Vortices and Strings and Vortex Strings

Khajuraho, Dec 2004

Plan

- new soliton sol^ns in YM theories
- classical + quantum dynamics
- embedding in string theory
- σ-model / gauge theory correspondence

Based on

with Amihay Hanany

- Vortices, Instantons + Branes 0306...
- Monopoles in the Higgs Phase 0307...
- Vortex strings + 4d gauge dynamics 0403...

The Theory

$\mathcal{N}=2$ $U(N_c)$ SQCD with $N_f = N_c$ flavors

We'll focus on the fields

$$(A_\mu)^a{}_b \qquad \phi^a{}_b \qquad q^a{}_i$$

$a = 1 \dots N_c$

$i = 1 \dots N_f$

$$\boxed{\mathcal{L} = \frac{1}{2e^2} \mathrm{Tr}\left[\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi \right] + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 + \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i + \frac{e^2}{2} \mathrm{Tr}\left(\sum_{i=1}^{N_c} q_i q_i^\dagger - v^2 \right)^2 + \dots}$$

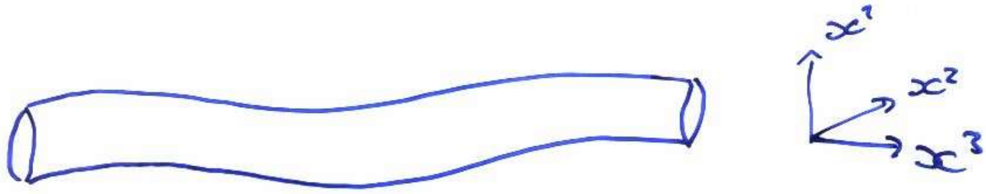
Vacuum $\quad q^a{}_i = v\delta^a{}_i \quad , \quad \phi = 0$

unique with mass gap

Symmetries $\quad U(N_c) \times SU(N_f) \longrightarrow SU(N)_{diag}$

$\Rightarrow$ vortices exist

The Vortices $\quad (\tfrac{1}{2}\text{-BPS})$



$$\boxed{\begin{aligned} (B_3)^a{}_b &= e^2\Big(\sum_i q^a{}_i\, q^{\dagger i}{}_b - v^2 \delta^a{}_b\Big) \\ \mathcal{D}_{\bar z}\, q_i &= 0 \end{aligned}}$$

$z = x^1 + i x^2$

These have tension

$$T = \mathrm{Tr}\Big[v^2 \int dx_1\, dx_2\, B_3\Big]$$

$$= 2\pi v^2 k$$

$k \in \mathbb{Z}^+$

What we know about Vortices

Solutions

- existence proven by Taubes for $U(1)$
- exact solutions unknown
- # parameters is

$$\boxed{\dim(\mathcal{V}_{k,N}) = 2kN}$$

moduli space of k vortices in $U(N)$

Low-Energy Dynamics

- $d = 1+1$ $\mathcal{N} = (2,2)$ σ-model on $\mathcal{V}_{k,N}$

$$L_{\text{vortex}} = \int d^2x \; g_{pq}\, \partial_\alpha X^p \partial^\alpha X^q$$

- metric g_{pq} on $\mathcal{V}_{k,N}$ is
  - smooth
  - kähler
  - unknown (beyond asymptotic regime)

[cf. monopoles + instantons]

An Example : 1 vortex in $U(N)$

Suppose B_* and q_* satisfy $U(1)$ vortex equ's

$$B = \begin{pmatrix} B_* & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} \qquad q^a{}_i = \overbrace{\begin{pmatrix} q_* & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}}^{\text{flavor}} \Big\} \text{color}$$

But now act with $SU(N)_{diag} / U(N-1) \times U(1) \cong \mathbb{CP}^{N-1}$

$$\Rightarrow \quad \boxed{\mathcal{V}_{1,N} \cong \mathbb{C} \times \mathbb{CP}^{N-1}}$$

centre of mass $\qquad$ internal orientation

$\mathcal{V}_{k,N}$ consists of suitably stitching together $\mathcal{V}_{1,N}$ .... how?

