Navier-Stokes equation in AdS/CFT

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based on collaboration with G. Mandal and S. Wadia

Introduction and Motivation

・BH Thermodynamics

BH system $\xleftrightarrow{\text{stationary}}$ Thermodynamics

Q. What does happen if non-stationary fluctuations are considered？

In the case of the small fluctuations

Thermodynamics ⟶ Fluid Dynamics

$$\begin{cases} E: & \textbf{Energy} \\ S: & \textbf{Entropy} \\ T: & \textbf{Temperature} \end{cases} \qquad \begin{cases} e(x): & \textbf{Energy density} \\ s(x): & \textbf{Local "Entropy"} \\ t(x): & \textbf{Local "Temperature"} \end{cases}$$

BH Thermodynamics $\xrightarrow{?}$ BH Fluid Dynamics

A. Such a correspondence was confirmed in black branes.

(Sayantani B., Hubeny, Minwalla, Rangamani 2007)

Introduction and Motivation

(Sayantani Bhattacharyya, et al. 2007)

・Dynamical black brane in AdS5

(static) black brane

$G_{MN}(u^\mu, T)$

$\left\{ \begin{array}{l} T \text{ : temperature} \\ u^\mu \text{ : constant motion along the brane} \end{array} \right.$

↓

Dynamical black brane

$\tilde{G}_{MN}(u^\mu(x), T(x))$

$\left\{ \begin{array}{l} T(x) \text{ : local temperature} \\ u^\mu(x) \text{ : velocity fields} \\ x^\mu \text{ : boundary coordinates} \end{array} \right.$

This metric satisfies Einstein Equation only if u^μ, T satisfy $\nabla_\mu T^{\mu\nu} = 0.$

$T^{\mu\nu}$: energy-momentum tensor for the boundary fluid

Einstein equation ⟶ Navier-Stokes equation

The boundary fluid dynamics controls the bulk gravity.

Q. Why does such a reduction happen?

What is the meaning of the Navier-Stokes equation?

A. We can understand them by considering Radial ADM formalism.

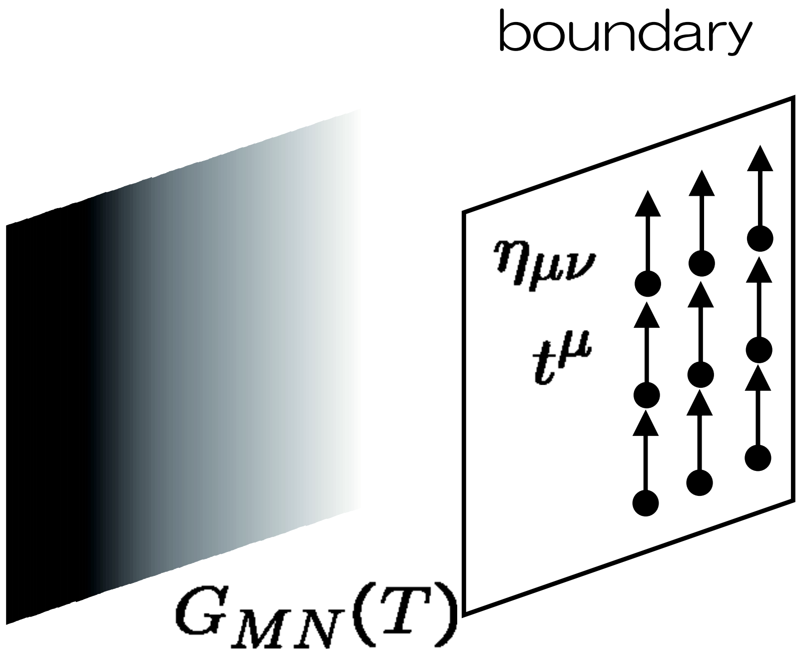
Plan of talk

1. Introduction and Motivation
2. Boundary Navier-Stokes equation
3. Radial ADM Formalizum
4. Conclusion

Boundary Navier-Stokes equation

(Sayantani Bhattacharyya. et al. 2007)

