Introduction

- We have seen at large N the free 3pt & 2pt functions can be reproduced using the δ fn overlap of Gunaydin variables

- The fact that 3pt fn at large N is described by δ fn overlap of bits indicates that it is possible to think of these correlation fns as splitting and joining of strings

- We would like to see if this picture still holds at finite λ (t'Hooft coupling). And, what it implies for Yang-Mills correlation functions at finite λ

- Finite λ in gauge theory is α' corrections in string theory

Units $R_{AdS} = 1$

$$\alpha' = \frac{1}{\sqrt{g_{YM}^2 N}} = \frac{1}{\sqrt{\lambda}}$$

$$G_N = \frac{1}{N^2}$$

$\lambda = 0$; $\alpha' \to \infty$

Tensionless string.

Making α' finite turns on nearest neighbour interactions between the bits

$$H_{ws} = \dot{x}^2 + \alpha' x'^2 \qquad \alpha' \to \infty$$

σ derivatives drop
string breaks up into bits

finite α': to leading order

$$H_{ws} = \sum_s \dot{x}^2(s) + \alpha' [x(s) - x(s-1)]^2$$

Thus $H_{ws} \to H_{ws} + H_{int}$ (nearest neighbour)

Given H_{int}

2 pt fn at λ order

3pt fn at $\frac{\lambda}{N}$ order is determined ?

- Previous lecture: we have seen how 2pt fns is modified at λ order

Aim: Focus on 3pt functions

→ Motivated from the bit picture we obtain a formula for 1-loop corrections to 3pt fns

→ Compare the resulting correction with gauge theory computations

Contents

- How α' corrections enter in the bit picture
- A formula for the correction to 3pt functions
- 3pt functions in the scalar SO(6) sector
- 3pt functions in a Non SO(6) sector
- Conclusions

Def : 1-loop correction to 3pt fn

Consider a diagonal set of operators O_i (primaries)

The general form for 2pt fn at 1-loop

$$\langle O_i \, O_i \rangle = \frac{1}{(x_{12})^{2\Delta_i}} \left(1 + \lambda \, g_{ii} - \lambda \, \nu_i \ln (x_{12}^2 \Lambda^2)\right)$$

3pt fn at 1-loop

$$\langle O_1 O_2 O_3 \rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}} \cdot \frac{1}{|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} \frac{1}{|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

$$\times \Big[C^{(0)}_{123} \Big(1 - \lambda \, \nu_1 \ln \frac{x_{12} x_{13}}{x_{23}} \Lambda - \lambda \nu_2 \ln \frac{x_{12} x_{23}}{x_{13}} \Lambda - \lambda \nu_3 \ln \frac{x_{13} x_{23}}{x_{12}} \Lambda \Big) + \lambda C^{(1)}_{123} \Big]$$

$C^{(0)}_{123} \sim \frac{1}{N}$

$C^{(1)}_{123} \sim \frac{1}{N}$

$C^{(1)}_{123}$ is Not Scheme Independent

$$\Lambda \rightarrow e^{\alpha} \Lambda$$

$$C^{(1)}_{123} \rightarrow C^{(1)}_{123} - (\nu_1 + \nu_2 + \nu_3) \, \alpha \, C^{0}_{123}$$

But Consider

$$\tilde{C}^{(1)}_{123} = C^{(1)}_{123} - \frac{1}{2} g_{11} C^{(0)}_{123} - \frac{1}{2} \lambda \, g_{22} C^{(0)}_{123} - \frac{1}{2} \lambda \, g_{33} C^{(0)}_{123}$$

under $\Lambda \rightarrow e^{\alpha} \Lambda$

$$g_{11} \rightarrow g_{11} - 2 \alpha \nu_1 \; ; g_{22} \rightarrow g_{22} - 2 \alpha \nu_2$$

$$g_{33} \rightarrow g_{33} - 2 \alpha \nu_3$$

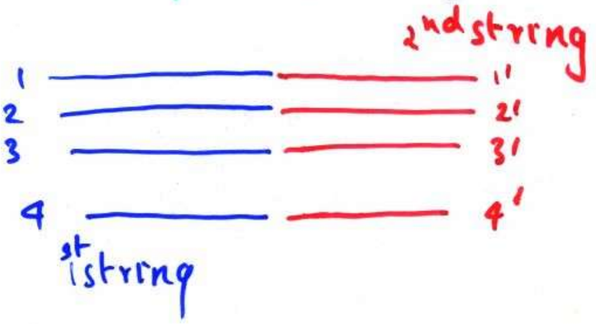
Therefore $\tilde{C}^{(1)}_{123}$ is scheme Independent.