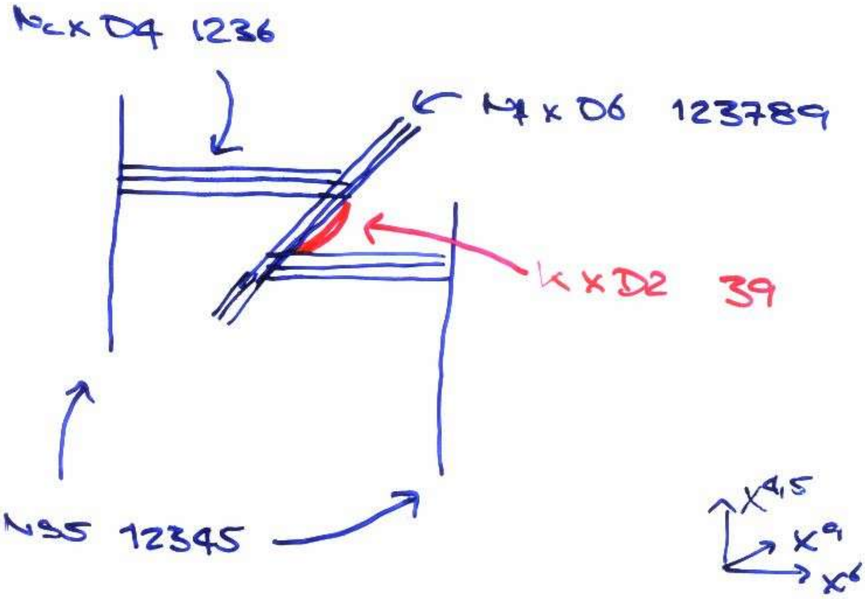
Vortices in String Theory

Plan

- Engineer theory on D-branes
- Identify vortices as other D-branes
- Read off vortex theory

We want $d=3+1$ $\mathcal{N}=2$ $U(N_c)$ + $N_f = N_c$ flavors

Type IIA



$$\Delta x^6\Big|_{NS5} \sim \frac{1}{e^2} \qquad \Delta x^9\Big|_{NS5} \sim v^2$$

Vortices are $k \times$ D2-branes in x^3-x^9 directions

Vortex Theory

(related to ADHM for instantons ... see later)

$d = 1+1 \quad \mathcal{N}=(2,2)$ $\quad U(k)$ vector multiplet $\quad (A_\alpha, \sigma)$

+ adjoint chiral $\quad Z$

+ N_c fundamental chirals $\quad \psi_a$

$$\boxed{\begin{aligned} L_{\text{vortex}} = \; & \frac{1}{2g^2} \mathrm{Tr}\left(F_{01}^2 + D\sigma^2\right) + |DZ|^2 + \sum_{a=1}^{N_c} |D\psi_a|^2 \\ & + |[\sigma, Z]|^2 + \sum_{a=1}^{N_c} \psi_a^\dagger \sigma^2 \psi_a + \frac{g^2}{2} \mathrm{Tr}\left(\sum_{a=1}^{N_c} \psi_a \psi_a^\dagger - [Z, Z^\dagger] - r\right)^2 \end{aligned}}$$

with $\quad g^2 \to \infty$

and $\quad \boxed{r = \frac{2\pi}{e^2}}$

and vacuum moduli space $\quad \mathcal{M}_{\text{Higgs}} \cong \mathcal{V}_{k,N}$

