \mathbb{Z})$ on the period matrix. We write an element g of $Sp(2, \mathbb{Z})$ as a (4×4) matrix in the block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.3}$$

where A, B, C, D are all (2×2) matrices with integer entries. They satisfy

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I\,, \tag{3.4}$$

so that $g^t J g = J$ where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the symplectic form. The action of g on the period matrix is then given by

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}. \tag{3.5}$$

The object of our interest is a Siegel modular form $\Phi_k(\Omega)$ of weight k which transforms as

$$\Phi_k[(A\Omega + B)(C\Omega + D)^{-1}] = \{\det(C\Omega + D)\}^k \Phi_k(\Omega), \tag{3.6}$$

under an appropriate congruence subgroup of $Sp(2, \mathbb{Z})$ [5]. The subgroup as well as the index k of the modular form are determined in terms of the order N of the particular CHL $\mathbb{Z}_N$ orbifold one is considering [5]. In a given CHL model, the inverse of the Φ_k is to be interpreted then as a partition function of dyons.

To see in more detail how the dyon degeneracies are defined in terms of the partition function, let us consider for concreteness the simplest model of toroidally compactified heterotic string as in the original proposal of Dijkgraaf, Verlinde, Verlinde [4]. Many of the considerations extend easily to the more general orbifolds with $\mathcal{N} = 4$ supersymmetry. In this case the relevant modular form is the well-known Igusa cusp form $\Phi_{10}(\Omega)$ of weight ten of the full group $Sp(2,\mathbb{Z})$. A dyonic state is specified by the charge vector $Q = (Q_e, Q_m)$ which transforms as a doublet of the S-duality group $SL(2,\mathbb{R})$ and as a vector of the T-duality group $O(22,6;\mathbb{Z})$. There are three T-duality invariant quadratic combinations Q_m^2, Q_e^2, and $Q_e \cdot Q_m$ that one can construct from these charges. Given these three combinations, the degeneracy $d(Q)$ of dyonic states of charge Q is then given by

$$d(Q) = g\left(\frac{1}{2}Q_m^2,\, \frac{1}{2}Q_e^2,\, Q_e \cdot Q_m\right) , \tag{3.7}$$

where $g(m,n,l)$ are the Fourier coefficients of $1/\Phi_{10}$,

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m\geq -1, n\geq -1, l} e^{2\pi i(m\rho+n\sigma+lv)} g(m,n,l)\,. \tag{3.8}$$

The parameters (ρ,σ,v) can be thought of as the chemical potentials conjugate to the integers $\left(\frac{1}{2}Q_m^2,\, \frac{1}{2}Q_e^2,\, Q_e \cdot Q_m\right)$ respectively. The degeneracy $d(Q)$ obtained this way satisfies a number of physical consistency checks. For large charges, its logarithm agrees with the Bekenstein-Hawking-Wald entropy of the corresponding black holes to leading and the first subleading order [4, 5, 35, 36, 37]. It is integral as expected for an object that counts the number of states. It is formally S-duality invariant [4, 5] but as we will see in the next section the formal proof is not adequate. An appropriate prescription is necessary as we explain in detail in the next section which also allows for a nontrivial moduli dependence.

3.1 S-Duality Invariance

The first physical requirement on the degeneracy $d(Q)$ given by (3.7) is that it should be invariant under the S-duality group of the theory. For the simplest case of toroidal compactification that we are considering, the S-duality group is $SL(2,\mathbb{Z})$ and more generally for $\mathbb{Z}_N$ CHL orbifolds its a congruence subgroup

$\Gamma_1(N)$ of $SL(2,\mathbb{Z})$. So, we would like to show for the $N=1$ example, that the degeneracy (3.7) is invariant under an S-duality transformation

$$Q_e \to Q'_e = aQ_e + bQ_m , \qquad Q_m \to Q'_m = cQ_e + dQ_m , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) . \tag{3.9}$$

A formal proof of S-duality following [4, 5] proceeds as follows. Inverting the relation (3.8) we can write

$$d(Q) = \int_{\mathcal{C}} d^3\Omega \, e^{-i\pi Q'^t \cdot \Omega \cdot Q} \, \frac{1}{\Phi_{10}(\Omega)} \tag{3.10}$$

where the integral is over the contours

$$0 < \rho \le 1, \quad 0 < \sigma \le 1, \quad 0 < v \le 1 \tag{3.11}$$

along the real axes of the three coordinates (ρ, σ, v). This defines the integration curve $\mathcal{C}$ as a 3-torus in the Siegel upper half plane. Now we would like to show

$$d(Q') = \int_{\mathcal{C}} d^3\Omega \, e^{-i\pi Q'^t \cdot \Omega \cdot Q'} \, \frac{1}{\Phi_{10}(\Omega)} \tag{3.12}$$

equals $d(Q)$. To do so, we define

$$\Omega' \equiv \begin{pmatrix} \rho' & v' \\ v' & \sigma' \end{pmatrix} = (A\Omega + B)(C\Omega + D)^{-1}, \tag{3.13}$$

for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix} \in Sp(2,\mathbb{Z}) . \tag{3.14}$$

We can change the integration variable from Ω to Ω'. Using these transformation properties and the modular properties of Φ_{10} we see that

$$d^3\Omega' = d^3\Omega \, , \tag{3.15}$$

$$\Phi_{10}(\Omega') = \Phi_{10}(\Omega) \, , \tag{3.16}$$

$$Q'^t \cdot \Omega' \cdot Q' = Q^t \cdot \Omega \cdot Q \tag{3.17}$$

Moreover, the integration contour $\mathcal{C}$ as defined in (3.11) is invariant under the duality transformation on the integration variables (3.13). We therefore conclude

$$d(Q') = \int_{\mathcal{C}} d^3\Omega'\, e^{-i\pi Q'^T \cdot \Omega' Q'} \frac{1}{\Phi_{10}(\Omega')} = d(Q)\,. \tag{3.18}$$

This formal proof is however not quite correct. The reason is that the partition function $1/\Phi_{10}$ has a double pole at $v = 0$ which lies on the integration contour $\mathcal{C}$. Thus the integral in (3.10) is not well-defined on the contour $\mathcal{C}$ and one must give an appropriate prescription for the integration. The non-invariance can also be seen explicitly from the Fourier expansion. To illustrate the point, let us look at states with

$$\frac{1}{2}Q_m^2 = -1, \quad \frac{1}{2}Q_e^2 = -1, \quad Q_e \cdot Q_m = N. \tag{3.19}$$

Then according to (3.10), the degeneracy of such states can be read off from the coefficient of y^N/qp in the Fourier expansion (3.8). From the product representation of Φ_{10} given for example in [4], we see that we need to pick the term that goes as $p^{-1}q^{-1}y^N$ in the expansion of

$$\frac{1}{qp(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2} = \frac{y}{qp}\frac{1}{(1-y)^2} = \sum_{N=1}^{\infty} N q^{-1} p^{-1} y^N \tag{3.20}$$

which implies that

$$d(-1,-1,N) = N. \tag{3.21}$$

Let us now look at what is required for invariance under $SL(2,\mathbb{Z})$ transformations. Consider, for example, the element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.22}$$

of the S-duality group which takes $(Q_e,\, Q_m)$ to $(Q_m, -Q_e)$. Hence $(\frac{1}{2}Q_m^2,\, \frac{1}{2}Q_e^2,\, Q_e \cdot Q_m)$ goes to $(\frac{1}{2}Q_e^2,\, \frac{1}{2}Q_m^2,\, -Q_e \cdot Q_m)$. Invariance of the spectrum under this element of the S-duality group would then predict $d(-1,-1,-N) = d(-1,-1,N) = N$. However, from the expansion (3.20) we see that there are no terms in the Laurent expansion that go as y^{-N} and hence an application of the formulae (3.10) and (3.8) would give $d(-1,-1,-N) = 0$ in contradiction with the prediction from S-duality.

This apparent lack of S-duality invariance is easy to fix with a more precise prescription. Note that the function $(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{-2}$ has a $\mathbb{Z}_2$ symmetry generated by the element S of the S-duality group that takes y to y^{-1}. Under this transformation the contour $|y| < 1$ is not left invariant but instead gets mapped to the contour $|y| > 1$. The new contour cannot be deformed to the original one without crossing the pole at $y = 1$ so if we are closing the contour around $y = 0$ then we need to take into account the contribution from this pole at $y = 1$. Alternatively, it is convenient to close the contour at $y^{-1} = 0$ instead of $y = 0$. Then we do not encounter any other pole and because of the symmetry of the function $(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{-2}$ under y going to y^{-1}, the Laurent expansion around y has the same coefficients as the Laurent expansion around y^{-1}. We then get,

$$\frac{1}{pq(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2} = \frac{y^{-1}}{pq} \frac{1}{(1 - y^{-1})^2} = \sum_{N=1}^{\infty} N p^{-1} q^{-1} y^{-N}. \tag{3.23}$$

If we now define $d(-1, -1, -N)$ as the coefficient of qpy^{-N} in the expansion (3.23) instead of in the expansion (3.20) then $d(-1, -1, -N) = N = d(-1, -1, N)$ consistent with S-duality.

States with negative N must exist if states with positive N exist not only to satisfy S-duality invariance but also to satisfy parity invariance. The $\mathcal{N} = 4$ super Yang-Mills theory is parity invariant. Under parity, our state with positive N goes to a state with negative N and the asymptotic values χ of the axion also flips sign at the same time. Hence if a state with N positive exists at $\chi = \chi_0$ then a state with N negative must exist at $\chi = -\chi_0$. Thus, the naive expansion (3.20) would give an answer inconsistent with parity invariance and one must use the prescription we have proposed, to satisfy parity invariance. Note that even though S-duality and parity both take the states $(-1, -1, N)$ to $(-1, -1, -N)$ they act differently on the moduli fields.

In either case, the important point is that to extract the degeneracies in an S-duality invariant way, we need to use different contours for different charges. The function $1/\Phi_{10}$ has many more poles in the (ρ, σ, v) space at various divisors that are the $Sp(2, \mathbb{Z})$ images of the pole at $y = 1$ and in going from one contour to the other these poles will contribute. Instead of specifying contours, a more practical way to state the prescription is to define the degeneracies $d(Q)$ by formulae (3.10) and (3.8) first for charges that belong to the 'fundamental cell' in the charge lattice satisfying the condition $\frac{1}{2}Q_m^2 \geq -1$, $\frac{1}{2}Q_e^2 \geq -1$, and $Q_e \cdot Q_m \geq 0$. For these charges $d(Q)$ can be represented as a contour integral for a contour of integration around $p = q = y = 0$ that avoids all poles arising as images of $y = 1$. This can be achieved by allowing (ρ, v, σ) to all have a large positive imaginary part as noted also in [38]. For other charges, the degeneracy is defined by requiring invariance under $SL(2, \mathbb{Z})$. The degeneracies so defined are manifestly S-duality invariant. This statement of S-duality invariance might appear tautologous, but its consistency depends on the highly nontrivial fact that an analytic function defined by $\Phi_{10}(\rho, \sigma, v)$ exists that is $SL(2, \mathbb{Z})$ invariant. Its pole structure guarantees that one gets the same answer independent of which way the contour is closed.

The choice of integration contour is possibly related to moduli dependence of the spectrum. To see this let us understand in some detail what precisely is required for S-duality. Given a state with charge Q that exists for the values of the moduli φ, the statement of S-duality only requires that the degeneracy $d(Q)$ at φ be the same as the degeneracy $d(Q')$ at φ' where Q' and φ' are S-duality transforms of Q and φ respectively. In many cases, one can then invoke the BPS property to assume that as we move around in the moduli space, barring phase transitions, the spectrum can be analytically continued from φ' to φ to deduce $d(Q') = d(Q)$ at φ. This argument is known to work perfectly for half-BPS states in theories with $\mathcal{N} = 4$ supersymmetric but with lower supersymmetry or for quarter-BPS states in $\mathcal{N} = 4$, generically there can be curves of marginal stability. In such cases, states with charges Q' may exist for moduli values φ' but not for φ and similarly states with charges Q may exist for moduli values φ but not for φ'. Therefore, there are two possibilities for extracting the dyon degeneracies.

- There are no curves of marginal stability in the dilaton-axion moduli space. In this case if two charge configurations Q and Q' are related by S-duality, then $d(Q) = d(Q')$.

- There are curves of marginal stability in the dilaton-axion moduli space. In this case one can say at most that $d(Q)$ at φ equals $d(Q')$ at φ'.

We will return later to a further discussion of these possibilities in the present context after considering explicit examples of moduli dependence and lines of marginal stability in 3.3.

3.2 Irreducibility Criterion and Higher Genus Contributions

One way to derive the dyon partition function is to use the representation of dyons as string webs wrapping the $\mathbf{T^2}$ factor in Type-IIB on $\mathbf{K3} \times \mathbf{T^2}$. Using M-theory lift, the partition function that counts the holomorphic fluctuations of this web can be related to the genus-two partition function of the left-moving heterotic string [3, 7, 39]. The appearance of genus-two is thus explained by the topology of the string web. Such a derivation immediately raises the possibility of contribution from higher genus Riemann surface because string webs with more complicated topology are certainly possible. In this section we address this question and show that the genus-two partition function correctly captures the dyon degeneracies if the charges satisfy certain irreducibility criteria. Otherwise, there are higher genus corrections to the genus-two formula.

There are various derivations of the dyon degeneracy formula, but often they compute the degeneracies for a specific subset of charges, and then use duality invariance to extend the result to generic charges. Such an application of duality invariance assumes in particular that under the duality group $SO(22, 6, \mathbb{Z})$ the only invariants built out of charges would be Q_e^2, Q_m^2, and $Q_e \cdot Q_m$. This assumption is incorrect. If two charges are in the same orbit of the duality group, then obviously they have the same value for these three invariants. However the converse is not true. In general, for arithmetic groups, there can be discrete invariants which cannot be written as invariants of the continuous group.

An example of a non-trivial invariant that can be built out of two integral charge vectors is $I = gcd(Q_e \wedge Q_m)$, *i.e.*, the gcd of all bilinears $Q_e^i Q_m^j - Q_e^j Q_m^i$.

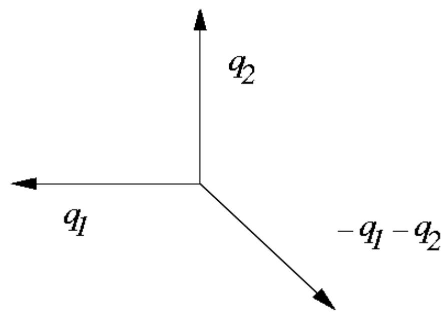

Figure 3.1: Charge conservation at a string junction

Our goal is to show that the genus-two dyon partition function correctly captures the degeneracies if $I = 1$. Note that half-BPS states have $I = 0$ and hence are naturally associated with a genus-one surface. If $I > 1$, then there are additional zero modes for the dyon under consideration and it would be necessary to correctly take them into account for counting the dyons.

The essential idea is to represent quarter-BPS states in the Type-IIB frame as a periodic string network wrapped on the two-torus. Type-IIB compactified on a **K3** has a variety of half-BPS strings that can carry a generic set of $(21, 5)$ charges arising from D5 and NS5 branes wrapped on the **K3**, D3-branes wrapped on some of the $(19, 3)$ two-cycles as well as D1 and F1-strings. Several half-BPS strings can join into a web that preserve a quarter of the supersymmetries [33, 34, 40, 41]. The supersymmetry condition requires that the strings lie in a plane, and that their central charge vectors also lie in a plane. The strands must be oriented at relative angles that mimic the relative angle of their central charge vectors. The condition on the angles between strings guarantees the balance of tensions at the junction of three strands of the web as shown in Fig.3.1.

The central charges are given in terms of the charges and the scalar moduli of the theory as $Z = Tq$. The matrix T contains the scalar moduli of the theory, that parameterize the way a vector in the $\Gamma^{(21,5)}$ Narain lattice of charges decomposes into a left-moving and a right-moving part. The five-dimensional right-moving part is the vector of central charges for the string. For generic values of the scalar moduli, one does not expect to have tensionless strings. Hence it follows that $TQ = 0$ implies $Q = 0$. The condition that all central charges TQ_i should lie in a plane, $TQ_i = a_i TQ_1 + b_i TQ_2$ is then equivalent to $Q_i = a_i Q_1 + b_i Q_2$, *i.e.*, the charge vectors Q_i of all strings should also lie in a plane. A periodic string web can be wrapped on the torus of a $\mathbf{K3} \times \mathbf{T^2}$ compactification as shown in Fig.3.2.

After compactification on the torus, the strands of the web can carry additional charges: momentum along the direction they wrap, and KK monopole charge for the circle they do not wrap. The charges are organized in a $(22, 6)$ charge vector. The result is a quarter-BPS dyon in the four dimensional theory. A dyon with generic charges Q_e, Q_m typically has a very simple realization as a web with three strands. A simple possible choice of charges on the strands would be Q_e, Q_m, $Q_e + Q_m$. This web comes from the periodic identification of a hexagonal lattice. As the shape of the T^2 or the moduli in T change, the size of one strand may become zero, and the web may degenerate into two cycles of the torus meeting at a point. On the other side, of the transition the intersection will open up in the opposite way and the configuration then smoothly become a new three-strands web. For example, the web with strands Q_e, Q_m, $Q_e + Q_m$ may go to a web with strands Q_e, Q_m, $Q_e - Q_m$. This process has interesting consequences on the stability of certain BPS states, that will be reviewed in 3.3.

It has been argued [7] that the partition function of supersymmetric ground states for such webs can be computed by an unconventional lift to M-theory that relates it to a chiral genus-two partition function of the heterotic string. The genus-two partition function computed using this lift for CHL-orbifolds [3, 39] indeed equals the dyon partition function obtained by other means.

A priori, the string junction need not to be stable against opening up into more complicated configurations. For example, a strand may split into two or more parallel strands, or the junction may open up into a triangle. Any complicated periodic network made out of strands with charges that are linear combinations of Q_e and Q_m, and such that the total charge flowing across one side of the fundamental cell is Q_e, and through the other side Q_m will give a possible realization of

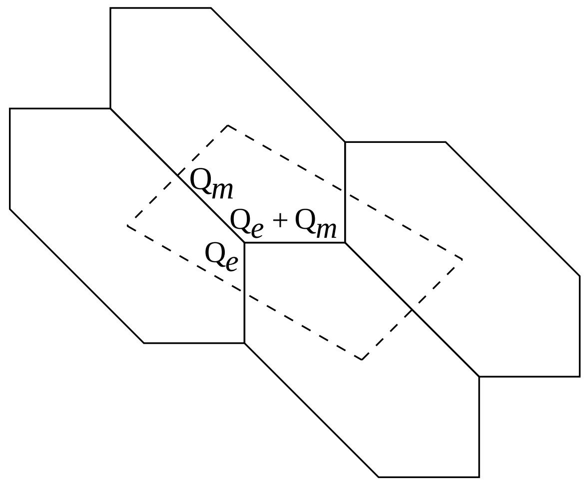

Figure 3.2: A quarter-BPS dyon carrying irreducible charges Q_e and Q_m with gcd($Q_e \wedge Q_m$)= 1

the dyon as shown in Fig.3.3. If that is possible, the M-theory lift would predict a more complicated expression for the dyon degeneracies. For simplicity, in the following analysis we restrict to configurations with no momentum or KK charge.

To understand the relation between the value of I and the possible variety of string webs that may describe a dyon with given charges, it is useful to consider a graph in the space of charges that is topologically dual to the string web. A dual graph is constructed as follows. For every face of the web associate a vertex in the dual graph. If two faces A and B in the web share an edge then the corresponding vertices A' and B' in the dual graph are connected by a vector that is equal in magnitude to the central charge of the edge but rotated by $\pi/2$ in orientation compared to the edge. Recall that each edge in the string web carries a central charge and that the relative angles between the edges mimic the angles between the corresponding central charge vectors. A junction has three faces and three edges which maps to a triangle in the dual graph with three vertices and three edges. Charge conservation at each junction means that the vector sum of the three edge vectors is zero. This then guarantees that the sides of the dual triangle actually close, as their vector sum is zero. A string web constructed from

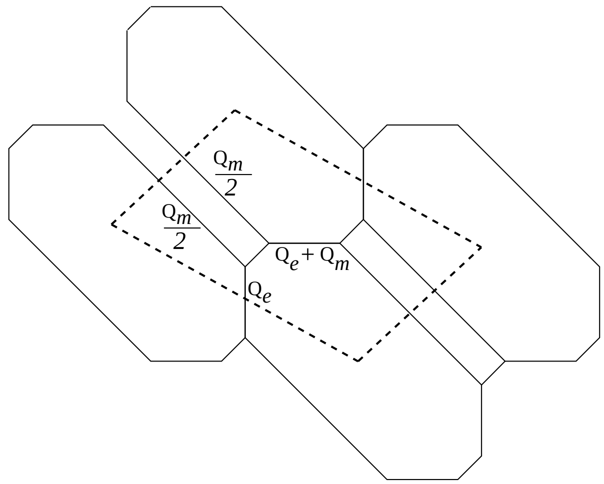

Figure 3.3: A quarter-BPS dyon carrying reducible charges Q_e and Q_m with gcd($Q_e \wedge Q_m$)=2

a period array of junctions then corresponds to a triangulation in the dual graph.

Now, the vertices of the dual graph will sit at integral points of the charge lattice, on the plane defined by the vectors Q_e and Q_m. The graph will have a fundamental cell with sides Q_e and Q_m. Our invariant I counts the number of integral points inside the fundamental cell. In this dual description, it is clear geometrically that $Q_e^i Q_m^j - Q_e^j Q_m^i$ are the various components of the area 2-form associated with the fundamental cell. If all the components do not have common factor then the fundamental parallelogram does not have any integral points either on the edges or inside [42].

Let us see in more detail that I counts the number of integral points inside the fundamental cell. If all $Q_e^i Q_m^j - Q_e^j Q_m^i$ are multiples of I, then consider any vector Q such that $Q \cdot Q_e$ is not a multiple of I. If such a vector does not exist, then Q_e is a multiple of I and there are extra integral points on the edges of the parallelogram. If on the other hand, such a vector exists, then $\frac{Q \cdot Q_e}{I} Q_m^i - \frac{Q \cdot Q_m}{I} Q_e^i$ is an integral charge vector that is a linear combination of Q_e and Q_m with fractional coefficients. Up to shifts by Q_e and Q_m it will lie inside the parallelogram. Conversely, if the lattice of integral points that are coplanar with Q_e and Q_m has a smaller fundamental cell than the parallelogram with sides

Q_e and Q_m, then $Q_e = aQ_1 + bQ_2$, $Q_m = cQ_1 + dQ_2$, $ad - bc > 1$. There will be $ad - bc$ points inside the parallelogram, and as $Q_e \wedge Q_m = (ad - bc) Q_1 \wedge Q_2$, $I = ad - bc$ is the number of points inside the parallelogram with sides Q_e and Q_m.

We thus see that if $I > 1$, then the fundamental cell in the dual graph has an internal integral point. Each of the internal points can be used as a vertex for a triangulation. A generic periodic triangulation subdivides a fundamental cell into at most $2I$ faces. [1] In the dual description, a string web on the torus that carries charges Q_e and Q_m will have at most $2I$ three-strand junctions, and I faces.

To put it differently, note that $I = 1$ without any internal faces is a genus two surface after M-theory lift. Adding a face increases the genus by one. Hence the genus of a M-theory lift of a string web with the invariant I will be a surface with genus $I + 1$.

