Integrability in $\mathcal{N} = 4$ Super Yang-Mills and Spinning Strings on $AdS_5 \times S^5$
Introduction

★ String/gauge dualities
  ● Phenomenological: Regge trajectories & stringy colour flux tubes.
  ● Qualitatively: ’t Hooft’s large-$N$ limit.
  ● Concrete: Maldacena’s AdS/CFT conjecture.
  ● Quantitative comparison feasible: BMN-limit, FT spinning strings.
  ● Or rather not: Three-loop problems.

★ What have we learned?
  ● Obtain higher-loop results (the cheap way).
  ● Integrability in gauge theory (even beyond one-loop) & strings.
  ● Not only to make mathematicians happy: Bethe Ansätze.
  ● Exact solution of the quantum planar spectrum within reach!
    Gauge theory: One-loop, higher-loops, all-loops, non-perturbative.
    String theory: Classical, quantum.
    Then compare...
Outline

★ Cast of Characters
• AdS/CFT, string sigma model, local operators, spin chains, BMN/FT.

★ How to compare, v1: The Sledgehammer Approach
• Classical spinning strings, scaling dimensions & dilatation operator.

★ How to compare, v2: Classical Hamiltonians
• Coherent states, light-like strings.

★ How to compare, v3: Classical/Quantum, Action Variables
• Integrability, Bethe ansätze, Lax monodromy, algebraic curves.

★ Assumptions
• Planar: No string interactions, strict large-$N$.
• Spectrum of states: Energies, scaling dimensions.
• Classical bosonic strings, perturbative gauge theory.
AdS/CFT Correspondence

AdS/CFT conjecture claims equivalence of

- IIB string theory on $AdS_5 \times S^5$ background and
- $\mathcal{N} = 4$ superconformal gauge theory.

Symmetry group: $\tilde{\text{PSU}}(2, 2|4)$

- String theory: Isometries of target space.
- Gauge theory: $\mathcal{N} = 4$ superconformal invariance.

Conserved charges: Spins $J_{1,2,3}$ on $S^5$, spins $S_{1,2}$ on $AdS_5$ and one non-compact generator $H/D$: Hamiltonian/dilatation generator.

AdS/CFT predicts the agreement of

- the spectrum of $H$ in string theory (energies $\{E\}$) with
- the spectrum of $D$ in gauge theory: (scaling dimensions $\{D\}$).
The Target Space

Target space $AdS_5 \times S^5$ is (the universal cover of) the coset

$$\frac{PSU(2, 2|4)}{Sp(1, 1) \times Sp(2)} \sim \frac{SO(2, 4)}{SO(1, 4)} \times \frac{SO(6)}{SO(5)} \times \text{ferm.}$$

Embedding into higher-dimensional flat space

$$S^5 \subset \mathbb{R}^6 \quad \text{and} \quad AdS_5 \subset \mathbb{R}^{2,4}$$

$$\bar{X}^2 = \bar{X}^T \bar{X} = 1. \quad \bar{Y}^2 = \bar{Y}^T \eta \bar{Y} = 1.$$
String Theory on $AdS_5 \times S^5$

Sigma model action (Green-Schwarz superstring)

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-\gamma} \left( \frac{1}{2} (\partial_a \vec{X})^2 - \frac{1}{2} (\partial_a \vec{Y})^2 \right) + \ldots .$$

Equations of motion

$$\partial_+ \partial_- \vec{X} + (\partial_+ \vec{X} \cdot \partial_- \vec{X}) \vec{X} = 0,$$

$$\partial_+ \partial_- \vec{Y} + (\partial_+ \vec{Y} \cdot \partial_- \vec{Y}) \vec{Y} = 0.$$

Virasoro constraint

$$\left( \partial_\pm \vec{X} \right)^2 = \left( \partial_\pm \vec{Y} \right)^2.$$
Symmetry Currents and Global Charges

Conserved currents of $SO(6)$ and $SO(2,4)$

\[ j = 2\tilde{X} d\tilde{X}^\top - 2d\tilde{X} \tilde{X}^\top, \quad \tilde{j} = 2\tilde{Y} d\tilde{Y}^\top \eta - 2d\tilde{Y} \tilde{Y}^\top \eta. \]

Flatness

\[ dj + j \wedge j = 0, \quad d\tilde{j} + \tilde{j} \wedge \tilde{j} = 0. \]

Global charges

\[ J = \frac{\sqrt{\lambda}}{4\pi} \oint *j + \ldots, \quad S = \frac{\sqrt{\lambda}}{4\pi} \oint *\tilde{j} + \ldots. \]

EOM (equivalent to conservation) & Virasoro constraints via currents

\[ d*j = 0, \quad d*\tilde{j} = 0, \quad \text{Tr} j_{\pm}^2 = \text{Tr} \tilde{j}_{\pm}^2 \]
\( \mathcal{N} = 4 \) Gauge Theory

\( \text{SU}(N) \) gauge field \( A_\mu \), 4 adjoint fermions \( \Psi^a_\alpha, \bar{\Psi}^{\dot{\alpha}a} \), 6 adjoint scalars \( \Phi_m \)

\[
S_{\mathcal{N}=4} = N \int \frac{d^4 x}{4\pi^2} \text{Tr}\left( \frac{1}{4} (F_{\mu \nu})^2 + \frac{1}{2} (D_\mu \Phi_m)^2 - \frac{1}{4} [\Phi_m, \Phi_n]^2 + \ldots \right).
\]