An Example $\quad$ 1 vortex in $U(N)$

Vortex theory is • $U(1)$ + neutral scalar Z

+ N charged scalars ψ_a

$$\mathcal{M}_{Higgs} = \left\{ Z, \psi_a \;\middle|\; \sum_a |\psi_a|^2 = r \right\} \Big/ U(1)$$

$$\cong \mathbb{C} \times \mathbb{CP}^{N-1}$$

$$\cong \mathcal{V}_{1,N}$$

with kähler class $r = 2\pi/e^2$

Generally $\quad k$ vortices in $U(N)$

$$\mathcal{M}_{Higgs} = \left\{ Z, \psi_a \;\middle|\; \sum_{a=1}^{N_c} \psi_a \psi_a^\dagger - [Z, Z^\dagger] = r \right\} \Big/ U(k)$$

Z: U(k) adjoint $\qquad \psi_a$: U(k) fundamental

This is symplectic quotient construction

- $\dim(\mathcal{M}_{Higgs}) = \dim(\mathcal{V}_{k,N}) = 2kN$
- Symmetries agree
- does <u>not</u> give Manton metric for $k \geq 2$

Generalization to

- $N_f \geq N_c$
- non-commutative vortices
- Chern-Simons Theories + Polychronakos-Susskind model of Quantum Hall Effect

Monopoles

To see how monopoles arise, we need to deform $d=3+1$ theory

- $\mathcal{N}=2$ $U(N_c)$ with $N_f = N_c$ massive flavors

$$\boxed{\mathcal{L} = \frac{1}{2e^2}\mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu\phi^2\right] + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2 + \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i + \frac{e^2}{2}\mathrm{Tr}\left(\sum_i q_i q_i^\dagger - v^2\right) + \dots}$$

Vacuum $\quad \phi = \mathrm{diag}(m_1 \dots m_N) \qquad q^a{}_i = v\delta^a{}_i$

Symmetries $\quad U(N_c)\times SU(N_f) \xrightarrow{v^2} SU(N)_{diag} \xrightarrow{m} U(1)^{N-1}_{diag}$

What happens to vortices?

Old solutions remain only if

$$\sum_i q_i^\dagger (\phi - m_i)^2 q_i \Big|_{\text{vortex}} = 0$$

$\Rightarrow$ vortices must lie diagonally in gauge group

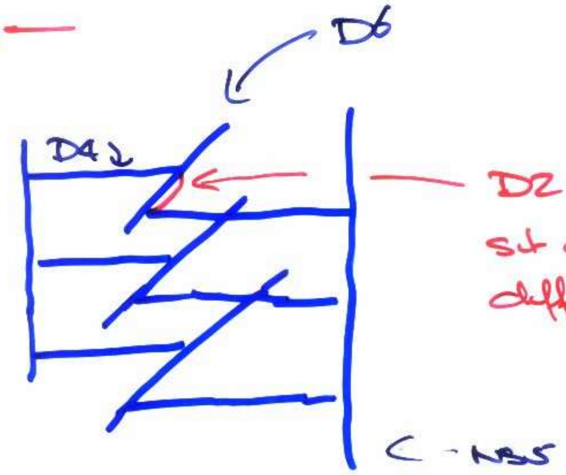
e.g. 1 vortex

$$B = \begin{pmatrix} B_* & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{array}{c} \text{flavor} \\ \overbrace{\begin{pmatrix} q_* & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}} \end{array} \Big\} \text{color}$$

and now theres no $SU(N)_{diag}$ symmetry to sweep out moduli space

$\Rightarrow$ $\boxed{N_c \text{ different types of vortices in the theory}}$

Vortex Theory



D2 is vortex: can sit in one of N_c different positions

$d = 1+1$ $\mathcal{N} = (2,2)$ $U(k)$ vector multiplet

+ adjoint chiral multiplet

+ N_c massive fundamental chirals

$$\mathcal{L}_{vortex} = \frac{1}{2g^2} \mathrm{Tr}\left(F_{01}^2 + \mathcal{D}\sigma^2\right) + |\mathcal{D}Z|^2 + \sum_{a=1}^{N_c} |\mathcal{D}\psi_a|^2$$

$$+ \sum_{a=1}^{N_c} \psi_a^\dagger (\sigma - m_a)^2 \psi_a + \frac{g^2}{2} \mathrm{Tr}\left(\sum_a \psi_a \psi_a^\dagger - [Z, Z^\dagger] - r\right)^2$$

