Hydrodynamics from the D1 brane

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References

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- *"Hydrodynamics from the Dp-brane,"* by J. Mas and J. Tarrio, JHEP **0705**, 036 (2007) (arXiv:hep-th/0703093).

Motivation

- No shear in 1+1 dimensions.
  Conformal fluid is a perfect fluid i.e.
  $T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu}$.
- Only viscous transport coefficient is bulk viscosity.
- Gauge invariant fluctuations found for the Dp brane for $p \geq 2$ can't be extended to $p = 1$.

Outline I

- Gauge/gravity duality for the D1 brane
  - D1 brane supergravity solution
  - Thermodynamics for the D1 brane
  - Regimes of validity for our analysis.
- Hydrodynamics in 1+1 dimensions.
  - Lorentz structure of correlators
  - Poles of the Green's function
- Sound channel in gravity
  - Effective 3-D system
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  - Diffeomorphism invariant sound mode
  - Dispersion relation
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Outline II

- Stress Tensor correlators
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  - ξ/s using the Kubo's formula
- Other Examples
  - F1 solution
  - Transverse space as Sasaki-Einstein Manifolds
- Summary

Summary

- For the $SU(N)$ gauge theory with 16 supercharges in $1+1$D on the D1-branes in range $\sqrt{\lambda}N^{-1} << T << \sqrt{\lambda}$,
  - $$\xi = \frac{2^6\pi^{\frac{7}{2}}}{3^3}\frac{N^2T^2}{\sqrt{\lambda}}.$$
  - $$\frac{\xi}{s} = \frac{1}{4\pi}.$$
- For $T << \sqrt{\lambda}N^{-1}$ and $T >> \sqrt{\lambda}$,
  the D1-brane gauge theory $\rightarrow$ a CFT.
  So, bulk viscosity vanishes.

D1 brane supergravity solution

Non extremal D1 brane solution in Einstein frame is

$$ds_{10}^2 = H^{-\frac{3}{4}}(r)(-f(r)dt^2 + dx_1^2) + H^{\frac{1}{4}}(r)\left(\frac{dr^2}{f(r)} + r^2 d\Omega_7^2\right),$$

$$\begin{aligned} e^{\phi(r)} &= H(r)^{\frac{1}{2}}, \\ F_7^{RR} &= 6L^6\omega_{S_7}, \end{aligned}$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^6 \qquad H = \left(\frac{L}{r}\right)^6$$

$$L^6 = 2^6\pi^3 g_{YM}^2 N\alpha'^4$$

D1 brane thermodynamics

- The temperature in terms of the non-extremal parameter r_0 is

$$T = \frac{3r_0^2}{2\pi L^3},$$

- The entropy density is

$$s = \frac{1}{4G_3}\left(\frac{r_0}{L}\right)^4 = \frac{2\pi^4}{4!G_{10}} r_0^4 L^3.$$

Regime of validity

- This supergravity solution is valid in the temperature range

$$\sqrt{\lambda} N^{-\frac{2}{3}} << T << \sqrt{\lambda}.$$

- Here, the t'Hooft coupling

$$\lambda = g_{YM}^2 N \qquad g_{YM}^2 = \frac{g_s}{2\pi\alpha'}$$

.

- Out side this range, curvature in supergravity grows large.
- For $T >> \sqrt{\lambda}$, Perturbative description of YM.
- For $T << \sqrt{\lambda} N^{-\frac{2}{3}}$, can dualize to fundamental string solution.

F1 brane solution

Now for $T << \sqrt{\lambda}N^{-2/3}$, the holographic dual of the Yang-Mills theory is given by the non-extremal fundamental string solution.

$$ds_{10}^2 = H^{-\frac{3}{4}}(r)(-f(r)dt^2 + dx_1^2) + H^{\frac{1}{4}}(r)\left(\frac{dr^2}{f(r)} + r^2 d\Omega_7^2\right),$$

$$\begin{aligned} e^{\phi(r)} &= H(r)^{-\frac{1}{2}}, \\ F_7^{NS} &= 6L^6\omega_{S_7}. \end{aligned}$$

Here, ω_{S_7} denotes the volume form on the 7-sphere.