In a general Basis: $\tilde{C}^{(1)}_{ijk} = C^{(1)}_{ijk} - \frac{1}{2} \lambda g_{ie} C^{e\,(0)}_{\;jk} - \frac{1}{2} \lambda g_{je} C^{0\;e}_{i\;k} - \frac{1}{2} \lambda g_{ke} C^{0\;\;k}_{ij}$

· How does α' corrections enter in the bit picture

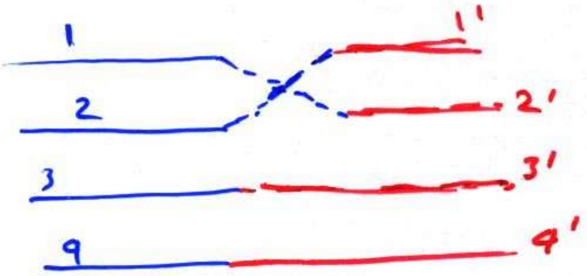
· Modification of 2 pt function

δ fn overlap implements matching of states bit by bit



Presence of Hint in world sheet alters free propagation of bits

e.g ∃ an amplitude for overlap



$\lambda\, U_{int}{}^{1'\,2'}_{\;\;1\,2}$

$\lambda = \frac{g_{YM}^2 N}{32\pi^2}$

Thus $|V_2\rangle \longrightarrow |V_2\rangle^{\lambda=0} + U_{int}|V_2\rangle^{\lambda=0}$

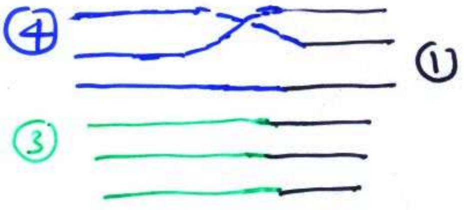
We saw $U_{int} = H_{int} \cdot U_1$

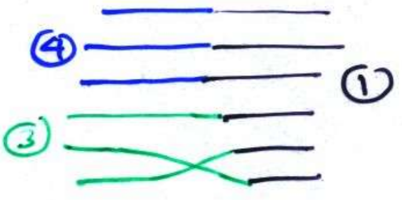
$$U_1 = \frac{-1}{\pi^2}\int d^4y \frac{e^{iy\cdot\hat{p}}}{(y^2+\varepsilon^2)^2}$$

$\lim_{\varepsilon \to 0}$

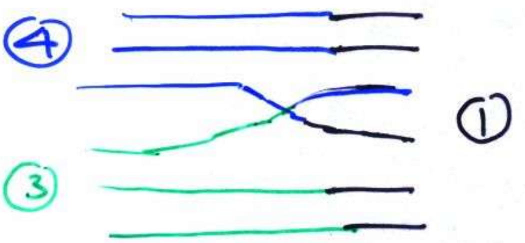
- Corrections to 3pt fn

free overlap $|V_3\rangle^{\lambda=0}$





λU_{14} λU_{13} (2 body)

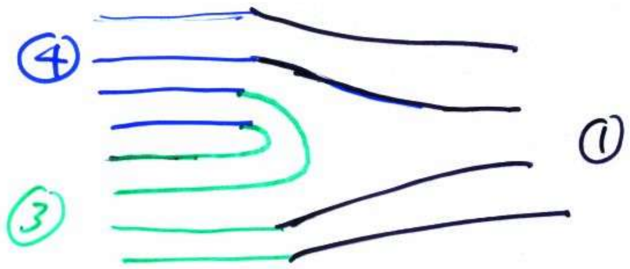


λU^{1}_{43} (3 body

Therefore length Consering Process

$$|V_3\rangle^{\lambda=0} \rightarrow |V_3\rangle^{\lambda=0} + \lambda (U_{14} + U_{13} + U^{1}_{43})|V_3\rangle$$

for length Non-Conserving



$\lambda \rightarrow$ corrections

$$\lambda \langle 1|\langle 3|\langle 4| \underbrace{U_{13} + U_{14} + U_{34}}_{\text{2 body terms}} + \underbrace{U^{3}_{14} + U^{1}_{43} + U^{4}_{13}}_{\text{3 body terms}} |V_3\rangle$$