"twisted" masses m_a are the same as $d=3+1$ theory

An Example

$k = 1$ vortex in $U(N)$

Vortex theory is

- $d = 1+1$, $\mathcal{N} = (2,2)$ $U(1)$ gauge theory
  - \+ neutral complex scalar
  - \+ N massive charged scalars

$$\boxed{V_{\text{vortex}} = \sum_{a=1}^{N_c} \varphi_a^\dagger (\sigma - m_a)^2 \varphi_a + \frac{g^2}{2}\left(\sum_a |\varphi_a|^2 - r\right)^2}$$

$\Rightarrow$ Theory on vortex string has N_c isolated vacua

$$\left.\begin{aligned} \sigma &= m_b \\ |\varphi_a|^2 &= r\,\delta_{ab} \end{aligned}\right\} \quad b = 1 \ldots N_c$$

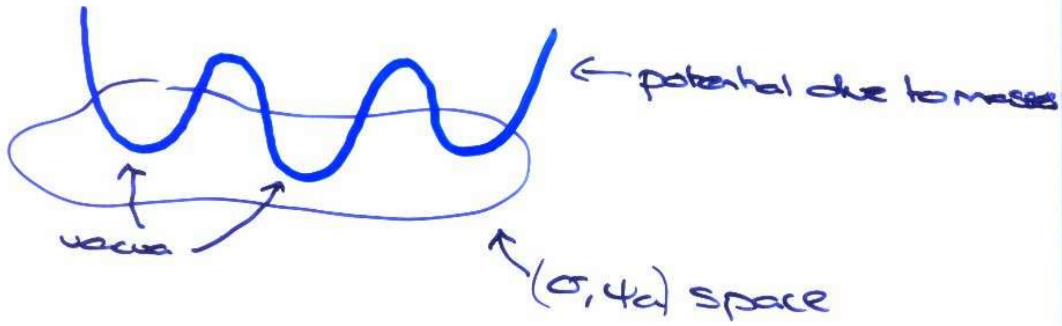
These correspond to the N_c different vortex solutions

Kinks on the Vortex String

Isolated vacua in $d=1+1$ $\Rightarrow$ kinks on string!

$$\boxed{\begin{aligned} \partial_3 \sigma &= g^2 \left(\sum_a |\psi_a|^2 - r \right) \\ \mathcal{D}_3 \psi_a &= (\sigma - m_a)\psi_a \end{aligned}}$$

$\frac{1}{2}$-BPS



The BPS kinks have mass

$$M_{\text{kink}} = r\,(m_{+\infty} - m_{-\infty})$$

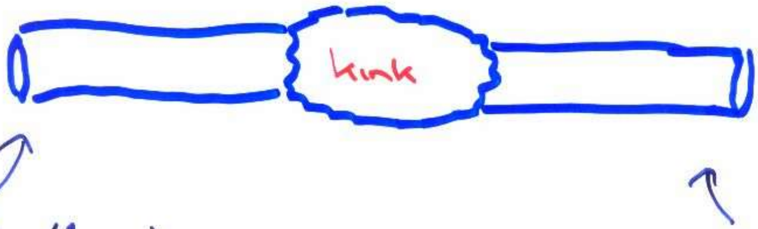
$$= \frac{2\pi}{e^2} \langle \phi \rangle$$

using $r = 2\pi/e^2$

and $\phi = \mathrm{diag}(m \quad m)$

What is the kink?