Field theory in position space:
Correlators of local operators \( \mathcal{O}_{1,2,\ldots}(x) \)

\[
\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \mathcal{O}_3(z) \ldots \rangle = F_{1,2,3,\ldots}(x, y, z, \ldots)
\]

Local operator: Gauge invariant combination of fields.
Perturbation theory at weak coupling: Feynman diagrams.
CFT: Scaling dimension \( D_\mathcal{O} \) of \( \mathcal{O} \) determines two-point function

\[
\langle \mathcal{O}(x) \mathcal{O}(y) \rangle \sim \frac{1}{|x - y|^{2D_\mathcal{O}(g)}}.
\]
Local Operators

Local, gauge-invariant combination of the fields, e.g.:

\[ O^a_{\mu m\alpha}(x) = 27 \text{Tr} \Phi_n(x) D^\nu \Psi^a_\alpha(x) F_{\nu\mu}(x) \text{Tr} \Phi_m(x) \Phi_n(x) \]
\[ + 13g \text{Tr} D_\mu D_\nu \Phi_m(x) \Phi_n(x) D^\nu \Phi_n(x) \Psi^a_\alpha(x) + \ldots \]

- Position-space representation of QFT.
- Do not identify \( O \) and descendants \( \partial_\mu O, \partial_\mu \partial_\nu O, \ldots \).
- Drop position \( (x) \): Local operators as abstract objects.
- Building blocks \( \mathcal{W}_A = \{ D^n \Phi, D^n \Psi, D^n F \} \).
  Canonical gauge transformation: \( \mathcal{W}_A \mapsto U \mathcal{W}_A U^{-1} \).
- Mixing problem: Lots and lots of similar operators.
Basis of Building Blocks

The building blocks are fields and their derivatives

\[ \mathcal{W}_A = \{ \mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F} \} . \]

Consider a scalar \( \Phi \). No problem with \( \Phi \) and \( \mathcal{D}_\mu \Phi \), but

\[ \mathcal{D}_\mu \mathcal{D}_\nu \Phi = \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi + \mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi + \frac{1}{4} \eta_{\mu\nu} \mathcal{D}^2 \Phi . \]

Jacobi identity

\[ \mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi = -ig[F_{\mu\nu}, \Phi] = O(gF\Phi) . \]

Equations of motion

\[ \mathcal{D}^2 \Phi = O(g\Psi^2) + O(g^2 \Phi^3) . \]

Only \( \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi \) is elementary; \( \mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi \) and \( \mathcal{D}^2 \Phi \) are reducible.
## Lorentz Multiplets

<table>
<thead>
<tr>
<th>$D$</th>
<th>$D^n\bar{F}$</th>
<th>$D^n\bar{\Psi}$</th>
<th>$D^n\Phi$</th>
<th>$D^n\Psi$</th>
<th>$D^n\mathcal{F}$</th>
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<tbody>
<tr>
<td>1</td>
<td></td>
<td>$(0,0)$</td>
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<tr>
<td>$\frac{3}{2}$</td>
<td>$(0,\frac{1}{2})$</td>
<td>$\left(\frac{1}{2},0\right)$</td>
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<tr>
<td>2</td>
<td>$(0,1)$</td>
<td>$\left(\frac{1}{2},\frac{1}{2}\right)$</td>
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<td>$\frac{5}{2}$</td>
<td>$\left(\frac{1}{2},1\right)$</td>
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<td>3</td>
<td>$\left(\frac{1}{2},\frac{3}{2}\right)$</td>
<td>$(1,1)$</td>
<td>$(\frac{3}{2},\frac{1}{2})$</td>
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<td>$\frac{7}{2}$</td>
<td>$(1,\frac{3}{2})$</td>
<td>$(\frac{3}{2},1)$</td>
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\[\ldots\quad \ldots\quad \ldots\quad \ldots\]

$\text{SL}(2, \mathbb{C})$: \[
\left(\frac{n}{2}, \frac{n+2}{2}\right), \quad \left(\frac{n}{2}, \frac{n}{2}\right), \quad \left(\frac{n+2}{2}, \frac{n}{2}\right), \quad \left(\frac{n}{2}, \frac{n+1}{2}\right), \quad \left(\frac{n+1}{2}, \frac{n}{2}\right)\]
# Poincaré/Conformal Multiplets

\[
\begin{array}{cccccc}
D & D^n \bar{F} & D^n \bar{\Psi} & D^n \Phi & D^n \Psi & D^n \bar{\Phi} \\
1 & (0, 0) & & & & \\
\frac{3}{2} & (0, \frac{1}{2}) & \text{P/K} & (\frac{1}{2}, 0) & & \\
2 & (0, 1) & (\frac{1}{2}, \frac{1}{2}) & (1, 0) & & \\
\frac{5}{2} & (\frac{1}{2}, 1) & & (1, \frac{1}{2}) & & \\
3 & (\frac{1}{2}, \frac{3}{2}) & (1, 1) & (\frac{3}{2}, \frac{1}{2}) & & \\
\frac{7}{2} & (1, \frac{3}{2}) & & (\frac{3}{2}, 1) & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
\end{array}
\]

\[\text{SU}(2, 2): \quad [0, -3, 2] \quad [0, -1, 0] \quad [2, -3, 0] \]

\[\ldots [0, -2, 1] \quad [1, -2, 0] \]