But The Scheme Independent λ - correction is given by normalizing the 2pt fn

$$\tilde{C}_{123} = \langle 1|\langle 2|\langle 3|\,\lambda\left(u_{13} + u_{14} + u_{34} + u^{3}_{14} + u^{1}_{43} + u^{4}_{13}\right)|V_3\rangle$$

$$-\frac{1}{2}\langle 1\,|\,u_{ij}\,|i\rangle\,\langle j\,|\,\langle 2\,\langle 3\,|V_2\rangle$$

$$-\frac{1}{2}\langle 2\,|u_{2\,j}|i\rangle\,\langle j\,|\,\langle 2\,\langle 3\,|V_2\rangle$$

$$-\frac{1}{2}\langle 3\,|\,u_{3\,j}|i\rangle\,\langle j\,|\,\langle 2\,\langle 3\,|V_3\rangle$$

Constants from 2 body terms in the 3pt function cancel with the constants due to normalization of the 2pt function. What is left over ??

- $\tilde{C}_{123} = H_{3\,\text{body}} - \frac{1}{2} H_{2\,\text{body}}.$

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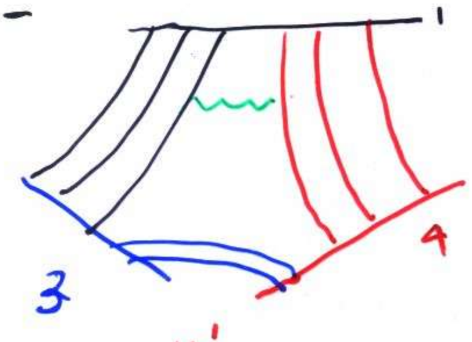
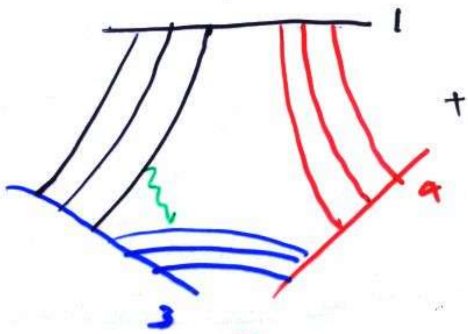