- black brane in AdS5



M, N : bulk(5 dim) indexes
μ, ν : boundary(4 dim) indexes

time-like killing $\quad t^{\mu} = (\overset{v}{1}, 0, 0, 0)$

$$ds^2 = 2dvdr - \frac{r^2}{l^2} f(br) dv^2 + \frac{r^2}{l^2} dx^{i2}$$: Eddington-Finkelstein coordinate

$$f(br) = 1 - \frac{1}{(br)^4}$$

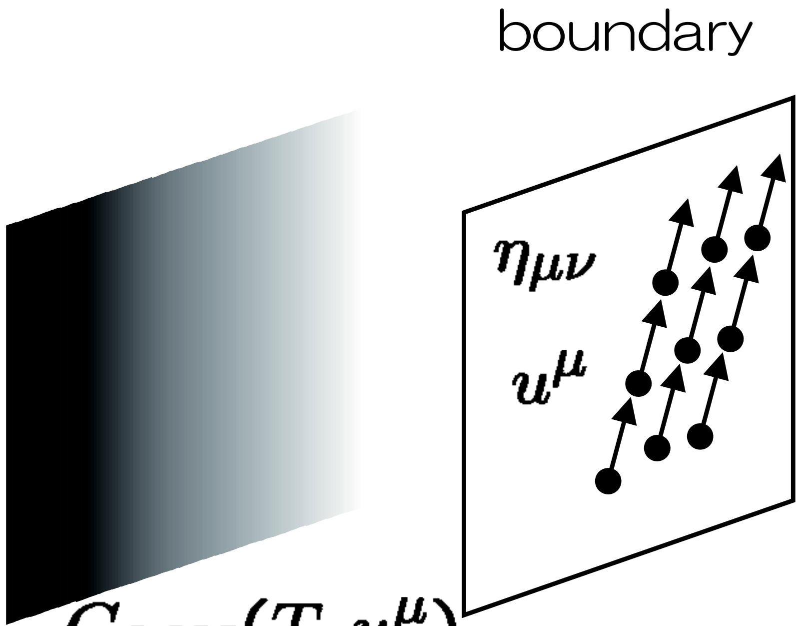
$$T = \frac{1}{\pi b}$$: temperature

Boundary Navier-Stokes equation

(Sayantani Bhattacharyya, et al. 2007)

·black brane in AdS5

$$\left\{ \begin{array}{l} M, N : \text{bulk(5 dim) indexes} \\ \mu, \nu : \text{boundary(4 dim) indexes} \end{array} \right.$$



$G_{MN}(T, u^{\mu})$

time-like killing $\quad t^{\mu} = (1, 0, 0, 0)$

↓ boost (transverse direction)

$u^{\mu}, \quad u \cdot u \equiv \eta_{\mu\nu} u^{\mu} u^{\nu} = -1$

u^{μ} : constant boost parameter

$$ds^2 = -2u_{\mu} dx^{\mu} dr - \frac{r^2}{l^2} f(br) u_{\mu} u_{\nu} dx^{\mu} dx^{\nu} + \frac{r^2}{l^2} (u_{\mu} u_{\nu} + \eta_{\mu\nu}) dx^{\mu} dx^{\nu}$$

: Eddington-Finkelstein coordinate

$$f(br) = 1 - \frac{1}{(br)^4}$$

$$\left\{ \begin{array}{ll} u^{\mu} & : \text{constant} \\ T = \frac{1}{\pi b} & : \text{temperature} \end{array} \right.$$

Boundary Navier-Stokes equation

- Boundary Energy momentum tensor (Balasubaramanian and Kraus 1999)

$$S = -\frac{1}{16\pi G_5}\int dx^5\sqrt{-G}\left(R - \frac{20}{l^2}\right) - \frac{1}{8\pi G_5}\int_B dx^4\sqrt{-h}K + \frac{1}{8\pi G_5}S_{ct}(h_{\mu\nu})$$

$$L_{ct} = -\frac{3}{l}\sqrt{-h}\left(1 - \frac{l^2}{12}\mathcal{R}\right)$$: counter term

Regularize the energy-momentum tensor

- $h_{\mu\nu}$: metric on the boundary
- $K_{\mu\nu}$: extrinsic curvature
- $\mathcal{R}_{\mu\nu}$: Ricci curvature on the boundary
- $\mathcal{G}_{\mu\nu}$: Einstein tensor on the boundary

