The fuzzy S^2 structure of M2-M5 systems in ABJM

Costis Papageorgakis

Tata Institute of Fundamental Research

String Theory and Fundamental Physics, Kanha

12 February 2009

(w/ H. Nastase and S. Ramgoolam, in progress)

Important developments over last year relating to effective action for multiple M2-branes. Following leads from Bagger-Lambert and Gustavsson involving 3-algebras, ABJM wrote a CS action in 3d with $\mathrm{U}(N) \times \mathrm{U}(\bar{N})$ gauge fields, coupled to bi-fundamental matter [Aharony-Bergman-Jafferis-Maldacena]

$$
\begin{aligned}
S = & \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\
& - \mathrm{Tr} D_\mu C_I^\dagger D^\mu C^I + \frac{4\pi^2}{3k^2} \mathrm{Tr} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \right. \\
& \left. + 4 C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6 C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \right)
\end{aligned}
$$

- $\mathcal{N} = 6$ superconformal, $\mathrm{SU}(4)$ R-symmetry
- Can use $\lambda = N/k$ as 't Hooft coupling

- For $\lambda \ll 1$ gauge theory weakly coupled
- For $\lambda \gg 1$ string theory dual in terms of near horizon limit of N M2 branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.
- For small k M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$
- The orbifold acts on the circle of the Hopf fibration $S^1 \hookrightarrow S^7 \xrightarrow{\pi} \mathbb{CP}^3$
- For large k Type IIA on $\mathrm{AdS}_4 \times \mathbb{CP}^3 \rightarrow$ ($\mathrm{AdS}_4/\mathrm{CFT}_3$)

Even though M2-brane physics emerge at strong coupling this is significant progress!

But what about the M5-brane?

M5-brane potentially emerges from M2-brane in several ways through a generalisation of the Myers effect:

- Via an M2$\perp$M5 intersection or 'fuzzy funnel' (see BIon) [Basu-Harvey]
- As a vacuum of a mass-deformed theory of M2s (see $\mathcal{N} = 1^*$) [Bena]

Both require that M2s blow up into a fuzzy three-sphere with the matriceal scalars becoming fuzzy directions on M5

Interesting mass-deformation of ABJM found by GRvV: Split complex scalars into $C^I = (R^\alpha, Q^{\dot\alpha})$ and introduce potential [Gomis-Rodríguez-Gómez-van Raamsdonk-Verlinde, Hosomichi-Lee^3-Park]

$$V = |M^\alpha|^2 + |N^\alpha|^2$$

where

$$\begin{aligned} M^\alpha =& \mu Q^\alpha + \frac{2\pi}{k}(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta + 2Q^\beta R^\dagger_\beta R^\alpha) \\ N^\alpha =& -\mu R^\alpha + \frac{2\pi}{k}(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta + 2R^\beta Q^\dagger_\beta Q^\alpha) \end{aligned}$$

- Breaks conformal invariance and R-symmetry $\mathrm{SU}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$
- Preserves $\mathcal{N} = 6$ supersymmetry

GRvV also found set of classical 1/2-BPS vacua for $Q^{\dot{\alpha}} = 0$. Need to solve

$$R^\alpha = \frac{2\pi}{k\mu}(R^\alpha R^\dagger_\beta R^\beta - R^\beta R^\dagger_\beta R^\alpha)$$

which leads to

$$R^\alpha = f G^\alpha \qquad \text{for} \qquad f^2 = \frac{k\mu}{2\pi}$$

where

$$(G^1)_{m,n} = \sqrt{m-1}\,\delta_{m,n}$$
$$(G^2)_{m,n} = \sqrt{(N-m)}\,\delta_{m+1,n}$$
$$(G^\dagger_1)_{m,n} = \sqrt{m-1}\,\delta_{m,n}$$
$$(G^\dagger_2)_{m,n} = \sqrt{(N-n)}\,\delta_{n+1,m}$$

(for $f^2 = \frac{k}{2\pi s}$ the above is 'funnel' solution, where s the direction along which M2s extend away from the M5)

The G^αs should encode all information about geometry. GRvV also notice that

$$G^\alpha G^\dagger_\alpha = X^1 X^1 + X^2 X^2 + X^3 X^3 + X^4 X^4 = N - 1$$

Looks like S^3. Extrapolating to $k = 1$ seem to get the M5-brane. At finite N the 'three-sphere' is fuzzy.
[Terashima, Hanaki-Lin]

However, the solution has $G^1 = G^\dagger_1$. Immediately reduces

$$G^\alpha G^\dagger_\alpha = X^1 X^1 + X^2 X^2 + X^3 X^3 = N - 1$$

Q: where is the (fuzzy) S^3? [Nastase, CP, Ramgoolam]

Immediately note that ‘usual’ $\mathrm{SO}(4)$-covariant fuzzy S^3 of Guralnik-Ramgoolam cannot be at work: solution has only got $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry.

Investigate algebra of G^αs: First construct bilinears $J^\alpha_\beta = G^\alpha G^\dagger_\beta$. These obey

$$[J^\alpha_\beta, J^\mu_\nu] = \delta^\mu_\beta J^\alpha_\nu - \delta^\alpha_\nu J^\mu_\beta$$

Further defining $J_i = (\sigma_i)^\alpha_\beta J^\beta_\alpha$ picks out traceless combinations and leads to

$$[J_i, J_j] = 2i\epsilon_{ijk} J_k$$

Similar things hold for $\bar{J}^{\alpha}_{\beta} = G^{\dagger}_{\beta} G^{\alpha}$

$$[\bar{J}_i, \bar{J}_j] = 2i\epsilon_{ijk}\bar{J}_k$$

Looks like there are two independent $\mathrm{SU}(2)$s.