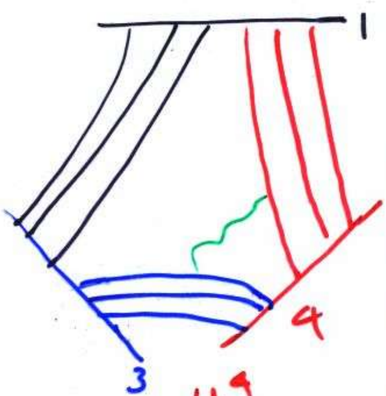
u^{1}_{43}

+



u^{3}_{14}

+

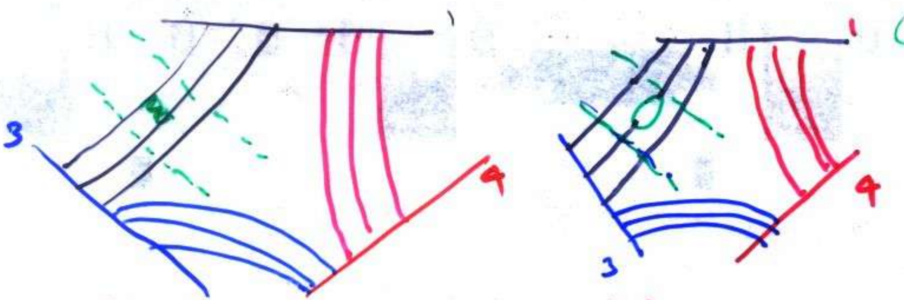


u^{4}_{13}

3 body terms in 3pt fn

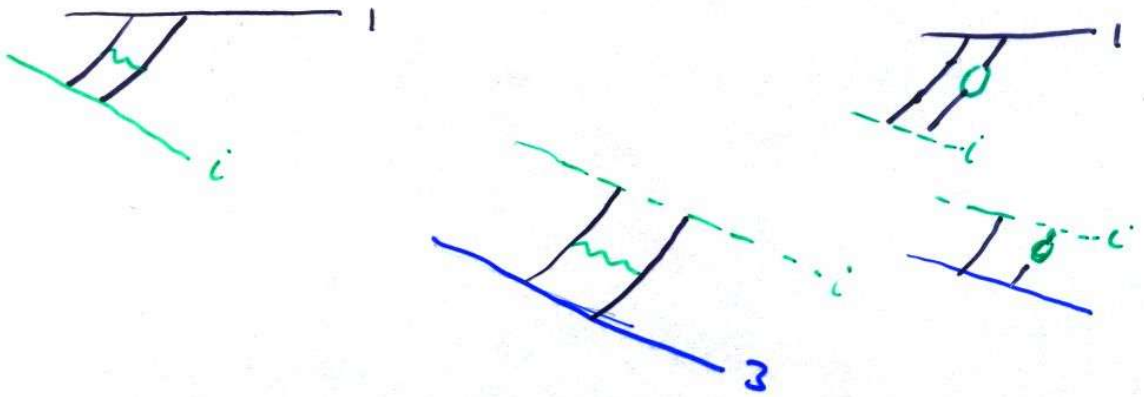
+ cyclic changes in bits

2 Body Terms e.g



u_{13} + cyclic permutation of bits

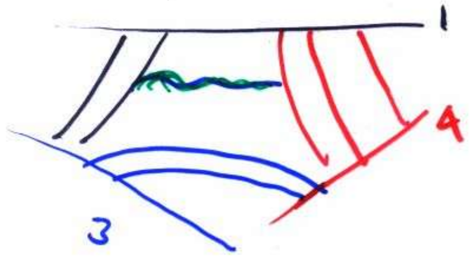
Constants from such 2 body terms in 3pt fn occurs in the Normalizations g_{1i} & g_{3i}



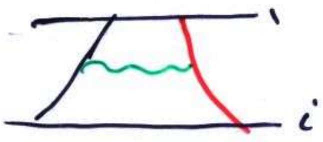
· These are weighted by a factor of ½

· Thus constants from the Normalizations cancel off Constants in the 2 body term
This includes constants from self energies.

· One is left with



$-\frac{1}{2}.$



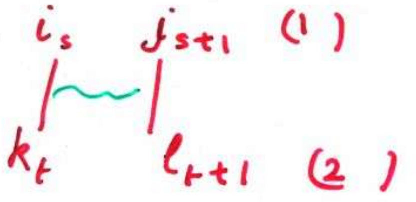
Constants from genuine 3 body term

$-\frac{1}{2}$ (constant from same term treated as 2 body)

λ Corrections to 3pt fn : SO(6) sector

- From the bit picture

Consider the 2 body term



The free Correlation function between these bits

$$\langle l_{t+1}|\langle k_t| u_0^2 e^{i\hat{p}(x_1-x_2)} |i_s\rangle |i_{s+1}\rangle$$

$$= \langle k_t|i_s\rangle \langle l_{t+1}|j_{s+1}\rangle \frac{1}{(x_{12}^2)^2}$$

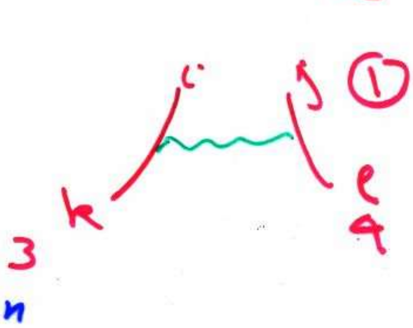
The λ- order Correction

$$\lambda \langle l|\langle k| u_0^2 \tilde{U}_1 H_{int} e^{i\hat{p}(x_1-x_2)} |i\rangle|j\rangle$$

$$= \lambda \langle l|\langle k| H_{int} |i\rangle|j\rangle \left(\frac{-1}{\pi^2}\right) \int d^4y \frac{1}{(y^2+\epsilon^2)} \frac{1}{[(y+x_{12})^2+\epsilon^2]^2}$$

$$= -\lambda \langle l|\langle k| H_{int}|i\rangle|j\rangle \left[2\ln\left(\frac{x_{12}^2}{\epsilon^2}\right) - 2 \right]$$

Now the 3 body term



Performing a similar calculation

$$-\lambda \langle l|\langle k| H_{int}| i\rangle |j\rangle \left[2 \ln \frac{x_{13}^2 x_{14}^2}{x_{34}^2 \epsilon^2} \right]$$

The Scheme Independent Contribution

$$H_{3\,body} - \frac{1}{2} H_{2\,body}$$

$$= -\lambda \langle e|\langle k|\ H_{int}\ |i\rangle |j\rangle$$

- Note: The SO(6) dependendence factors out & is identical to the Anamalous dimension Hamiltonian

- Gauge Theory Calculation

Puzzle ?? The anomalous dimension Hamiltonian in SO(6) sector arises after including Self Energy diagrams

In a 3pt fn the 3 body term does not have Self Energy pieces

How does Hint control the λ corrections?