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\( \mathcal{N} = 4 \) Supersymmetry/Superconformal Multiplet

\[
\begin{array}{ccccccc}
D & D^n \bar{F} & D^n \bar{\Psi} & D^n \Phi & D^n \Psi & D^n F \\
1 & & & & & \bar{Q}/\bar{S} & Q/S \\
\frac{3}{2} & (0, \frac{1}{2}) & \quad & P/K & (\frac{1}{2}, 0) \\
2 & (0, 1) & (\frac{1}{2}, \frac{1}{2}) & (1, 0) \\
\frac{5}{2} & (\frac{1}{2}, 1) & \quad & (1, \frac{1}{2}) \\
3 & (\frac{1}{2}, \frac{3}{2}) & (1, 1) & (\frac{3}{2}, \frac{1}{2}) \\
\frac{7}{2} & (1, \frac{3}{2}) & \quad & (\frac{3}{2}, 1) \\
& \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\( \text{PSU}(2|4|2): \quad [0; 0; 0, 1; 0; 0] \)
Gauge Theory and Spin Chains

Single trace operator, two complex scalars $\phi_1, \phi_2$ (a.k.a. $Z, \phi$ or $Z, X$)

$$\mathcal{O} = \text{Tr} \phi_1 \phi_1 \phi_2 \phi_1 \phi_1 \phi_2$$

Length $L$: # of fields

Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow\rangle$$

Length $L$: # of sites

Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

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How to compare in AdS/CFT?

AdS/CFT: String energies & gauge dimensions match: \( \{ E \} = \{ D \} \).

Supergravity/protected states: \( \{ E \} = \{ D \} = \{ 2, 3, 4, \ldots \} \). √

Generic states: Classical strings vs. e.g. Konishi \( \text{Tr} \Phi_m \Phi_m \)

- String theory at \( \lambda \approx \infty \): \( E \sim 4\sqrt{\lambda} \).
- Gauge theory at \( \lambda \approx 0 \): \( D = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \ldots \).

Cannot compare. ×

Large spin \( S \) on \( AdS^5 \): Classical strings vs. twist-two \( \text{Tr}(D^pW)(D^{S-p}W) \)

- String theory at \( \lambda \approx \infty \): \( E \sim \sqrt{\lambda} \log S + \mathcal{O}(S^0) \).
- Gauge theory at \( \lambda \approx 0 \): \( D = f(\lambda) \log S + \mathcal{O}(S^0) \).

Qualitative agreement, cannot compare quantitatively. √/×

Dynamical tests prevented by strong/weak nature of the duality.
Large Spin Limits of AdS/CFT

Proposal: Consider states with large spin $J$ on $S^5$

- BMN limit; non-planar and near $O(1/J)$ extensions.

$O \sim \text{Tr} \phi_1 \phi_1 \ldots \phi_2 \ldots \phi_2 \ldots \phi_2 \ldots \phi_1 \phi_1 \longleftrightarrow$ short quantum strings.

- Semiclassical Spinning Strings.

$O \sim \text{Tr} \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \longleftrightarrow$ long classical strings.

Effective coupling constant

$$\lambda' = \frac{\lambda}{J^2}.$$

- String theory: Expansion in $\lambda'$ and $1/J \sim 1/\sqrt{\lambda},$
- Gauge theory: $\ell$-loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in $\lambda'$ apparently equivalent to expansion in $\lambda$. Compare!
Three-Loop Discrepancies

BMN state with 2 excitations

\[ O_n \approx \sum_{p=0}^{J} \exp \frac{2\pi i n p}{J} \text{Tr} \phi_1^p \phi_2 \phi_1^{-p} \phi_2 \]

\[ D - J \approx 2 \sqrt{1 + \frac{\lambda n^2}{J^2}} \]

Gauge theory dimension in near BMN limit \( O(1/J) \)

\[ D - J = 2 + \frac{\lambda n^2}{J^2} \left( 1 - \frac{2}{J} \right) - \frac{\lambda^2 n^4}{J^4} \left( \frac{1}{4} + \frac{0}{J} \right) + \frac{\lambda^3 n^6}{J^6} \left( \frac{1}{8} + \frac{1}{2J} \right) + \ldots \]

Energy of near plane-wave string

\[ E - J = 2 + \lambda' n^2 \left( 1 - \frac{2}{J} \right) - \lambda'^2 n^4 \left( \frac{1}{4} + \frac{0}{J} \right) + \lambda'^3 n^6 \left( \frac{1}{8} + \frac{0}{J} \right) + \ldots \]

Three-loop mismatch also for 3 excitations.
Similar disagreement for spinning strings.
Spinning Strings

Many examples investigated:

- Folded
- Circular
- Pulsating
- Higher modes
- Plane waves

Example: Ansatz for Spinning string on $\mathbb{R}_t \times S^2$:

$t(\tau, \sigma) = \mathcal{E} \tau$, \quad \vec{X}(\tau, \sigma) = \begin{pmatrix} \sin \vartheta(\sigma) \cos \mathcal{J} \tau \\ \sin \vartheta(\sigma) \sin \mathcal{J} \tau \\ \cos \vartheta(\sigma) \end{pmatrix}$.

