Quantum Entropy Function

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One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH}(\vec{Q}) = \ln d_{micro}(\vec{Q})$$

Strominger, Vafa

$d_{micro}(\vec{Q})$: degeneracy of microstates carrying a given set of charges $\vec{Q}$

$$S_{BH}(\vec{Q}) = A/4G_N$$

A= Area of event horizon of a black hole of charge $\vec{Q}$

This formula is quite remarkable since it relates a geometric quantity in space-time to a counting problem.

However the Bekenstein-Hawking formula for the entropy receives α' and g_s corrections.

Our goal is to search for an exact relation of the form

$$d_{macro}(\vec{Q}) = d_{micro}(\vec{Q})$$

$d_{macro}(\vec{Q})$: Some generalization of the Bekenstein-Hawking formula taking into account α' and g_s corrections.

We shall focus on extremal, BPS black holes.

Extremality: essential for the separation between the horizon degrees of freedom and those living outside the horizon by an infinite throat

Supersymmetry: (probably) needed for ensuring stability of extremal black holes.

Also we shall work in some fixed duality frame.

Take a single centered black hole of charge $\vec{Q}$

Proposal for $d_{macro}(\vec{Q})$:

$$\sum_{\substack{\vec{Q}_{hor},\vec{Q}_{hair} \\ \vec{Q}_{hor}+\vec{Q}_{hair}=\vec{Q}}} d_{hor}(\vec{Q}_{hor})\, d_{hair}(\vec{Q}_{hair};\vec{Q}_{hor})$$

$d_{hor}(\vec{Q}_{hor})$: contribution from the horizon with charge $\vec{Q}_{hor}$

d_{hair}: contribution from the hair of the single-centered black hole, with the horizon carrying charge $\vec{Q}_{hor}$, and the hair carrying charge $\vec{Q}_{hair}$.

Hair: smooth normalizable deformations of the black hole solution with support outside the horizon and satisfying BPS properties.

$d_{hair}(\vec{Q}_{hair};\vec{Q}_{hor})$: the degeneracy of the hair carrying total charge $\vec{Q}_{hair}$, obtained by Crnkovic-Witten type quantization of the hair degrees of freedom.

Our main focus on this talk will be on $d_{hor}(\vec{Q})$.

Our goal: Find a macroscopic prescription for computing $d_{hor}(\vec{Q})$

To leading order in g_s but all orders in α', $d_{hor}(\vec{Q})$ is given by the exponential of the Wald entropy

– can be computed using the entropy function formalism

We shall begin with a lightening review of the results of the entropy function formalism.

Postulate: An extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

Regarding all other directions (including angular coordinates) as compact we can regard the near horizon geometry of an extremal black hole as

$$AdS_2 \times \text{a compact space (fibered over } AdS_2\text{)}$$

Note: Magnetic charges are encoded in the fluxes through the compact space.

Consider string theory in such a background containing two dimensional metric $g_{\mu\nu}$ and $U(1)$ gauge fields $A_\mu^{(i)}$ among other fields.

The most general field configuration consistent with $SO(2,1)$ isometry:

$$ds^2 \equiv g^{(2)}_{\mu\nu} dx^\mu dx^\nu = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i, \quad \cdots\cdots\cdots$$

$\mathcal{L}^{(2)}(v, \vec{e}, \cdots)$: The Lagrangian density evaluated in this background.

For black hole with electric charge $\vec{q}$, define

$$\mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left(e_i \, q_i - v \, \mathcal{L}^{(2)} \right)$$

One finds that

1. All the near horizon parameters are obtained by extremizing $\mathcal{E}$ with respect to v, e_i and the other near horizon parameters.

2. $S_{wald}(\vec{q}) = \mathcal{E}$ at this extremum.

Thus in the classical limit

$$d_{hor}(\vec{q}) = e^{S_{wald}(\vec{q})} = e^{\mathcal{E}}$$

We shall propose an expression for $d_{hor}(\vec{q})$ in the full quantum theory as a path integral over the Euclidean continuation of the near horizon geometry.