When a face opens up at a string junction, its size is a zero mode in that the mass of dyon is independent of the size of the additional face. These zero modes and the invariant I have been discussed earlier in a related context of quarter-BPS dyons in field theory using their realization as string junctions going between a collection D3-branes [43]. In that context, the zero mode is one of the collective coordinates that must be quantized to determine the ground state wavefunction. Similar comments might apply in our case. More work is need to obtain a definite interpretation of the higher genus contribution.

3.3 States with Negative Discriminant

An important test of the dyon degeneracy formula is that for large charges, the logarithm of the predicted degeneracy $\log d(Q)$ matches with Bekenstein-Hawking entropy. To make this comparison, for a given a BPS dyonic state with electric and magnetic charges (Q_e, Q_m), one would like to find a supersymmetric black

[1] This follows from Euler formula on the torus: a triangulation has $3/2$ edges for each face, hence the number of vertices is $1/2$ the number of faces.

hole solution of the effective action with the same charges and mass and then compute its entropy. It is useful to define the 'discriminant' Δ by

$$\Delta(Q) = Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2. \tag{3.24}$$

which is the unique quartic invariant of the full U-duality group $SO(22, 6; \mathbb{Z}) \times SL(2, \mathbb{Z})$. For a black hole with charges (Q_e, Q_m), the attractor value of the horizon area is proportional to the square root of the discriminant and the entropy is given by

$$S(Q) = \pi \sqrt{\Delta(Q)} \ . \tag{3.25}$$

On the microscopic side also, the discriminant is a natural quantity. It is useful to think of $SL(2, \mathbb{Z})$ as $SO(1, 2; \mathbb{Z})$ which has a natural embedding into $Sp(2, \mathbb{Z}) \sim SO(2, 3; \mathbb{Z})$. The dyon degeneracy formula depends on the T-duality invariant vector of $SO(1, 2; \mathbb{Z})$

$$\begin{pmatrix} Q_m^2/2 \\ Q_e^2/2 \\ Q_e \cdot Q_m \end{pmatrix} \tag{3.26}$$

The discriminant is the norm of this vector with the Lorentzian metric

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.27}$$

With this norm, for a given state $(Q_m^2/2,\ Q_e^2/2,\ Q_e \cdot Q_m)$,the vector (3.26) is spacelike, lightlike, or timelike depending on whether Δ is positive, zero, or negative. We can accordingly refer to the state as spacelike, lightlike, or timelike.

Clearly, to obtain a physically sensible, nonsingular, supersymmetric, dyonic black hole solution in supergravity, it is necessary that the discriminant defined in (3.24) is positive and large so that the entropy defined in (3.25) is real. The vector in (3.26) in this case is spacelike. This fact seems to lead to the following puzzle regarding the dyon degeneracy formula. The formula predicts a large number of states that can have vanishing or negative discriminant. Since there are no big black holes in supergravity in that case, there does not appear to be a supergravity realization of these states predicted dyon degeneracy. This raises the following question. *Do the lightlike and timelike states predicted by the dyon degeneracy formula actually exist in the spectrum and if so what is their macroscopic realization?* It is important to address this question to determine the range of applicability of the dyon degeneracy formula.

3.3.1 Microscopic Prediction

To start with, let us emphasize that the lightlike or timelike states are not necessarily pathological even though there is no supergravity solution corresponding to them. The simplest example of a lightlike state is the half-BPS purely electric state in the heterotic frame with winding w along a circle and momentum n along the same circle [26, 27]. For such a state, $Q_e^2 = 2nw$ is nonzero but since it carries no magnetic charge, both Q_m^2 and $Q_e \cdot Q_m$ are zero and hence the discriminant is zero. The supergravity solution is singular but higher derivative corrections generate a horizon with the correct entropy [44, 45, 46]. We would like to know if similarly there exist quarter-BPS states that are timelike or lightlike in accordance with the predictions of the dyon degeneracy formula and what their supergravity realization is.

In general, it is not easy to extract closed form asymptotics from the degeneracy formula in this regime when the discriminant is negative or zero. But we have already encountered a simple example of a timelike state in 3.1. The states with $(Q_m^2/2, Q_e^2/2, Q_e \cdot Q_m)$ equal to $(-1, -1, N)$ have discriminant $1 - N^2$ which can be arbitrarily negative and we have determined the degeneracy of this state to be $d(-1, -1, N) = N$. Do such states exist in the physical spectrum, and if so what is their supergravity realization that can explain the degeneracy?

It is easy to construct such a state from a collection of winding, momentum, KK5, NS5 states in heterotic description. We choose a convenient representative that makes the supergravity analysis in the following section simpler. We consider heterotic string compactified on $\mathbf{T^4} \times \mathbf{S^1} \times \mathbf{\tilde{S}^1}$. Let the winding and momentum around the circle S^1 be w and n and around the circle $\mathbf{\tilde{S}^1}$ be $\tilde{w}$ and $\tilde{n}$. Similarly, K and W are the KK-monopole and NS5-brane charges associated with the circle

$\mathbf{S^1}$ whereas $\tilde{K}$ and $\tilde{W}$ are the KK-monopole and NS5-brane charges associated with the circle $\tilde{S}^1$. Note that the state with charge W can be thought of as an NS5 brane wrapping along $\mathbf{T^4} \times \mathbf{\tilde{S}^1}$ whereas the states with charges $\tilde{W}$ is wrapping along $\mathbf{T^4} \times \mathbf{S^1}$. While the state that magnetically dual to n is K in terms of Dirac quantization condition, the state that is S-dual to n is W. Similar comment holds for other states. With this notation, we then choose the charges $\Gamma = (Q_e|Q_m) = (n, w; \tilde{n}, \tilde{w}|W, K; \tilde{W}, \tilde{K})$ to be

$$\Gamma = (1, -1; 0, N|0, 0; 1, -1). \tag{3.28}$$

This state clearly has $(Q_m^2/2, Q_e^2/2, Q_e \cdot Q_m) = (-1, -1, N)$. We will show in the appendix 3.3.2 that the supergravity solution corresponding to this state with the required degeneracy has two centers instead of one. One center is purely electric with charge vector

$$\Gamma_1 = (1, -1; 0, N|0, 0; 0, 0), \tag{3.29}$$

and the other purely magnetic with charge vector

$$\Gamma_2 = (0, 0; 0, 0|0, 0; 1, -1), \tag{3.30}$$

both separated by a distance L. The corresponding supergravity solution exists for charge configuration with a positive, nonzero value for the distance L both for positive and negative N in a large regions of the moduli space but not for all values of the moduli. We discuss the explicit solution and as well as the moduli dependence and lines of marginal stability in the next subsections.

It is easy to see that such a two-centered solution has the desired degeneracy in agreement with the prediction from the dyon partition function. Each center individually contributes no entropy because for example the electric center by itself has $Q_e^2/2 = -1$ and hence carries no left-moving oscillations. However, because the charges are not mutually local, there is a net angular momentum $j = N/2$ in the electromagnetic field. For large N, the angular momentum multiplet has $2j + 1$ or N states in agreement with the dyon degeneracy formula. We thus see that at least some of the states with negative discriminant predicted by the dyon degeneracy formula can be realized physically but as multi-centered configurations.

3.3.2 Supergravity Analysis

For the supergravity analysis of the dyonic configurations, it is convenient to use the $\mathcal{N} = 2$ special geometry formalism. Consider Type-II string compactified on a Calabi-Yau threefold with Hodge numbers $(h^{1,1}, h^{2,1})$ which results in a $\mathcal{N} = 2$ supergravity in four dimensions with $h^{1,1}$ vector multiplets and $h^{2,1} + 1$ hypermultiplets. The hypermultiplets will not play any role in our analysis. The vector multiplet moduli space is a special Kähler geometry parameterized by $h^{1,1} + 1$ complex projective coordinates $\{X^I\}$ with $I = 0, 1, \ldots, h^{1,1}$ and $\{\lambda X^I\} \sim \{X^I\}$. The low energy effective action of the vector multiplets is completely summarized by a prepotential which is a homogeneous function $F(X^I)$ of degree two,

$$F(\lambda X^I) = \lambda^2 F(X^I). \tag{3.31}$$

In particular, the Kähler potential $\mathcal{K}$ is determined in terms of the prepotential by

$$e^{-\mathcal{K}} = i(\bar{X}^I F_I - X^I \bar{F}_I), \tag{3.32}$$

where

$$F_I = \frac{\partial F}{\partial X^I}. \tag{3.33}$$

In our case, since we have a special Calabi-Yau $\mathbf{K3} \times \mathbf{T^2}$, we actually get $\mathcal{N} = 4$ supersymmetry which has two additional gravitini multiplets. With our charge assignment, the vector fields in the gravitini multiplets are not excited and we can restrict our attention to the $\mathcal{N} = 2$ sector. In the heterotic frame, we have excited electric and magnetic charges (3.28) which couple only to gauge fields associated with the $\mathbf{T^2}$ part and to the metric and the dilaton-axion. As a result, the sigma model corresponding to the black hole configuration in $\mathbf{R^4}$ is completely factorized into the $\mathbf{T^4}$ conformal field theory and the sigma model involving $\mathbf{T^2} \times \mathbf{R^4}$ parts. This implies that for analyzing our charge configuration, we can restrict our attention to the moduli fields associated with $\mathbf{T^2}$ and the dilaton-axion. The prepotential in this case can be chosen to be

$$F(X^I) = -\frac{X^1 X^2 X^3}{X^0}, \tag{3.34}$$

which corresponds to the so called STU model. Here

$$S = X^1/X^0 = a + ie^{-2\Phi} \tag{3.35}$$

is the dilaton-axion field, where a is the axion and Φ is the dilaton in the heterotic frame. Similarly $T = X^2/X^0$ is the complex structure modulus of the $\mathbf{T^2}$ and $U = X^3/X^0$ is the Kähler modulus of the $\mathbf{T^2}$ in the heterotic frame. All other moduli fields do not vary in the geometry corresponding to our charge configuration. Restricting to the STU model greatly simplifies the analysis. Indeed this motivates the choice of the charges as in (3.28).

Given the prepotential (3.34) specifying the special geometry, there is a natural symplectic action $Sp(4,\mathbb{R})$ on (X^I, F_I). Similarly, the charges (p^I, q_I) transform as a symplectic vector. These charges are more naturally defined in the Type-IIA frame, where q_I are the electric charges arising from D0-brane and wrapped D2-branes, and p^I are the magnetic charges arising from D6-brane and wrapped D4-branes.

A general supersymmetric multi-centered dyonic solution has a metric of the form

$$-e^{-2G(\vec{r})}(dt + \omega_i dx^i)^2 + e^{2G(\vec{r})}(dr^2 + r^2 d\Omega_2^2). \tag{3.36}$$

The four complex moduli fields X^I that solve the equations of motion are determined in terms of the function G and harmonic functions H^I and H_I by the eight real equations

$$\begin{aligned} e^{-G}(X^I - \bar{X}^I) &= H^I(\vec{r}) &(3.37)\\ e^{-G}(F_I - \bar{F}_I) &= H_I(\vec{r}), &(3.38) \end{aligned}$$

in the gauge

$$e^{-\mathcal{K}} = \frac{1}{2} \tag{3.39}$$

with the Kähler potential given by (3.32). For a configuration with s charge centers with charges $\Gamma_a = (p^I_a, q_{Ia}), a = 1, \ldots s$ localized at the centers $\vec{r} = \vec{r}_a$ respectively, the harmonic functions H_I and H^I are given by [47]

$$H^I = h^I + \sum_{a=1}^{s} \frac{p^I_a}{|\vec{r} - \vec{r}_a|}, \qquad H_I = h_I + \sum_{a=1}^{s} \frac{q_{Ia}}{|\vec{r} - \vec{r}_a|}. \tag{3.40}$$

The constants of integration h_I and h^I will be determined in terms of the moduli fields shortly. Let $\Sigma(Q)$ be the entropy of the black hole which in our case equals $\pi\sqrt{\Delta(Q)}$. Then geometry of the solution is completely determined in terms of these harmonic functions [47]. The moduli are given by

$$\frac{X^A}{X^0} = \frac{\partial_A \Sigma(H) - iH^A}{\partial_0 \Sigma(H) - iH^0} \tag{3.41}$$

with $A = 1, 2, 3$ and $\partial_A = \partial/\partial H^A$. The metric is given by

$$e^{-2G} = \Sigma(H), \tag{3.42}$$

$$\nabla \times \omega = H^I \nabla H_I - H_I \nabla H^I. \tag{3.43}$$

Taking divergence of both sides then implies the Denef's constraint [48]

$$H^I \nabla^2 H_I - H_I \nabla^2 H^I = 0. \tag{3.44}$$

This is a consistency condition for a solution with s centers to exist, where ∇^2 is the flat space Laplacian in $\mathbf{R^3}$. This implies the following s equations

$$(h_I p_a^I - h^I q_{Ia}) + \sum_{b=1}^{s} \frac{(p_a^I q_{Ib} - q_{Ia} p_b^I)}{|\vec{r}_a - \vec{r}_b|} = 0, \tag{3.45}$$

where sum over repeated I index is assumed. Summing over the index a in the equation above gives the summed constraint

$$(h_I p^I - h^I q_I) = 0, \tag{3.46}$$

where $p^I = \sum p_a^I$ and $q_I = \sum q_{Ia}$ are the total charges.

The values of the moduli fields $S = S_1 + iS_2$, $T = T_1 + iT_2$ and $U = U_1 + iU_2$ at asymptotic infinity are specified by six real constants. The solutions on the other hand are determined by eight real constants of integration $(h^I, h_I), I = 0, 1, 2, 3$ which however must satisfy two real constraints (3.39) and (3.46). Thus, they can be determined in terms of the six asymptotic values of the moduli fields and the complete supersymmetric solution for all fields is then determined by (3.37), (3.36), and (3.40).

Specializing to our case, we will consider a two-centered solution so $s = 1, 2$. We restrict ourselves to a region of moduli space where $\mathbf{T^2}$ is factorized into two circles $\mathbf{S^1} \times \tilde{\mathbf{S}}^{\mathbf{1}}$ and there is no B field on the torus. In other words, we work on the submanifold of the moduli space with $T_1 = U_1 = 0$. Let R_1 and R_2 be the radii of the circles $\mathbf{S^1}$ and $\tilde{\mathbf{S}}^{\mathbf{1}}$ respectively, χ be the asymptotic expectation value of the axion, and g^2 be the string coupling given by the asymptotic value of $e^{2\Phi}$. A nonzero value of χ will be essential to obtain a well defined solution. Given this asymptotic data

$$S_\infty = \chi + \frac{i}{g^2}, \qquad T_\infty = i\frac{R_1}{R_2}, \qquad U_\infty = iR_1R_2, \tag{3.47}$$

we now proceed to determine the constants of integration (h^I, h_I).

At asymptotic infinity, $G(\vec{r})$ vanishes, so the solutions (3.37) reduce to

$$2\mathrm{Im}(X^I) = h^I, \qquad 2\mathrm{Im}(F_I) = h_I. \tag{3.48}$$

Let $X^0_\infty = \alpha + i\beta$. Then from (3.37) and (3.47) we see that the constants of integration are given by

$$h^0 = 2\beta \qquad h_0 = -2\alpha\frac{R_1^2}{g^2} - 2R_1^2\beta\chi \tag{3.49}$$

$$h^1 = 2\alpha\frac{1}{g^2} + 2\beta\chi \qquad h_1 = 2\beta R_1^2 \tag{3.50}$$

$$h^2 = 2\alpha\frac{R_1}{R_2} \qquad h_2 = 2\beta\frac{R_1R_2}{g^2} - 2\alpha\chi R_1R_2 \tag{3.51}$$

$$h^3 = 2\alpha R_1R_2 \qquad h_3 = 2\beta\frac{R_1}{R_2g^2} - 2\alpha\chi\frac{R_1}{R_2}\,. \tag{3.52}$$

The two constants α and β that we have introduced are in turn determined in terms of the charges by plugging (3.49) into the two constraint equations (3.39) and (3.46). Equation (3.39) in particular implies

$$|X^0|^2 = \alpha^2 + \beta^2 = \frac{1}{16S_2T_2U_2} = \frac{g^2}{16R_1^2}. \tag{3.53}$$

3.3.3 Moduli Dependence and Lines of Marginal Stability

So far our analysis is valid for any charge assignment but with the specific choice of the asymptotic moduli as in (3.47). The remaining equations (3.46) as well as (3.49) depend on the specific charge assignment of the configuration under study. To use the attractor equations to analyze the geometry for our charge configuration (3.29) and (3.30), we first translate the charges given in the heterotic frame to purely D-brane charges in the Type-IIA frame. The charges (p^I, q_I) in the Type-IIA arise from various D-branes wrapping even-cycles. We label charges so that q_0 is the number of D0-branes, q_1 is the number of D2-branes wrapping the $\mathbf{T^2}$, q_2 is the number of D2-branes wrapping a 2-cycle Σ_2 in $\mathbf{K3}$, q_3 is the number of D2-branes wrapping a 2-cycle $\tilde{\Sigma}_2$ that has intersection number one with the cycle Σ_2. Similarly, p^0 is the number of D6-branes wrapping $\mathbf{K3} \times \mathbf{T^2}$, p^1, p^2, p^3 are the number of D4-branes wrapping $\mathbf{K3}$, $\tilde{\Sigma}_2 \times \mathbf{T^2}$ and $\Sigma_2 \times \mathbf{T^2}$ respectively. By the duality chain in appendix B, these charges in the Type-IIA frame are related to the electric and magnetic charges (Q_e, Q_m) in the heterotic frame by

$$Q_e = (n, w; \tilde{n}, \tilde{w}) \equiv (q_0, -p^1, q_2, q_3) \tag{3.54}$$

$$Q_m = (W, K; \tilde{W}, \tilde{K}) \equiv (q_1, p^0, p^3, p^2)\,. \tag{3.55}$$

Now we are ready to apply the $\mathcal{N} = 2$ formalism to our two-centered configuration with the charge assignment (3.29) and (3.30). The electric center has charges

$$\Gamma_1 = (1, -1, 0, N|, 0, 0, 0, 0) \tag{3.56}$$

and the magnetic center has charges

$$\Gamma_2 = (0, 0, 0, 0|0, 0, 1, -1). \tag{3.57}$$

The constraint (3.46) then reads

$$h_1 - h^0 - Nh^3 + h_3 - h_2 = 0\,. \tag{3.58}$$

Substituting the values of the integration constants h^I and h_I in terms of α and β from (3.49) into this equation, we obtain one equation for the two unknowns α and β in terms of charges and asymptotic moduli

$$\beta(R_1^2 - 1 + \frac{R_1}{R_2 g^2}(1 - R_2^2)) + \alpha(-NR_1R_2 - \chi\frac{R_1}{R_2} + \chi R_1 R_2) = 0 \tag{3.59}$$

Combining this with the second equation (3.53) that comes from the gauge fixing constraint $e^{-\mathcal{K}} = \frac{1}{2}$ (3.39), we can now solve for the two unknowns to obtain

$$\alpha = \frac{(R_1^2 - 1 + \frac{R_1}{R_2 g^2}(1 - R_2^2))g}{4R_1(NR_1R_2 + \chi\frac{R_1}{R_2} - \chi R_1R_2)\Lambda}, \qquad \beta = \frac{g}{4R_1\Lambda}\,, \tag{3.60}$$

where

$$\Lambda^2 = 1 + \left(\frac{R_1^2 - 1 + \frac{R_1}{R_2 g^2}(1 - R_2^2)}{-NR_1R_2 - \chi\frac{R_1}{R_2} + \chi R_1 R_2}\right)^2 . \tag{3.61}$$

We have thus determined the integrations constants (3.49) that appear in the solution (3.40) completely in terms of the asymptotic moduli and the charges. The geometry of the solution is in tern determined entirely in terms of the harmonic functions. In particular the separation L between the two centers can be obtained by solving Denef's constraint (3.45), which for our configuration becomes

$$h_2 - h_3 = \frac{N}{L} \tag{3.62}$$

we have,

$$2\frac{R_1}{R_2}({R_2}^2 - 1)(\frac{\beta}{g^2} - \alpha\chi) = \frac{N}{L} \tag{3.63}$$

Since L is the separation between the two centers, it must be positive. This requires that $(\frac{\beta}{g^2} - \alpha\chi)$ must be positive. It is clear that this can be ensured for a large region of moduli space. The locus in the moduli space where this quantity becomes negative determines the line of marginal stability in the upper half S plane by the equation

$$\frac{1}{g^2} - (\frac{R_1^2 - 1 + \frac{R_1}{R_2 g^2}(1 - R_2^2)}{NR_1R_2 + \chi\frac{R_1}{R_2} - \chi R_1 R_2})\chi = 0, \tag{3.64}$$

which simplifies to

$$\chi = N\frac{R_1 R_2}{R_1^2 - 1}\frac{1}{g^2}. \tag{3.65}$$

This equation defines a straight line in the complex S_∞ plane with $S_\infty = \chi + i/g^2$. Note that the slope of the line is proportional to N. For fixed R_1 and R_2, this defines a curve of marginal stability in the complex S_∞ plane. For positive N, the desired two-centered solution exists if $\chi + i/g^2$ lies to the left of the line defined by the equation (3.65). In this region, the distance between the two centers determined by Denef's constraint (3.63) is positive and finite. After crossing the line of marginal stability, the solution ceases to exist because then there is no solution with positive L to the constraint (3.63). As one approaches the line of marginal stability from the left, the distance L between the electric and magnetic centers goes to infinity. In other words, the total state with charge vector Γ decays into two fragments with charge vectors Γ_1 and Γ_2. The mass M of the state with charge Γ is given in terms of the central charge by the BPS formula $M = |Z(\Gamma)|$ with

$$Z = e^{\mathcal{K}/2}(p^I F_I - q_I X^I). \tag{3.66}$$

At the curve of marginal stability, it is easy to check that $Z(\Gamma) = Z(\Gamma_1) + Z(\Gamma_2)$. Hence the state with charge vector Γ can decay into its fragments with charge vectors Γ_1 and Γ_2 by a process that is marginally allowed by the energetics and charge conservation.

Similarly, if N is negative, the straight line defined by (3.3.3) has negative slope and a solution with positive L exists only to the right of this line. As we have noted, the S-transformation maps the configuration with N positive to N negative. Hence the line with positive slope gets mapped to a line with negative slope and thus the curves of marginal stability move under S-duality. The fact that a two centered solution exists for both signs and with the correct degeneracy is consistent with our prescription for extracting an S-duality invariant spectrum. In the wedge between the two lines defined the two lines of marginal stability for N positive and N negative, both states coexist. In other regions, only one or the other state exists.

The simplicity of the line of marginal stability defined by (3.65) has a simple and beautiful interpretation from the string web picture. Indeed a string web made out of strands with certain charges exists only if these charges can be carried by a supersymmetric string in six dimensions. If one crosses a line of degeneration in the moduli space, across which a strand with charges, say, $Q_e + Q_m$ shrinks to zero length and is replaced by a strand with charge $Q_e - Q_m$, the quarter-BPS state will decay if no supersymmetric string with charge $Q_e - Q_m$ exists. The line of degeneration is simply the line at which a string of charge Q_e along one cycle of the torus and a string of charge Q_m along the other can be simultaneously supersymmetric. This is equivalent to the requirement that the phase of S is the same as the angle between the central charge vectors for Q_e and Q_m, that defines a straight line in the S plane. In the present case $Q_e = (1, -1, 0, N)$ and $Q_m = (0, 0, 1, -1)$, hence $Q_e \pm Q_m = (1, -1, \pm 1, N \mp 1)$. $\frac{1}{2}(Q_e \pm Q_m)^2 = -1 \pm N$, but a BPS string with charge Q must have $Q^2/2 \geq -1$. Hence the line of degeneration of the string web is indeed a line of marginal stability.