Equation of motion and Virasoro constraint

$\vartheta'' + \mathcal{J}^2 \sin \vartheta \cos \vartheta = 0$, \quad $\vartheta'^2 + \mathcal{J}^2 \sin^2 \vartheta = \mathcal{E}^2$. 
Periodicity

Solve equations of motion and Virasoro constraint

\[ \vartheta(\sigma) = \text{am}(\mathcal{E}(\sigma - \sigma_0), \eta), \quad \eta = \mathcal{E}/J. \]

Folded string: \( \vartheta(0) = 0 \) and \( \vartheta'(\pi/2n) = 0 \)

\[ J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n}{\pi} K(1/\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\eta \pi} K(1/\eta). \]

Circular string: \( \vartheta(0) = 0 \) and \( \vartheta(2\pi/n) = 2\pi \)

\[ J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n\eta}{\pi} K(\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\pi} K(\eta). \]
Expansion

Some solutions admit expansion in $\lambda' = \lambda/J^2$.

E.g., particular two-spin solution, $J = J_1 + J_2$

$$J_1 = \sqrt{\lambda} J_1(\eta_{1,2}), \quad J_2 = \sqrt{\lambda} J_2(\eta_{1,2}), \quad E = \sqrt{\lambda} E(\eta_{1,2}).$$

Solve $J_1, J_2$ for $\eta_1, \eta_2$, substitute in energy $E$ and expand around $J = \infty$

$$E = \sqrt{\lambda} E(J_1/\sqrt{\lambda}, J_2/\sqrt{\lambda})$$

$$= J + \frac{\lambda}{J} E_1(J_2/J) + \frac{\lambda^2}{J^3} E_2(J_2/J) + \ldots$$

Similar to weak coupling expansion in gauge theory. Want to compare!
Scaling Dimensions

Consider a local operator, e.g. Konishi operator: \( \mathcal{O} = \text{Tr} \Phi_m \Phi_m \).

Compute two-point function (in \( 4 + 2\epsilon \) dimensions)

\[
\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{12(1 - 1/N^2)}{|x - y|^{4+4\epsilon}} \left(1 + \ldots + \frac{6g^2}{\epsilon \mu^{2\epsilon}|x - y|^{2\epsilon}} + \ldots\right)
\]

\[
\to \quad \frac{12(1 - 1/N^2)}{\mu^{-12g^2+\ldots} |x - y|^4} \left(1 + \frac{6g^2 \log \frac{1}{\mu^2|x - y|^2}}{\mu^2|x - y|^2} + \ldots\right)
\]

\[
= \frac{12(1 - 1/N^2)}{|x - y|^{2(2+6g^2+\ldots)}}.
\]

Scaling dimension of \( \mathcal{O} \)

\[
D_{\mathcal{O}} = 2 + 6g^2 - 12g^4 + \ldots = 2 + \frac{3g_Y^2 N}{4\pi^2} - \frac{3g_Y^4 N^2}{16\pi^4} + \ldots.
\]
Dilatation Generator

Scaling dimensions $D_\mathcal{O}(g)$ as eigenvalues of the dilatation generator $D(g)$

$$D(g) \mathcal{O} = D_\mathcal{O}(g) \mathcal{O}.$$ 

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$

$$D(g) = D_0 + g^2 D_2 + g^3 D_3 + g^4 D_4 + \ldots .$$

Local action along spin chain (homogeneous). Spin chain Hamiltonian
Tree Level

Classical dilatation acts on one field at a time:

\[ D(0) = D_0 \]

Fields have definite classical dimension \( D_0 \mathcal{W} = \dim(\mathcal{W}) \mathcal{W} \) with

\[
\dim(D^n \Phi) = 1 + n, \quad \dim(D^n \Psi) = \frac{3}{2} + n, \quad \dim(D^n \mathcal{F}) = 2 + n.
\]

For composites: \( D_0 \text{ Tr } \mathcal{W}_1 \ldots \mathcal{W}_L = \dim(\mathcal{W}_1 \ldots \mathcal{W}_L) \text{ Tr } \mathcal{W}_1 \ldots \mathcal{W}_L \) with

\[
\dim(\mathcal{W}_1 \ldots \mathcal{W}_L) = \sum_{p=1}^{L} \dim(\mathcal{W}_p).
\]
One-Loop

One-loop $\mathcal{O}(g^2)$ dilatation operator $D_2$:

\[
D_{2(12)} = D_2 = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} + \frac{1}{2}
\]

Extract logarithmic (divergent) piece of Feynman diagrams.

* Scalars without derivatives: $D_{2(12)} = \mathcal{I}_{(12)} - \mathcal{P}_{(12)} + \frac{1}{2}\mathcal{K}_{(12)}$. [Minahan Zarembo]

**Example:** Konishi descendant $\text{Tr} \phi_1^2\phi_2^2 + \ldots$:

\[
D_2 \text{Tr} \phi_1\phi_1\phi_2\phi_2 = +2 \text{Tr} \phi_1\phi_1\phi_2\phi_2 - 2 \text{Tr} \phi_1\phi_2\phi_1\phi_2,
\]

\[
D_2 \text{Tr} \phi_1\phi_2\phi_1\phi_2 = -4 \text{Tr} \phi_1\phi_1\phi_2\phi_2 + 4 \text{Tr} \phi_1\phi_2\phi_1\phi_2.
\]

Eigenvalue $D_2 = 0$: Eigenstate: $\mathcal{O} = 2 \text{Tr} \phi_1\phi_1\phi_2\phi_2 + \text{Tr} \phi_1\phi_2\phi_1\phi_2$,

Eigenvalue $D_2 = 6$: Eigenstate: $\mathcal{O} = \text{Tr} \phi_1\phi_1\phi_2\phi_2 - \text{Tr} \phi_1\phi_2\phi_1\phi_2$. 