$\longrightarrow$ Quantum entropy function

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$
$$F^{(i)}_{rt} = e_i$$

Euclidean continuation:

$t = -i\theta, \quad r = \cosh\eta, \quad \theta \equiv \theta + 2\pi, \quad 0 \leq \eta < \infty$

This gives

$$\begin{aligned} ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \\ F^{(i)}_{\theta\eta} &= i e_i \sinh\eta \end{aligned}$$

$$\rightarrow \quad A^{(i)}_\theta = -i\, e_i\, (\cosh\eta - 1) = -i\, e_i\, (r - 1)\,.$$

Proposal for the quantum entropy function $d_{hor}(\vec{q})$

$$d_{hor}(\vec{q}) = \left\langle \text{exp}[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

$\langle \ \rangle_{AdS_2}$: unnormalized path integral over various fields of string theory on euclidean global AdS_2.

$\oint$: a closed contour at the boundary of AdS_2.

'finite': Infrared finite part of the amplitude.

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

The superscript '$finite$' refers to the finite part of the amplitude defined by expressing it as

$$e^{CL} \times \text{finite part}$$

L: length of the boundary of AdS_2.

C: A constant

The definition can be shown to be independent of the choice of $f(\theta)$.

We shall work with $f(\theta) = 1$.

The role of

$$\exp[-iq_i \oint d\theta\, A_\theta^{(i)}]$$

We could absorb this into the boundary terms in the action.

However we have displayed it explicitly since it plays a special role.

It is the only term in the boundary action that involves the gauge field and not its field strength.

Why do we need this term?

In AdS_d the Maxwell's equation has two solutions in the asymptotic region:

$A_\theta^{(i)} \sim r^{-d+3}$: electric field mode

$A_\theta^{(i)} \sim$ constant: constant mode

Thus for $d \geq 4$ the constant mode of the gauge field is dominant at infinity.

We fix the constant mode by a boundary condition and integrate over the electric field mode.

However for $d = 2$,

Electric field mode: $A_\theta^{(i)} \sim r$

Constant mode: $A_\theta^{(i)} \sim$ constant

Thus the electric field mode is dominant

$\longrightarrow$ we must work in a sector with fixed asymptotic electric field i.e. fixed charge, and allow the constant mode to fluctuate.

However now the extremization of the action no longer gives the classical equations of motion.

The variation of the action contains boundary terms proportional to $\delta A_\theta^{(i)}$ which are no longer constrained to vanish by boundary condition.

$\to$ we need to add new boundary term in the action to cancel the boundary terms proportional to $\delta A_\theta^{(i)}$.

The $\exp[-iq_i \oint d\theta\, A_\theta^{(i)}]$ precisely achieves this task.

$$d_{hor}(\vec{q}) = \left\langle \text{exp}[-iq_i \oint d\theta\, A^{(i)}_\theta] \right\rangle^{finite}_{AdS_2}$$

We shall try to justify this proposal by showing that

1. In the classical limit

$$\ln d_{hor}(\vec{q}) \to S_{wald}(\vec{q})$$

2. This fits in with the usual rules of AdS/CFT correspondence.

$$\left\langle \text{exp}[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2}$$

In the classical limit this reduces to

$$e^{-\mathcal{S}} \, \text{exp}[-iq_i \oint d\theta \, A_\theta^{(i)cl}]$$

$$A_\theta^{(i)cl} = -i \, e_i \, (r_0 - 1)$$

$\mathcal{S}$ = Euclidean action = $\mathcal{S}_{bulk} + \mathcal{S}_{boundary}$

$$\mathcal{S}_{bulk} = -\int_1^{r_0} dr \sqrt{\det g}\, d\theta\, \mathcal{L}^{(2)} = -(r_0 - 1)\, 2\pi v\, \mathcal{L}^{(2)}$$

$$-iq_i \oint d\theta\, A_\theta^{(i)cl} = -2\pi\, \vec{q} \cdot \vec{e}\, (r_0 - 1)$$

$$\mathcal{S}_{boundary} = -2\pi\, K r_0 + \mathcal{O}(r_0^{-1})$$

K: some constant which depends on the details of the boundary terms.

The length of the boundary is

$$L = 2\pi\sqrt{v} r_0 + \mathcal{O}(r_0^{-1})\,.$$

This gives

$$\left\langle \text{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}$$

$$= \left[e^{L(v\,\mathcal{L}^{(2)}+K-\vec{e}\cdot\vec{q})/\sqrt{v}+2\pi(\vec{e}\cdot\vec{q}-v\,\mathcal{L}^{(2)})+\mathcal{O}(r_0^{-1})} \right]$$

Extracting the finite part we get

$$d_{hor}(\vec{q}) \simeq \text{exp}\left[2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)})\right] = \text{exp}\left[S_{wald}(\vec{q})\right]$$

Note: A change in the boundary action changes K but the finite part is insensitive to such a change.