Regime of validity

- F1 solution can be trusted in the following temperature range

$$\sqrt{\lambda}N^{-1} << T << \sqrt{\lambda}N^{-\frac{2}{3}}.$$

- To conclude, for $T >> \sqrt{\lambda}$ and $T << \sqrt{\lambda}N^{-1}$, the YM theory $\rightarrow$ free conformal field theory. So, bulk viscosity vanish.
- Supergravity description available in region

$$\sqrt{\lambda}N^{-1} \ll T \ll \sqrt{\lambda}.$$

- Fairly large domain for large N.
- Bulk viscosity non trivial in this regime.

Two point correlator: structure

- We define the retarded Green's function of the stress tensor as

$$G_{\mu\nu,\alpha\beta}(x-y) = -i\theta(x^0-y^0)\langle[T_{\mu\nu}(x), T_{\alpha\beta}(y)]\rangle.$$

- Its Fourier transform is denoted as $G_{\mu\nu,\alpha\beta}(k)$.
- It is symmetric by definition under
  - interchange of indices (μ,ν).
  - interchange of indices (α,β).
-

$$G_{\mu\nu,\alpha\beta}(k) = G_{\alpha\beta,\mu\nu}(k).$$

  due to CPT invariance.

Two point correlator: structure

Conservation of Stress energy tensor → Ward identity

$$k^\mu G_{\mu\nu,\alpha\beta}(k) = 0.$$

This suggests a useful tensor which forms a basis to write down the correlator is

$$P_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}.$$

Note that $k^\mu P_{\mu\nu} = 0$. Can split it as

$$G_{\mu\nu,\alpha\beta}(k) = P_{\mu\nu} P_{\alpha\beta} G_B(k^2) + H_{\mu\nu,\alpha\beta} G_S(k^2),$$

where

$$H_{\mu\nu,\alpha\beta} = \frac{1}{2}(P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha}) - P_{\mu\nu} P_{\alpha\beta}.$$

Two point correlator: structure

$$H_{\mu\nu,\alpha\beta} = \frac{1}{2}(P_{\mu\alpha}P_{\nu\beta} + P_{\mu\beta}P_{\nu\alpha}) - P_{\mu\nu}P_{\alpha\beta}.$$

- Note that $\eta^{\mu\nu}H_{\mu\nu,\alpha\beta} = 0$.
- The two tensors above are orthogonal

$$P_{\mu\nu}P_{\alpha\beta}H^{\mu\nu}{}_{,\alpha'\beta'} = 0.$$

- By substituting the value of $k_\mu = (-\omega, q)$, we find that all components of $H_{\mu\nu,\alpha\beta}$ vanish.
- So, the two point function of the stress tensor in a $1+1$ dimensional theory is entirely dependent on just one function $G_B(k^2)$.

Two point correlator: structure

- Thus the two point function can be written as

$$G_{\mu\nu,\alpha\beta}(\omega, q) = P_{\mu\nu} P_{\alpha\beta} G_B(\omega, q).$$

- Writing it explicitly, we obtain

$$G_{tt,tt} = \frac{q^4}{(\omega^2 - q^2)^2} G_B, \quad G_{tt,tx} = \frac{q^3\omega}{(\omega^2 - q^2)^2} G_B,$$
$$G_{tt,xx} = \frac{\omega^2 q^2}{(\omega^2 - q^2)^2} G_B, \quad G_{tx,tx} = \frac{\omega^2 q^2}{(\omega^2 - q^2)^2} G_B,$$
$$G_{tx,xx} = \frac{\omega^3 q}{(\omega^2 - q^2)^2} G_B, \quad G_{xx,xx} = \frac{\omega^4}{(\omega^2 - q^2)^2} G_B.$$

- Thus all components of the thermal Green's function of the stress tensor are determined by a single function G_B.