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Tensor Product

⋆ Find result for generic building blocks: \( \{D^n\Phi, D^n\Psi, D^nF\} \).

Too many field combinations to conveniently compute in field theory.
But: Symmetry implies \( D_2 \) classically invariant: \( [J_0, D_2] = 0 \).
Consider tensor product of two spins \( V_F = \langle D^n\Phi, D^n\Psi, D^nF \rangle \)

\[
V_F \otimes V_F = \sum_{j=0}^{\infty} V_j.
\]

Two-parton multiplets \( V_j \) irreducible and distinct:

\[
D_{2(12)} = \sum_{j=0}^{\infty} D_j P_{(12),j}.
\]

Only one sequence of unknown coefficients \( D_j \) remains.
Complete One-Loop

★ Compute $D_j$ from Feynman diagrams. (Boring!)
★ Can do better: Use symmetry.

Outline of algebraic derivation of $D_j$:

- Superconformal algebra: $\{Q, S\} \sim L + \bar{L} + R + D$.
- For particular states: $\{Q_1, S_1\} \sim D_2$.
- Supercharges $Q_1, S_1$ at $\mathcal{O}(g)$ tightly constrained.
- Find $Q_1, S_1$ and thus $D_2$ up to one overall constant ($g$).

Result:

$$D_j = 2h(j), \quad h(j) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j}.$$ 

Complete dilatation operator of $\mathcal{N} = 4$ SYM: $(\mathcal{J}_{(12)}$ “total s.c. spin” op.)

$$D_{2(12)} = \sum_{j=0}^{\infty} 2h(j) \mathcal{P}_{(12),j} = 2h(\mathcal{J}_{(12)}).$$
Higher-Loops

Contribution to dilatation generator at $\mathcal{O}(g^k)$ has (up to) $k + 2$ legs

$D_3 = \begin{array}{c}
\text{Diagram 1} + \\
\text{Diagram 2}
\end{array}$

$D_4 = \begin{array}{c}
\text{Diagram 1} + \\
\text{Diagram 2} + \\
\text{Diagram 3} + \ldots
\end{array}$

At higher loops: Length $L$ fluctuates, dynamic spin chain.

Also (super)momenta $Q, P$ & (super)boosts $S, K$ are corrected, e.g.

$P_1, K_1, Q_1, S_1 = \begin{array}{c}
\text{Diagram 1} + \\
\text{Diagram 2}
\end{array}$
Algebraic Construction

Assume most general form and use closure of symmetry algebra

\[ [J_A(g), J_B(g)] = F^{C}_{AB} J_C(g). \]

**Sector of \( \phi_{1,2,3}, \psi_{1,2} \): At three loops (using BMN scaling)**

- Dimension of Konishi

\[ D_K = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \ldots \]

- Computation in QCD and lift to \( \mathcal{N} = 4 \) SYM.
- Two-loop computation and lift by multiplet shortening.
- BMN matrix model.

**Sector of \( \phi_{1,2} \): At five loops (using BMN scaling & integrability)**

- Reproduces BMN energy formula \( D - J = \sum_k \sqrt{1 + \lambda' n_k^2} \).
Stringing Spins

Spectrum of $\text{Tr} \phi_1^{14-K} \phi_2^K$, $K = 0, \ldots, 6$, red: $K = 7$.

How to identify states? Lowest state for $K = 7$ okay.
Comparison of Hamiltonians

Direct comparison works fine up to two loops; but...

- Need to find suitable ansätze for string theory.
- Can solve only a few analytically.
- Extremely hard to identify gauge theory states.
- Diagonalisation tedious.
- Numerical accuracy low at small length $L \approx 20$.
- No proof...

Idea: Do not compare spectrum, but Hamiltonian

Problems:

- Classical vs. quantum Hamiltonian.
- Analytic in $\lambda'$ vs. first few loop orders.
Coherent States

Want to reconstruct classical wave function $\mathbf{X}(\tau, \sigma)$ from spin chain.

- Quantum spin $\frac{1}{2}$: Up $|\uparrow\rangle = |\phi_1\rangle$, down $|\downarrow\rangle = |\phi_2\rangle$ or superposition.
- Classical spin on $S^2$: $\vartheta$ (angle), $\varphi$ (phase).

Quantum state has no phase information (Heisenberg).
Cannot reconstruct $\mathbf{X}$!

Coherent spin $\frac{1}{2}$: (over-complete basis)

$$|\vartheta, \varphi\rangle = \exp(-\frac{i}{2}\varphi) \cos(\frac{1}{2}\vartheta) |\uparrow\rangle + \exp(+\frac{i}{2}\varphi) \sin(\frac{1}{2}\vartheta) |\downarrow\rangle$$

Spins $|\vartheta, \varphi\rangle$ map directly to $S^2$
Coherent Hamiltonian

Spin chain for discretised wave-function $\vec{X}(\tau, \frac{2\pi}{L}s) = \vec{X}_s = \vec{X}(\vartheta_s, \varphi_s)$.

$|\vec{X}\rangle = |\vartheta_1, \varphi_1\rangle \cdots |\vartheta_L, \varphi_L\rangle$.