AdS_2/CFT_1 **correspondence**

Euclidean AdS_2 is the Poincare disk.

$\rightarrow$ its boundary is a circle of circumference L.

Thus AdS/CFT correspondence $\rightarrow$

$$\left\langle \text{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2} = Z_{CFT_1} = Tr_{\vec{q}} e^{-LH}$$

$Tr_{\vec{q}}$: trace over states of charge $\vec{q}$ in CFT_1

H: Hamiltonian of dual CFT_1

Thus we have, for large L,

$$\left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2} = Tr_{\vec{q}} e^{-LH}$$
$$= d_{CFT}(\vec{q}) e^{-E_0 L}\,.$$

E_0, $d_{CFT}(\vec{q})$: ground state energy, degeneracy

Taking the finite part we get

$$d_{hor}(\vec{q}) = d_{CFT}(\vec{q})$$

Note: In the more conventional units we take the length of the boundary to be finite, but scale energies by L.

Only the ground states of the CFT survive.

What can we say about CFT_1?

It should be identified as the infrared limit of the quantum mechanics that describes the black hole solution, after stripping off the hair contribution.

Thus d_{CFT} together with the hair contribution should give us the microscopic degeneracies.

– agrees with our proposal.

A consequence

In the computation of d_{hor} we must include contribution from all saddle points preserving the asymptotic boundary conditions.

Take a $\mathbb{Z}_N$ orbifold that acts on AdS_2 by

$$\theta \to \theta + 2\pi/N$$

and an appropriate $\mathbb{Z}_N$ transformation in the compact directions.

After a change of coordinates we find that the new configuration satisfies the asymptotic b.c.

$$
\begin{aligned}
ds^2 &= v\left((r^2 - N^{-2})d\theta^2 + \frac{dr^2}{r^2 - N^{-2}}\right), \\
F^{(i)}_{r\theta} &= -i\, e_i, \quad \cdots, \qquad \theta \equiv \theta + 2\pi\,.
\end{aligned}
$$

Its classical contribution to d_{hor} is $e^{S_{wald}/N}$.

Such contributions are indeed present in the known microscopic degeneracies of black holes in $\mathcal{N} = 4$ supersymmetric string theories.

Summary

We have a concrete proposal for relating the extremal black hole entropy to the microscopic degeneracy

$$d_{hor}(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

should agree with the microscopic degeneracies after removing the hair contribution.

1. It reduces to the relation between wald entropy and statistical entropy in the classical limit.

2. It is in the spirit of AdS/CFT correspondence.

Degeneracy or Index?

Often in the microscopic theory we compute the index rather than degeneracy.

– protected against quantum corrections.

e.g. in $D = 4$ we calculate the helicity trace index

$$B_{2n} = (-1)^n \, Tr_{\vec{Q}} \left[(-1)^{2h} \, (2h)^{2n} \right]$$

$4n$: Number of broken SUSY generators

Thus on the black hole also we should compute the index.

The $(2h)^{2n}$ factor is needed to absorb the fermion zero modes associated with broken SUSY.

For a black hole solution these zero modes form part of hair degrees of freedom

Since in $D = 4$ the black hole horizons always have $h = 0$ we get the following formula for the index on the macroscopic side

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} B_{2n;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

– can be computed using quantum entropy function.

A consistency check:

Suppose two black holes have identical near horizon geometry but different asymptotic geometries.

Suppose further that we know the appropriate index for both these black holes, and can compute the hair contribution for both the black holes.

Then by stripping off the hair contribution we can get the 'microscopic result' for $d_{hor}(\vec{Q})$ for both the black holes.

They must agree.

An example:

System 1: BMPV black hole

– A five dimensional rotating black hole in type IIB on $K3 \times S^1$.

System 2: A four dimensional black hole in type IIB on $K3 \times T^2$ obtained by placing the BMPV black hole in Taub-NUT

They have identical near horizon geometries but different index and different d_{hair}

But d_{hor} computed by stripping off d_{hair} from the index gives the same result for both.