It is not surprising that the existence of quarter-BPS dyons depends on the moduli and that there are lines of marginal stability which separate the regions where the state exists from where it does not exist. This phenomenon is well known in the field theory context [49]. Moduli dependence of the spectrum of quarter-BPS dyons and the lines of marginal stability in the present string-theoretic context have been observed and analyzed from a different perspective also in the forthcoming publication [50].

Chapter 4

Degeneracy of Decadent Dyons.

A quarter-BPS dyon in $\mathcal{N} = 4$ super Yang-Mills theory is generically 'decadent' in that it is stable only in some regions of the moduli space and decays on submanifolds in the moduli space. Using this fact, and from the degeneracy of the system close to the decay, a new derivation for the degeneracy of such dyons is given. The degeneracy obtained from these very simple physical considerations is in precise agreement with the results obtained from index computations in all known cases. Similar considerations apply to dyons in $\mathcal{N} = 2$ gauge theories. The relation between the $\mathcal{N} = 4$ field theory dyons and those counted by the Igusa cusp form in toroidally compactified heterotic string is elucidated.

In this chapter, we consider the exact degeneracies of quarter-BPS dyons in $\mathcal{N} = 4$ supersymmetric gauge theories. For a gauge group of rank r, the gauge group is broken to $U(1)^r$ on the Coulomb branch which is $6r$-dimensional for $\mathcal{N} = 4$. At a generic point in this Coulomb branch moduli space, there is a rich spectrum of such dyons in this theory whose degeneracy is known exactly in many cases from index computations and vanishing theorems as well as from direct computations. Unlike the half-BPS dyons in $\mathcal{N} = 4$ gauge theories which are stable in all regions of the moduli space, these dyons exist as stable single particle states only in some regions of moduli space. These dyons are prone to decay, or are 'decadent', on certain submanifolds of the moduli space, which can be of real codimension one or higher in $\mathcal{N} = 4$ theories. We would like to know how 'degenerate' these decadent dyons are.

The stability criterion for the decadent dyons follows from the usual considerations of charge and energy conservation using the BPS mass formula. For a dyon of electric charge vector Q and magnetic charge vector P we denote the total charge vector by $\Gamma = [Q; P]$. The BPS mass formula then gives the mass M of such a state, in the $\mathcal{N} = 2$ notation, by the relation

$$M = |Z(\Gamma)| \,. \tag{4.1}$$

where $Z(\Gamma)$ is the central charge that depends on the moduli fields and linearly on the charge vector Γ . If the dyon with charge Γ decays into two dyons with smaller charges Γ_1 and Γ_2 then one has $Z(\Gamma) = Z(\Gamma_1) + Z(\Gamma_2)$ which by triangle inequality implies that

$$M = |Z(\Gamma)| \leq |Z(\Gamma_1)| + |Z(\Gamma_2)| = M_1 + M_2. \tag{4.2}$$

Hence, by energy conservation, the only way the decay can proceed is if M becomes equal to $M_1 + M_2$ at some point in the moduli space saturating the bound above. In $\mathcal{N} = 2$ theories, this defines a codimension one surface or a 'wall' in the moduli space. On one side of the wall where $M < M_1 + M_2$, the dyon with charge Γ is stable. At the wall it is marginally unstable and decadent. Upon crossing the wall it no longer exists as a single particle stable state. In $\mathcal{N} = 4$ theories, there are more than one central charges which have to be aligned for the decay to occur and hence the submanifold of decadence can have codimension one or higher. As a result, this submanifold is not a wall since one can just avoid it by going around it and access other regions of the moduli space. It is therefore more accurate it to call it the 'surface of decadence' in the $\mathcal{N} = 4$ case.

Given such a dyon of charge Γ that is stable in some region of the moduli space, we would like to know its degeneracy $\Omega(\Gamma)$ in that region. One can compute it applying standard methods of semiclassical quantization of solitons in gauge theories, viewing the dyon as a charged excitation of a monopole system. Collective coordinate quantization then reduces the problem of computing the degeneracy $\Omega(\Gamma)$ to counting the number of eigenvalues of the Hamiltonian of supersymmetric quantum mechanics of the bosonic and fermionic collective coordinates. This counting problem then becomes roughly equivalent to a cohomological problem of counting harmonic forms on the monopole moduli space which can be handled using index formulae[1]. Applying these methods, the degeneracy of quarter-BPS dyons has been computed by Stern and Yi [51, 52, 53, 54] for a special class of charge assignments. The same formula has been derived from another quiver dynamics in [55].

We will give here a new derivation of the degeneracy of these dyons using a very simple physical argument that makes use of the fact that the dyons are decadent near the surface of decadence. We will utilize the known degeneracies of half-BPS dyons and an argument similar to the one used by Denef and Moore [56] in their discussion of the wall-crossing formula. The results are in perfect agreement with the known degeneracies of Stern-Yi dyons computed using much more sophisticated techniques mentioned above. Moreover, this method can be naturally generalized to more complicated charge assignments as well as to arbitrary gauge groups giving predictions for situations that have not hitherto been considered using the index methods.

This chapter is organized as follows.After we derive the degeneracies of these dyons from their behavior near the surface of decadence, we present the basic physical argument, discuss the case of $SU(3)$ Stern-Yi dyons and of $SU(N)$ Stern-Yi dyons and show that the degeneracy obtained using these arguments precisely agrees with the results known in these cases from index computations both in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ case. We then discuss the relation of these dyons to the dyons in the field theory limit of string theory. Finally, we explain in particular, why only some of the decadent dyons considered here are accounted for by the partition function for string theory dyons given by the inverse of the Igusa cusp form [4, 6, 7, 22, 25, 38, 57].

[1]For quarter-BPS dyons, unlike in the half-BPS dyons, the problem is a little more subtle involving a potential on the moduli space as discussed in [51].

4.1 Computing degeneracies near the surface of decadence

For simplicity and also for comparison with known results, we consider in this section dyons in $SU(N)$ gauge theories but these considerations are more general and would apply to other groups.

It is well known that dyons in $SU(N)$ gauge theory have a nice geometric realization in terms of (p,q) strings stretching between N D3-branes. The low energy world volume theory of N D3-branes is a $U(N)$ Yang-Mills theory with $\mathcal{N}=4$ supersymmetry. Factoring out an overall center-of-mass $U(1)$ degree of freedom, one obtains an $SU(N)$ gauge theory. Simple roots of $SU(N)$ are $\{\alpha_i\}$ with $i=1,\ldots N-1$ with the usual Cartan inner product $\alpha_i\cdot\alpha_i=2$, $\alpha_i\cdot\alpha_j=-1$ for $i=j\pm 1$, and 0 otherwise. Giving expectation values to the six Higgs scalars in the adjoint representation corresponds to placing the D-branes at non-coincident positions in the transverse $\mathbb{R}^6$ space which breaks the gauge symmetry to $U(1)^{N-1}$.

Consider a dyon with electric charge Q and magnetic charge P expanded in the basis of simple roots as

$$Q=q_i\alpha_i, \quad P=p_i\alpha_i. \tag{4.3}$$

If the electric and magnetic charge vectors are parallel to each other then the dyonic configuration preserves half the supersymmetries. Since it breaks eight supersymmetries, there are four complex fermionic zero modes for the center of mass motion giving rise to a 16-dimensional ultra-short multiplet. If the electric and magnetic charge vectors are nonparallel, the dyon preserves only a quarter of the supersymmetries. Since now it breaks twelve supersymmetries, there are six complex fermionic zero modes for the center of mass motion giving rise to a 64-dimensional short multiplet. In $\mathcal{N}=2$ theories by contrast, in both cases, the dyon is half-BPS and there are four broken supersymmetries. Hence there are always two complex fermionic zero modes giving rise to a 4-dimensional half-hypermultiplet for the center of mass motion.

4.1.1 Basic physical argument

Let us first summarize the argument for $\mathcal{N} = 2$ dyons of the type considered by Stern and Yi [51]. Given a dyon with charge $\Gamma = [Q; P]$ of the Stern-Yi type, we would like to compute its degeneracy in a region of moduli space where it exists. Now, as we will discuss in the next sections, there exist surfaces of decadence for such a dyon where it decays into two dyons with charges $\Gamma_1 = [Q_1; P_1]$ and $\Gamma_2 = [Q_2; P_2]$ respectively.

Very close to the surface of decadence, the products of the decay are arbitrarily far away. In this case, one would expect that the degeneracy of the total configuration would be just the product of the degeneracies of individual fragments if the interactions between them were short-ranged. However, this configuration has angular momentum in the long-ranged electromagnetic field

$$J = \frac{1}{2}(\langle \Gamma_1, \Gamma_2 \rangle - 1), \tag{4.4}$$

from the Saha effect as for a electron in the magnetic field of a magnetic monopole, where $\langle \Gamma_1, \Gamma_2 \rangle = Q_1 \cdot P_2 - Q_2 \cdot P_1$ is a symplectic product of charges that is invariant under $SL(2, \mathbb{Z})$ electric-magnetic duality. Note that there is a shift of $-1/2$ to the angular momentum of the electromagnetic field above, which has to do with the contribution of fermion zero modes [55]. Taking into account this additional degeneracy of $(2J + 1)$ one concludes that the degeneracy of the original dyon is given by

$$\Omega(\Gamma) = | \langle \Gamma_1, \Gamma_2 \rangle | \, \Omega(\Gamma_1) \, \Omega(\Gamma_2). \tag{4.5}$$

Note that the formula above counts the internal degeneracies, and hence does not include the overall multiplicity of four coming from the fermionic oscillators associated with the center of mass coordinate. To get the total number of states, we must multiply (4.5) by this factor of 4.

In the $\mathcal{N} = 4$ case there is an additional complication. In this case, we will be considering a decay in which one center, say Γ_1, is half-BPS. This center breaks eight supersymmetries. Since the overall state is quarter-BPS, the total configuration must break twelve supersymmetries. This can happen in two ways. Either, the center Γ_2 is quarter-BPS and breaks twelve supersymmetries by itself which includes the eight supersymmetries broken by the first center. Or, the center Γ_2 is half-BPS but breaks a different half of the supersymmetries such that altogether there are twelve broken supersymmetries. In either case, additional four supersymmetries are broken in the internal theory of the two charge centers Γ_1 and Γ_2. These broken supersymmetries give rise to two complex fermion zero modes that furnish a 4-dimensional multiplet with the same spin content as the half hypermultiplet of $\mathcal{N} = 2$. The degeneracy then is similar to (4.5) with an additional multiplicative factor of 4:

$$\Omega(\Gamma) = 4 |\langle \Gamma_1, \Gamma_2 \rangle| \, \Omega(\Gamma_1) \, \Omega(\Gamma_2). \tag{4.6}$$

To deduce this by a slightly different argument, one can think of the total angular momentum of the system to be given by the tensor product of the half hypermultiplet with the spin $\mathbf{j}$ of the electromagnetic field given by (4.4). The half hypermultiplet has spin content of one $(\mathbf{\frac{1}{2}}) + 2(\mathbf{0})$. The total system of the electromagnetic field and the relative zero modes has spins $(\mathbf{j+1}) + 2(\mathbf{j}) + (\mathbf{j-1})$ with $\mathbf{j}$ given by (4.4). The multiplicity from these four representation is then $4|\langle \Gamma_1, \Gamma_2 \rangle|$.

Let us now see how these formulae can be applied to compute the degeneracies of decadent dyons, for example, in the $\mathcal{N} = 4$ case. The formula (4.6) effectively reduces the task of finding the degeneracy of $\Omega(\Gamma)$ of a state with charge Γ to finding the degeneracies $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ of the subsystems. This in itself would not be useful in general unless we knew how to compute $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ independently which is indeed the problem at hand. However, we will be considering the situation when at least one of the dyons with charge Γ_1 is half-BPS and stable so that its electric and magnetic charges are parallel and are relatively prime. Such a dyon we call irreducible, otherwise it is reducible.

Now, an irreducible dyon can be shown to have unit degeneracy using duality as follows. Since the electric and charge vectors are parallel, we must have $Q_1 = aV_1$ and $P_1 = cV_1$ a primitive charge vector V. Further, since the dyon is an absolutely stable single particle half-BPS state, the integers a and c must be relatively prime for otherwise the dyon can split into subsystems without costing any energy. Now, a primitive vector V_1 corresponds to a purely electric state and hence is proportional to the charge vector of a massive gauge boson of the theory. In this case, by an $SL(2,\mathbb{Z})$ electric-magnetic duality transformation, the state is dual to a purely electric gauge boson of the theory

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} V_1 \\ 0 \end{pmatrix}. \tag{4.7}$$

Since a massive gauge boson of the theory is known to have unit degeneracy, by duality it then follows that the half-BPS dyon with charge Γ_1 also has unit degeneracy. This conclusion can be explicitly checked also by a calculation similar to the one in [58].

The other decay product with charge Γ_2 can be either reducible or irreducible. If it is irreducible, then no further decay is possible. We then know the degeneracy of both decay products and hence of the original dyon using (4.6). An example of such a decay when a quarter-BPS dyons goes directly into irreducible fragments will be discussed in for $SU(3)$ dyons.

If the dyon with charge Γ_2 is reducible, then its degeneracy is *a priori* not known. However, one can now apply the reasoning in the previous paragraph iteratively. We can consider the surface of decadence of this dyon with charge Γ_2 where at least one of the decay products is irreducible. Continuing in this manner, one can relate the degeneracy of the original dyonic configurations to the degeneracies of the irreducible fragments up to factors coming from angular momentum degeneracies. An example of such a decay will be discussed in the subsection for $SU(N)$ dyons with $N > 3$.

The reasoning outlined here is similar to the one used by Denef and Moore to derive the wall crossing formula for dyons in $\mathcal{N} = 2$ string compactifications [56]. But there are differences. First, here we are using an additional input in the $\mathcal{N} = 2$ case that on one side of the wall the degeneracy is zero. This can be ascertained for these field theory dyons from their realization as string webs. Second, for $\mathcal{N} = 4$ dyons, the surface of decadence is generically surface of codimension bigger than one and is not really a wall. So we are not crossing any wall but merely approaching a surface of decadence. In $\mathcal{N} = 2$ string theories, the dyon degeneracies are not known explicitly for a generic compactifications and there is no independent way of checking the validity of this reasoning. Here, in the context of supersymmetric gauge theories, explicit formulae are known for the degeneracies in the work of Stern and Yi. Our rederivation of the Stern-Yi degeneracies that we now describe in the following sections can thus be viewed as a check of the heuristic reasoning outlined above.

4.1.2 Two-Centered Stern-Yi Dyons in $SU(3)$ Gauge Theories

Consider an $SU(3)$ dyon in an $\mathcal{N} = 4$ theory which has electric and magnetic charge vectors given by

$$Q = q_1\alpha_1 + q_2\alpha_2 \tag{4.8}$$

$$P = p_1\alpha_1 + p_2\alpha_2 \tag{4.9}$$

Following earlier work of [52] and [53], Stern and Yi [51] considered a simple charge configuration with magnetic charge vector $P = \alpha_1 + \alpha_2$. A quarter-BPS dyon can be viewed as a quantum charged excitation of a half-BPS monopole configuration. Now if $q_1 = q_2$, then the electric and magnetic charge vectors of the dyon would be parallel, both along $\alpha_1 + \alpha_2$. Such a configuration would give a half-BPS state. To break the supersymmetry further and obtain a quarter-BPS dyon it is necessary that $s = q_1 - q_2$ is nonzero so that the electric and magnetic charge vectors are misaligned. It is then useful to write the electric charge vector as

$$Q = (n + s)\alpha_1 + (n - s)\alpha_2. \tag{4.10}$$

Dirac quantization condition then demands that $n \pm s$ must be integral although n and s could individually be half-integral[52]. At some point in moduli space these states could decay into dyonic states into irreducible states

$$[(n + s)\alpha_1 + (n - s)\alpha_2; \alpha_1 + \alpha_2] \to [(n + s)\alpha_1; \alpha_1] + [(n - s)\alpha_2; \alpha_2]\,, \tag{4.11}$$

so that $V_1 = \alpha_1$ and $V_2 = \alpha_2$ in the notation of the discussion in and both decay products are irreducible.

Indeed in the string web picture [33, 34, 40, 41, 42, 43, 49, 59, 60, 61, 62], the dyons are realized as a two-centered configuration. Near the surface of decadence the distance between the two centers becomes very large. Note that the decay process across the wall is well described by semi-classical field configurations purely in terms of the low energy effective action on the Coulomb branch even when it occurs at strong coupling as would be the case for $\mathcal{N}=2$ dyons [60].

Now since, both centers are half-BPS dyons, they have unit degeneracy. The contribution from the angular momentum degeneracy factor is given by

$$|\langle\Gamma_1,\Gamma_2\rangle| = |(n+s)\alpha_1\cdot\alpha_2-(n-s)\alpha_2\cdot\alpha_1| = 2|s| \tag{4.12}$$

Hence the degeneracy of a $SU(3)$ quarter-BPS dyon with charge vectors $P = \alpha_1+\alpha_2$ and $Q=(n+s)\alpha_1+(n-s)\alpha_2$ is given by an application of the formula (4.6)

$$4\cdot 2|s|\cdot 1\cdot 1 = 8|s|, \tag{4.13}$$

in precise agreement with the results of Stern and Yi. To get the total number of states, we multiply by a factor of 16 coming from the center of mass multiplicity.

4.1.3 Multi-centered Stern-Yi Dyons in $SU(N)$ Gauge Theory

We now consider more general Stern-Yi dyons in a $SU(N)$ $\mathcal{N}=4$ gauge theory where a cascade of decays is necessary to get to decay products that are all half-BPS. The charge vector is $\Gamma=[Q;P]$ with

$$Q=(n+s_1+\ldots+s_{n-2})\alpha_1+(n-s_1+\ldots+s_{n-2})\alpha_2+\ldots(n-s_1\ldots s_{n-2})\alpha_{n-1} \tag{4.14}$$

$$P=\alpha_1+\alpha_2+\ldots+\alpha_{n-1}. \tag{4.15}$$

In the string web picture, these dyons are realized as multi-centered configurations.

We now approach the surface of decadence in the moduli space where the dyon breaks up into half-BPS dyon with charge Γ_1 and a quarter-BPS dyon with charge Γ_2 given by

$$\Gamma_1 = [(n + s_1 + s_2 + \ldots + s_{n-2})\alpha_1; \alpha_1] \,, \quad \Gamma_2 = [Q - Q_1; P - P_1] \,. \tag{4.16}$$

The angular momentum factor $\langle \Gamma_1, \Gamma_2 \rangle$ equals $2s_1$. The Γ_2 charge center can further decay and we can iterate the process until we are left as the decay products with irreducible dyons of unit multiplicities. This iteration gives the degeneracy to be

$$16 \cdot \prod_{i<j}^{N-2} |8s_i| \,, \tag{4.17}$$

precisely what Stern and Yi obtained using their index computation.

For the $\mathcal{N} = 2$ dyons, similar reasoning using the formula (4.5) gives

$$4 \cdot \prod_{i<j}^{N-2} |2s_i|, \tag{4.18}$$

once again in agreement with Stern and Yi.

4.2 Relation to string theory dyons

The partition function that counts the degeneracies of quarter-BPS dyons in heterotic string theory compactified on a six-torus $\mathbf{T}^6$ is given in terms of the Igusa cusp form which is a modular form of weight ten of the group $Sp(2, \mathbb{Z})$. It depends on three complex variables with a Fourier expansion given by

$$\frac{1}{\Phi_{10}(p, q, y)} = \sum c(m, n, l) p^m q^n y^l, \tag{4.19}$$

where the sum is over $m, n \geq -1$ and $l \in \mathbb{Z}$. A quarter-BPS dyons in this theory is specified by a charge vector $\Gamma = (Q_e; Q_m)$ where here both Q_e and Q_m are Lorentzian vectors that take values in the $\Gamma^{22,6}$ Narain lattice. There are three quadratic combinations $Q_e^2, Q_m^2, Q_e \cdot Q_m$ with respect to a Lorentzian inner product invariant under the $O(22, 6, ; \mathbb{Z})$. For a given vector Q_e in this lattice, one can define the right-moving part Q_{eR} to be the projection onto the 22 space-like directions and Q_{eL} to be the projection onto the 6 time-like directions. The inner product is then defined by

$$Q_e^2 = Q_{eR}^2 - Q_{eL}^2. \tag{4.20}$$

The degeneracy $d(\Gamma)$ is then given in terms of the Fourier coefficients by

$$d(\Gamma) = c(Q_e^2/2, Q_m^2/2, Q_e \cdot Q_m). \tag{4.21}$$

This formula was proposed in [63] and derived in [25, 38] using the 4d-5d lift and using a genus-two partition function in [7]. Generalization to CHL orbifolds have been discussed in [3, 5, 39, 64, 65, 66]. Note that according to the prescription above, we can have nonzero degeneracies apparently only for states that have

$$Q_e^2 \geq -2, \quad Q_m^2 \geq -2. \tag{4.22}$$

A more careful treatment of the degeneracy formula extends them by analytic continuation to all other states related by electric-magnetic duality to those that satisfy $Q_e^2 \geq -2$ and $Q_m^2 \geq -2$ in a way that the spectrum is duality invariant [6, 57, 67].

Since the low energy effective action for the heterotic string contains the action for supersymmetric nonabelian Yang-Mills theory, it is natural to ask if the dyon partition function above also counts degeneracies of these decadent dyons that we have considered in the previous sections. Indeed, our work was partly motivated by this question. If this is true, it would give a nontrivial check of the degeneracies predicted by the dyon partition function.

If the dyon partition function could count the field theory dyons like the Stern-Yi dyons then it would lead to many puzzles. Firstly, the degeneracies derived from the dyon partition function depend only on the three integers $Q_e^2, Q_m^2, Q_e \cdot Q_m$ and not on the components of the charges as the Stern-Yi degeneracy (4.17) seems to depend on. Second, the Stern-Yi degeneracies only grow polynomially as a function of charges, whereas the stringy dyon degeneracy grows exponentially if the discriminant

$$\Delta = Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2, \tag{4.23}$$

is positive.

We will show that these puzzles get resolved by the fact that the field theory dyons are in a different duality orbit than the ones that are counted by the dyon partition function. Hence one cannot apply the dyon partition function to count the Stern-Yi dyons except for special ones when the gauge group is $SU(3)$.

To see this clearly, we need to think more carefully about the field theory limit of string theory. To be able to analyze a dyon in field theory we would like to decouple stringy states and gravity from the consideration. At the same time, we would like to have a nonabelian structure in the gauge theory so that we do not have to deal with a Dirac monopole which is a singular field configuration but have instead a t'Hooft-Polykov monopole. In this case, the monopole is smooth solitonic configuration with a finite core which can be analyzed in field theory using semiclassical quantization. Such a limit is easily achieved if we consider the gauge group like $SU(3)$ to be embedded in the left-moving $E_8 \times E_8$ symmetry for example and consider Higgs expectation value v that is much smaller compared to the string mass scale Λ. In this case massive string states can be ignored. Moreover, the mass M of dyons will go as v/g^2 where g is the string coupling and gravitation backreaction will go as $GM^2 = v^2/\Lambda^2$ using the fact Newton's constant G goes as g^2/Λ^2. Thus, gravitational back reaction can also be ignored as long as v is much smaller than Λ and one can analyze the dyons in a field theory limit.