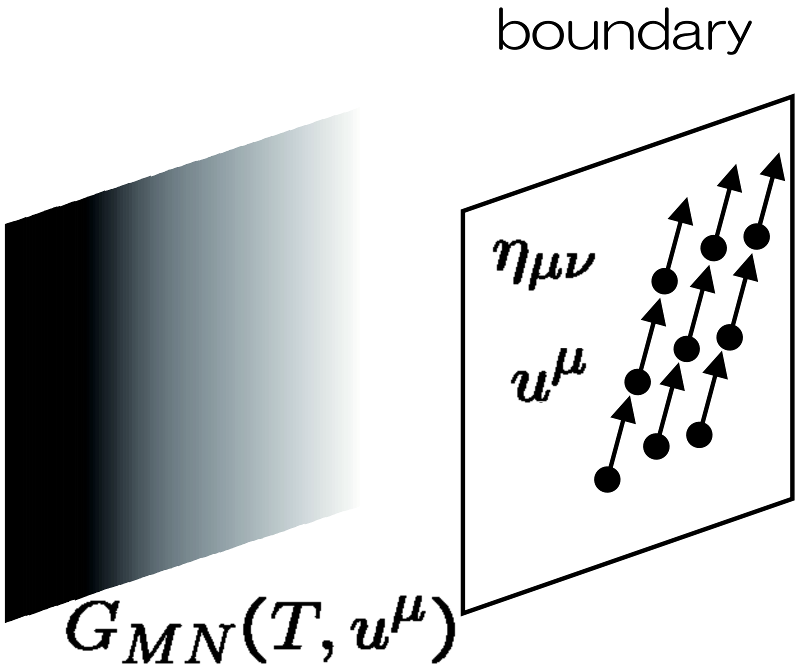
Then, we can calculate the energy-momentum tensor.

$$T^{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{\mu\nu}} = \frac{1}{8\pi G_5}\left(K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{l}h^{\mu\nu} - \frac{l}{2}\mathcal{G}^{\mu\nu}\right)$$

Boundary Navier-Stokes equation

· Boundary Energy momentum tensor

$$\begin{cases} M, N : \text{bulk(5 dim) indexes} \\ \mu, \nu : \text{boundary(4 dim) indexes} \end{cases}$$



·Metric

$$ds^2 = -2u_{\mu}dx^{\mu}dr - \frac{r^2}{l^2}f(br)u_{\mu}u_{\nu}dx^{\mu}dx^{\nu} + \frac{r^2}{l^2}(u_{\mu}u_{\nu} + \eta_{\mu\nu})dx^{\mu}dx^{\nu}$$

$$\begin{cases} u^{\mu} & : \text{constant} \\ T = \dfrac{1}{\pi b} & : \text{temperature} \end{cases}$$

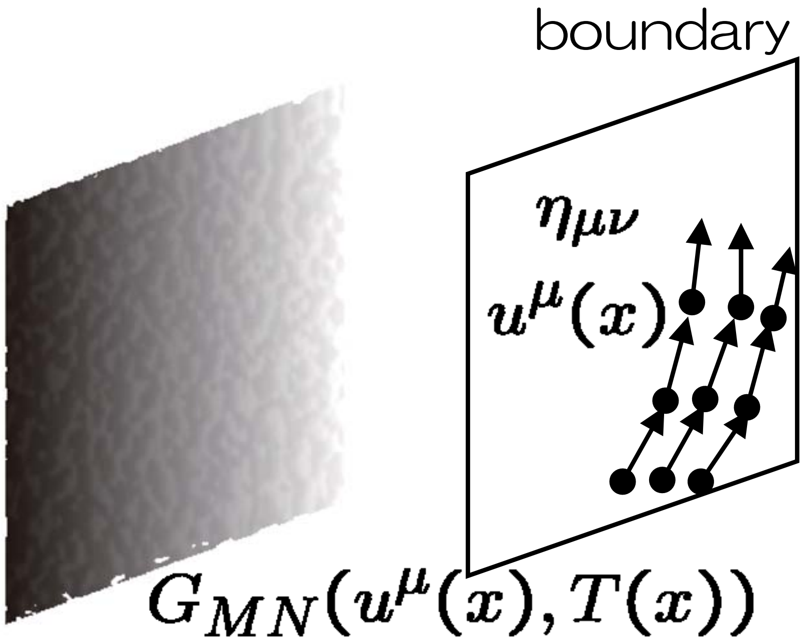
$$T^{\mu\nu} = \frac{(\pi T)^4}{16\pi G_5}(4u^{\mu}u^{\nu} + \eta^{\mu\nu})$$