Ingredients in the proof.

- Rewrite the gauge Exchange diagram as a quartic + certain collapsed diagrams
- Cancellations of the collapsed diagrams

For simplicity consider the following SO(6) configuration

$$Z^1 \quad \bar{Z}^2 \quad \lim_{2\to 1}$$

$$\bar{Z}_3 \quad Z_4$$

- Part of the 3pt function of the Konishi Scalar
- Using point split regularization

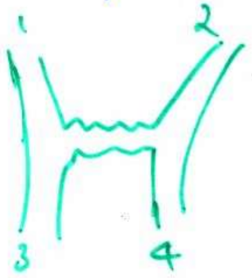
Quartic diagram

$$= \lim_{2\to 1} -\frac{\varphi(r,s)}{x_{13}^2 x_{24}^2} = -\int d^4y \frac{1}{(y-x_1)^2}\frac{1}{(y-x_2)^2}\frac{1}{(y-x_3)^2}\frac{1}{(y-x_4)^2}$$

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \; ; \; s = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$= -\frac{1}{x_{13}^2 x_{14}^2}\left[\ln\left(\frac{x_{13}^2 x_{14}^2}{x_{34}^2 \varepsilon^2}\right) + 2\right]$$

The Gauge Exchange



$$= (r-s)\frac{\varphi(r,s)}{x_{13}^2 x_{24}^2} + (s'-r')\frac{\varphi(r',s')}{x_{13}^2 x_{24}^2}$$

$1\to\infty$ collapse

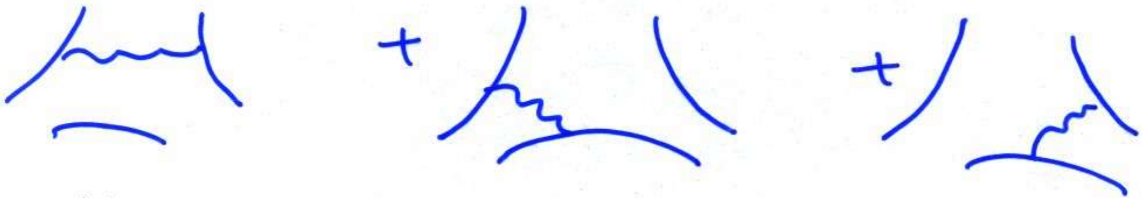
$$r' = \frac{x_{34}^2}{x_{24}^2} \; ; \; s' = \frac{x_{23}^2}{x_{24}^2}$$

$$\underbrace{+(2\to\infty) + (3\to\infty)}_{\text{collapse diagrams}} + (4\to\infty)$$

- Dangerous collapse $(1\to\infty)$; $(2\to\infty)$

Seem to destroy conformal Invariance : these terms are functions of (r',s') other than constants or logs

But in Considering all 3 body Interactions



they Cancel

The other 2 collapse $(2\to\infty)$ & $(3\to\infty)$.

give
$$\frac{1}{x_{13}^2}\frac{1}{x_{24}^2}\left[\ln\frac{x_{14}^2}{\epsilon^2}+\ln\frac{x_{13}^2}{\epsilon^2}+2\times 2\right]$$

To find the scheme Independent constant we consider the 2 body term

All the 4 collapses contribute ; No dangerous collapses . Therefore the constants is twice that of the 3 body collapse

→ Constants from the collapsed diagrams Cancel in the Combination $H^{3\,body}-\frac{1}{2}H^{2\,body}$

What is left behind

Constants from the $\frac{1}{2}$(Quartic term + $\overset{\text{Quartic part}}{\text{Gauge Exchange}}$)

$$= \frac{2.}{x_{13}{}^2 x_{14}{}^2}$$

Now Compare: The Anomalous dimension Calculation

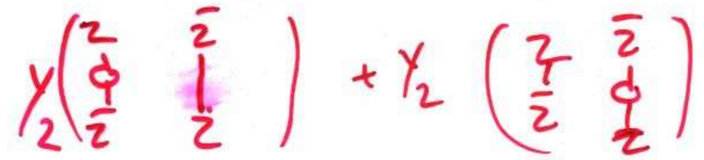
→ We need to look at log terms

Quartic Term $= \frac{2.}{(x_{12}^2)^2} \ln \frac{x_{12}{}^2}{\varepsilon^2}$

+

Gauge Exchange $= \frac{2.}{(x_{12}^2)^2} \ln \left(\frac{x_{12}{}^2}{\varepsilon^2}\right)$ + 4 Collapsed diagrams