It is crucial for a useful field theory limit that the charges are purely left-moving, that is $Q_e^2 < 0$ and $Q_m^2 < 0$. This is because, in the heterotic string, which consists of a right-moving superstring and a left-moving bosonic string, only the left-moving $U(1)$ gauge symmetries can get enhance at special points in the moduli space of toroidal compactification. For example, for a circle compactification, at a generic radius of the circle we have $U(1)_L \times U(1)_R$ which couples to the charges

$$q_{L,R} = \sqrt{\frac{\alpha'}{2}}\left(\frac{m}{R} \pm \alpha' w R\right), \tag{4.24}$$

where m is the Kaluza-Klein momentum and w the winding number along the circle. At the self-dual radius of the circle however where $R^2 = \alpha'$, only $U(1)_L$ gets enhanced to a nonabelian $SU(2)_L$ but the $U(1)_R$ remains abelian. This is a consequence of the fact that the left-moving ground state energy is -1 as the bosonic string whereas the right-moving ground state energy is 0 as for the superstring. As a result, while certain states carrying left-moving momentum become massless at the self-dual radius, all states carrying right-moving momentum remain massive.

This implies that dyons coupling to both right-moving and left-moving $U(1)$ fields cannot be analyzed in a field theory limit as nonsingular solitonic configuration and stringy corrections would have to taken into account. For this reason we should embed our field theory gauge group into the purely left-moving symmetry.

Following, this reasoning, we can embed an $SU(N)$ gauge group into the $SO(32)$ gauge group of the heterotic string for $N \leq 16$. In this case $Q_e = Q$ and $Q_m = P$ with $Q_e^2 = -Q^2$ and $Q_m^2 = -P^2$ with the understanding that the Q_e^2 and Q_m^2 are defined using the Lorentzian inner product (4.20) whereas Q^2 and P^2 are defined using the positive definite Euclidean Cartan metric on the root space of the gauge group as we have used in the previous sections [1]. We refer to the charge vector as spacelike, timelike, or lightlike depending on whether the Lorentzian norm is positive, negative, or zero respectively. With the embedding above, we conclude that the field theory dyons must correspond to states with timelike charge vectors in the Narain lattice.

To understand the main issues, let us first focus on the $SU(3)$ Stern-Yi dyons. For the degeneracy of a string theory dyon that satisfies the bound(4.22) to match with a Stern-Yi dyon, the two charge configurations must lie in the same U-duality orbit. Now, the U-duality group $G(\mathbb{Z})$ of the string theory is

$$G(\mathbb{Z}) = O(22, 6, ; \mathbb{Z}) \times SL(2, \mathbb{Z}). \tag{4.25}$$

The U-duality orbit of the charges can be characterized by various invariants. To start with, we have the discriminant defined in (4.23) which is the unique quartic invariant of the continuous duality group $G(\mathbb{R})$. In addition, as noted in [6], there

[1] Once we turn on Wilson lines to break the gauge group we will have more accurately $Q_e = Q + k$ where k is a light-like vector with $Q \cdot k = 0$ so that Q_e^2 still equals $-Q^2$. Moreover, the charge vector is not strictly left-moving. This does not change the main point of the argument and hence we will ignore it.

is a discrete invariant

$$I = gcd(Q_e \wedge Q_m). \tag{4.26}$$

The wedge product gives the antisymmetric area tensor of the parallelogram bounded by the vectors Q_e and Q_m. The invariant I then counts the number of lattice points inside this parallelogram [6]. See also [68, 69].

For an $SU(3)$ Stern-Yi dyon with charge vectors are

$$P = \alpha_1 + \alpha_2, \quad Q = (n+s)\alpha_1 + (n-s)\alpha_2. \tag{4.27}$$

Using the embedding described above, we see that the two invariants for such an configuration are given by

$$\Delta = 12s^2, \quad I = 2s \tag{4.28}$$

We note that $\Delta > 0$ as it must be for a BPS configuration. Now, starting from spacelike Q_e^2 and Q_m^2, one can show that its impossible to go by U-duality to a configuration with both electric and magnetic charges timelike. To prove this we consider a general S-duality transformation acting on Q_e and Q_m as

$$Q'_e = aQ_e + bQ_m \tag{4.29}$$

$$Q'_m = cQ_e + dQ_m \tag{4.30}$$

Now, if Q_e and Q_m are positive norm vectors then $aQ_e \pm bQ_m$ is a positive norm vector. So, ${Q'_e}^2 \geq 0$ and similarly ${Q'_m}^2 \geq 0$. Thus, the only string dyonic configurations which can be U-dual to a field theory dyon will be those with timelike Q_e and Q_m[1]. By definition of Φ_{10}, the only such charges it counts are those with $Q_e^2 = -2$ and $Q_m^2 = -2$. Taking $Q_e \cdot Q_m = M$ to be arbitrary, we obtain a dyonic charge configuration with invariants $I = 1$ and $\Delta = 4 - M^2$. Hence the two sets of invariants match only for $s = \frac{1}{2}$ and $M = \pm 1$ which corresponds to $Q_e^2 = -2$, $Q_m^2 = -2$ and $|Q_e \cdot Q_m| = 1$. Consequently, only these string dyonic configurations lie in the duality orbit of SU(3) Stern-Yi dyons.

[1] The other possibility is having the electric(magnetic) charge to be timelike and the magnetic(electric) charge to be spacelike. But this will yield $\Delta < 0$ and breaks supersymmetry.

It is easy to see that in fact *all* Stern-Yi dyons with electric charge 4.10 are counted by the dyon degeneracy formula for all values n with $s = 1/2$. This follows from the fact that one can change the value of n by a duality transformation of the form

$$\begin{pmatrix} 1 & n - \frac{1}{2} \\ 0 & 1 \end{pmatrix} \tag{4.31}$$

Note that n must be half-integral for configuration with $s = 1/2$.

For an $SU(N)$ Stern-Yi dyonic configuration with $N > 3$ given by

$$Q_e = \sum_{i=1}^{N-1}(n + \sum_{j=1}^{N-2} P_{ij} s_j)\alpha_i \tag{4.32}$$

$$Q_m = \sum_{i=1}^{N-1} \alpha_i \tag{4.33}$$

where $P_{ij} = -1$ for $j < i$ and $P_{ij} = 1$ for $i \geq j$, the U-duality invariants are $\Delta = 4(2\sum_{i=1}^{N-2} {s_i}^2 + \sum_{i,j=1}^{N-2} s_i s_j)$ and $I = gcd(2s_1, 2s_2, ..2s_{N-2})$. For matching to the configurations whose degeneracy is counted by Φ_{10} we must have $I = 1$ which translates to the condition that the various $2s_i$ are mutually coprime. Further, we can easily see that $\Delta > 3$ for these Stern-Yi dyons[1]. Hence the string theory dyons do not lie in the U-duality orbit of any field theory SU(N) dyon with $N > 3$.

We therefore conclude that with the exception of the SU(3) dyons with $I = 1$, the field theoretic dyons considered earlier are outside the realm of applicability of the dyon partition function of string theory dyons in terms of the Igusa cusp form. A similar analysis has been carried out independently in [70, 71]. For other values of $I > 1$, a different partition function is required. For a recent proposal for the dyons with I=2 see [72].

[1]This follows easily from the inequality $({s_i}^2 + {s_j}^2) > 2s_i s_j$.

Chapter 5

Duality Symmetry and Cardy Limit.

In this chapter, we study supersymmetric and non-supersymmetric extremal black holes obtained in Type IIA string theory compactified on $K3 \times T^2$, with duality group $O(6, 22, \mathbb{Z}) \times SL(2, \mathbb{Z})$. In the Cardy limit an internal circle combines with the AdS_2 component in the near horizon geometry to give a BTZ black hole whose entropy is given by the Cardy formula. We study black holes carrying $D0 - D4$ and $D0 - D6$ brane charges. We find, both in the supersymmetric and non-supersymmetric cases, that a generic set of charges cannot be brought to the Cardy limit using the duality symmetries. In the non-supersymmetric case, unlike the supersymmetric one, we find that when the charges are large, a small fractional change in them always allows the charges to be taken to the Cardy limit. These results could lead to a microscopic determination of the entropy for extremal non-supersymmetric black holes, including rotating cases like the extreme Kerr black hole in four dimensions.

5.1 Motivations.

The microscopic descriptions of blackholes that have been developed so far are usually in terms of a $1+1$ dim. Conformal Field Theory (CFT). Furthermore the microscopic counting has been done most reliably in the thermodynamic limit of the CFT, see e.g., [1], [73], [74], and the reviews, [75], [76], [77], [78], and references therein; some papers which discus the microscopic counting for non-supersymmetric black holes are[1], [82], [83], [84]. In terms of the energy, L_0, and central charge of the CFT, C, the condition for the thermodynamic limit to be valid takes the form,

$$L_0 \gg C. \tag{5.1}$$

For a supersymmetric or non-supersymmetric extremal black hole, L_0 and C are determined by the charges carried by the black hole. The entropy in this limit is given by the well known Cardy formula,

$$S = 2\pi\sqrt{\frac{CL_0}{6}}. \tag{5.2}$$

In the discussion below, we will often refer to the thermodynamic limit as the Cardy limit. We see from eq.(5.2) that in this limit a knowledge of the central charge and the energy, L_0, is sufficient to determine the entropy. Moreover, the central charge is a robust quantity which can often be determined quite easily by anomaly considerations. This makes it easy to carry out a microscopic calculation of the entropy, [85], [86], [87].

In addition, when the condition, eq.(5.1) is valid subleading corrections to the entropy can also often be easily calculated. These continue to have the form, eq.(5.2). The subleading corrections arise due to corrections to the central charge, C and can be determined by anomaly considerations[88], [89], [90], [91].

Since so much can be understood in the Cardy limit, it is natural to ask whether any charge configuration can be put in the Cardy limit using the duality symmetries of string theory. This is the main question we will explore in this paper. Our focus is on big black holes. These carry large charges, $Q \gg 1$, and have a horizon radius which is large compared to the Planck and string scales, so that their horizon geometry is well described by the supergravity approximation. We are interested in both supersymmetric and non-supersymmetric extremal black holes of this type.

[1]For recent developments on rotating black holes, see, [79], [80],[81].

One comment is worth making at this stage [1]. Sometimes the condition eq.(5.1) is not necessary and a much weaker condition suffices. This happens for example in the D1-D5-P system when the CFT is at the orbifold point. At this point in the moduli space the twisted sectors can be thought of as multiply wound strings. In the singly wound sector the relevant condition is given by eq.(5.1). In contrast in the maximally wound sector the effective central charge is order one and energy is given by replacing L_0 by,

$$L_0 \to L_0 Q_1 Q_5, \tag{5.3}$$

where Q_1, Q_5 are the $D1, D5$ brane charges. Thus the condition, eq.(5.1), is automatically met for large charges in the maximally wound sector.

Away from the orbifold point though the different twisted sectors mix. The only condition which can now guarantee the validity of the Cardy formula is eq.(5.1), which ensures that the system is in the thermodynamic limit. It is well known that the CFT dual to the Black hole is not at the orbifold point. Thus a microscopic calculation of the entropy using the Cardy formula would require this condition to be valid. In the supersymmetric case, where one is calculating an index, one can still justify working at the orbifold point, where the dominant contribution comes from the maximally wound sector, and hence one would not need to impose the condition, eq.(5.1). However, for non-supersymmetric black holes, which are the ones of primary interest in this paper, the entropy can change as one moves in moduli space. A legitimate microscopic calculation in this case would have to be done away from the orbifold point and would require the condition, eq.(5.1), to hold for the Cardy formula to be valid.

It should be mentioned that the mass gap for excitations above the BTZ black hole can be calculated in the gravity side and is well known to go like,

$$E_{gap} \sim 1/(LC), \tag{5.4}$$

where L is the length of the circle on which the CFT lives. This shows that an effective picture in terms of one multiply wrapped long string must continue to

[1] We thank S. Mathur and A. Strominger for emphasising this point to us.

hold even away from the orbifold point. However a first principles argument of why this happens is still missing especially in the non-supersymmetric case. In the absence of such an argument it is appropriate to require, at least in a first principles calculation of the microscopic entropy, that for the Cardy formula to be valid the condition, eq.(5.1), holds. This chapter explores how restrictive this condition is, once the duality symmetries of string theory are taken into account.

5.2 Cardy Limit.

We now turn to discussing the Cardy limit. Consider a Black hole carrying $D0-D4$ brane charge. In our notation the non-zero charges are, $q_0, p^1, p^i, i = 2, \cdots 23$. This solution can be lifted to M-theory, and the near horizon geometry in M-theory is given by a BTZ black hole in $AdS_3 \times S^2$. The AdS_3 space-time admits a dual description in terms of a 1+1 dim. CFT living on its boundary. The central charge, C, of the CFT can be calculated from the bulk, it is determined by the curvature of the AdS_3 spacetime. For large charges we get,

$$C = 3|p^1 d_{ij} p^i p^j|, \tag{5.5}$$

where d_{ij} is the matrix η_{ij}, eq.(1.1), restricted to the 22 dimensional subspace of charges given by $D4$-branes wrapping two-cycles of $K3$ and T^2. This corresponds to the second, third and fourth factor of $\mathcal{H}$ and the two $\mathcal{E}_8$'s in eq.(3.27).

The BTZ black hole is a quotient of AdS_3 obtained by identifying points separated by a space-like direction. The symmetry of AdS_3 is $SO(2,2)$; this is broken by the identification of points in the BTZ black hole to $SO(2,1) \times U(1)$. The size of the circle obtained by this identification, L, is given in terms of the radius of AdS_3, R_{AdS}, by

$$\frac{L}{R_{AdS}} \sim \frac{|q_0|}{C}, \tag{5.6}$$

where q_0 is the zero-brane charge carried by the Black hole.

In the Cardy limit the condition,

$$|q_0| \gg C, \tag{5.7}$$

is satisfied. From eq.(5.6) we see that this leads to the condition, $\frac{L}{R_{AdS}} \gg 1$. From, eq.(5.5) we see for this limit to be valid, the condition,

$$|q_0| \gg |p^1 d_{ij} p^i p^j|, \tag{5.8}$$

must hold. Since, $\frac{L}{R_{AdS}} \gg 1$, in the Cardy limit, the distance between points which are identified in the BTZ background is much bigger than R_{AdS}. As a result, the effect of the reduced symmetry in the BTZ background, due to taking the quotient, can be neglected in the Cardy limit. The partition function in the bulk can then be calculated using the full symmetries of AdS_3. The resulting answer is the well known Cardy formula,

$$S = 2\pi\sqrt{\frac{C|q_0|}{6}}. \tag{5.9}$$

The Cardy limit corresponds to the thermodynamic limit of the microscopic $1+1$ dim. CFT. In this limit the dimensionless temperature T of the CFT satisfies the condition,

$$T \gg 1. \tag{5.10}$$

Away from the Cardy limit the breaking of $SO(2,2)$ to $SO(2,1)$ becomes important and there is no way to calculate the partition function or entropy without knowing more details of the bulk, or the dual boundary conformal field theory.

So far we have considered a system with $D0 - D4$ brane charge. What about including other charges? If a $D6$-brane charge is also present, we show in §5, that on lifting to M-theory one does not get an AdS_3 space-time. All other charges are allowed by the requirement that the M-theory lift gives an AdS_3 spacetime in the near-horizon limit. So a general configuration which admits an AdS_3 lift can also include $D2$-brane charges, and non-zero values for $n_1, n_2, w_1, w_2, NS_1, NS_2, KK_1, KK_2$, besides having $D0-D4$ brane charges. The resulting central charge of the $1+1$ dim. CFT after lifting to M-theory is [1]

$$C = 3|p^1 \vec{Q}_m^2|. \tag{5.11}$$

[1] The central charge is determined by all the branes which are extended strings in the AdS_3. One can see from eq.(1.4), eq.(1.5), that this formula gives a dependence on all of them. Localised excitations, like momentum modes or wrapped 2-branes, correspond to states and do not change the central charge.

In the more general case, the condition for the Cardy limit is,

$$|\hat{q}_0| \gg C. \tag{5.12}$$

Where, $|\hat{q}_0|$ is,

$$|\hat{q}_0| = \frac{|\vec{Q}_e^2\vec{Q}_m^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2|}{2|p^1\vec{Q}_m^2|}. \tag{5.13}$$

Using eq.(1.7), eq.(5.11) and eq.(5.13) this can be written in the form,

$$I \gg 6(p^1)^2(\vec{Q}_m^2)^2. \tag{5.14}$$

To summarise, for a charge configuration to be in the Cardy limit, two conditions must hold. First the $D6$-brane charge, p^0, must vanish. Second, eq.(5.12) or equivalently, eq.(5.14), must be valid. We refer to these two conditions as the Cardy conditions below.

Before proceeding let us note that we are neglecting $1/Q$ corrections in the formula for the central charge, eq.(5.11). For these to be small, the BTZ black hole should be a state in a weakly coupled AdS_3 background. The Radius of the AdS_3 space, R_{AdS}, in units of the three dimensional Planck scale, $l_{Pl}^{(3)}$, is given by,

$$\frac{R_{AdS}}{l_{Pl}^{(3)}} \sim C. \tag{5.15}$$

For the BTZ black hole to be a state in a weakly coupled AdS_3 spacetime, $\frac{R_{AdS}}{l_{Pl}^{(3)}} \gg 1$, yielding the condition[1],

$$C \gg 1. \tag{5.16}$$

The conditions on the charges for the Cardy limit are not duality invariant. This raises the question, when can a charge configuration be brought to the Cardy limit after a duality transformation? This is the central question we address in this paper. In §3 we first address this question for the case where the starting

[1]The stronger conditions are, $\frac{R_{AdS}}{l_{11}} \gg 1$, $\frac{R_{S2}}{l_{11}} \gg 1$, and $\frac{V_6}{l_{11}^6} \gg 1$, where R_{S^2}, V_6 are the Radius of the S^2 and volume of the internal space respectively. From these, and the relation, $l_{Pl}^{(3)} = \frac{l_{11}^9}{R_{S^2}^2 V_6}$, the condition, $\frac{R_{AdS}}{l_{Pl}^{(3)}} \gg 1$, follows.

configuration, has $D0 - D4$ brane charges. Our analysis includes both the supersymmetric and non-supersymmetric cases. Following this in §4, we address this question when the starting configuration carries $D0 - D6$ brane charges.

There is one potentially confusing point that we would like to address before going further. In asking whether a system of charges can be brought to the Cardy limit, we are really asking whether any of the internal circles of the compactification can combine with the AdS_2 component of the near horizon geometry and give rise to a three-dimensional BTZ black hole and whether this black hole has charges which lie in the Cardy limit. There are six internal circles for example in the Heterotic description, corresponding to the 6 Hyperbolic lattices, $\mathcal{H}$ in eq.(3.27), and we allow for the internal circle to be any one of them. Our results, mentioned in the introduction, which say that generically this is not possible, mean that for generic charges there is no internal circle which can combine in this manner, yielding the Cardy limit.

There are two ways to carry out the analysis. We can keep the charges fixed and ask whether a suitable circle can be found. This corresponds to a passive transformation, under which the charges are kept fixed but the basis in the charge lattice, with respect to which the components were written in eq.(1.4), eq.(1.5), is changed. Alternatively, we can keep the basis fixed and change the charges, and ask whether the transformed charges meet the required conditions. This corresponds to an active transformation. We will adopt this latter active of point of view in the paper. In this point of view the internal circle which combines and potentially gives rise to a BTZ black hole is kept fixed and in our conventions is the M-theory circle in the IIA description.

5.3 The $D0 - D4$ System

In this section we analyse the $D0 - D4$ system. Subsection 3.1 discusses the supersymmetric case, and subsections 3.2, 3.3, discuss the non-supersymmetric case. In both cases we find that a generic set of charges cannot be brought to the Cardy limit. Subsection 3.4, discuss what happens if starting with generic charges we now allow the charges to vary. We find that in the non-supersymmetric case a near-by charge configuration can always be found which can be brought to the Cardy limit. Additional relevant material is in appendices A and B.

Our starting configuration for the $D0-D4$ case has non-zero values for q^0, p^1, p^i, in the notation of eq(1.4), eq.(1.5), and all other charge are vanishing. It is easy to see from eq.(1.6) that

$$\vec{Q}_e \cdot \vec{Q}_m = 0, \tag{5.17}$$

in this case.

In our analysis we are interested in the case of large charges, $|q_0|, |p^1|, |p^i| \gg 1$. The Cardy condition for the starting configuration takes the form, eq.(5.8). We see that for a generic set of initial charges this condition will not be met. Generically all charges will be roughly comparable, $|q_0| \sim |p^1| \sim |p^i| \sim Q \gg 1$ Now the LHS of eq.(5.8) is linear in Q while the RHS is cubic in Q, so generically, for $Q \gg 1$, the inequality, eq.(5.8), will not be met.

Below we formulate a set of necessary condition which must be met, for the final configuration to be in the Cardy limit. For generic initial charges, we find that these conditions are not met. And so we learn that generically a system with $D0-D4$ charge cannot be brought to the Cardy limit. In some special, non-generic cases, these necessary conditions are met. We construct some examples of this type and explicitly find a duality transformation bringing them to the Cardy limit [1].

Let us denote the final configuration which is obtained after carrying out a duality transformation on the initial $D0-D4$ charges by $(\vec{Q}'_e, \vec{Q}'_m)$. As was pointed out above, the $D6$-brane charge, $p^{0'}$, in the final configuration must vanish for this to happen, and eq.(5.14) must be met.

We can restate eq.(5.14) in the slightly weaker form as,

$$|I| \gg (p^{1'}(\vec{Q}'_m)^2)^2. \tag{5.18}$$

[1] Of course a trivial way in which this could happen is if the initial configuration, while being non-generic, is itself in the Cardy limit, and meets condition, eq.(5.8). In the example we construct, the initial charges while being rather special are not in the Cardy limit. We find explicitly the duality transformation bringing them to this limit.

This gives rise to the condition,

$$\left|\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}}\right| \ll \frac{1}{|p^{1'}|}. \tag{5.19}$$

Since $|p^{1'}| > 1$ eq.(5.19) leads to the condition,

$$\left|\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}}\right| \ll 1. \tag{5.20}$$

The final configuration, $(\vec{Q}'_e, \vec{Q}'_m)$ is obtained from the initial one, by the action of a combined $SL(2,\mathbb{Z})$ transformation and an $O(6,22,\mathbb{Z})$ transformation. Denote the element of $SL(2,\mathbb{Z})$ by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.21}$$

By definition, $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$. The $SL(2,\mathbb{Z})$ transformation acts on the charges as follows,

$$\begin{aligned} \vec{Q}_e &\rightarrow a\vec{Q}_e + b\vec{Q}_m \\ \vec{Q}_m &\rightarrow c\vec{Q}_e + d\vec{Q}_m. \end{aligned} \tag{5.22}$$

The $O(6,22)$ transformation does not change the value of the bilinears, eq.(1.6), also the initial charges satisfy the condition, $\vec{Q}_e \cdot \vec{Q}_m = 0$. This leads to,

$$(\vec{Q}'_m)^2 = c^2\vec{Q}_e^2 + d^2\vec{Q}_m^2. \tag{5.23}$$

Using eq.(5.20), now gives,

$$\left|c^2\frac{\vec{Q}_e^2}{\sqrt{|I|}} + d^2\frac{\vec{Q}_m^2}{\sqrt{|I|}}\right| \ll 1. \tag{5.24}$$

This condition will play an important role in the discussion below.