⟶ Conformally invariant perfect fluid stress tensor

Boundary Navier-Stokes equation

- Dynamical brane and Boundary Energy momentum tensor

$\left\{ \begin{array}{l} M, N : \text{bulk(5 dim) indexes} \\ \mu, \nu : \text{boundary(4 dim) indexes} \end{array} \right.$



$$ds^2 = -2u_\mu(x)dx^\mu dr - \frac{r^2}{l^2}f(b(x)r)u_\mu(x)u_\nu(x)dx^\mu dx^\nu + \frac{r^2}{l^2}(u_\mu(x)u_\nu(x) + \eta_{\mu\nu})dx^\mu dx^\nu + \cdots$$

$u^\mu(x), T(x)$ depending terms

$\left\{ \begin{array}{ll} u^\mu(x) & \text{: velocity field} \\ T(x) = \frac{1}{\pi b(x)} & \text{: local temperature} \end{array} \right.$

$$T^{\mu\nu} = \frac{(\pi T)^4}{16\pi G_5}(4u^\mu u^\nu + \eta^{\mu\nu}) - 2(\pi T)^3\sigma^{\mu\nu} + \cdots$$

shear $\sim \nabla_\mu u_\nu + \nabla_\nu u_\mu$

Note: The metric does not satisfy Einstein equations if u^μ and b are arbitrary functions.

$\Longrightarrow$ $\nabla_\mu T^{\mu\nu} = 0 \supset$ Navier-Stokes equation

$$\partial_v b^{(1)} = \sigma_{ij}\sigma^{ij}$$

Boundary Navier-Stokes equation

- Dynamical brane and Boundary Energy momentum tensor

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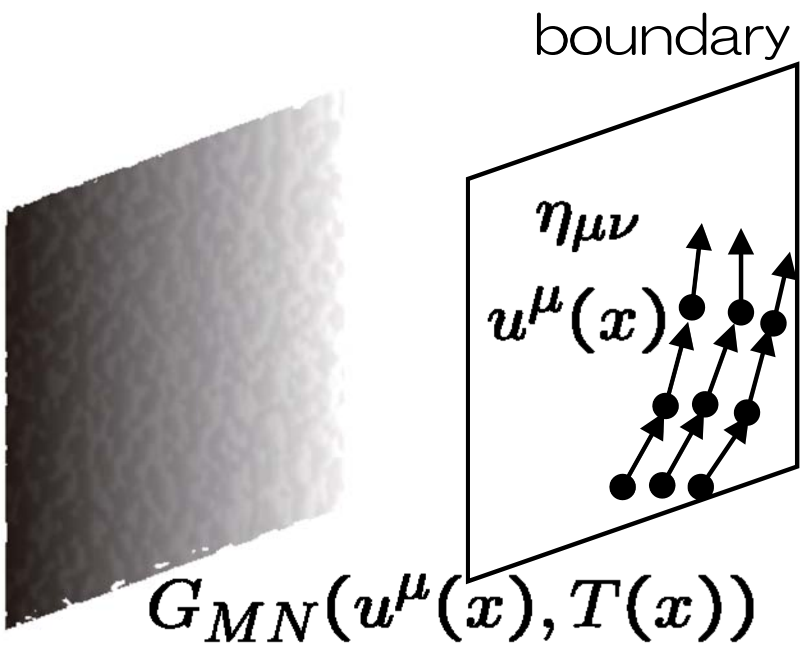

$$ds^2 = -2u_\mu(x)dx^\mu dr - \frac{r^2}{l^2}f(b(x)r)u_\mu(x)u_\nu(x)dx^\mu dx^\nu + \frac{r^2}{l^2}(u_\mu(x)u_\nu(x) + \eta_{\mu\nu})dx^\mu dx^\nu + \cdots$$