Classical Hamiltonian (off-diagonal elements suppressed)

$$\langle \vec{X} | g^2 D_2 | \vec{X} \rangle = \frac{g^2}{4} \sum_{s=1}^{L} (\vec{X}_s - \vec{X}_{s+1})^2 \rightarrow \frac{g^2}{4L} \int_0^{2\pi} d\sigma (\partial_\sigma \vec{X})^2 = H[\vec{X}]$$

Canonical brackets for spin $\frac{1}{2}$ model

$$\{X_{s,k}, X_{s',l}\} = \frac{1}{2} \delta_{s,s'} \varepsilon_{klm} X_m.$$  

Equations of motion

$$\partial_\tau \vec{X} = \{H, \vec{X}\} = \frac{g^2}{4L} (\partial_\sigma \partial_\sigma \vec{X}) \times \vec{X}.$$
Relativistic Expansion

Expansion of string on $\mathbb{R}_t \times S^3$ around “light-cone”

$$t(\tau, \sigma) = \mathcal{E} \tau, \quad \vec{X}(\tau, \sigma) = \begin{pmatrix}
\cos\left(\frac{1}{2} \vartheta\right) \cos(\mathcal{E} \tau + \phi + \frac{1}{2} \varphi) \\
\cos\left(\frac{1}{2} \vartheta\right) \sin(\mathcal{E} \tau + \phi + \frac{1}{2} \varphi) \\
\sin\left(\frac{1}{2} \vartheta\right) \cos(\mathcal{E} \tau + \phi - \frac{1}{2} \varphi) \\
\sin\left(\frac{1}{2} \vartheta\right) \sin(\mathcal{E} \tau + \phi - \frac{1}{2} \varphi)
\end{pmatrix}.$$ 

Relativistic limit $\mathcal{E} \to \infty$ while $\dot{\vartheta}, \dot{\varphi}, \dot{\phi} \sim 1/\mathcal{E}$.

Can solve for EOM for $\phi$ (Hopf fibration) and get ($\vec{X} \in S^2$ as before)

$$\partial_\tau \vec{X} = \frac{g^2}{4\mathcal{E}} (\partial_\sigma \partial_\sigma \vec{X}) \times \vec{X} + \ldots,$$ 

$$H[\vec{X}] = \frac{g^2}{4\mathcal{E}} \int_0^{2\pi} d\sigma (\partial_\sigma \vec{X})^2 + \ldots$$

In leading order $L = \mathcal{E}$ and we get generic agreement.
Integrability

Coherent approach works fine in the investigated cases, but...

- Relies on perturbation theory. Complexity increases with order.
- Needs explicit form of dilatation operator as input. Gets very complicated beyond one-loop or for larger sectors.
- Relativistic limit subtle beyond one-loop.

Idea: Compare integrable structure.

- Gauge theory: Integrability at one loop and higher loops.
- Bethe ansätze.
- Thermodynamic limit.
- String theory Lax connection and monodromy.
- Analytic properties of the monodromy.
- Algebraic curve.
One-Loop Integrability

Only in planar limit!

Existence of higher charges $Q_2, Q_3, Q_4, \ldots$ (scalar, commuting),

$$[J_0, Q_r] = [Q_r, Q_s] = 0, \quad D_2 = Q_2.$$ 

Structure of charges

$$Q_2 = D_2, \quad Q_3 = \frac{i}{2} D_2 - \frac{i}{2} D_2 \quad \text{etc.}$$

One-loop integrability found for

- **Sector of scalars** $\{\Phi_m\}$: $\mathfrak{so}(6)$ spin chain.
- **Complete $\mathcal{N} = 4$ SYM**: $\mathfrak{psu}(2,2|4)$ super spin chain.
- **Sectors of $\mathcal{N} < 4$ theories**.

References:

- Minahan Zarembo
- NB Staudacher
- Braun, Derkachov, Korchemsky, Manashov
- Belitsky
- Braun, Derkachov, Korchemsky
- Wang
- Belitsky
- NB, Ferretti, Heise, Zarembo
- Wang

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Test for Integrability

Consider parity $\mathcal{P}$ (charge conjugation/spin chain/world sheet):

$$\text{Tr} \, \phi_1 \phi_1 \phi_2 \phi_1 \phi_1 \phi_2 \phi_2 \xleftarrow{\mathcal{P}} \text{Tr} \, \phi_2 \phi_2 \phi_1 \phi_1 \phi_1 \phi_2 \phi_1 \phi_1$$

Even/odd charges have even/odd parity

$$\mathcal{P} Q_r \mathcal{P}^{-1} = (-1)^r Q_r.$$  

Implies degenerate pairs of opposite parity: (only in planar limit!)

$$D_+ = D_-.$$  

Test for integrability.
Higher-Loop Integrability

No formalism yet (R-matrix, Yang-Baxter equation, . . .).

Existence of higher charges $Q_r(g)$,

$$[J(g), Q_r(g)] = [Q_r(g), Q_s(g)] = 0,$$

$$D(g) = D_0 + g^2 Q_2(g).$$

Test: Planar parity pairs preserved at higher loops

$$D_+(g) = D_-(g).$$

Higher-loop integrability for

- **Sector of scalars $\{\Phi_m\}$**: Not closed at higher loops due to mixing.
- **Sector of $\{\phi_{1,2,3}, \psi_{1,2}\}$**: Observed at three-loops (pairs).
  - Even through length fluctuates.
- **Sector of $\{\phi_{1,2}\}$**: Construct five-loops dilop. via integrability.
Bethe Ansatz

Bethe equations for sector of \( \{ \phi_1, \phi_2 \} \). Bethe roots: \( u_k \in \mathbb{C} \).