Consider 1 vortex in $U(N)$



vortex $B \sim \begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$ vortex $B \sim \begin{pmatrix}0 & & & \\ & 1 & & \\ & & 0 \ddots & \\ & & & 0\end{pmatrix}$

$\Rightarrow$ kink is source for magnetic field $B \sim \begin{pmatrix}1 & & & \\ & -1 & & \\ & & 0 \ddots & \\ & & & 0\end{pmatrix}$

So kink has
- quantum numbers of magnetic monopole
- mass $M_{kink} = \frac{2\pi}{e^2}\langle\phi\rangle = M_{monopole}$

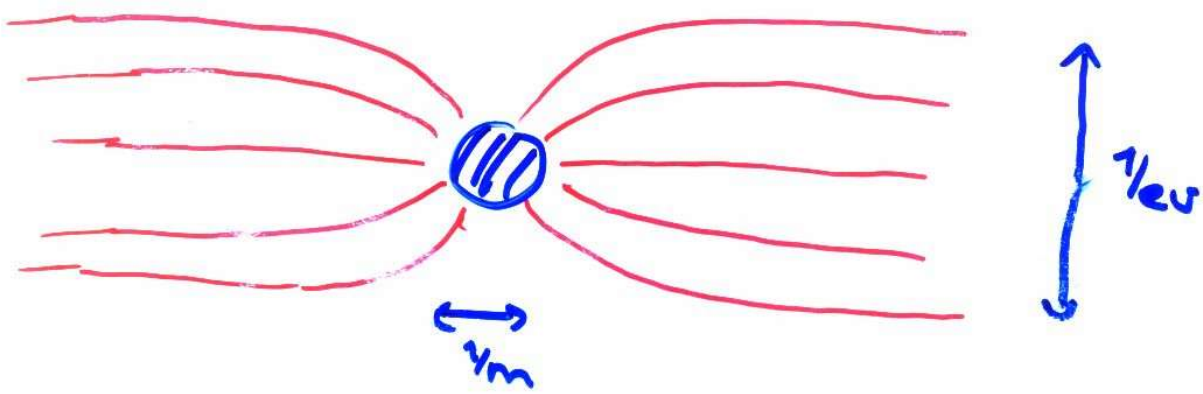
$\Rightarrow$ should probably identify it with magnetic monopole!

Meissner effect $\Rightarrow$ monopole is confined

Go back to $d = 3+1$

Find ¼-BPS equations describing monopole-vortex string

$$\boxed{\begin{aligned} &B_1 = \mathcal{D}_1\phi \;,\quad B_2 = \mathcal{D}_2\phi \;,\quad B_3 = \mathcal{D}_3\phi - e^2\Big(\sum_i q_i q_i^\dagger - v^2\Big) \\ &\mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \;,\quad \mathcal{D}_3 q_i = (\phi - m_i)\, q_i \end{aligned}}$$



Note: These equ's have much larger class of solutions including D-branes + more

Instantons

Can show using similar arguments that

$$\boxed{\text{Vortex in vortex string} = \text{YM instanton in vortex string}}$$

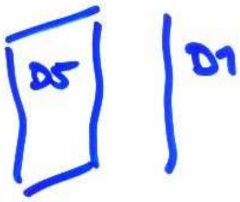
There is a related important fact

$$\boxed{\text{Vortex moduli space } \mathcal{V}_{k,N} \cong \text{Complex middle dimensional submanifold of instanton moduli space } \mathcal{I}_{k,N}}$$

To see this . . . . .

Vortex Theory: $\mathcal{N}=(2,2)$ $U(k)$ + adjoint chiral
+ N fund. chirals

ADHM Instanton Theory:



$\mathcal{N}=(4,4)$ $U(k)$ + adjoint hypermultiplet
+ N fundamental hypermultiplets

$$\Rightarrow \quad \boxed{V_{k,N} \cong Z_{k,N}\Big|_{h=0}}$$

S^1 action arising from rotating instantons in plane

Note • works at level of moduli space ... what about so...?