Poles in the correlators

The stress tensor of a fluid in $1+1$ dimensions is given by

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} - \xi(u^\mu u^\nu + \eta^{\mu\nu})\partial_\lambda u^\lambda,$$

where u^μ is the 2-velocity with

$$u_\mu u^\mu = -1$$

and ξ is the bulk viscosity.
Now consider small fluctuations from the rest frame of the fluid.

$$T^{00} = \epsilon + \delta T^{00}, \qquad T^{0x} = \delta T^{0x},$$
$$T^{xx} = P + \delta T^{xx},$$
$$u^0 = 1, \qquad u^x = \delta u^x.$$

Poles in the correlators

Putting them into the conservation equation $\nabla^\mu T_{\mu\nu} = 0$ and after some manipulations, one gets

$$\left(-i\omega^2 + iq^2 v_s^2 + \frac{\xi}{\epsilon + P}\omega q^2\right)\delta T^{0x} = 0.$$

where

$$v_s^2 = \frac{\partial P}{\partial \epsilon}$$

is the speed of sound.
We thus get dispersion relation for δT^{0x} upto leading order as

$$\omega = \pm v_s q - i\frac{\xi}{2(\epsilon + P)}q^2.$$

Linear response theory says G_B has a pole at this ω.

Effective 3-D system

First consistently truncate the 10D NH geometry to 3D by reducing on the S^7. Ansatz is

$$\begin{aligned} ds_{10}^2 &= e^{-14B(r)} ds_3^2 + e^{2B(r)} L^2 d\Omega_7^2, \\ ds_3^2 &= -c_T(r)^2 dt^2 + c_X(r)^2 dz^2 + c_R(r)^2 dr^2. \end{aligned}$$

We take $B(r) = -\frac{1}{24}\phi(r)$ and keep the RR flux on the 7-sphere constant. The effective action

$$S = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left[R - \frac{\beta}{2} \partial_\mu \phi \partial^\mu \phi - \mathcal{P}(\phi) \right].$$

Here, $\beta = \frac{16}{9}$, and $\mathcal{P} = -\frac{24}{L^2} e^{\frac{4}{3}\phi}$.

Effective 3-D system

The D1-brane in 10-dimensions reduces to

$$\begin{aligned} ds_3^2 &= -c_T(r)^2 dt^2 + c_X(r)^2 dz^2 + c_R(r)^2 dr^2, \\ \phi &= -3\log\left(\frac{r}{L}\right), \end{aligned}$$

with the components of the metric given by

$$c_T^2 = \left(\frac{r}{L}\right)^8 f, \qquad c_X^2 = \left(\frac{r}{L}\right)^8, \qquad c_R^2 = \frac{1}{f}\left(\frac{r}{L}\right)^2,$$

with $f = 1 - \frac{r_0^6}{r^6}$.

Linear perturbations and gauge fixing

- We take small perturbations in the above background.

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \qquad \text{and} \qquad \phi \to \phi + \delta\phi.$$

- We make a Fourier transformation

$$\delta g_{\mu\nu} = e^{-i(\omega t - qz)} h_{\mu\nu}(r), \quad \delta\phi = e^{-i(\omega t - qz)} \varphi(r).$$

- We further parametrize the metric perturbations as

$$h_{tt} = c_T^2 H_{tt}, \quad h_{tz} = c_X^2 H_{tz}, \quad h_{zz} = c_X^2 H_{zz}.$$

- We fix the gauge by choosing

$$\delta g_{rt} = \delta g_{rz} = \delta g_{rr} = 0.$$

- We have 4 dynamical equations and 3 constraints.

Diffeomorphism invariant sound mode

- One is left with the residual gauge freedom under the infinitesimal diffeomorphisms

$$x^\mu \to x^\mu + \xi^\mu \text{ with } \mu \in \{t, z, r\}.$$

- The metric changes as

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu .$$

- We next form a diffeomorphism invariant quantity out of the perturbations.
- Unique in our case.
- Differs from such quantity constructed for Dp systems for $p \geq 2$.

Diffeomorphism invariant sound mode

We find the following combination gauge invariant.

$$\begin{aligned} Z_P &= Z_0 + A_\varphi \varphi \\ \text{where} \quad Z_0 &= q^2 \frac{c_T^2}{c_X^2} H_{tt} + 2q\omega H_{tz} + \omega^2 H_{zz} \\ A_\varphi &= \frac{2}{\phi'} \left(q^2 \frac{c_T^2}{c_X^2} \ln' c_T - \omega^2 \ln' c_X \right). \end{aligned}$$

Using linear perturbations of Einstein equations and the dilaton equation of motion, we get

$$Z_P'' + \left[\ln' \left(\frac{c_T c_X}{c_R} \right) - 2 \frac{A_\varphi'}{A_\varphi} \right] Z_P' + \mathcal{G} Z_P = 0.$$

with $\mathcal{G} = - \left[\frac{c_R^2}{c_T^2} \left(q^2 \frac{c_T^2}{c_X^2} - \omega^2 \right) + 2 \frac{A_\varphi'}{A_\varphi} \ln' \left(\frac{c_X}{c_T} \right) \right].$

Dispersion relation for the sound mode

We define a variable $u = \frac{r_0^2}{r^2}$,

so that $u \to 1$ is the horizon.
and $u \to 0$ is the boundary.