\[
\prod_{j=1}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} = \left( \frac{u_k + i/2}{u_k - i/2} \right)^L.
\]

Momentum constraint and one-loop scaling dimension:

\[
\prod_{j=1}^{K} \frac{u_j - i/2}{u_j + i/2} = 1, \quad D_2 = \sum_{j=1}^{K} \frac{1}{u_j^2 + 1/4}.
\]

Example: Konishi \( \mathcal{O} = \text{Tr} \, \phi_1^2 \phi_2^2 + \ldots \), \( L = 4 \), \( K = 2 \). Find

\[
u_{1,2} = \pm \frac{1}{\sqrt{12}}, \quad D_2 = \frac{1}{1/12 + 1/4} + \frac{1}{1/12 + 1/4} = 6.
\]
Higher-Loop Bethe Ansatz

Higher-loops for sector of $\{\phi_1, \phi_2\}$. Modified Inozemtsev.

$$\prod_{j=1, j\neq k}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} = \frac{x(u_k + i/2)^L}{x(u_k - i/2)^L}, \quad x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2 - 2g^2}.$$ 

Momentum constraint and one-loop scaling dimension:

$$\prod_{j=1}^{K} \frac{x(u_j - i/2)}{x(u_j + i/2)} = 1, \quad D = L + g^2 \sum_{j=1}^{K} \left( \frac{i}{x(u_j + i/2)} - \frac{i}{x(u_j - i/2)} \right).$$

Example: Konishi $L = 4$, $K = 2$. Find

$$u_{1,2} = \pm \frac{1}{\sqrt{12}} \left( 1 + 4g^2 - 5g^4 + \ldots \right), \quad D = 4 + 6g^2 - 12g^4 + 42g^6 + \ldots.$$
Bethe Ansätze

★ For conformal $\mathcal{N} = 4$ gauge theory
- Complete at one loop: Seven types of Bethe roots. ✓
- Sector of $\phi_1, \phi_2$: Up to very high loop order. ✓
- Sector of $\phi_1, \psi_1$: Up to three loops. ✓
- Sector of $\Phi_m$: Higher loops in the thermodynamic limit. ✓
- States with fluctuating length $L$. ×
- Higher rank algebras $L$ beyond one loop. ×

★ For related models
- Near-BMN strings: Up to $O(1/J)$ and possibly beyond? Reproduces $\sqrt[4]{\lambda}$ for generic states.
- Large $N_c$ QCD, closed & open chains.
- Sectors of $\mathcal{N} = 1, \mathcal{N} = 2$ gauge theory, etc..
Thermodynamic Limit

- Long spin chains, $L \to \infty$.
- Large number of Bethe roots $K \sim L$.
- Low energy, $D_2 \sim 1/L$.

Roots $u_k$ condense on (disconnected) contours $C = C_1 \cup \ldots \cup C_A$:

Discrete sums turn into integrals ($\prod = \exp \sum \log$) with density $\rho(u)$

$$\sum_{k=1}^{K} f(u_k) \to \int_{C} du \rho(u) f(u).$$
Bethe Equations in the Thermodynamic Limit

Bethe equations in thermodynamic limit become integral equations

\[ 2 \int_{C} \frac{dv}{v-u} \rho(v) + \frac{1}{u} = 2\pi n_a \quad \text{for } u \in C_a \]

Momentum constraint and one-loop scaling dimension:

\[ \int_{C} \frac{du}{u} \rho(u) = 2\pi n_0, \quad D_2 = \frac{2}{L} \int_{C} \frac{du}{u^2} \rho(u) \cdot \]

Example: Two-cut solution \( n_{1,2} = \pm n \)

\[ D = L + \frac{2\lambda n^2}{\pi^2 L} K(t)(E(t) + (t-1)K(t)) + \ldots, \quad \frac{K}{L} = 1 - \frac{E(t)}{K(t)}. \]

Agrees with folded string at one loop (two loops)
Algebraic Curve

Define a function $p(u)$ (quasi-momentum) with two sheets $p_k(u)$

$$p_{1,2}(u) = \pm \int_C \frac{dv \rho(v)}{v-u} \pm \frac{1}{2u},$$

Bethe equations: $\dot{p}(u) := \frac{1}{2}p(u + \epsilon) + \frac{1}{2}p(u - \epsilon)$

$$\dot{p}_1(u) - \dot{p}_2(u) = 2\pi n_\alpha \quad \text{for } u \in C_\alpha$$

Quasi-momentum $p(u)$ is continuous (modulo $2\pi$).

Derivative $p'(u)$ has the following properties

- Analytic and single-valued. ✓
- Finitely many branch cuts. ✓
- Double-pole at $u = 0$. ✓

Algebraic curve: $F(u, p') = P_2(u)p'^2 + P_0(u) = 0$. 
Bosonic String on $\mathbb{R} \times S^3$

$S^3$ is group manifold of $SU(2)$.
Coordinates $h(\sigma, \tau) \in SU(2)$ with $h^\dagger h = 1$, $\det h = 1$ and $t(\sigma, \tau) \in \mathbb{R}$.
Standard sigma model action

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int \left( -dt \wedge *dt + \text{Tr} \, dh^\dagger \wedge *dh + \Lambda (\det h - 1) \right).$$

Equations of motion

$$h^{-1} d*dh - h^{-1} dh \wedge h^{-1} *dh = 0, \quad \det h = 1, \quad d*dt = 0.$$ 