5.3.1 The Supersymmetric Case

Since eq.(5.17) is true for the $D0 - D4$ system, it follows from eq.(1.7) that the duality invariant, I, is,

$$I = \vec{Q}_e^2\vec{Q}_m^2. \tag{5.25}$$

For a supersymmetric system, $I > 0$, so we see that $\vec{Q}_e^2, \vec{Q}_m^2$ have the same sign. From, eq.(5.23) it follows that $(\vec{Q}'_m)^2$ must also have the same sign as $\vec{Q}_e^2, \vec{Q}_m^2$.

Thus eq.(5.20) takes the form,

$$c^2 \frac{|\vec{Q}_e^2|}{\sqrt{I}} + d^2 \frac{|\vec{Q}_m^2|}{\sqrt{I}} \ll 1. \tag{5.26}$$

Now by doing an $SL(2, \mathbb{Z})$ transformation if necessary we can always take the initial charges to satisfy the condition,

$$\left| \frac{\vec{Q}_e^2}{\vec{Q}_m^2} \right| \geq 1. \tag{5.27}$$

(Either this condition is already met or we do the $SL(2, \mathbb{Z})$ transformation $(\vec{Q}_e, \vec{Q}_m) \rightarrow (-\vec{Q}_m, \vec{Q}_e)$ after which it is true).

Using the expression for I in eq.(5.25), eq. (5.27) leads to,

$$\frac{|\vec{Q}_e^2|}{\sqrt{I}} \geq 1. \tag{5.28}$$

Now since c, d are integers, we see that the only way, eq.(5.26) can be met is if, $c = 0$. The resulting $SL(2, \mathbb{Z})$ matrix must then take the form,

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{5.29}$$

From eq.(5.23) it now follows that,

$$(\vec{Q}'_m)^2 = \vec{Q}_m^2. \tag{5.30}$$

The condition, eq.(5.20), using eq.(5.25), eq.(5.30) then leads to,

$$|\vec{Q}_e^2| \gg |\vec{Q}_m^2| \tag{5.31}$$

A few points are now worth making. Eq.(5.31) is a necessary condition on the initial charges $(\vec{Q}_e, \vec{Q}_m)$ which must be met, to be able to go to the Cardy limit. It is easy to see that this condition will not be met generically. If all the initial charges, q_0, p^1, p^i are of the same order, $Q \gg 1$, then, $\vec{Q}_m^2 = 2d_{ij}p^ip^j$ and

$\vec{Q}_e^2 = -2q_0 p^1$ are both quadratic in Q and will generically be roughly comparable, so that eq.(5.31) is not met. On the other hand this condition is somewhat less non-generic than the condition required for the initial configuration to be in the Cardy limit, since both sides of the inequality scale like Q^2 in eq.(5.31), while in eq.(5.8) the rhs scales relative to the lhs by a factor of Q^2. Thus one can find initial charges which are not in the Cardy limit, but which meet the condition eq.(5.31). We will present some explicit examples below and show that they can be sometimes brought to the Cardy limit by duality transformations.

Before doing so let us comment that the eq.(5.31) can in fact be somewhat tightened. Let $\gcd(\vec{Q}_e)$ stand for the greatest common divisor of all the integer charges in $\vec{Q}_e$. Then the stronger form of this condition is,

$$|\vec{Q}_e^2| \gg (\text{gcd}\vec{Q}_e)^2 |\vec{Q}_m^2| \tag{5.32}$$

In the appendix , we discuss how eq.(5.32) can be derived.

In the example we present next, the starting configuration is not in the Cardy limit, but condition, eq.(5.32) is met. We will present the explicit duality transformation that brings this configuration to the Cardy limit.

5.3.1.1 An Explicit Example

We start with the charges,

$$\vec{Q}_e = (-p^1 + 1, -p^1, 0, 0, 0, 0, 0, \cdots, 0) \tag{5.33}$$

$$\vec{Q}_m = (0, 0, p^2, p^2, 0, 0, 0, \cdots, 0) \tag{5.34}$$

with,

$$(p^1)^2 \gg 3(p^2)^2 \gg 1. \tag{5.35}$$

The quadratic bilinears, eq.(1.6), take the values,

$$\begin{aligned} \vec{Q}_e^2 &= 2(p^1 - 1)p^1 \\ \vec{Q}_m^2 &= 2(p^2)^2 \\ \vec{Q}_e \cdot \vec{Q}_m &= 0 \end{aligned} \tag{5.36}$$

The invariant, I, eq.(1.7), takes the value,

$$I = 4p^1(p^1 - 1)(p^2)^2 \tag{5.37}$$

Note that this starting configuration is not in the Cardy limit as these charges do not satisfy the condition, eq.(5.14). But the starting configuration does satisfy eq.(5.32) since, $\gcd(\vec{Q}_e) = \gcd(p, p-1) = 1$, and eq.(5.35) holds.

Now we carry out the transformation, $B \in O(3,3,\mathbb{Z}) \subset O(6,22,\mathbb{Z})$, given by,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} . \tag{5.38}$$

B acts non-trivially on the 6 dimensional sublattice of $\Gamma^{6,22}$, with an inner product given by first three $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ factors in eq.(1.1), and acts trivially on the rest of the lattice. The resulting charges are given by,

$$\vec{Q}'_e = (-p^1, 1, p^1, 0, p^1, p^1, 0, \cdots, 0) \tag{5.39}$$

$$\vec{Q}'_m = (0, 0, p^2, p^2, 0, -p^2, 0, \cdots 0) \tag{5.40}$$

Since the second entry in $\vec{Q}'_m$ vanishes, there is no $D6$-brane charge. From, eq.(5.39) we see that $p^{1'} = -1$. Also,

$$\vec{Q}'_m{}^2 = 2(p^2)^2. \tag{5.41}$$

Now the Cardy condition requires that,

$$I \gg 6\left(p^{1'}\vec{Q}'_m{}^2\right)^2 . \tag{5.42}$$

Using eq.(5.37), eq.(5.41) and eq.(5.35), we see that this condition is indeed met.

An example where all the final charges are much bigger than unity can be obtained by scaling all the charges above, by $\lambda \gg 1$ and taking

$$(p^1)^2 \gg 3(\lambda)^2(p^2)^2. \tag{5.43}$$

5.3.2 The Non-supersymmetric Case

In the Non-supersymmetric $D0-D4$ system, I also takes the form, eq.(5.25). By doing an $SL(2,\mathbb{Z})$ transformation if necessary we can assume, without loss of generality that

$$\frac{|\vec{Q}_m^2|}{|\vec{Q}_e^2|} \leq 1. \tag{5.44}$$

For subsequent discussion it is useful to define the parameter, α, as follows,

$$\alpha = \frac{|\vec{Q}_m^2|}{\sqrt{|I|}} = \frac{\sqrt{|I|}}{|\vec{Q}_e^2|} = \sqrt{\frac{|\vec{Q}_m^2|}{|\vec{Q}_e^2|}}. \tag{5.45}$$

where the last two equalities follows from eq.(5.25). We see from eq.(5.44) that

$$\alpha \leq 1. \tag{5.46}$$

Since I is negative, we learn from eq.(5.25) that $\vec{Q}_e^2, \vec{Q}_m^2$ must have opposite signs. There are then two possibilities, either $\vec{Q'}_m^{\,2}$ has the same sign as $\vec{Q}_e^2$, or it has the opposite sign as $\vec{Q}_e^2$. In both cases, eq.(5.24) takes the form,

$$0 < \left| -d^2\alpha + \frac{c^2}{\alpha} \right| \ll 1. \tag{5.47}$$

The requirement $|-d^2\alpha + \frac{c^2}{\alpha}| > 0$ arises from the condition that $\vec{Q'}_m^{\,2}$ is non-vanishing, and this in turn arises from the requirement that the central charge, C, eq.(5.11), does not vanish.

The analysis and conclusions are similar in the two cases. Below we give details for the case when $\vec{Q'}_m^{\,2}$ and $\vec{Q}_e^2$ have the same sign and also state the conclusions for the case when $\vec{Q}_m'^2$ and $\vec{Q}_e^2$ have the opposite sign.

In the case when $\vec{Q}_m'^2, \vec{Q}_e^2$, have the same sign, eq.(5.47) takes the form,

$$0 < -d^2\alpha + \frac{c^2}{\alpha} \ll 1. \tag{5.48}$$

It is interesting to compare this with the condition that arose in the susy case, eq.(5.26). This constraint required the charges to be non-generic and to satisfy the condition, eq.(5.31), in the susy case. In terms of α, defined in eq.(5.45), this condition takes the form,

$$\alpha^2 \ll 1. \tag{5.49}$$

At first sight it might seem that the difference in relative sign between the two terms makes eq.(5.48) easier to satisfy in the non-susy case. To explore this question we will take, $\alpha < 1$, but not much less than unity and ask whether such a set of charges can be brought to the Cardy limit. We will find that in fact eq.(5.48) cannot be met for generic initial charges. Also, we will see that the nature of the non-genericity which allows eq.(5.48) to be met is interestingly different from the susy case, and this has interesting consequences which we will discuss further in the next subsection.

Conditions, eq.(5.46) and eq.(5.47), and the fact that c takes integer values, imply that d cannot vanish. We can then write eq.(5.48) as follows,

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \ll 1. \tag{5.50}$$

Since $d^2 \geq 1$ and $\alpha \leq 1$, this gives rise to a weaker condition,

$$0 < \left(-\alpha + \left|\frac{c}{d}\right|\right)\left(\alpha + \left|\frac{c}{d}\right|\right) \ll 1. \tag{5.51}$$

Now if α is not very much less than unity, as we are assuming, then $(\alpha + |\frac{c}{d}|)$ cannot be very much less than unity. Thus the only way to meet the condition, eq.(5.51), is for

$$0 < \left|\frac{c}{d}\right| - \alpha \ll 1. \tag{5.52}$$

In general we see from eq.(5.45) that α is an irrational number and $|\frac{c}{d}|$ is a rational number. We know that any irrational number can be approximated arbitrarily well by a rational number, therefore one can meet condition eq.(5.52) for a general α.

Let us however go back to the stronger condition, eq.(5.50), we will see that this cannot be met generically. We state the condition in eq.(5.50) as follows:

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) < \delta, \tag{5.53}$$

where, δ is a small number satisfying,

$$\delta \ll 1. \tag{5.54}$$

Eq.(5.52) then takes the form,

$$0 < \left|\frac{c}{d}\right| - \alpha < \delta. \tag{5.55}$$

As was mentioned above, since any irrational number can be approximated arbitrarily well by a rational number, c, d can always be found so that eq.(5.55) is met. However, for a generic irrational number, α, the integers, d, c that satisfy eq.(5.55) will have to be of order $O(1/\delta)$ [1]. Approximating,

$$\alpha + \left|\frac{c}{d}\right| \sim 2\alpha, \tag{5.56}$$

we see that

$$\frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \simeq 2d^2\left(-\alpha + |\frac{c}{d}|\right) \sim O(1/\delta). \tag{5.57}$$

It then follows that eq.(5.50) will not be generically met, since δ satisfies the condition, eq.(5.54).

In other words, while α can be approximated arbitrarily well by the ratio of two integers, $|c/d|$, in general doing so to better accuracy by choosing δ to be smaller will make the condition, eq.(5.50), harder to meet.

The condition in eq.(5.50) can be met if α is a non-generic irrational number for which eq.(5.55) can be met by taking

$$c, d \sim O\left(\frac{1}{\delta^{1/2-\epsilon}}\right). \tag{5.58}$$

with $\epsilon > 0$. In this case one finds that,

$$\frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \sim O(\delta^{2\epsilon}), \tag{5.59}$$

and thus eq.(5.50) can be met if $\delta \ll 1$.

An example is provided by

$$\alpha = \sqrt{\frac{p-1}{p}}. \tag{5.60}$$

[1] For example to approximate $1/\sqrt{2} = 0.707106...$, to n significant figures, c, d would have to be $O(n)$.

It is easy to see that eq.(5.50) is met in this case if $c = d = 1$ and $p \gg 1$. This example, fits in with the discussion above. The irrational number α, in this case, is well approximated to $O(1/p)$ by two integers which are unity, and which therefore satisfies the condition, eq.(5.58).

The example above can be easily generalised to the case,

$$\alpha = \frac{m}{n}\sqrt{\frac{p-1}{p}} \tag{5.61}$$

where $m < n$ and $mn \ll p$. Once again eq.(5.50) can be met, by taking, $c = m, d = n$. We will have more to say about what these examples are teaching us in the following subsection, where we consider varying the charges.

To summarise the discussion above, we have learned that eq.(5.50) can be met, but only for rather special values of the initial charges. These charges are such that α is of the form,

$$\alpha = \frac{m}{n} - \epsilon, \tag{5.62}$$

where $0 < \epsilon \ll 1$, and the integers, m, n are not very big, and meet the condition,

$$2n^2\epsilon \ll 1. \tag{5.63}$$

In this case, by taking, $c = m, d = n$ eq.(5.50) can be met [1].

There is another way to characterise the non-genericity of α. Suppose we choose the initial charges such that α took a special value, eq.(5.62), and integers, c, d exist meeting conditions, eq.(5.50). We could ask by how much can the initial charges be varied so that integers c, d continue to exist, meeting the condition eq.(5.53). If all the initial charges are of order Q and they are varied by a small amount ΔQ, we have that,

$$\frac{\Delta\alpha}{\alpha} \sim \frac{\Delta Q}{Q}. \tag{5.64}$$

Using, eq.(5.56), we can write the condition, eq.(5.53) as,

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \simeq 2d^2\left(-\alpha + \frac{c}{d}\right) < \delta. \tag{5.65}$$

[1] For the matrix eq.(5.21) to exist c, d must be coprime. This requires that we cancel off any common factors in m, n and take them to be coprime.

Now, when

$$\Delta\alpha \sim \frac{\delta}{2d^2}, \tag{5.66}$$

c, d will have to change from their initial values for, the inequality, eq.(5.65) to continue to hold. But for a generic small variation, new integers, c, d, cannot be found meeting condition, eq.(5.58), rather the new integers will be of order $O(1/\delta)$ and as a result eq.(5.53) will not be met. Therefore the maximum variation for the initial charges is of order,

$$\frac{\Delta Q}{Q} \sim \frac{\delta}{2d^2}. \tag{5.67}$$

Since δ satisfies eq.(5.54), and d is a non-vanishing integer, we see that this variation is small.

To summarise, in this subsection we have seen that a non-supersymmetric system carrying generic $D0 - D4$ brane charges cannot be brought to the Cardy limit after a duality transformation. The case when α is rational needs to be treated somewhat differently, we analyse this case below. Some examples, of non-generic charges, which can be brought to the Cardy limit using the duality symmetry are discussed in the appendix.

5.3.3 Rational α

Since we saw that α had to be close to a rational number for the integers c, d to exist meeting the condition in eq.(5.50), it might seem at first that for any α which is rational one can always meet this condition. We show here that this is not true, eq.(5.50) can be met by rational α but again of a rather special form.

Suppose that

$$\alpha = \frac{m}{n} \tag{5.68}$$

so that ϵ in eq.(5.62) vanishes. We will again take the case where $\alpha < 1, \alpha \not\ll 1$ [1]. Without loss of generality, we can take m, n to be co-prime. One could now choose $d = m, c = n$ so that

$$\left|\frac{d}{c}\right| - \alpha = 0 \ll 1. \tag{5.69}$$

[1] We impose this restriction since if $\alpha \ll 1$, the charges are be non-generic to start with.

However in this case we see that eq.(5.50) is not met at the other end, since, $(|\frac{d}{c}| - \alpha) \ngtr 0$.

We need to find integers, c, d such that $|\frac{d}{c}|$ is close to α, but does not exactly cancel it. This will not be generically possible for exactly the same reason as the case of irrational α. To meet the condition eq.(5.55), c, d will generically be of order $1/\delta$, while to meet eq.(5.50) they would need to meet condition eq.(5.58). These two requirements are not compatible.

To understand when the condition in eq.(5.50) can be met more precisely, let us write this equation as,

$$0 < \frac{1}{\alpha}(\alpha|d| + |c|)(-\alpha|d| + |c|) \ll 1. \tag{5.70}$$

Now since, $|c| > \alpha|d|$ we have, $|c| + |d|\alpha > 2|d|\alpha$, and it follows from eq.(5.70) that,

$$0 < \frac{2|d|}{n}(n|c| - m|d|) \ll 1. \tag{5.71}$$

Since, the minimum non-vanishing value of $(n|c| - m|d|)$ is unity, one consequence of eq.(5.71) is that, $n/|d| \gg 1$. Given that α is not much smaller than unity it follows then that,

$$m, n \gg 1. \tag{5.72}$$

Also since, $2|d| > 1$, it follows from eq.(5.71) that

$$0 < \frac{n|c| - m|d|}{n} \ll 1. \tag{5.73}$$

In summary, if α is a rational number, $\alpha = m/n$, an $SL(2,\mathbb{Z})$ transformation can be found bringing the charges to a form where condition, eq.(5.50) is met, if two integers, c, d exist which are coprime, and which satisfy the condition, eq.(5.71). Generically, we have argued above, such integers do not exist, and thus eq.(5.50) will not be met.

One final comment before we move on. In the analysis above we considered the case where $\vec{Q}_m'^2$ had the same sign as $\vec{Q}_e^2$. If instead $\vec{Q}_m'^2$ has the opposite sign as $\vec{Q}_e^2$, the condition, eq.(5.48) is replaced by,

$$0 < d^2\alpha - \frac{c^2}{\alpha} \ll 1. \tag{5.74}$$

The discussion above, for the irrational and rational values of α, then goes through essentially unchanged leading to similar conclusions. For generic values of the charges, condition eq.(5.47) will not be satisfied. The condition in eq.(5.62) in this case is replaced by the requirement that

$$\alpha = \frac{m}{n} + \epsilon, \tag{5.75}$$

with $\epsilon > 0$, such that,

$$2n^2\epsilon \ll 1. \tag{5.76}$$

If this requirement is met, eq.(5.74) can be met by taking, $c = n, d = m$. For rational, α, eq.(5.71) is replaced by,

$$0 < \frac{2|c|}{m}(m|d| - n|c|) \ll 1. \tag{5.77}$$

5.3.4 Changing The Charges

In our discussion above for the non-supersymmetric case we saw that for rather special values of α the condition, eq.(5.50) can be met. An example is given in eq.(5.61). This prompts one to ask the following question: Although a generic charge configuration cannot be brought to the Cardy limit, can we find a charge configuration lying near by, which can be brought to the Cardy limit ? In this subsection we will answer the question. For large charges, $Q \gg 1$, we show that such a near-by charge configuration does exist in the non-supersymmetric case. In contrast, in the supersymmetric case, such a near-by configuration does not exist.

Before proceeding let us state more clearly what we mean by a charge configuration lying near the starting $D0-D4$ configuration. Suppose we carry out a change in the charges,

$$\vec{Q}_e \rightarrow \vec{Q}_e + \Delta\vec{Q}_e \tag{5.78}$$

$$\vec{Q}_m \rightarrow \vec{Q}_m + \Delta\vec{Q}_m. \tag{5.79}$$

The change is small, and the new charge configuration is near the original one, if the conditions,

$$\begin{aligned} \left|\frac{\vec{Q}_e \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| &\ll 1 \\ \left|\frac{\vec{Q}_m \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| &\ll 1 \\ \left|\frac{\Delta\vec{Q}_{e,m} \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| &\ll 1, \end{aligned} \tag{5.80}$$

are met [1]. In these inequalities, $\Delta\vec{Q}_{e,m}$ in the numerator stands for either, $\Delta\vec{Q}_e$, or $\Delta\vec{Q}_m$, the inequality holds in both cases. Similarly, $\vec{Q}_{e,m}$ in the denominator stands for either $\vec{Q}_e$ or $\vec{Q}_m$. Note that it follows from these conditions that the change in the duality invariant, I, eq.(1.7), and therefore also the change in the entropy, eq.(1.8), eq.(1.9), is small.

Let us first consider the supersymmetric case. The required condition for an $SL(2,\mathbb{Z})$ transformation, eq.(5.21), to exist is that α, eq.(5.45), satisfies the condition, eq.(5.49). Suppose we start with generic charges, where $\alpha \leq 1$, but where condition eq.(5.49) is not met, and now carry out the change in the charges, eq.(5.78). The initial charges, $\vec{Q}_e, \vec{Q}_m$, are both either space-like or time-like, and since condition eq.(5.49) is not met, are roughly comparable in magnitude. It is then clear, and straightforward to verify explicitly, that small changes, meeting conditions, eq.(5.80), will not allow, eq.(5.20) to be met. We learn then that in the supersymmetric case there is no near by configuration - obtained by a small change in charges- which brings the charges to the Cardy limit.

Next we come to the non-supersymmetric case. Here one of the two vectors, $\vec{Q}_e, \vec{Q}_m$ is space-like and the other time-like, and this makes the analysis more involved, as we have already seen above. We will explicitly construct a new set of charges, close to the original one and show that it can be taken to the Cardy limit after a duality transformation. The construction will be based on the example, eq.(5.61), and will proceed in two steps. We will first find an

[1] These conditions are manifestly invariant under the $O(6,22,\mathbb{Z})$ group. Once we choose a particular basis to write the initial charges as, $(\vec{Q}_e, \vec{Q}_m)$, there is no residual $SL(2,\mathbb{Z})$ invariance left. The conditions, eq.(5.80), are written in this basis, and are in-effect also $SL(2,\mathbb{Z})$ invariant.

altered set of charges for which an $SL(2,\mathbb{Z})$ transformation meeting condition, eq.(5.50), exists. Then in the second step we will further alter these charges so that the $SL(2,\mathbb{Z})$ transformation we have identified in the first step, followed by an appropriate $O(6,22,\mathbb{Z})$ transformation, brings this final set of altered charges to the Cardy limit. At both stages we will ensure that the changes in the charges are small and that the conditions, eq.(5.80), are met.

In the starting configuration, the $D0-D4$ brane charges are large, of order, Q, and roughly comparable, so that α satisfies condition, eq.(5.46), but $\alpha \not\ll 1$.