Matrices in the adjoint of $\mathrm{U}(N)$ ($\mathrm{U}(\bar{N})$) admit a decomposition in terms of J_is ($\bar{J}_i$s). This is an expansion in terms of fuzzy spherical harmonics

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}(J_i)$$

with

$$\hat{Y}_{lm}(J_i) = \sum_{i} \alpha_{lm}^{(i_1 \dots i_l)} J_{i_1} \dots J_{i_l}$$

However, the algebra for odd products of G^α or $G^\dagger_\alpha$ will combine the two. Define

$$\mathbf{G}^\alpha = \begin{pmatrix} 0 & G^\alpha \\ G^\dagger_\alpha & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & \bar{J}_i \end{pmatrix}$$

Then we can summarise

$$[\mathbf{J}_i, \mathbf{G}^\alpha] = \begin{pmatrix} 0 & J_i G^\alpha - G^\alpha \bar{J}_i \\ \bar{J}_i G^\dagger_\alpha - G^\dagger_\alpha J_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(\sigma_i)^\alpha{}_\beta G^\beta \\ G^\dagger_\beta (\sigma_i)^\beta{}_\alpha & 0 \end{pmatrix}$$

The $\mathbf{G}^\alpha$s have good transformations under combined $\mathrm{SU}(2)$.

The bifundamental fields also admit harmonic decomposition under the $\mathrm{SU}(2)$

$$R^\alpha = R^\alpha_\beta G^\beta \qquad \text{with} \qquad R^\alpha_\beta = \sum_{l,m} (a_{lm})^\alpha_\beta \hat{Y}_{lm}(J_i)$$

Next look at quadratic fluctuation spectrum around GRvV vacuum

$$R^\alpha = fG^\alpha + r^\alpha \ , \qquad Q^{\dot\alpha} = q^{\dot\alpha} \ , \qquad A_\mu = a_\mu \ , \qquad \hat{A}_\mu = \hat{a}_\mu$$

The r^α further decompose into trace and traceless parts

$$r^\alpha = rG^\alpha + s_i \frac{1}{2}(\sigma_i)^\alpha_\beta G^\beta \ , \qquad r^\dagger_\alpha = G^\dagger_\alpha r + G^\dagger_\beta s_i \frac{1}{2}(\sigma_i)^\beta_\alpha$$

Use Matrix Theory technology to convert into fields on smooth (classical) S^2 at large N.

[Iso-Kimura-Tanaka-Wakatsuki, CP-Ramgoolam-Toumbas]

There is a standard dictionary between matrices on the fuzzy sphere at large N and functions on the sphere

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}(J_i) \to a(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x_i)$$

$$\frac{1}{N} \mathrm{Tr} \to \int d\Omega$$

$$\frac{J_i}{N} \to x_i$$

$$[J_i, \hat{a}] \to -2iK_i^a \partial_a a(\theta, \phi)$$

At large N the mode s_i can be further decomposed into radial and angular components on S^2

$$s_i = x_i \phi + K_i^a A_a$$

Expanding around the GRvV solution also triggers version of Higgs mechanism for gauge fields, present in ABJM (CS-matter) theories.

When a single scalar gets a large, trivial vev the diagonal subgroup of the two non-dynamical gauge fields becomes dynamical. Matter gets promoted from bifundamental to adjoint. [Mukhi-CP]

In M-theory corresponds to moving all M2-branes far away from the $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

However, for the case at hand things significantly more complicated: four fields get a vev and that has nontrivial matrix structure (proportional to G^α)

To cut a (very) long story short:

- The calculation involves combination of Matrix Theory techniques + glorified Higgsing
- Through the above: $(r, s_i) \to (r + \phi, A_a)$
- Non-dynamical $(A_\mu, \hat{A}_\mu)$ become a dynamical $\tilde{A}_\mu = A_\mu + \hat{A}_\mu$
- The combination $(r - \phi)$ plays role of Goldstone mode and does not appear
- Still going but appears inevitable that one will recover part of the abelian theory of a single D4 brane wrapping an S^2

Thus:

- D4-brane - no M5-brane! Since calculation is at weak coupling this makes sense
- However, S^2 structure is now explicit

How does this tally with initial geometric description in terms of G^αs?

- Relation between J_i and G^α is precisely the Hopf map associated with fibration $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$
- The D4 is wrapping the S^2 base of the above bundle. At large N this base is smooth. At finite N it becomes fuzzy.
- At weak 't Hooft coupling (large k) the S^1 fibre has shrunk

Summary & To do:

- Classical GRvV solutions do not describe (fuzzy) S^3; instead describe S^2 base of Hopf fibration
- If M5-brane is to emerge at small k, as expected, finite-k effects should modify the G^α, even though solution is BPS
- Starting from M5 wrapped on S^3 the D4 on S^2 is obtained by a double-dimensional reduction
- In terms of ABJM geometry the shrinking of the S^1 fibre should be related to particular action of $\mathbb{Z}_k$ orbifold
- Would be interesting to make this more precise