Finally ... The Quantum Theory

[see also Shifman + Yung]

The $d=1+1$ vortex theory and $d=3+1$ theory have the same BPS spectrum

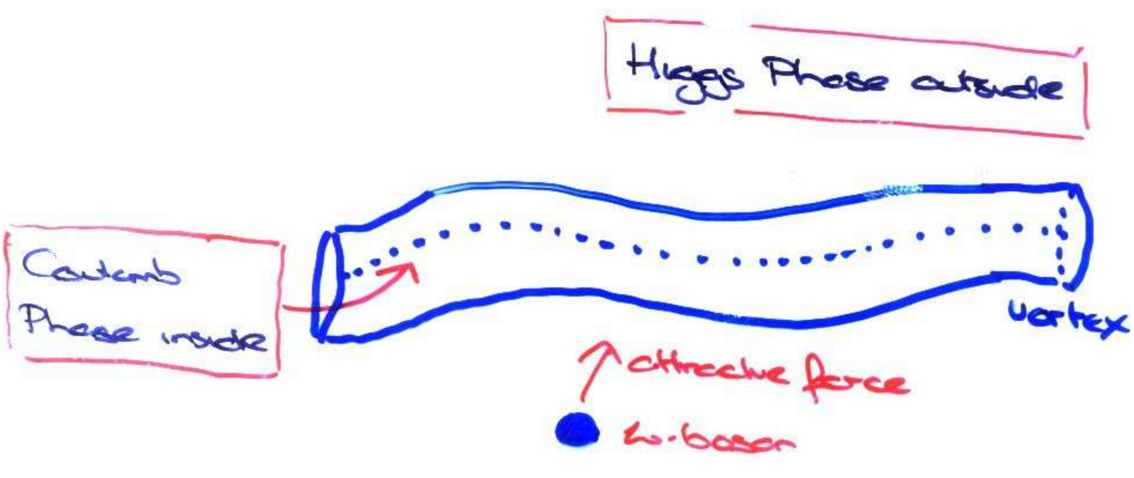
(Dorey; Dorey, Hollowood, Tong '98; Puri '99)

monopoles $\longleftrightarrow$ kinks

W-bosons + quarks $\longleftrightarrow$ elementary string excitations

$d=3+1$: $\mathcal{N}=2$ $U(N_c)$ with $N_f = N_c$ flavors in vacuum $\phi = \mathrm{diag}(m_1 \dots m_{N_c})$ in Coulomb branch

$d=1+1$: $\mathcal{N}=(2,2)$ $U(1)$ with N massive chiral multiplets



What does this mean?

$$\boxed{M_{kink} = M_{monopole}}$$

holds in the full quantum theory

To see what this means we can work in weak coupling regime

$$M = M_{classical} + M_{1\text{-loop}} + \sum_{n=1}^{\infty} M_{n\text{-instanton}}$$

Calculated in { d=1+1 for kinks; d=3+1 for monopole

Sum over { d=2+0 vortices for kinks; d=4+0 instantons for monopoles

Some Aspects of Quantum Theory

The relationship

$$\boxed{r = \frac{2\pi}{e^2}}$$

FI parameter of $d=1+1$ vortex theory

gauge coupling of $d=3+1$ theory

This is preserved under RG flow

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{\mu_{uv}}{\mu}\right)$$

Exact Spectrum : The theories are solved by

- Seiberg-Witten sol^n for $d=3+1$
- Exact twisted superpotential for $d=1+1$

Generalizations :

- anomalies
- dyons $\to$ monopoles + kinks
- Witten effect

Summary

- new solutions for BPS confined monopoles
- Vortices related to instantons
- Spectrum of vortex string = spectrum of $d=3+1$ gauge theory

Future Directions

- $\mathcal{N}=1$ SQCD $\Rightarrow$ $\mathcal{N}=(0,2)$ vortex string
  - phases ?
  - condensates ?
  - duality ?
- $\mathcal{N}=0$ but large N $\Rightarrow$ weak coupling