We are interested in a solution which is ingoing at the horizon
Taking appropriate limits, we find the ingoing solution behaves as

$$Z_p = \frac{1}{A_\varphi}(1-u)^{-\frac{i}{3}\alpha\omega}.$$

where

$$\alpha = \frac{L^3}{2r_0^2} = \frac{3}{4\pi T}.$$

Dispersion relation for the sound mode

We next consider a solution of the type

$$Z_p = \frac{1}{A_\varphi}(1-u^3)^{-\frac{i}{3}\alpha\omega} Z(u).$$

We also consider the hydrodynamic limit

$$\omega << T \quad \text{and} \qquad q << T$$

and we ignore terms of order q^2/T^2, ω^2/T^2, $\omega q/T^2$ and higher, but keep terms of order ω/T, q/T .
In this limit,

$$\partial_u^2 Z - \frac{\{2+(1-2i\alpha\omega)u^3\}}{u(1-u^3)}\partial_u Z - \frac{18u^4(4\lambda-3)}{(1-u^3)(4-4\lambda-u^3)^2} Z = 0.$$

Dispersion relation for the sound mode

Its well behaved solution at the horizon is

$$Z = \frac{6\lambda - 2(1-\lambda)(3-4i\alpha\omega) - u^3(3+2i\alpha\omega)}{12(3-2i\alpha\omega)(4-4\lambda+u^3)}.$$

Next impose Dirichlet condition, $Z = 0$ at the boundary,

$$-4i\alpha\omega^3 + 6\omega^2 + 4i\alpha\omega q^2 - 3q^2 = 0.$$

We assume $\omega \sim q$, and solve this equation perturbatively.

$$\omega = \pm\frac{1}{\sqrt{2}}q - i\frac{\alpha}{6}q^2 + ...$$

with

$$\alpha = \frac{L^3}{2r_0^2} = \frac{3}{4\pi T}.$$

Viscosity by entropy ratio

Dispersion relation from general hydrodynamics considerations

$$\omega = v_s - \frac{i}{2}\frac{1}{\epsilon + P}\xi q^2, \qquad v_s^2 = \frac{\partial P}{\partial \epsilon}.$$

Dispersion relation from linearized supergravity analysis

$$\omega = \pm\frac{1}{\sqrt{2}}q - i\frac{\alpha}{6}q^2 + \ldots$$

By comparing, we get

$$v_s^2 = \frac{1}{2} \qquad \frac{\xi}{\epsilon + P} = \frac{\alpha}{3} = \frac{1}{4\pi T}.$$

$$v_s^2 = \frac{\partial P}{\partial \epsilon} \qquad \rightarrow \qquad \epsilon = 2P.$$

The medium seems to behave as a conformal fluid in 2+1D.

Viscosity by entropy ratio

$$\frac{\xi}{\epsilon + P} = \frac{1}{4\pi T}.$$

$$\epsilon + P = Ts \qquad \rightarrow \qquad \frac{\xi}{s} = \frac{1}{4\pi}.$$

$$s = \frac{2^4 \pi^{\frac{5}{2}}}{3^3} \frac{N^2 T^2}{\sqrt{\lambda}} \qquad \rightarrow \qquad \xi = \frac{2^6 \pi^{\frac{7}{2}}}{3^3} \frac{N^2 T^2}{\sqrt{\lambda}}.$$

For Dp-branes with $p \geq 2$ (Mas and Tarrio hep-th/0703093),

$$\frac{\xi}{s} = \frac{\xi}{\eta} \frac{\eta}{s} = \frac{1}{4\pi} \frac{2(3-p)^2}{p(9-p)}.$$

The general expression continues to hold also for $p = 1$.