$u^\mu(x), T(x)$ depending terms

$$\begin{cases} u^\mu(x) & \text{: velocity field} \\ T(x) = \dfrac{1}{\pi b(x)} & \text{: local temperature} \end{cases}$$

Einstein equations for the fluctuations around the black brane

↓

Boundary Navier-Stokes equation with respect to u^μ and b .

$$\nabla_\mu T^{\mu\nu} = 0$$

Plan of talk

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3. Radial ADM Formalizum
4. Conclusion

Radial ADM Formalizum

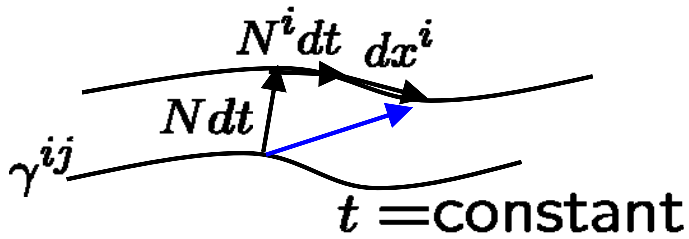
・Boundary energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{\mu\nu}}$$
$$= \frac{1}{8\pi G_5}\left(K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{l}h^{\mu\nu} - \frac{l}{2}G^{\mu\nu}\right)$$

・ADM Formalizm

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

$$\pi^{ij} = \frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma_{ij}}$$
$$= \frac{1}{8\pi G_5}\left(K^{ij} - K\gamma^{ij}\right)$$



$\longrightarrow$ $\nabla_i \pi^{ij} = 0$ ：constraint equation

Radial ADM Formalizum

・Boundary energy-momentum tensor

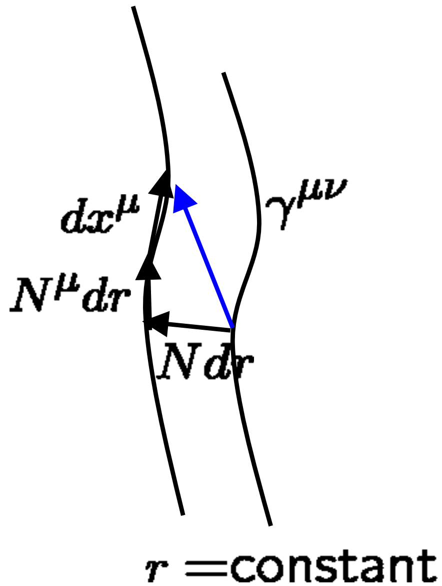
$$T^{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{\mu\nu}}$$
$$= \frac{1}{8\pi G_5}\left(K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{l}h^{\mu\nu} - \frac{l}{2}G^{\mu\nu}\right)$$

・Radial ADM Formalism

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}\left(dx^\mu + N^\mu dr\right)\left(dx^\nu + N^\nu dr\right)$$

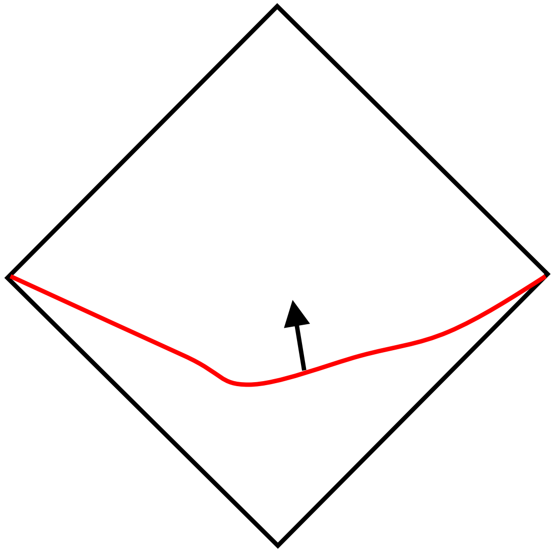
$$\pi^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma_{\mu\nu}}$$
$$= \frac{1}{8\pi G_5}\left(K^{\mu\nu} - K\gamma^{\mu\nu}\right)$$