+ Self Energies



4 Collapsed diagrams + Self Energy = 0

What is left Behind : Quartic term + Quartic part of Gauge Exchange

$$= \frac{4}{(x_{12}^2)^2} \ln \frac{x_{12}{}^2}{\varepsilon^2}$$

This argument can be carried out for All possible SO(6) configurations

$Z\ u \quad Z\ \bar{Z} \quad Z\bar{Z}$

$\bar{Z}\ \bar{u} \quad u\ \bar{u} \quad w\ \bar{w}$ etc

Thus the Anomalous dimension Hamiltonian controls corrections to 3pt fn.

$\underline{\lambda\text{-corrections}}$ to 3pt $\underline{\text{fn in a Non SO(6) sector}}$

Motivation:

Simplification in the SO(6) sector:

- Log terms were proportional to constants
- The SO(6) spin dependent term factorizes in the calculation

This factorization is not there if there are derivatives in the operators

We wish to see if Hint, the anomalous dimension Hamiltonian determines the correction to Structure constants

• The Gauge Theory Calculation

- Specification of Operators.

- Consider the set of single trace operators

$$\mathrm{Tr}\left(D^{k_1} Z \ldots D^{k_2} u \ldots \right)$$

where $D_{x^3+ix^4}$ is the holomorphic covariant derivative

- There is a class of primaries of the form

$$\sum_{k=0}^{n} (-1)^k \, {}^{n}C_k \, \mathrm{Tr}\left[\frac{\partial^k Z}{k!} \ldots \frac{\partial^{n-k} u}{(n-k)!} \ldots \right]$$

use $[K_z, \partial^k Z] = k^2 \partial^{k-1} Z$

Simplifications due to the choice of these operators

- Performing differentiation in a holomorphic direction is easy

- In calculation of 3pt fn (or 2pt fn) of 3 primary operators

e.g $$\sum_{h=0}^{n} (-1)^k \, {}^{n}C_k \, \mathrm{Tr}\left(\frac{\partial^k Z}{k!} \, \frac{\partial^{n-k} Z}{(n-k)!} \right)$$

$$\sum_{k=0}^{m} (-1)^k \, {}^{m}C_k \, \mathrm{Tr}\left[\frac{\bar\partial^k u}{k!} \, \frac{\bar\partial^{m-k} \bar{z}}{(m-k)!} \right] \qquad \sum_{k=0}^{l} (-1)^k \, {}^{l}C_k \, \mathrm{Tr}\left[\frac{\bar\partial^k \bar{u}}{k!} \, \frac{\bar\partial^{l-k} \bar{z}}{(l-k)!} \right]$$

$$m + l = n$$

- It is sufficient to focus on terms proportional to $\delta_{\mu\nu}$ (or here 1)
  - As correlation function of primaries have a definite tensor structure

- This implies it is sufficient to look at only the combination

$$\begin{matrix} D^k Z & D^{n-k} Z \\ \bar{D}^{k'} \bar{Z} & \bar{D}^{n-k'} \bar{Z} \end{matrix} = H_{kk';\, n-k\, n-k'}$$

Total No: of holomorphic derivatives = Total no: of Anti holomorphic derivatives

- Anomalous dimension Hamiltonian is known for this sector, and it is sufficiently simple.

sketch of the Calculation

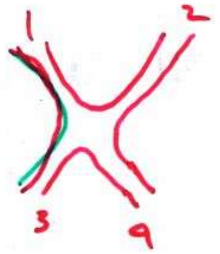
Goal: Calculate Constants in

$$H^{3pt}_{kk';\, n-k\, n-k'} - \frac{H^{2pt}_{kk';\, n-k\, n-k'}}{2}$$

Check if $H_{int} U_1$ (H_{int}; anomalous dimension hamiltonian) in the bit picture determines the constants