Stress-Energy correlator

We first need to expand the action along with the Gibbons-Hawking boundary term to second order in the metric and dilaton fluctuations.

$$\begin{aligned} S &= S_{\text{bulk}} + S_{GH}, \\ S_{\text{bulk}} &= \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left[R - \frac{\beta}{2}\partial_\mu\phi\partial^\mu\phi - \mathcal{P}(\phi)\right] \\ S_{GH} &= \frac{1}{8\pi G_3}\int d^2x\sqrt{-h}K|_{r\to\infty}. \end{aligned}$$

where

$$\beta = \frac{16}{9} \quad \mathcal{P} = -\frac{24}{L^2}e^{4\phi/3}.$$

Stress-Energy correlator

After algebraic manipulations, we obtain

$$S^{(2)} = \frac{1}{16\pi G_3} \int \frac{d\omega dq}{(2\pi)^2} \mathcal{A}(\omega, q, r) Z'_P(r, \vec{k}) Z_P(r, -\vec{k}) + S^{(2)}_{CT},$$

where

$$\mathcal{A}(\omega, q, r) = -\frac{\beta}{2A_\varphi^2} \frac{c_T c_X}{c_R}.$$

Metric fluctuation - stress energy tensor coupling is

$$S_{\text{coupling}} = \frac{1}{2} \int d^4x [H^0_{tt} T^{tt} + H^0_{zz} T^{zz} + 2H^0_{tz} T^{tz}].$$

We evaluate two point fn(s) using the AdS/CFT correspondence

$$\langle \exp(iS_{\text{coupling}}) \rangle = \exp[iS^{(2)}(H^0_{\mu\nu})].$$

Stress-Energy correlator

Consider

$$G_{tt,tt} = -i\langle[T_{tt}, T_{tt}]\rangle = -4\frac{\delta S^{(2)}}{\delta H^0_{tt}(\vec{k})\delta H^0_{tt}(-\vec{k})}.$$

Note $\quad Z^0_P = q^2 H^0_{tt} + 2\omega q H^0_{tz} + \omega^2 H^0_{zz} + A^0_\varphi \varphi^0.$

Near $\quad u \to 0, \quad Z_P = C(1+\ldots) + Du^3(1+\ldots),$

$$= Z^0_P\left[1+\ldots+\frac{D}{C}\frac{r_0^6}{r^6}(1+\ldots)\right].$$

$$\Rightarrow G_{tt,tt} = -4\frac{\delta S^{(2)}}{\delta H^0_{tt}(\vec{k})\delta H^0_{tt}(-\vec{k})} = -\frac{1}{16\pi G_3}\frac{q^4}{(\omega^2-q^2)^2}\frac{6r_0^6}{L^7}\frac{D}{C}.$$

Stress-Energy correlator

Recall $$G_{\mu\nu,\alpha\beta}(\omega, q) = P_{\mu\nu}P_{\alpha\beta}G_B(\omega, q).$$

- We can read out the holographic value of G_B from the expression of $G_{tt,tt}$ as

$$G_B(\omega, q) = -\frac{1}{16\pi G_3}\frac{6r_0^6}{L^7}\frac{D}{C}.$$

- The poles in the Green's function are therefore same as the zeros of the factor C.
- The *Dirichlet boundary condition* for the mode $Z_P = 0$ at the horizon is equivalent to setting $C = 0$.
- Poles in G_B $\rightarrow$*dispersion relation of the sound mode.*
- Evalation of remaining two point functions reproduces the same expression for $G_B(\omega, q)$.