⟶ $\nabla_\mu \pi^{\mu\nu} = 0$: "constraint" equation

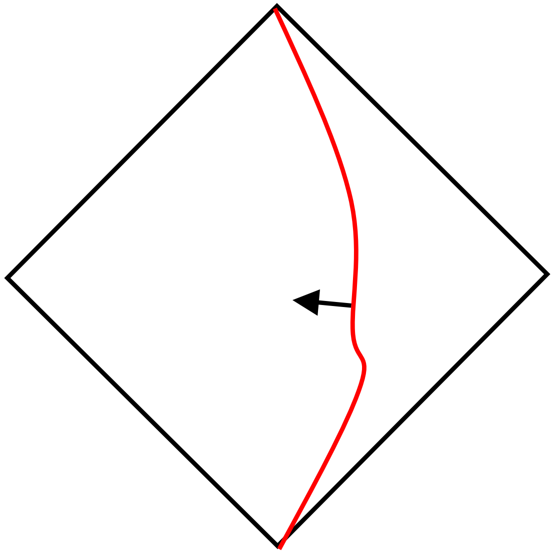


Radial ADM Formalizum

・ ADM Formalism and Radial ADM Formalism



ADM Formalism



Radial ADM Formalism

We can regard that Einstein equations describe an evolution along radial direction instead of time direction.

Radial ADM Formalizum

・Correspondence between the two energy-momentum tensors.

$$\pi^{\mu\nu} = \frac{1}{8\pi G_5}(K^{\mu\nu} - K\gamma^{\mu\nu})$$

$\downarrow \; r \to \infty$

$$T^{\mu\nu} = \frac{1}{8\pi G_5}\left(K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{l}h^{\mu\nu} - \frac{l}{2}\mathcal{G}^{\mu\nu}\right)$$

$\nabla_\mu \pi^{\mu\nu} = 0$: "constraint" equation

? $\downarrow \; r \to \infty$

$\nabla_\mu T^{\mu\nu} = 0$: Navier-Stokes equation

We need to confirm that the counter terms do not prevent the constraint equations.

Balasubaramanian and Kraus added the counter terms
such that the equation of the motions are not changed.

$$\nabla_\mu h^{\mu\nu} = \nabla_\mu \mathcal{G}^{\mu\nu} = 0$$

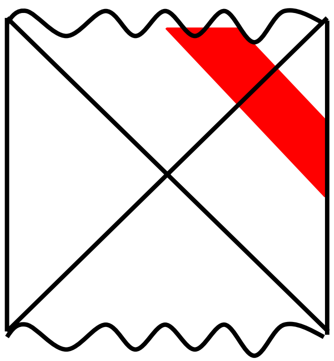
We can regard the Navier-Stokes equations as the constraint equations
in the radial ADM formalism

Conclusion

We can regard the Navier-Stokes equations as the constraint equations in the radial ADM formalism.

Discussion

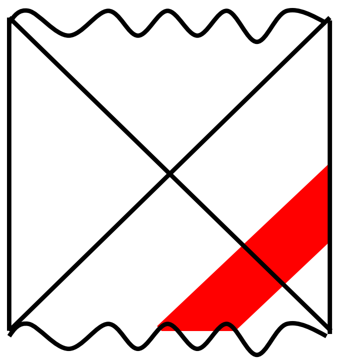
- R^2 corrections, non-relativistic limit, $\cdots$
- Meaning of the regularity condition



boundary

gluon plasma

$\delta S \geq 0, \ \ \eta > 0$



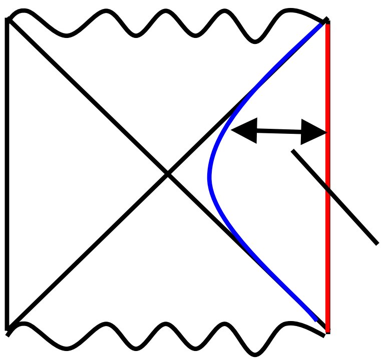
boundary

???

$\delta S \leq 0, \ \ \eta < 0$

fine tuned process?

- Relation to the membrane paradigm



The near horizon membrane also has fluid picture.

radial evolution(?)

cf. Iqbal and Liu (2008), Brustein and Gorbonos (2009)