The First Step:

In the first step, we then change the $D0-D4$ charges (no new charges are excited at this stage) so that the new value of α is a rational, m/n. The change in α can be kept small,

$$\left|\alpha - \frac{m}{n}\right| < \epsilon, \tag{5.81}$$

with,

$$\epsilon < 1, \tag{5.82}$$

if we take the integers, m, n to be sufficiently large,

$$m, n \sim O(1/\epsilon). \tag{5.83}$$

The required change in the charges is of order ΔQ where,

$$\frac{\Delta Q}{Q} \sim \frac{\Delta\alpha}{\alpha} \sim \epsilon \tag{5.84}$$

Next, we change one of the $D4$-brane charges by order unity, this gives rise to a final value of[1] α,

$$\alpha = \frac{m}{n}\sqrt{1 - \frac{1}{Q}}. \tag{5.85}$$

Now choosing,

$$c = m, d = n, \tag{5.86}$$

eq.(5.53) is met, if the condition,

$$\frac{mn}{Q} < \delta, \tag{5.87}$$

[1] For example, if only , $p^2, p^3 \neq 0$, in the basis, eq.(1.5), then changing p^2 by unity would give, $\alpha = \frac{m}{n}\sqrt{1 - \frac{1}{p^2}} = \frac{m}{n}\sqrt{1 - \frac{1}{Q}}$, if $p^2 = Q$.

is valid. Using eq.(5.83) this gives,

$$\epsilon > \frac{1}{\sqrt{\delta Q}}. \tag{5.88}$$

We will see below, that δ which was introduced first in eq.(5.53), can be taken to be a fixed small number, meeting condition, eq.(5.54), and not scaling like an inverse power of Q. Then by taking Q to be sufficiently big, so that

$$Q \gg \frac{1}{\delta} \gg 1, \tag{5.89}$$

condition eq.(5.88) can be made compatible with eq.(5.82). To keep the shift in the charges small, it is best to take ϵ to be as small as possible, subject to the condition, eq.(5.88). We will take,

$$\epsilon \sim \frac{1}{\sqrt{\delta Q}}. \tag{5.90}$$

It is useful in the subsequent discussion to distinguish between the altered charges obtained at this stage and the original charges we started with. We denote the altered charges by the tilde superscript. In the basis, eq.(1.4), eq.(1.5), we have,

$$\begin{aligned} \vec{\tilde{Q}}_e &= (\tilde{q}_0, -\tilde{p}^1, 0, 0, \cdots, 0) \\ \vec{\tilde{Q}}_m &= (0, 0, \tilde{p}^i, 0, 0, 0, 0). \end{aligned} \tag{5.91}$$

Before proceeding further it is worth examining condition eq.(5.87) more carefully. The inequality, eq.(5.50), arose from eq.(5.20). It's stronger form is given by the condition in eq.(5.19). Here, $p^{1'}$ is the charge that arised due to the $D4$-branes wrapping the K3, in the final configuration which lies in the Cardy limit and which is obtained by starting with the altered charges and doing the duality transformation. From eq.(5.19), eq.(5.53) we see that δ must satisfy the condition,

$$\delta \ll \frac{1}{|p^{1'}|}. \tag{5.92}$$

Now if $p^{1'} \sim Q$ we see that eq.(5.92), eq.(5.88), together imply that the condition in eq.(5.82) cannot be met. We will see below that the final charge configuration has a value for $p^{1'}$ which is much smaller than Q. In fact $p^{1'}$ can be taken to be $O(1)$ and not $O(Q)$. Thus, as was mentioned above, δ can be taken to be a small number not scaling like an inverse power of Q. One can then choose Q to meet the condition, eq.(5.89), and this will then suffice to meet eq.(5.88) and eq.(5.82).

From eq.(5.84) and eq.(5.90) we see that the required change in the charges are of the order,

$$\frac{\Delta Q}{Q} \sim \epsilon \sim \frac{1}{\sqrt{\delta Q}}. \tag{5.93}$$

This gives,

$$\Delta Q \sim \sqrt{\frac{Q}{\delta}}. \tag{5.94}$$

We see that while, $\Delta Q \gg 1$, from eq.(5.93), eq.(5.89), it follows that,

$$\frac{\Delta Q}{Q} \sim \frac{1}{\sqrt{\delta Q}} \ll 1, \tag{5.95}$$

so that the fractional change in the charges are small. Condition eq.(5.95) ensures that the requirements in eq.(5.80) are met, so that the changes in charge are small.

We have now completed the first step. The $SL(2,\mathbb{Z})$ transformation that takes the altered charges to the Cardy limit has the form,

$$A = \begin{pmatrix} a & b \\ m & n \end{pmatrix} \tag{5.96}$$

The integers m, n have been determined in terms of α for the altered charges above eq.(5.85). As discussed in the appendix , a, b, can be chosen so that they satisfy the conditions,

$$\begin{aligned} a &\sim O(m) \\ b &\sim O(n). \end{aligned} \tag{5.97}$$

The relations in eq.(5.97) will be important in the following discussion.

The Second Step:

We now proceed to the second step and construct the $O(6,22,\mathbb{Z})$ transformation. This will require a further change in the charges. We will excite extra charges which lie in the last two $\mathcal{H} \oplus \mathcal{H}$ subspaces in eq.(3.27). These are charges which arises from the T^2. The altered charges at the first stage are given in eq.(5.91). We now change them further, so that the final altered charges take the form,

$$\begin{aligned}\vec{\tilde{Q}}_e &= (\tilde{q}_0, -\tilde{p}^1, 0, 0, \cdots, -b, 0, n, 0) \\ \vec{\tilde{Q}}_m &= (0, 0, \tilde{p}^i, a, 0, -m, 0).\end{aligned} \tag{5.98}$$

Here a, b, m, n are elements of the $SL(2,\mathbb{Z})$ matrix, eq.(5.96). Note that, $\tilde{q}_0, \tilde{p}^i \sim O(Q)$. From eq.(5.83), eq.(5.97), we see that $a, b, m, n \sim 1/\epsilon$. From, eq.(5.90) we then learn that

$$a, b, m, n \sim \frac{1}{\epsilon} \sim \sqrt{\delta Q}. \tag{5.99}$$

The changes in charges that give eq.(5.98) then meet the condition

$$\frac{\Delta Q}{Q} \sim \sqrt{\frac{\delta}{Q}} \ll 1, \tag{5.100}$$

where the last inequality follows from the fact that the charge Q meets the condition, eq.(5.89). This ensures that the conditions in eq.(5.80) are met.

The $SL(2,\mathbb{Z})$ transformation, eq.(5.96), followed by an $O(6,22,\mathbb{Z})$ transformation that we describe explicitly in the appendix , now brings the charges, eq.(5.98) to the form,

$$\begin{aligned}\vec{Q}'_e &= (a\tilde{q}_0, 1, b\tilde{p}^i, 0, -ma\tilde{q}_0\tilde{p}^1, 1, -a\tilde{q}_0(a\tilde{p}^1 + 1)) \\ \vec{Q}'_m &= (m\tilde{q}_0, 0, n\tilde{p}^i, 1, -m^2\tilde{q}_0\tilde{p}^1, 0, -m(a\tilde{p}^1 + 1)\tilde{q}_0).\end{aligned} \tag{5.101}$$

These charges are in the Cardy limit. Since the second entry in $\vec{Q}'_m$ vanishes, the $D6$-brane charge vanishes. From the second entry in $\vec{Q}'_e$ we see that $|p^{1'}|$ is unity, as was promised above. Finally, the extra charges excited in going from eq.(5.91) to eq.(5.98) does not change the value of $(\vec{\tilde{Q}}_m)^2$. Thus,

$$\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}} \simeq \left(\frac{mn}{Q}\right) \simeq \delta \ll 1, \tag{5.102}$$

where we have used eq.(5.83) for m, n and eq.(5.90) for ϵ. It then follows that eq.(5.19) is met and the final charges are in the Cardy limit.

Two comments before we end. First, there is some leeway in the $O(6, 22, Z)$ transformation which acting on the charges, eq.(5.98), brings them to the Cardy limit. For example, an $O(6, 22, \mathbb{Z})$ transformation can be found that results in $p^{1'}$ being a number much large than unity, but not scaling with Q. Second, we have seen in subsection 3.2 that in the vicinity of one set of charges which can brought to the Cardy limit, are other near by charges meeting condition, eq.(5.67), which can also be taken to the Cardy limit. Using, eq.(5.86), eq.(5.83), we see that eq.(5.67) takes the form,

$$\frac{\Delta Q}{Q} \sim \delta\epsilon^2. \tag{5.103}$$

Since, $\delta \ll 1, \epsilon < 1$, the size of this variation, $\frac{\Delta Q}{Q} \ll \epsilon$. Thus starting from one of the special charge configurations which can be brought to the Cardy limit, a variation of order, eq.(5.103), takes us to charges of the generic kind which can no longer be taken to the Cardy limit by a duality transformation. These charges will have to be changed by an amount of order, eq.(5.84), to be able to bring them to the Cardy limit.

5.4 The $D0 - D6$ System

In this section we consider the $D0-D6$ system, where only $q_0, p^0 \neq 0$, and all other charges vanish, eq.(1.4), eq.(1.5). We show that such a charge configuration can never be brought to the Cardy limit. For this set of charges we have the following relations,

$$\begin{aligned} \vec{Q}_e^2 &= 0 \\ \vec{Q}_m^2 &= 0 \\ \vec{Q}_e \cdot \vec{Q}_m &= q_0 p^0. \end{aligned} \tag{5.104}$$

The invariant I, eq.(1.7), is,

$$I = -(q_0 p^0)^2, \tag{5.105}$$

It is negative, and the state breaks supersymmetry.

Let us assume that there is an $SL(2, \mathbb{Z})$ transformation, eq.(5.21) which followed by an $O(6, 22, \mathbb{Z})$ transformation brings the charges to the Cardy limit.

Denoting the final charges by $\vec{Q}'_e, \vec{Q}'_m$, we have that,

$$\vec{Q}'^2_m = 2cdq_0p^0. \tag{5.106}$$

If the final charges are in the Cardy limit, it follows from eq.(5.14), and the fact that $|p^{1'}| \geq 1$ that,

$$\frac{|(\vec{Q}'_m)^2|}{\sqrt{|I|}} \ll 1. \tag{5.107}$$

From, eq.(5.106) and eq.(5.105), this leads to the condition,

$$|cd| \ll 1. \tag{5.108}$$

Now note that c, d are integers. Thus the only way in which eq.(5.108) can be met is if $cd = 0$. This will mean that $\vec{Q}'^2_m = 0$ and hence the central charge, eq.(5.11), for the final charges vanishes. We do not want the central charge to vanish since the resulting AdS_3 space-time would not be described by weakly coupled supergravity. As a result we find that there is no duality transformation which can bring the $D0 - D6$ system to the Cardy limit.

In parallel with our discussion of section 3.4 we now ask if there are near by charges which can be brought to the Cardy limit. The following construction shows that such a set of charges does exits, as in the non-supersymmetric $D0-D4$ system. The $D0-D6$ system we start with has charges which in the basis, eq.(1.4), eq.(1.5), are given by,

$$\begin{aligned} \vec{Q}_e &= (q_0, 0, \cdots, 0) \\ \vec{Q}_m &= (0, p^0, 0, \cdots, 0). \end{aligned} \tag{5.109}$$

The charges meet the condition,

$$|\vec{Q}_e \cdot \vec{Q}_m| = |q_0p^0| \gg 1. \tag{5.110}$$

For the change in the charges to be small the condition, analogous to eq.(5.80) in the $D0 - D4$ case, is given by,

$$\left| \frac{\vec{Q}_e \cdot \Delta\vec{Q}_{e,m}}{\vec{Q}_e \cdot \vec{Q}_m} \right| \ll 1$$

$$\begin{aligned} \left|\frac{\vec{Q}_m\cdot\Delta\vec{Q}_{e,m}}{\vec{Q}_e\cdot\vec{Q}_m}\right| &\ll 1 \\ \left|\frac{\Delta\vec{Q}_{e,m}\cdot\Delta\vec{Q}_{e,m}}{\vec{Q}_e\cdot\vec{Q}_m}\right| &\ll 1. \end{aligned} \tag{5.111}$$

Now consider the altered charges,

$$\begin{aligned} \vec{Q}_e &= (q_0, 0, 1, 0, \cdots, 0) \\ \vec{Q}_m &= (0, p^0, -1, 1, \cdots, 0). \end{aligned} \tag{5.112}$$

It is easy to see that conditions, eq.(5.111), are met and the changes in the charges are small.

In eq.(5.112), we have activated additional charges lying in the second Hyperbolic sublattice, $\mathcal{H}$, defined in eq.(3.27). We could have instead activated the additional charges to lie in any of the other Hyperbolic sublattices (or infact the $\mathcal{E}_8$ sublattices), and a similar discussion would go through.

Now consider an $O(2,2)$ transformation acting on the two $\mathcal{H}$ sublattices in which the charges lie, of the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & p^0 & 0 \\ 0 & 0 & 1 & 0 \\ -p^0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.113}$$

This brings the altered charges, eq.(5.112), to the form,

$$\begin{aligned} \vec{Q}'_e &= (q_0, p^0, 1, -p^0 q_0, 0, \cdots, 0) \\ \vec{Q}'_m &= (0, 0, -1, 1, 0, \cdots 0). \end{aligned} \tag{5.114}$$

These charges are in the Cardy limit. The second entry in $\vec{Q}'_m$ vanishes, therefore, $p^{0'} = 0$. Also, $p^{1'} = p^0, (\vec{Q}'_m)^2 = -2$, so that the condition, eq.(5.14), is met, as long as

$$|q_0| \gg 1. \tag{5.115}$$

Note that the central charge, $C \sim |p^{1'}(\vec{Q}'_m)^2| \sim (p^0)^2$. This meets the condition, $C \gg 1$ if $|p^0| \gg 1$. Alternatively, if $p^0 \sim O(1)$, we can excite additional charges in eq.(5.112) so that, for example, $p^{1'} \gg 1$, and thus $C \gg 1$.

5.5 Absence of Magnetic Monopole Charge

We have mentioned above that lifting a configuration with $D6$ brane charge to M-theory cannot give a locally AdS_3 spacetime in the near-horizon limit. We prove this statement here.

We start with a general extremal black hole, carrying charges given in eq.(1.4), eq.(1.5), in four dimensions in IIA theory. The near horizon geometry is $AdS_2 \times S^2$. An AdS_2 space-time has $SO(2,1)$ symmetry. This gets enhanced to $SO(2,2)$ in the AdS_3 case [1]. In the special case where the black hole carries no $D0$-brane charge, N units of $D6$-brane charge, and arbitray values of the other charges, it is well known that one does not get the $SO(2,2)$ symmetry of AdS_3 in the near horizon limit geometry. The $D6$-brane charge is KK monopole charge along the M direction. This charge results in the M-direction being fibered over the S^2 resulting in the near horizon geometry of form, $AdS_2 \times S^3/Z_N$.

Here we will examine what happens if the black hole carries both $D0$ and $D6$ brane charges, besides having arbitrary values of the other charges, and find that the symmetries of the near horizon geometry are $SO(2,1) \times SO(3) \times U(1)$ and are therefore not enhanced to $SO(2,2)$. This proves that the only way to get a locally AdS_3 geometry on lifting to M-theory is for the $D6$-brane charge to vanish.

Lifting the $AdS_2 \times S^2$ near-horizon geometry to M-theory, gives,

$$\begin{aligned} ds^2 &= R^2(-\cosh^2\theta_1 d\phi_1^2 + d\theta_1^2) + R^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ g_{\psi\psi}(d\psi + \alpha\sinh\theta_1 d\phi_1 + \beta\cos\theta_2 d\phi_2)^2 \end{aligned} \tag{5.116}$$

Here we are using Global coordinates θ_1, ϕ_1 for AdS_2, polar coordinates, θ_2, ϕ_2 for the S^2, and denoting the M-theory direction as ψ. The metric component, $g_{\psi\psi}$, is a constant. α, β are proportional to the $D0$ and $D6$ brane charges and are non-vanishing if these charges are non-vanishing. We seek the Killing vectors for this metric.

[1] Our analysis of the symmetries in this section will be local. So the breaking of $SO(2,2)$ symmetry due to identifications which are made in the BTZ geometry will not be relevant.

It is convenient to analytically continue the AdS_2 metric to that of S^2 as follows,

$$\begin{aligned}
\theta_1 &\rightarrow i\left(\frac{\pi}{2} - \theta_1\right) \\
(R^2)_{AdS} &\rightarrow -R^2 \\
\alpha &\rightarrow -i\alpha. \qquad (5.117)
\end{aligned}$$

This gives,

$$\begin{aligned}
ds^2 &= R^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + R^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\
&+ g_{\psi\psi}(d\psi + \alpha\cos\theta_1 d\phi_1 + \beta\cos\theta_2 d\phi_2)^2. \qquad (5.118)
\end{aligned}$$

We show that the isometry group of this metric is, $SO(3) \times SO(3) \times U(1)$, it will then follow by analytic continuation that the isometry group of eq.(5.116) is, $SO(2,1) \times SO(3) \times U(1)$.

By rescaling the ψ coordinate, α and β, this metric can be written as,

$$\begin{aligned}
ds^2 &= R^2[(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + (d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\
&+ (d\psi' + \alpha'\cos\theta_1 d\phi_1 + \beta'\cos\theta_2 d\phi_2)^2]. \qquad (5.119)
\end{aligned}$$

α', β' are proportional to α, β and only vanish when the latter do. Next we drop the overall factor of R^2, and rescale ϕ_1, ϕ_2 as follows,

$$\alpha'\phi_1 \rightarrow \phi_1, \quad \beta'\phi_2 \rightarrow \phi_2. \qquad (5.120)$$

Note this rescaling is well defined only if α', β', and hence α, β, are non-vanishing. This gives for the metric,

$$\begin{aligned}
ds^2 &= d\theta_1^2 + d\theta_2^2 + (1 + (\tilde{\alpha})^2\sin^2\theta_1)d\phi_1^2 + (1 + (\tilde{\beta})^2\sin^2\theta_2)d\phi_2^2 + d\psi^2 \\
&+2\cos\theta_1 d\psi d\phi_1 + 2\cos\theta_2 d\psi d\phi_2 + 2\cos\theta_1\cos\theta_2 d\phi_1 d\phi_2, \qquad (5.121)
\end{aligned}$$

where,

$$(\tilde{\alpha})^2 = \frac{1}{\alpha'^2} - 1 \qquad (5.122)$$

$$(\tilde{\beta})^2 = \frac{1}{\beta'^2} - 1. \qquad (5.123)$$

To save clutter we will henceforth drop the tildes on α, β and denote the metric in eq.(5.121) as,

$$\begin{aligned} ds^2 &= d\theta_1^2 + d\theta_2^2 + (1+\alpha^2\sin^2\theta_1)d\phi_1^2 + (1+\beta^2\sin^2\theta_2)d\phi_2^2 + d\psi^2 \\ &\quad +2\cos\theta_1 d\psi d\phi_1 + 2\cos\theta_2 d\psi d\phi_2 + 2\cos\theta_1\cos\theta_2 d\phi_1 d\phi_2. \end{aligned} \tag{5.124}$$

The reader should note that α, β, in eq.(5.124) are different from α, β, as appearing in eq.(5.118).

We now turn to studying the isometries of the metric, eq.(5.124). First note that $\partial_{\phi_1}, \partial_{\phi_2}, \partial_\psi$, are commuting isometries of this metric. They can be taken to be part of the Cartan generators of the full isometry group. Any other killing vector, ξ, can then be taken to carry definite charges with respect to these generators, and satisfies the relations,

$$[\partial_{\phi_1}, \xi] = im_1\xi, \tag{5.125}$$

$$[\partial_{\phi_2}, \xi] = im_2\xi, \tag{5.126}$$

$$[\partial_\psi, \xi] = im_3\xi, \tag{5.127}$$

where m_1, m_2, m_3 are the eigenvalues with respect to these three isometries.

The killing vector, ξ, must satisfy the Killing conditions,

$$\partial_\alpha\xi^\gamma g_{\gamma\beta} + \partial_\beta\xi^\gamma g_{\gamma\alpha} + \xi^\gamma\partial_\gamma g_{\alpha\beta} = 0 \tag{5.128}$$

for all values of α, β.

These Killing conditions are studied in more detail in the appendix . One finds that there are only four more non-trivial Killing vectors, corresponding to $m_1 = \pm\sqrt{1+\alpha^2}, m_2 = m_3 = 0$ and $m_2 = \pm\sqrt{1+\beta^2}, m_1, m_3 = 0$. Altogether there are then seven Killing vectors, given by,

$$\begin{aligned} \xi_1 &= e^{i\sqrt{1+\alpha^2}\phi_1}\left[\partial_{\theta_1} + \frac{i}{\sqrt{1+\alpha^2}}\cot\theta_1\partial_{\phi_1} - \frac{i}{\sqrt{1+\alpha^2}}\frac{1}{\sin\theta_1}\partial_\psi\right] \\ \xi_2 &= e^{-i\sqrt{1+\alpha^2}\phi_1}\left[\partial_{\theta_1} - \frac{i}{\sqrt{1+\alpha^2}}\cot\theta_1\partial_{\phi_1} + \frac{i}{\sqrt{1+\alpha^2}}\frac{1}{\sin\theta_1}\partial_\psi\right] \\ \xi_3 &= \partial_{\phi_1} \end{aligned}$$

$$\begin{aligned}
\xi_4 &= e^{i\sqrt{1+\beta^2}\phi_2}\left[\partial_{\theta_2} + \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} - \frac{i}{\sqrt{1+\beta^2}}\frac{1}{\sin\theta_2}\partial_\psi\right] \\
\xi_5 &= e^{-i\sqrt{1+\beta^2}\phi_2}\left[\partial_{\theta_2} - \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} + \frac{i}{\sqrt{1+\beta^2}}\frac{1}{\sin\theta_2}\partial_\psi\right] \\
\xi_6 &= \partial_{\phi_2} \\
\xi_7 &= \partial_\psi
\end{aligned} \qquad (5.129)$$

The first three give rise to an $SO(3)$ isometry, the second three to another $SO(3)$ and the last to an $U(1)$ isometry, giving the total symmetry group, $SO(3) \times SO(3) \times U(1)$. After analytic continuation this implies that the metric we started with has isometries, $SO(2,1) \times SO(3) \times U(1)$.

We refer the reader to the appendix for more details.

Chapter 6

Appendix.

6.1 Hecke Operators and the Multiplicative Lift

In this section we summarise the construction of Hecke operators and the multiplicative lift, following [16]. Let us define $\Delta_N(t)$ as

$$\Delta_N(t) = \{g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}; \quad a,b,c,d \in \mathbb{Z}, \quad \det(g) = t\}. \tag{6.1}$$

The action of the Hecke operator T_t on a weak Jacobi form $\phi_{k,m}$ is then given by

$$T_t(\phi_{k,m})(\tau,z) = t^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\backslash\Delta_N(t)} (c\tau+d)^{-k} \exp{(-\frac{2\pi imcz^2}{c\tau+d})}\phi_{k,m}(\frac{a\tau+b}{d}, az). \tag{6.2}$$

To compute everything concretely, we need to define representatives of $\Gamma_0(N)\backslash\Delta_N(t)$. Choose the complete set of cusps $\{s\}$ of $\Gamma_0(N)$ represented by the set of representative matrices $\{g_s\}$. Let

$$g_s \,\epsilon\, SL(2,\mathbb{Z}) = \begin{pmatrix} x_s & y_s \\ z_s & w_s \end{pmatrix} \tag{6.3}$$

Define a natural number h_s by

$$g_s^{-1}\Gamma_0(N)g_s \cap P(\mathbb{Z}) = \{\pm \begin{pmatrix} 1 & h_s n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z}\}$$

where $P(\mathbb{Z})$ is the set of all upper-triangular matrices over integers with unit determinant. We can then write

$$\Gamma_0(N)\backslash\Delta_N(t) = \cup_s\{g_s \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; \; a, b, d \in \mathbb{Z}, ad = t, \; az_s = 0 \, mod \, N, \; b = 0, ..., h_s d - 1\}. \tag{6.4}$$

For each cusp we define $n_s = \frac{N}{\text{g.c.d}(z_s, N)}$. We define

$$\phi_s(\tau, z) = \phi(\frac{x_s\tau + y_s}{z_s\tau + \omega_s}, \frac{z}{z_s\tau + \omega_s}), \tag{6.5}$$

with Fourier expansion

$$\phi_s(\tau, z) = \sum_{n,l} c_s(n, l) \exp(2\pi i(n\tau + lz)). \tag{6.6}$$

As usual, one can show that $c_s(n, l)$ depends only on $4n - l^2$ and $l \, mod \, 2$ so we write $c_s(n, l) = c_{s,l}(4n - l^2)$ following the notation in [16]. In general $n \in h_s^{-1}\mathbb{Z}$ need not be an integer. If 4 does not divide h_s, which is true for all cases of our interest, then $l \, mod \, 2$ is determined only by $4n - l^2$ and in that case we can write simply $c_s(4n - l^2) = c_{s,l}(4n - l^2)$.