ξ/s using the Kubo's formula

We next extract Bulk viscosity from the Green's function. This is done using the Kubo's formula. In $1+1$D, Kubo's formula for bulk viscosity is given by

$$\begin{aligned}\xi &= \lim_{\omega\to 0}\frac{1}{\omega}\int_0^\infty dt\int dz e^{i\omega t}\langle[T_{zz}(x),T_{zz}(0)]\rangle.\\ &= \lim_{\omega\to 0}\frac{i}{\omega}G_{zz,zz}(\omega,q=0).\end{aligned}$$

Recall

$$G_{zz,zz}=\frac{\omega^4}{(\omega^2-q^2)^2}G_B(\omega,q),\qquad G_B=-\frac{1}{16\pi G_3}\frac{r_0^6}{L^7}\frac{D}{C}.$$

ξ/s using the Kubo's formula

$$C = \frac{(-4i\alpha\omega^3 + 4i\alpha\omega q^2 + 6\omega^2 - 3q^2)}{9(2i\alpha\omega - 3)},$$

$$D = \frac{\omega[9i\omega q^2 + 8i\alpha^2\omega(\omega^2 - q^2)^2 - 12\alpha(2q^4 - 3q^2\omega^2 + \omega^4)]}{54(3i + 2\alpha\omega)(q^2 - \omega^2)}.$$

$$G_{zz,zz}(\omega, 0) = -\frac{1}{16\pi G_3}\left[\frac{r_0^6}{L^7}\frac{D}{C}\right]_{q=0}.$$

$$G_{zz,zz}(\omega, 0) = -\frac{1}{16\pi G_3}\frac{6r_0^6}{L^7}\frac{i\alpha\omega}{3} = -i\frac{\omega s}{4\pi}.$$

Using Kubo's formula, we get $$\frac{\xi}{s} = \frac{1}{4\pi}.$$

Fundamental string solution

In the temperature range $\sqrt{\lambda}N^{-1} << T << \sqrt{\lambda}N^{-\frac{2}{3}}$, we have fundamental string solution.
Consistent truncation leads to a similar effective action

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left[R - \frac{8}{9}\partial_\mu\phi\partial^\mu\phi + \frac{24}{L^2}e^{-\frac{4}{3}\phi}\right].$$

The 10D string solution reduces to

$$\begin{aligned} ds^2 &= -c_T(r)^2dt^2 + c_X(r)^2dz^2 + c_R(r)^2dr^2, \\ \phi &= 3\log\left(\frac{r}{L}\right). \end{aligned}$$

Only change is $\phi \to -\phi$.

ξ/s for the F1-brane case

The equation of sound mode

$$Z_P'' + \left[\ln'\left(\frac{c_T c_X}{c_R}\right) - 2\frac{A_\varphi'}{A_\varphi}\right] Z_P + \mathcal{G} Z_P = 0.$$

depends on dilaton only by the ratio

$$\frac{A_\varphi'}{A_\varphi} = \frac{[A_H \ln'(c_X)]'}{A_H \ln'(c_X)} - \frac{\phi''}{\phi'},$$

which remains unchanged. So the result $\frac{\xi}{s} = \frac{1}{4\pi}$ remains valid in temperature range

$$\sqrt{\lambda} N^{-1} << T << \sqrt{\lambda} N^{-\frac{2}{3}}.$$

Sasaki-Einstein 7 manifolds as transverse spaces

We start from the 10 dimensional solution

$$ds^2 = H^{-\frac{3}{4}}(r)\left(-f(r)dt^2 + dx_1^2\right) + H^{\frac{1}{4}}(r)\left(\frac{dr^2}{f(r)} + r^2 dS_{X_7}^2\right),$$

$$\begin{aligned} e^{\phi(r)} &= H(r)^{\frac{1}{2}}, \\ F_7 &= 6L^6\omega_{X_7}. \end{aligned}$$

where $H(r) = \left(\frac{L}{r}\right)^6$, $\qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^6$.
Leads to the same 3D effective action as in D1 case, but with

$$\frac{1}{\tilde{G}_3} = \frac{L^7 \text{Vol}(X_7)}{G_{10}}.$$

So, the viscosity to entropy ratio remains same.

Summary

- For the $SU(N)$ gauge theory with 16 supercharges in $1+1$D on the D1-branes in range $\sqrt{\lambda}N^{-1} << T << \sqrt{\lambda}$,
  - $$\xi = \frac{2^6\pi^{\frac{7}{2}}}{3^3}\frac{N^2T^2}{\sqrt{\lambda}}.$$
  - $$\frac{\xi}{s} = \frac{1}{4\pi}.$$
- For $T << \sqrt{\lambda}N^{-1}$ and $T >> \sqrt{\lambda}$,
  the D1-brane gauge theory $\rightarrow$ a CFT.
  So, bulk viscosity vanishes.