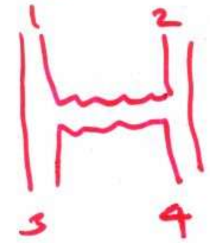
Process Involved

- Quartic Interaction



· Point split allows action of derivatives

- Gauge Exchange



Evaluated by equation

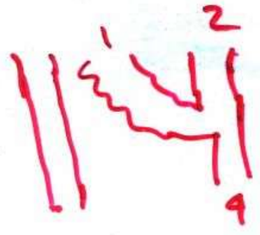
$$= (r-s)\,\frac{\varphi(r,s)}{x_{13}^2 x_{24}^2} + (s'-r')\,\frac{\varphi(r',s')}{x_{13}^2 x_{24}^2} + \cdots$$

$1 \to \infty$ collapse

collapsed diagrams

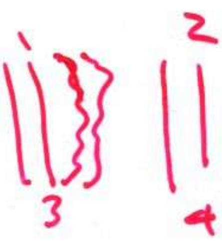
- Gauge Boson on one leg.

Note $D^k Z = \partial^k Z + g_{YM} \sum_{j=1}^{k} {}^kC_j \left[\partial^{j-1} A, \partial^{k-j} Z \right]$

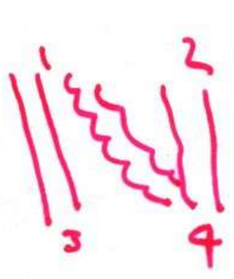


- Gauge Boson on 2 legs

e.g



or



Tree Contractions.

Important Mechanisims restoring Conformal invariance in the 3pt Calculation

There is an apparent difficulty.

Consider the $1 \to \infty$ or the $2 \to \infty$ collapse

$$(s'-r')\,\frac{\varphi(r',s')}{x_{13}^2\,x_{2q}^2} \qquad r' = \frac{x_{3q}^2}{x_{2q}^2} \qquad s' = \frac{x_{23}^2}{x_{2q}^2}$$

$1 \to \infty$

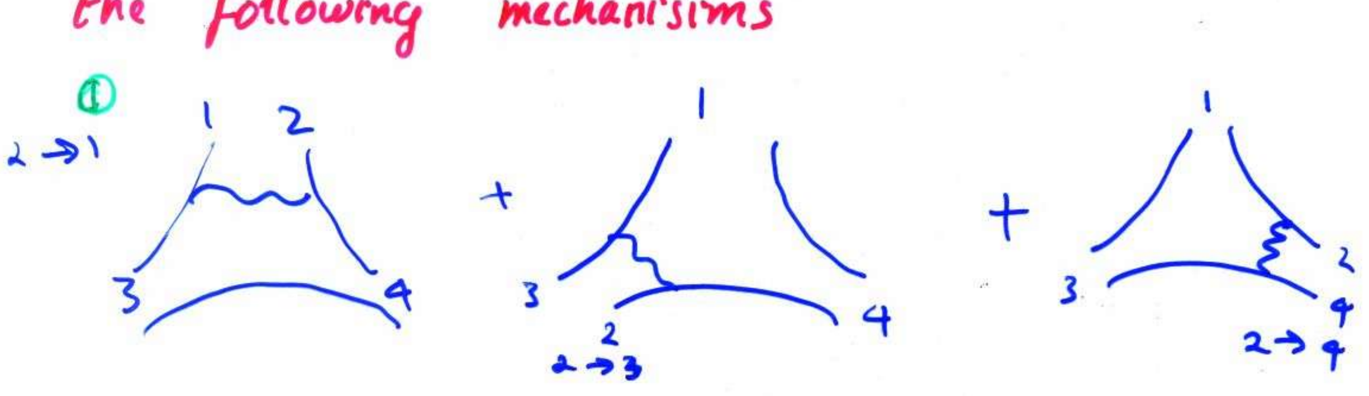
or $r' = \frac{x_{3q}^2}{x_{13}^2}$; $s' = \frac{x_{1q}^2}{x_{13}^2}$

$2 \to \infty$ collapse

- These are Non trivial functions of r', s'. Not just logs or constants
- Conformal Invariance demands the presence of only logs or constants