For $\Gamma_0(N)$ with N prime, there are only two cusps, one at $i\infty$ and the other at 0 in the fundamental domain. Hence the index s runs over 1 and 2. For this case, various objects with the subscript s defined in the formula for the lift above take the following values:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_1 = 1, \quad z_1 = 0, \quad n_1 = 1 \tag{6.7}$$

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad h_2 = N, \quad z_2 = 1, \quad n_2 = N \tag{6.8}$$

In this case we can then write

$$\Gamma_0(N)\backslash\Delta_N(t) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{Z}); ad = t, b = 0, ..., d - 1\} \tag{6.9}$$

$$\cup \quad \{g_2 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{Z}); ad = t, a \equiv 0 \, mod \, N, b = 0, ..., Nd - 1\} \tag{6.10}$$

Given a weak Jacobi form ϕ of weight 0 and index 1, we can define

$$L\phi(\rho,\nu,\sigma) = \sum_{t=1}^{\infty} T_t(\phi)(\rho,\nu)\exp(2\pi i\sigma t). \tag{6.11}$$

Using the explicit representation of the Hecke operators, one can then show [16]

$$\begin{aligned}
L\phi &= \sum_s \sum_{t=1}^{\infty} \sum_{\substack{ad=t \\ az_s=0 \, mod \, N}} \sum_{b=0}^{h_s d-1} \phi_s(\frac{a\rho+b}{d}, a\nu)\exp(2\pi i t\sigma) &(6.12)\\
&= \sum_s \sum_{t=1}^{\infty} \sum_{\substack{ad=t \\ a\in n_s\mathbb{Z}}} (ad)^{-1} dh_s \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nd-l^2)\exp(2\pi i(an\rho + al\nu + t\sigma)) &(6.13)\\
&= \sum_s h_s \sum_{a=1}^{\infty} \frac{1}{an_s} \sum_{m=1}^{\infty} \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nd-l^2)\exp(2\pi i(an\rho + al\nu + m\sigma)) &(6.14)\\
&= \sum_s \frac{h_s}{n_s} \log\left(\prod_{\substack{l,m,n\,\in\,\mathbb{Z} \\ m\geq 1}} (1-e^{n_s(n\rho+l\nu+m\sigma)})^{c_{s,l}(4mn-l^2)} \right). &(6.15)
\end{aligned}$$

6.2 Consistency Check

As a consistency check we compare the coefficients of the leading powers of p, q, y in the multiplicative lift with the Fourier expansion of Φ_6 obtained using the additive lift in [5]. The leading terms, corresponding to a single power of p, in the expansion are

$$-pqy\prod_n (1-q^n)^{c_1(0)}(1-q^n y)^{c_1(-1)}(1-q^n y^{-1})^{c_1(-1)}(1-q^{2n})^{c_2(0)}. \tag{6.16}$$

Substituting the values of the c_1 and c_2 coefficients and collecting terms with the same powers in q and y together, we obtain

$$\Phi_6(\Omega) = [(2-y-\frac{1}{y})q + (-4+\frac{2}{y^2})q^2 + (-16-\frac{1}{y^3}) - \frac{4}{y^2} + \frac{13}{y} + 13y - 4y^2 - y^3)q^3]p + \ldots \tag{6.17}$$

To compare, we now read off the coefficients from its sum representation derived in [5] by the additive lift. The seed for the additive lift is

$$\phi_{6,1} = \eta^2(\tau)\eta^8(2\tau)\theta_1^2 = \sum_{l,n\geq 0} C(4n-l^2)q^n y^l \tag{6.18}$$

The lift is then given by

$$\Phi_6(\Omega) = \sum_{m\geq 1} T_m[\phi_{6,1}(\rho,\nu)]p^m, \tag{6.19}$$

with the Fourier expansion

$$\Phi_6(\Omega) = \sum_{\substack{m>0,\\ n\geq 0, r\in\mathbb{Z}}} a(n,m,r)q^n p^m y^r. \tag{6.20}$$

Given the action of the Hecke operators, $a(n,m,r)$ can be read off from this expansion knowing $C(N)$ as in (6.18). These are in precise agreement with the same coefficients in the expansion of the product representation given above in (6.17).

6.3 Tightening the Conditions in the Supersymmetric Case

A supersymmetric $D0-D4$ system, which can be taken to the Cardy limit, must meet the condition, eq.(5.31). In this appendix we show that this condition can be somewhat strengthened, leading to eq.(5.32).

This comes about as follows. In general the $SL(2,\mathbb{Z})$ transformation, eq.(5.29), will be followed by an $O(6,22,\mathbb{Z})$ transformation, $B \in O(6,22,\mathbb{Z})$, to obtain the final configuration, $(\vec{Q}'_e, \vec{Q}'_m)$ which is given by,

$$\vec{Q}'_e = B\vec{Q}_e + bB\vec{Q}_m \tag{6.21}$$

$$\vec{Q}'_m = B\vec{Q}_m. \tag{6.22}$$

We will see shortly that this final configuration is in the Cardy limit if and only if the configuration, $(\tilde{\vec{Q}}_e, \tilde{\vec{Q}}_m)$, defined by,

$$(\tilde{\vec{Q}}_e, \tilde{\vec{Q}}_m) = (B\vec{Q}_e, B\vec{Q}_m) \tag{6.23}$$

is in the Cardy limit. Note that the charges, $(\tilde{\vec{Q}}_e, \tilde{\vec{Q}}_m)$, are obtained by applying only the transformation, $B \in O(6,22,\mathbb{Z})$ on $(\vec{Q}_e, \vec{Q}_m)$. Applying condition eq.(5.18) to the charges, $(\vec{\tilde{Q}}_e, \vec{\tilde{Q}}_m)$, we learn that for them to be in the Cardy limit,

$$|I| \gg \left(\tilde{p}^1 \left(\vec{\tilde{Q}}_m\right)^2\right)^2. \tag{6.24}$$

From eq.(6.23) we see that $\left(\vec{\tilde{Q}}_m\right)^2 = \vec{Q}_m^2$. Now since $\vec{\tilde{Q}}_e$ is obtained by applying an $O(6,22,\mathbb{Z})$ transformation to $\vec{Q}_e$, the minimum value $\tilde{p}^1$ can take is $\gcd(\vec{Q}_e)$. Eq.(5.32) then follows, after using eq.(5.25) for I.

To complete the argument let us show that $(\vec{Q}'_e, \vec{Q}'_m)$ can be in the Cardy limit if an only if $(\vec{\tilde{Q}}_e, \vec{\tilde{Q}}_m)$ is in the Cardy limit. To see this we note that from eq.(6.21) and eq.(6.23) it follows that,

$$\vec{Q}'_e = \vec{\tilde{Q}}_e + b\vec{\tilde{Q}}_m, \tag{6.25}$$

and,

$$\vec{Q}'_m = \vec{\tilde{Q}}_m. \tag{6.26}$$

If $\vec{Q}'_m$ is in the Cardy limit the $D6$-brane charge for this configuration must vanish, so, $p^{0'} = 0$. From eq.(6.26) we see this implies that $\tilde{p}^0$ also vanishes. Eq.(6.26) also implies that $(\vec{Q}'_m)^2 = (\vec{\tilde{Q}}_m)^2$. And eq.(6.25) implies that $p^{1'} = \tilde{p}^1$. The second condition for the Cardy limit, eq.(5.14), is

$$I \gg 6(p'_1 \vec{Q}'^2_m)^2. \tag{6.27}$$

Since I is a duality invariant, it then follows that the condition eq.(6.27) is the same as the corresponding condition in terms of the tilde variables,

$$I \gg 6\left(\tilde{p_1}\left(\vec{\tilde{Q}}_m\right)^2\right)^2. \tag{6.28}$$

6.4 Some Non-supersymmetric Examples

In this appendix we present some examples of charges in th non-supersymmetric case, which can be brought to the Cardy limit after a duality transformation.

We take,

$$\vec{Q}_e = (p-1, -1, 0, 0, 0, \cdots 0) \tag{6.29}$$

$$\vec{Q}_m = (0, 0, 1, p, 0, \cdots 0), \tag{6.30}$$

with,

$$p \gg 1. \tag{6.31}$$

The quartic invariant, I, eq.(1.7) is,

$$I = -4p(p-1). \tag{6.32}$$

The value of $p^1 = 1$, and $\vec{Q}_m^2 = 2p$, so we see that condition, eq.(5.14) is not met and the starting configuration is not in the Cardy limit. In this example, $|\vec{Q}_e^2| < |\vec{Q}_m^2|$, so that $\alpha > 1$ to begin, we therefore carry out the $SL(2,\mathbb{Z})$ transformation, $(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$, which gives,

$$\begin{aligned} \vec{Q}_e &= (0, 0, 1, p, 0, \cdots) \qquad (6.33) \\ \vec{Q}_m &= -(p-1, -1, 0, 0, 0, \cdots, 0). \qquad (6.34) \end{aligned}$$

The resulting value of α is,

$$\alpha = \sqrt{\frac{p-1}{p}}. \tag{6.35}$$

This is of the form discussed above in eq.(5.60). Starting with the charges, eq.(6.33), we now carry out $SL(2,\mathbb{Z}) \times O(6,22,\mathbb{Z})$ transformations which bring it in the Cardy limit. The $SL(2,\mathbb{Z})$ transformation is,

$$A = (\begin{pmatrix} (p-1) & -p \\ 1 & -1 \end{pmatrix}) \tag{6.36}$$

with resulting charges,

$$\begin{aligned} \tilde{\vec{Q}}_e &= (p(p-1), -p, p-1, (p-1)p, 0 \cdots, 0) \qquad (6.37) \\ \tilde{\vec{Q}}_m &= (p-1, -1, 1, p, 0, \cdots, 0) \qquad (6.38) \end{aligned}$$

This is followed by an $O(6,22,\mathbb{Z})$ transformation,

$$B = (\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}) \tag{6.39}$$

By this we mean that B acts non-trivially on the 4 dimensional sublattice of charges where the inner product is given by the first two factors of $\mathcal{H}$ in eq.(1.1), and acts trivially on the rest of the lattice. The transformation B gives the final charges,

$$\begin{aligned} \vec{Q}'_e &= (p(p-1), -1, p-1, 0, 0, \cdots, 0) && (6.40) \\ \vec{Q}'_m &= (p-1, 0, 1, 1, 0, \cdots 0). && (6.41) \end{aligned}$$

Since the second entry in $\vec{Q}'_m$ vanishes, the $D6$ brane charge in the final configuration vanishes as is needed for the Cardy limit. From the second entry in $\vec{Q}'_e$ we see that $|p^{1'}| = 1$, and we also have that, $|\vec{Q}'^2_m| = 2$. Since I is given by, eq.(6.32), we see that condition eq.(5.14) is now met and the final set of charges are in the Cardy limit.

To obtain an example with all final charges which are non-zero being much bigger than unity we can scale the initial charges, so that $(\vec{Q}_e, \vec{Q}_m) \to (\lambda\vec{Q}_e, \lambda\vec{Q}_m), \lambda \gg 1$, and now take,

$$p \gg \lambda. \qquad (6.42)$$

Another example is as follows. We take,

$$\begin{aligned} \vec{Q}_e &= (q_0, -p^1, 0, 0, \cdots, 0) && (6.43) \\ \vec{Q}_m &= (0, 0, p^2, p^2, 0, \cdots, 0), && (6.44) \end{aligned}$$

with

$$|q_0| \sim |p^1|. \qquad (6.45)$$

This system is not in the Cardy limit.

Applying the $O(6,22)$ transformation which acts non-trivially only on the 4 dimensional sublattice gives by the first two factors of $\mathcal{H}$ in eq.(1.1) and has the form,

$$B = (\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}), \qquad (6.46)$$

gives the final charges,

$$\vec{Q}'_e = (q_0, q_0 - p^1, q_0, -q_0, 0 \cdots 0) \tag{6.47}$$
$$\vec{Q}'_m = (0, 0, p^2, p^2, 0, \cdots). \tag{6.48}$$

As long as the condition,

$$|q_0 p^1| \gg 6(p_1 - q_0)^2 (p^2)^2 \tag{6.49}$$

is met this final configuration satisfies eq.(5.14) and is in the Cardy limit.

6.5 More Details on Changing the Charges

Two results of relevance to section 3.4 will be derived here.

First, we show that an $SL(2,\mathbb{Z})$ matrix of the form, eq.(5.96), can always be found where a, b meet the conditions, eq.(5.97).

The integers, m, n are determined in terms of the value of α for the altered charges, eq.(5.85). These can be taken to be coprime. Thus an $SL(2,\mathbb{Z})$ matrix can always be found of the form,

$$A' = (\begin{pmatrix} a' & b' \\ m & n \end{pmatrix}) \tag{6.50}$$

The integers, a', b' satisfy the condition,

$$\det(A) = a'n - b'm = 1. \tag{6.51}$$

From here it follows that,

$$\left[\frac{a'}{m}\right] = \left[\frac{b'}{n}\right] \tag{6.52}$$

where $[\frac{a'}{m}]$ denotes the integer part of $|\frac{a'}{m}|$, and similarly for $[\frac{b'}{n}]$. Now, the allowed values of integers, a', b', which satisfy eq.(6.51) are not unique. One can see that if a', b' satisfy eq.(6.51) then so do,

$$a = a' - \left[\frac{a'}{m}\right] m \tag{6.53}$$
$$b = b' - \left[\frac{a'}{m}\right] n \tag{6.54}$$

From eq.(6.52) it follows that the relations in eq.(5.97) are valid. The resulting $SL(2,\mathbb{Z})$ transformation is then given in eq.(5.96).

Next we show that starting with the charges, eq.(5.98), and applying the $SL(2,\mathbb{Z})$ transformation, eq.(5.96), followed by an $O(6,22,\mathbb{Z})$ transformation, gives rise to the charges, eq.(5.101). The $SL(2,\mathbb{Z})$ transformation acting on eq.(5.98) gives the charges,

$$\begin{aligned}\vec{\tilde{Q}}_e &= (a\tilde{q}_0, -a\tilde{p}^1, b\tilde{p}^i, 0, 0, 1, 0)\\ \vec{\tilde{Q}}_m &= (m\tilde{q}_0, -m\tilde{p}^1, n\tilde{p}^i, 1, 0, 0, 0).\end{aligned} \tag{6.55}$$

Next, we determine the $O(6,22,\mathbb{Z})$ transformation. Consider a four dimensional subspace of the charge lattice, where the metric, eq.(3.27), is, $\mathcal{H}\oplus\mathcal{H}$. The following matrix is an element of $O(2,2,\mathbb{Z})$,

$$(\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & q & 0\\ 0 & 0 & 1 & 0\\ -q & 0 & 0 & 1\end{pmatrix}), \tag{6.56}$$

for any $q\in\mathbb{Z}$. Now starting with the charges, eq.(6.55), consider such a transformation, with $q = m\tilde{p}^1$, acting on the charges lying in the first Hyperbolic subspace and the second last Hyperbolic subspace, as defined in eq.(3.27). And next such a transformation, with $q = (a\tilde{p}^1 + 1)$, acting on the charges in the first Hyperbolic subspace and the last Hyperbolic subspace, as defined in eq.(3.27). This takes the charges in eq.(6.55) to their final values in eq.(5.101).

6.6 Some more details on the Isometry Analysis of Section 5

In this section we will derive all the isometries preserved by the metric eq.(5.124). The Killing vectors must satisfy the conditions given by eq.(5.128). The $(\theta_1,\theta_1), (\theta_2,\theta_2), (\theta_1,\theta_2)$ components of this equation take the form,

$$\partial_{\theta_1}\xi^{\theta_1} \;=\; 0$$

$$\begin{aligned} \partial_{\theta_2}\xi^{\theta_2} &= 0 \\ \partial_{\theta_1}\xi^{\theta_2} + \partial_{\theta_2}\xi^{\theta_1} &= 0. \end{aligned} \tag{6.57}$$

The $(\phi_1, \phi_1), (\phi_2, \phi_2), (\phi_1, \phi_2)$, components are,

$$\begin{aligned} im_1\xi_{\phi_1} + \alpha^2\xi^{\theta_1}\sin\theta_1\cos\theta_1 &= 0 \\ im_2\xi_{\phi_2} + \beta^2\xi^{\theta_2}\sin\theta_2\cos\theta_2 &= 0 \\ im_1\xi_{\phi_2} + im_2\xi_{\phi_1} - \sin\theta_1\cos\theta_2\xi^{\theta_1} - \sin\theta_2\cos\theta_1\xi^{\theta_2} &= 0 \end{aligned} \tag{6.58}$$

The $(\psi, \psi), (\psi, \phi_1), (\psi, \phi_2)$, components are,

$$\begin{aligned} im_3\xi_\psi &= 0 \\ im_1\xi_\psi + im_3\xi_{\phi_1} - \sin\theta_1\xi^{\theta_1} &= 0 \\ im_2\xi_\psi + im_3\xi_{\phi_2} - \sin\theta_2\xi^{\theta_2} &= 0 \end{aligned} \tag{6.59}$$

The $(\theta_1, \phi_1), (\theta_2, \phi_2), (\theta_1, \phi_2), (\theta_2\phi_1)$, components are,

$$\begin{aligned} \partial_{\theta_1}\xi^\gamma g_{\gamma\phi_1} + im_1\xi^{\theta_1} &= 0 \\ \partial_{\theta_2}\xi^\gamma g_{\gamma\phi_2} + im_2\xi^{\theta_2} &= 0 \\ \partial_{\theta_1}\xi^\gamma g_{\gamma\phi_2} + im_2\xi^{\theta_1} &= 0 \\ \partial_{\theta_2}\xi^\gamma g_{\gamma\phi_1} + im_1\xi^{\theta_2} &= 0 \end{aligned} \tag{6.60}$$

Finally the $(\theta_1, \psi), (\theta_2, \psi)$, components are,

$$\begin{aligned} \partial_{\theta_1}\xi^\gamma g_{\gamma\psi} + im_3\xi_1^\theta &= 0 \\ \partial_{\theta_2}\xi^\gamma g_{\gamma\psi} + im_3\xi_2^\theta &= 0 \end{aligned} \tag{6.61}$$

Setting $m_1 = m_2 = m_3 = 0$ we have from the (ψ, ϕ_1) and (ψ, ϕ_2) components that, $\xi_1^\theta = \xi_2^\theta = 0$. It then follows from the remaining equations that there are only three Killing vectors of this type. These are, $\partial_\psi, \partial_{\phi_1}, \partial_{\phi_2}$, which have already been identified above.

Next setting $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ we have, from the equation for (ψ, ψ), (ϕ_1, ϕ_1) and (ψ, ϕ_1) components that,

$$-\frac{\alpha^2\cos\theta_1}{m_1}\xi_1^\theta = \frac{1}{m_3}\xi_1^\theta, \tag{6.62}$$

from which we conclude that

$$\xi_1^\theta = 0. \tag{6.63}$$

Similarly we learn that $\xi_2^\theta = 0$. From the $(\phi^1, \phi^1), (\phi^2, \phi^2), (\psi, \psi)$, components it then follows that,

$$\xi_\mu = 0 \ \ \forall \ \mu, \tag{6.64}$$

leading to the conclusion that there is no Killing vector of this type.

We will now set $m_1 = m_2 = 0$ and $m_3 \neq 0$. The (ϕ_1, ϕ_1) and (ϕ_2, ϕ_2) components give, respectively, $\xi^{\theta_1} = 0$ and $\xi^{\theta_2} = 0$. The (ψ, γ) components for $\gamma = \psi, \phi_1$ and ϕ_2 give $\xi_\psi = 0, \xi_{\phi_1} = 0$ and $\xi_{\phi_2} = 0$ respectively. Thus we have no killing vector with $m_1 = m_2 = 0$ and $m_3 \neq 0$.

Let us now set $m_2 = m_3 = 0$ and $m_1 \neq 0$. Considering the (ϕ_2, ϕ_2) component, we get $\xi^{\theta_2} = 0$. From the $(\phi_1, \phi_1), (\phi_1, \phi_2)$ and (ψ, ϕ_1) components we get,

$$\begin{aligned}
\xi_{\phi_1} &= -\left(\frac{\xi^{\theta_1}}{im_1}\right)\alpha^2 \sin\theta_1 \cos\theta_1 \\
\xi_{\phi_2} &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\sin\theta_1 \cos\theta_2 \\
\xi_\psi &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\sin\theta_1 \ .
\end{aligned} \tag{6.65}$$

The contravariant components of ξ can be shown to be

$$\begin{aligned}
\xi^{\phi_1} &= -\left(\frac{\xi^{\theta_1}}{im_1}\right)\cot\theta_1 \\
\xi^\psi &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\text{cosec } \theta_1 \ ,
\end{aligned} \tag{6.66}$$

and $\xi^{\phi_2} = 0$. We still have to satisfy the remaining nontrivial equations. The (θ_1, ϕ_1) component of the killing equation

$$\partial_{\theta_1}\xi^{\phi_1} g_{\phi_1\phi_1} + \partial_{\theta_1}\xi^\psi g_{\psi\phi_1} + im_1\xi^{\theta_1} = 0 \ , \tag{6.67}$$

gives

$$-\frac{1}{m_1}(1+\alpha^2) + m_1 = 0 \ . \tag{6.68}$$

Thus we must have

$$m_1 = \pm\sqrt{1+\alpha^2} \ . \tag{6.69}$$

It is straightforward to check that the (θ_1, ϕ_2) and (θ_1, ψ) components of the killing equation are satisfied. All other components are satisfied trivially provided ξ^{θ_1} is independent of θ_1, θ_2. As a result we get two linearly independent killing vectors corresponding to the two roots of m_1:

$$\begin{aligned} \xi_1 &= e^{i\sqrt{1+\alpha^2}\phi^1}\left(\partial_{\theta_1} + \frac{i}{\sqrt{1+\alpha^2}}\cot\theta_1\partial_{\phi_1} - \frac{i}{\sqrt{1+\alpha^2}}\text{cosec }\theta_1\partial_\psi\right) , \\ \xi_2 &= \xi_1^* . \end{aligned} \tag{6.70}$$

In a similar way we can obtain two more linearly independent killing vectors upon setting $m_1 = m_3 = 0$ and $m_2 \neq 0$. We find

$$\begin{aligned} \xi_3 &= e^{i\sqrt{1+\beta^2}\phi^1}\left(\partial_{\theta_2} + \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} - \frac{i}{\sqrt{1+\beta^2}}\text{cosec }\theta_2\partial_\psi\right) , \\ \xi_4 &= \xi_3^* . \end{aligned} \tag{6.71}$$

Let us now set $m_1 \neq 0, m_2 \neq 0$ and $m_3 = 0$. The (ψ, ϕ_1) and (ψ, ϕ_2) components together gives

$$\begin{aligned} im_1\xi_\psi - \sin\theta_1\xi^{\theta_1} &= 0 \\ im_2\xi_\psi - \sin\theta_2\xi^{\theta_2} &= 0 . \end{aligned} \tag{6.72}$$

Eliminating ξ_ψ from the above two equations, we find

$$\frac{\xi^{\theta_1}}{\xi^{\theta_2}} = \frac{m_1\sin\theta_2}{m_2\sin\theta_1} . \tag{6.73}$$

Since ξ^{θ_1} is independent of θ_1 and ξ^{θ_2} is independent of θ_2, the above equation can be met only if ξ^{θ_1} is proportional to $\sin\theta_2$ and vice versa. From $\partial_{\theta_1}\xi^{\theta_2} + \partial_{\theta_2}\xi^{\theta_1} = 0$ we find $\partial_{\theta_1}\partial_{\theta_2}\xi^{\theta_2} = 0$, indicating the proportionality constants must be zero. From the above discussion, we get $\xi^{\theta_1} = \xi^{\theta_2} = \xi_\psi = 0$. It is now easy to see from the (ϕ_1, ϕ_1) and (ϕ_2, ϕ_2) components of the killing equation that $\xi_{\phi_1} = \xi_{\phi_2} = 0$. And hence we don't have any killing vector for the above choice of m_1, m_2, m_3. In a similar manner, we can show hat we don't have any nontrivial solution to the killing equations when $m_1 \neq 0, m_3 \neq 0$ and $m_2 = 0$ as well as when $m_2 \neq 0, m_3 \neq 0$ and $m_1 = 0$.