Such Non trivial functions are removed by the following mechanisims

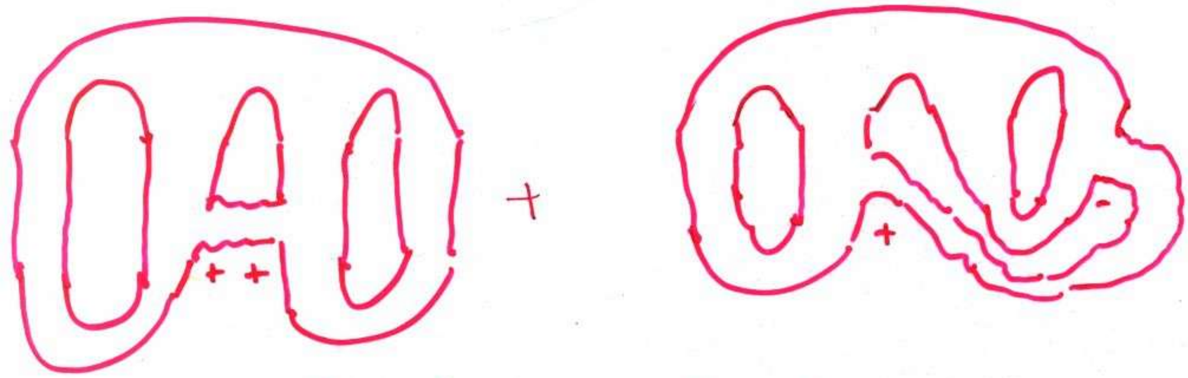
①



In the sum of such diagrams the dangerous Collapse s cancel.

② In a length Conserving Process

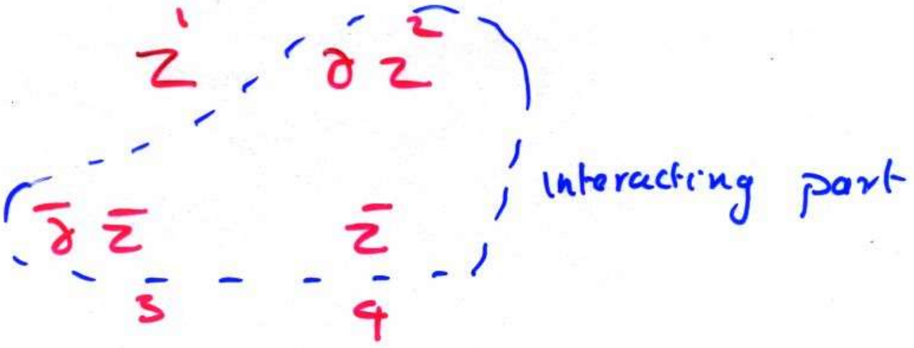
There are Non-nearest neighbour interactions which are planar



such diagrams Cancel.

③ Differential Equation Satisfied by $\varphi(r,s)$

e.g simplest Case



The $1 \to \infty$ collapse combines with the gauge boson on the 3rd leg using the differential Equation. satisfied by $\varphi(r,s)$

$$\varphi(r,s) + (s+r-1)\,\partial_s \varphi(r,s) + 2r\,\partial_r \varphi(r,s) = -\frac{\log r}{s}$$

LHS: Nontrivial function of r, s.

RHS: log

Similarly

$$\varphi(r,s) + (s+r-1)\,\partial_r \varphi(r,s) + 2s\,\partial_s \varphi(r,s) = -\frac{\log s}{r}$$

Some results:

$$H_{0k;k0}\,(3pt) - \frac{1}{2} H_{0k;k0}\,(2pt) = \frac{1}{2k^2}$$

log piece $\frac{1}{k}$

$$H_{kn;00}\,(3pt) - \frac{1}{2} H_{kn;00}\,(2pt) = \frac{1}{2}\left[h^2(n) - \sum_{l=1}^{k} \frac{h(n-l)}{l} + h_2(n)\right]$$

$$h(n) = \sum_{j=1}^{n} \frac{1}{j} \qquad \text{log piece: } h(n)$$

$$h_2(n) = \sum_{j=1}^{n} \frac{1}{j^2}$$

$$H_{kn;n-k0} - \frac{1}{2} H_{kn;n-k0}$$

$k \neq 0$
$k \neq n$

$$= \frac{1}{2}\left[2\,{}^{n}C_{k} \sum_{l=0}^{k} {}^{k}C_{l} \frac{(-1)^{n-l}}{(n-l)^2} - \frac{1}{(n-k)^2}\right]$$

log piece $\frac{1}{n-k}$

Conclusions

- One can use the bit picture to write 3pt & 2pt correlation function of $\mathcal{N}=4$ SYM at $\lambda=0$

- At finite λ (1-loop) the bit picture gives a simple formula for the corrections to 3pt fn.

- The gauge theory calculation of corrections to structure constants in the SO(6) sector agrees with the formula from the bit picture

- The gauge theory calculation in a non-SO(6) sector has been done

  [We need to check if the bit picture agrees in this sector]