In summary, the metric, eq.(5.124), has seven Killing vectors, given in eq.(5.129).

6.7 General canonical form of charge vector in $\Gamma^{6,6}$

We start with a charge vector $\vec{Q} \in \Gamma^{2,2}$, where $\Gamma^{2,2} = \mathcal{H} \oplus \mathcal{H}$, is the 4-dimensional lattice made out of two 2-dimensional Hyperbolic lattices, $\mathcal{H}$. In components, $\vec{Q}$ takes the form,

$$\vec{Q} = (a, -b, c, d). \tag{6.74}$$

The lattice, $\Gamma^{2,2}$, is invariant under the action of $O(2,2,\mathbb{Z})$. We show that using an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \in O(2,2,Z)$ the vector, $\vec{Q}$, can be brought to the form,

$$\vec{Q} = (gcd(\vec{Q}), \frac{\vec{Q}^2}{gcd(\vec{Q})}, 0, 0), \tag{6.75}$$

where,

$$gcd(\vec{Q}) = gcd(a, b, c, d), \tag{6.76}$$

and

$$\vec{Q}^2 = \vec{Q} \cdot \vec{Q}. \tag{6.77}$$

Note that the only non-vanishing components in eq.(6.75) lie in the first $\mathcal{H}$ sub-lattice.

It is useful for this purpose to represent $\vec{Q}$ as a 2×2 matrix,

$$Q = (\begin{pmatrix} a & -b \\ c & d \end{pmatrix}). \tag{6.78}$$

The first $SL(2,\mathbb{Z})$, which we denote as $SL(2,\mathbb{Z})_T$, acts on the left and performs row operations, while the second $SL(2,\mathbb{Z})$, which we denote as $SL(2,\mathbb{Z})_U$, acts on the right and carries out column operations. If $A \in SL(2,\mathbb{Z})_T, B \in SL(2,\mathbb{Z})_U$, then under their action,

$$Q \to AQB. \tag{6.79}$$

Note that $\vec{Q}^2 = det(Q)$. We will show that A, B can be found which bring Q to the form,

$$Q = (\begin{pmatrix} gcd(\vec{Q}) & 0 \\ 0 & \frac{det(Q)}{gcd(\vec{Q})} \end{pmatrix}) \tag{6.80}$$

This is equivalent to $\vec{Q}$ taking the form, eq.(6.75).

It is enough to prove this result for the case when $gcd(\vec{Q}) = 1$, in which case, eq.(6.80) becomes,

$$Q = (\begin{pmatrix} 1 & 0 \\ 0 & det(Q) \end{pmatrix}). \tag{6.81}$$

The more general result, eq.(6.80), then follows, by considering the vector, $\frac{1}{gcd(\vec{Q})}\vec{Q}$, which has unit value for its gcd. In the discussion below we will sometimes use to the notation,

$$gcd(Q) \equiv gcd(\vec{Q}) = gcd(a, b, c, d). \tag{6.82}$$

The proof is as follows. Given any 2 integers, Euclid gives us an algorithm to arrive at their gcd in the following fashion. Subtract the smaller of the 2 numbers from the larger and then if the result is still larger than the smaller number continue this operation till the result becomes otherwise. Then start subtracting the new smaller number from the new larger number and continue this set of steps till one of the numbers becomes zero at which point the other number is the gcd. If the two integers are a, c, the two elements of the first column of matrix, Q, eq.(6.78), then this sequence of operations can be implemented by an element of $Sl(2, \mathbb{Z})_T$ which acts on the left and carries out row operations. The resulting form of Q is,

$$Q = (\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}), \tag{6.83}$$

where $a' = gcd(a, c)$. Note that $gcd(Q)$ is preserved by this operation. Since $gcd(Q) = 1$, to begin with, we learn that,

$$gcd(a', b', d') = 1. \tag{6.84}$$

Now we come to the crucial step. Let $\{p_1, \cdots p_r\}$, be the set of distinct primes which divide d' but do not divide b'. Let $m = \Pi p_i$, be the product of all these primes. One can show that the two numbers, d', and, $a'm + b'$, are coprime. Let p' be a prime that divides d', then if it does not divide b' it must divide m (by construction) and thus cannot divide $a'm + b'$. If on the other hand p' divides b', it cannot divide m (again by construction) and also it cannot divide a' (since eq.(6.84) is valid), and therefore p' cannot divide $a'm + b'$. Thus, we learn that $gcd(d', a'm + b') = 1$ and these two numbers are coprime.

We use this result to bring Q, eq.(6.83), to the form, eq.(6.81). First, an $SL(2,\mathbb{Z})_U$ transformation can be carried out,

$$Q \rightarrow Q(\begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix}) = (\begin{pmatrix} a' & a'm + b' \\ 0 & d' \end{pmatrix}). \tag{6.85}$$

Since $gcd(a'm + b', d') = 1$, we can use Euclid's algorithm as in the discussion above to now find an $SL(2,\mathbb{Z})_U$ transformation which bring Q to the form,

$$Q = (\begin{pmatrix} a'' & 1 \\ c'' & d'' \end{pmatrix}). \tag{6.86}$$

Next, further $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U$ tranformations can be carried out to subtract the second column from the first a'' times, and the first row from the second d'' times. This followed by a row- column interchange operation gives Q in the form, $(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix})$. Since these operations preserve the determinant, we learn that $u = det(Q)$, leading to eq.(6.81).

We end by making a few points. First, note that this argument holds for space-like, time-like and null charge vectors, Q. Second, it follows from our analysis that there are two independent invariants for $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$. These are $det(Q)$ and $gcd(Q)$. Of these $det(Q)$ is an invariant of the continuous group, while $gcd(Q)$ is a discrete invariant. Third, if instead of $\Gamma^{2,2}$ we start with a lattice which is the direct sum of more than two copies of $\mathcal{H}$, a similar argument can be used sequentially on the first two $\mathcal{H}$ sublattices, then the first and third $\mathcal{H}$ sublattices etc, to finally bring the charge vector to the form,

$$\vec{Q} = (gcd(\vec{Q}), \frac{\vec{Q}^2}{gcd(\vec{Q})}, 0, 0, \cdots, 0, 0). \tag{6.87}$$

In particular this is true for $\Gamma^{6,6}$ which consists of six copies of $\mathcal{H}$. Finally, if there are two charge vectors, $\vec{Q}_e, \vec{Q}_m$, then the above argument can be used to put one of them, say $\vec{Q}_e$, in the form, eq.(6.87). Further transformations which act trivially on the first Hyperbolic sublattice will keep $\vec{Q}_e$ invariant. Using these transformations $\vec{Q}_m$ can now be brought to the form,

$$\vec{Q}_m = (\alpha, \beta, \gamma, \delta, 0, 0 \cdots, 0, 0), \tag{6.88}$$

so that only the components in the first two Hyperbolic sublattices are non-vanishing. These results apply in general to the cases when $\vec{Q}_e^2, \vec{Q}_m^2$, have space-like, time-like or null norms.

Chapter 7

Conclusions and Open Questions

Herein, we list the conclusions of the various lines of inquiry that have been pursued in the four chapters after the introduction, with each point in the list labeled by the corresponding chapter number:

1. The exact spectrum of dyons in four dimensions and of spinning black holes in five dimensions in CHL compactifications can be determined using a Borcherds product representation of level N Siegel modular forms of $Sp(2, \mathbb{Z})$. Various elements in the Borcherds product have a natural interpretation from the perspective of 4d-5d lift. The Hodge anomaly is identified the contribution of bound states of Type-IIB KK5-brane with momentum. The remaining piece is interpreted as arising from the symmetric product of the orbifolded D1-D5-P system. The appearance of an underlying chiral bosonic string on a genus two Riemann surface in this construction has a natural interpretation as the Euclidean worldsheet of the **K3** wrapped M5 brane on a string web in orbifolded theory. By factorization, this connection with the Siegel modular form can be made precise. Further, we have seen that a very rich and interesting mathematical structure underlies the counting of BPS dyons and black holes. Given the relation of Siegel modular forms to Generalized Kac-Moody algebras [18, 19, 23, 92], their appearance in the counting is perhaps indicative of a larger underlying symmetry of string theory. If so, investigating this structure further

might prove to be a fruitful avenue towards uncovering the full structure of M-theory.

2. the interpretation of the proposed dyon degeneracy formula presents many subtleties. It is unlikely that the formula is valid in all regions of moduli space for all charges in a way envisioned in [4, 5] that depends only on the three invariants $Q_e^2/2$, $Q_m^2/2$, and $Q_e \cdot Q_m$. We summarize below our observations and what we believe would be the consistent physical interpretation of the dyon degeneracy formula.

   - It is clear that the three invariants $(Q_e^2/2, Q_m^2/2, Q_e \cdot Q_m)$ do not uniquely specify the state and the degeneracy will depend on additional data. This is natural because the arithmetic duality group has many more invariants than the continuous duality group. We have identified a particular invariant I which determines when the genus-two partition function is adequate but this is not the end of the story. To illustrate this point, let us consider an even more striking example of a quarter-BPS lightlike state for which additional data is required to specify the degeneracy of states.[1] Consider a perturbative BPS state that is purely electric in the Type-IIA frame carrying winding w along a circle of the $\mathbf{T^2}$ factor and momentum n along the same circle. In the heterotic frame it corresponds to a state with w NS5-branes wrapping $\mathbf{T^4} \times \mathbf{S^1}$ with momentum n along the $\mathbf{S^1}$. For nonzero n and w the state carries arbitrary left-moving oscillations $N_L = nw$ and has entropy $2\pi\sqrt{2}\sqrt{nw}$. Unlike a similar heterotic electric state which is half-BPS, these states are quarter-BPS because both right and left movers carry supersymmetry for the Type-II string. Now, for all such states, all three invariants $(Q_e^2/2, Q_m^2/2, Q_e \cdot Q_m)$ vanish and so does the discriminant. Thus there is a large set of legitimate quarter-BPS states with the same values for the three invariant, namely zero, but very different entropy depending on the values of n and w. The de-

[1] We thank Boris Pioline for discussions on this point.

generacy of such states cannot possibly be captured by the genus-two partition function. This example illustrates that additional data might be required to determine the degeneracy of states, although alternative explanations are possible. The difference might also be attributed to a difference between the absolute degeneracy of states and the supersymmetry index computed by the dyon degeneracy formula.

- The states with negative discriminant appear problematic at first because there is no black hole corresponding to them. We have seen that they can nevertheless have a sensible physical realization. In the specific example considered here the states are described as a two-centered configuration in supergravity. These configuration have the right degeneracy coming from the angular momentum multiplicity consistent with the prediction of the dyon degeneracy formula. We would like to propose that other negative discriminant states also exist and can be realized as complicated multi-centered configurations. The supergravity analysis also indicates that existence of these states is moduli dependent. The states exist over a large region of the moduli space but cannot exist in certain regions of the moduli space because the distance between the two centers determined by Denef's constraint goes to infinity. This shows that generically there are walls of marginal stability in the moduli space that separate regions where the states exist from regions where they do not. This is not surprising since even in field theory, quarter-BPS states in $\mathcal{N} = 4$ theories are known to have curves of marginal stability [49, 61]. It is possible that this moduli dependence is related to the need to change the choice of contour to obtain an S-duality invariant answer. As these lines of marginal stability have a simple description in the string web picture, it might be possible to understand the change of contour from the M-theory lift of the string web.

- Despite these subtleties, it is also true that the dyon partition function has been derived from various points of views for specific charge configurations and in specific regions of moduli space. Considering the caveats above, a conservative interpretation of these results in our view is that the dyon degeneracy formula given in terms of the genus-two Siegel modular forms is exact and valid for specific charges in the specific regions of moduli spaces as well as for all charges related by a duality transformations in the dual regions of the moduli space. This already contains highly nontrivial information about the degeneracies of quarter-BPS bound states of various branes in the theory. This can be seen quite generally from the point of view of the string web picture. For a given charge configuration, and in a given region of the moduli space, if a string web is stable and can be lifted to a wrapped **K3**-wrapped M5-brane with a genus-two world sheet, then one can derive the degeneracy from the genus-two partition function of the left-moving heterotic string as has been done in [3, 7, 39]. However, as one moves around in the moduli space, the string web can become unstable. Once the string web is unstable, the dyon degeneracies can no longer be obtained from the genus-two partition function. Thus the derivation of the dyon partition function is valid in only a certain region of the moduli space for a given charge configuration. Moreover, for some quarter-BPS state, it may not be possible at all to represent the state as a string web that lifts to a **K3**-wrapped M5-brane. For example, the Type-II perturbative states discussed above lift to a circle-wrapped M2-brane with genus-one topology and not to a **K3**-wrapped M5-brane with genus-two topology. A circle-wrapped M2-brane is nothing but the Type-II string and hence for these states the counting is correctly done using the genus-one partition function of the Type-II string and not using a genus-two partition function of the heterotic string. These examples clearly delineate the range of applicability of the dyon degeneracy formula.

3. We have seen that a simple physical argument allows one to compute the degeneracies of decadent dyons in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-

Mills theory with little work. These results are in agreement with the known results obtained using much more elaborate and sophisticated index computations. Our results could also be viewed as a test of the reasoning underlying the wall-crossing formula in $\mathcal{N} = 2$ theories and of the degeneracy formula near the curve of decay in $\mathcal{N} = 4$ theories. This method of course allows one to count decadent dyons with more general charges in general gauge groups not hitherto considered in the field theory literature. It would be interesting to test such predictions using index computations.

It may seem surprising that this almost classical computation is capable of capturing the quantum degeneracies precisely. In this context, we note that a number of essentially quantum ingredients have implicitly gone into our reasoning. First, the shift of $-1/2$ to the classical field angular momentum from the fermionic zero modes in 4.4 is essentially quantum. Second, the angular momentum multiplicities of $2J + 1$ are also quantum. What is interesting is that after incorporating this information into an almost classical reasoning, one can determine the degeneracies exactly.

Finally, we have also seen that the dyons counted in field theory are not accounted for by the dyon partition functions recently derived in the context of string theory dyons except for one special case. This is because they lie in a different duality orbit than the dyons for which the dyon partition function has been derived.

4. This chapter dealt with two main results. First, we have shown that a generic supersymmetric or non-supersymmetric system of charges cannot be brought to the Cardy limit using the duality symmetries. Second, we have found that the required non-genericity to be able to bring a set of charges to the Cardy limit is interestingly different in the supersymmetric and the non-supersymmetric cases. For large charge, in the non-supersymmetric case but not the supersymmetric one, we can always find a set of charges lying close by which can be brought to the Cardy limit. The required shift

in the charges satisfy the condition [1],

$$\frac{\Delta Q}{Q} \sim \frac{1}{\sqrt{Q}}. \tag{7.1}$$

These results were proved for the $D0-D4$ system and the $D0-D6$ system. We expect them to be more general.

For example, our analysis of the $D0-D4$ system, leading to the conclusion that generic charges cannot be brought to the Cardy limit, immediately applies to all charges which satisfy the condition,

$$\vec{Q}_e \cdot \vec{Q}_m = 0. \tag{7.2}$$

Similarly, the analysis of the $D0-D6$ system applies to all charges meeting the condition,

$$\vec{Q}_e^2 = \vec{Q}_m^2 = 0. \tag{7.3}$$

with the conclusion that all such charges can never be brought to the Cardy limit. Also, all the results immediately apply to other charges which lie in the same duality orbit as the D0-D4 or D0-D6 systems.

In our analysis we did not determine all the necessary and sufficient conditions that need to be met to be able to bring a set of charges to the Cardy limit. To obtain a more complete understanding of these conditions, for a general set of charges, it would be useful to start with a classification of all the discrete invariants of $SL(2,\mathbb{Z}) \times O(6,22,\mathbb{Z})$. It should be possible to express the required conditions, for any charge configuration to be brought to the Cardy limit, in terms of these invariants. We leave such an analysis for the future.

Another approach would be to bring the charges to a canonical form and then carry out the analysis for general charges of this form. As long as the charges lie in the $\Gamma^{(6,6)}$ sublattice, made out of the 6 Hyperbolic sublattices,

[1] More correctly, the condition in the $D0-D4$ case is given in eq.(5.93), where δ is a small number that does not scale with Q, and in the $D0-D6$ case, with $q_0, p^0 \gg 1$, it is given by, $\frac{\Delta Q}{Q} \sim \frac{1}{Q}$.

$\mathcal{H}$ in eq.(3.27), one can show using the duality symmetries that the electric charges, $\vec{Q}_e$, can always be made to lie only in first hyperbolic sublattice, while the magnetic charges, $\vec{Q}_m$, take non-trivial values in the first two hyperbolic sublattices. These results are discussed in appendix E. One expects these results to be further generalized, when charges lying in the $\mathcal{E}_8 \times \mathcal{E}_8$ sublattice are also excited. For example, it has shown that a general time-like vector can always be made to lie in one Hyperbolic sublattice, (see the discussion in [1] [93]). Further analysis along these lines is also left for the future.

Our conclusions in the supersymmetric case are in accord with recent results obtained for the subleading corrections to the entropy, going like $1/Q$. If the system could be brought to the Cardy limit these corrections would be of the form, eq.(5.9), with the central charge receiving $1/Q$ corrections. The results for the first subleading corrections, which have been obtained by directly counting the dyonic degeneracy and computing the four derivative corrections using the Gauss-Bonnet term, are now known not to be generally of this form- See [35], [94], [95].

One of the main motivations of this investigation was to ask how far the AdS_3/CFT description can take us in understanding the entropy of non-supersymmetric black holes. If the charges lie in the Cardy limit, then at least in some region of moduli space, the black hole with these charges can be viewed as a BTZ black hole in AdS_3 space. The microscopic states which account for the black hole entropy can then be understood as states in a $1+1$ dim. CFT, and their entropy can be easily found in terms of the Cardy formula. Our result, that in the non-supersymmetric case a generic set of charges, after a small shift, can be brought to the Cardy limit is quite promising in this context. It tells us that such a microscopic counting for the leading order entropy is available for generic charges, at least in some region of moduli space.

The main complication in determining the entropy microscopically is then

[1]Also, V.V.Nikulin, Math.USSR Izvestija,14(1980),pg.103.

it's possible moduli dependence. This is a particularly important issue in the non-supersymmetric case. In the Cardy formula the entropy is determined by the central charge. Now, the central charge is protected by anomaly considerations and is therefore moduli independent. Thus for the charges which can be brought to the Cardy limit, the entropy must be moduli independent, at least for small shifts of moduli [1]. Since the required fractional shift to get to such a configuration is small, of order, $O(1/\sqrt{Q})$, eq.(7.1), one would hope that this is enough to prove that the leading entropy is generally moduli independent.

Once the moduli independence of the entropy is established, it is easy to furnish an argument, as follows, leading to the determination of the entropy microscopically. The entropy must now be a function only of the charges. And the dependence on the charges must enter through invariants of the discrete duality group, which is an exact symmetry of string theory. For the case we are studying here, one of these invariants, I, eq.(1.7), is also an invariant of the full continuous group, $SL(2,\mathbb{R}) \times O(6,22,\mathbb{R})$. The others are discrete invariants. Now the discrete invariants are not continuous functions of charge and typically undergo big jumps when the charges are changed only slightly [2] It is physically reasonable to demand that for large charges the leading order entropy does not undergo such discontinuous jumps. This would mean that any dependence on the discrete invariants must be sub-dominant at large charge [3]. The resulting functional dependence on the continuous invariant can then be determined by taking any convenient set of charges, which gives rise to a non-vanishing value for this invariant. In particular one can always find charges in the Cardy limit for which this in-

[1] Larger shifts might result in a jump, akin to a phase transition, where the formula for the entropy gets significant corrections.

[2] For example consider the discrete invariant, $\gcd(Q_e^i Q_m^j - Q_e^j Q_m^i, Q_e^k Q_m^l - Q_e^l Q_m^k), \forall i,j,k,l \in \{1,2,\cdots,28\}$. Since the gcd can vary discontinuously, this invariant can change by big jumps.

[3] This argument was given to us by Shiraz Minwalla, we thank him for the discussion on this point and related issues.

variant does not vanish. For such a set of charges a microscopic calculation of the entropy is often possible as was mentioned above, and this would then determine the entropy for all general charges.

These arguments should also apply when one includes angular momentum in four dimensions, $\vec{J}$. In this case there are now two invariants of the continuous duality symmetries, and the Rotation group, I and $\vec{J}^2$. An argument along the above lines would fix the dependence on both these invariants. Note that the resulting expression for the entropy would then also be valid when I, and more generally all the charges, $\vec{Q}_e, \vec{Q}_m$ vanish, leading to microscopic determination of the entropy of an extreme Kerr black hole in four dimensions. It is easy to check that the resulting answer is in agreement with the Bekenstein-Hawking entropy in this case.

The arguments above, whose purpose is to provide a microscopic understanding of the entropy, are already known to have counterparts on the gravity side. This makes us hopeful that they can be more fully fleshed out on the microscopic side as well. We end with a brief discussion of these issues from the gravity point of view.

Recent advances have now established that the attractor mechanism is valid for all extremal black holes, supersymmetric as well as non-supersymmetric ones (See [96], [97], [98], [99], for early work. More recent advances are in, e.g, [100],[101], [102], [103], [104], [105], [106], [107], [108], [109], [110], [111], [112], [113], [114], [115], see also, [116], and references therein). This shows that the entropy is not dependent on the moduli [1]. Once the moduli independence is established the duality symmetries allow the entropy for general charges to be related to the entropy which arise for a set of charges in the Cardy limit. In the supergravity approximation, which is valid at large charge, the duality group is enhanced to the full continuous group,

[1]More correctly this shows that the entropy is independent of small shifts in the moduli. There can be discontinuous jumps in the entropy as the moduli are varied, see ref Moore and Denef for related recent developments. However, this might be less of a worry if we are interested in the entropy of a single-centered black hole.

in the case we are considering here to $SL(2,\mathbb{R}) \times O(6,22,\mathbb{R})$. A duality transformation will act on both the charges and the moduli, and to begin with the entropy could have been a duality invariant function of the moduli and charges. However, once we have established that the entropy is moduli independent it must be an invariant of the charges alone. Since there is only one duality invariant of the continuous group [1], I, the entropy for a general set of charges can be related to the entropy for charges in the Cardy limit, with the same value of this invariant.

In sum, in this dissertation, we have explored various aspects of dyonic black hole entropy counting in $\mathcal{N} = 4$ string theories. We have arrived at an exact counting formula for certain classes of supersymmetric black holes and this has enabled us to investigate non-perturbative aspects such as lines of marginal stability in these theories. We have also made preliminary advances in arriving at a full microscopic understanding of non-supersymmetric extremal black holes in these theories. Needless to say, interesting new questions are ripe in this field and offer the promise of rich new areas of research.

[1] We are neglecting angular momentum, $\vec{J}$, here.

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