Gauge-fix time $t = E\tau/\sqrt{\lambda}$. Virasoro constraint

$$\text{Tr}(h^{-1} \partial_\pm h)^2 = -2(\partial_\pm t)^2 = -2E^2/\lambda.$$
Lax Pair

Introduce left $\mathfrak{su}(2)$ current

$$j = h^{-1} dh.$$  

Flatness, conservation (equations of motion) & Virasoro constraints

$$dj + j \wedge j = 0, \quad d*j = 0, \quad \text{Tr} j^2_{\pm} = -2E^2/\lambda.$$  

Lax pair: Family of flat connections ($\sim$ integrability)

$$a(x) = \frac{1}{1 - x^2} j + \frac{x}{1 - x^2} \ast j.$$  

Flatness of $a(x)$ for all $x$ equivalent to flatness and conservation of $j$:

$$da(x) + a(x) \wedge a(x) = 0.$$  

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Monodromy

Monodromy of Lax connection around closed string

\[ \omega(x) = P \exp \oint \gamma (-a(x)). \]

Independent of path \( \gamma \), but not of end point \( \gamma(0) = \gamma(2\pi) = (\tau, \sigma) \)

\[ d\omega(x) + [a(x), \omega(x)] = 0. \]

Shift generates similarity transformation. Eigenvalues preserved

\[ \omega(x) \simeq \text{diag}(e^{ip(x)}, e^{-ip(x)}). \]

The quasi-momentum \( p(x) \) is a conserved, gauge-invariant quantity. Complete set of action variables in Hamilton-Jacobi formalism.
Global Charges

Expansion of Lax connection at \( x = \infty \):

\[
a(x) = -\frac{1}{x} \ast j + \mathcal{O}(1/x^2), \quad J = \frac{\sqrt{\lambda}}{2\pi} \oint \ast j.
\]

Global \( su(2)_R \) charges \( J_R \) can be read off from monodromy at \( x = \infty \)

\[
\omega(x) = I + \frac{1}{x} \frac{2\pi J_R}{\sqrt{\lambda}} + \mathcal{O}(1/x^2), \quad p(x) = \frac{1}{x} \frac{2\pi J_{R,3}}{\sqrt{\lambda}} + \mathcal{O}(1/x^2).
\]

Fix \( p(\infty) = 0 \).

Global \( su(2)_L \) charges \( J_L \) can be read off from monodromy at \( x = 0 \)

\[
g\omega(x)g^{-1} = I - x \frac{2\pi J_L}{\sqrt{\lambda}} + \mathcal{O}(x^2), \quad p(x) = 2\pi n_0 - x \frac{2\pi J_{L,3}}{\sqrt{\lambda}} + \mathcal{O}(x^2).
\]

Winding number \( n_0 \).
Local Charges

Poles at $x = \pm 1$: Can diagonalise Lax connection perturbatively

$$u(x, \sigma)(\partial_{\sigma} + a_{\sigma}(x, \sigma))u(x, \sigma)^{-1} = \partial_{\sigma} - i \sum_{k=-1}^{\infty} (x \mp 1)^k \delta Q_k^\pm(\sigma).$$

Leading charge density $\delta Q_{-1}^\pm(\sigma)$ related to current $j_\pm$

$$\delta Q_{-1}^\pm \simeq -\frac{i}{2} j_\pm \simeq \frac{1}{2}(E/\sqrt{\lambda}) \text{diag}(+1, -1).$$

Diagonalised current gives conserved local charges $Q_k^\pm$

$$p(x) = \sum_{k=-1}^{\infty} (x \mp 1)^k Q_k^\pm, \quad Q_k^\pm = \oint_0^{2\pi} \delta Q_{k,11}^\pm(\sigma).$$

Residue of $p$ at $x = \pm 1$ equals $Q_{-1}^\pm = \pi E/\sqrt{\lambda}$ by Virasoro constraint.
Analyticity

Monodromy $\omega(x)$ is analytic in $x$ except at $x = \pm 1$: $\omega(x) \sim \exp \frac{iQ_x}{x \mp 1}$.

Diagonalisation introduces new singularities $\{x_a^*\}$ (eigenvalue crossing)

One full turn around $x_a^*$ interchanges eigenvectors/values (labelling).

Generic behaviour at degenerate eigenvalues $e^{+ip(x_a^*)} = e^{-ip(x_a^*)}$:

$$e^{+ip(x)}, e^{-ip(x)} = e^{ip(x_a^*)} \left(1 \pm \alpha \sqrt{x - x_a^*} + \mathcal{O}(x - x_a^*)\right).$$
Algebraic Curve

Two eigenvalues $p_{1,2}(x)$ (defined modulo $2\pi$) of $\omega(x)$

$$\omega(x) \simeq \text{diag}(e^{ip_1(x)}, e^{ip_2(x)})$$

as one function $p$ on a Riemann surface $\mathbb{M}_2$ with two sheets

- Single poles at $x = \pm 1$.
- Branch points are square-root singularities $\sqrt{x - x^*_a}$.
- Function continued analytically across cuts on other sheet (modulo $2\pi$).
- Assume finitely many $(2A)$ branch points. Other solutions as $A \to \infty$.

Derivative $p'(x)$ defines algebraic curve $F(x, p') = P_2(x)p'^2 + P_0(x) = 0$.

String configuration $h(\tau, \sigma) \longrightarrow$ Algebraic curve $F(x, p') = 0$. 
Admissible Curves

Not all algebraic curves can arise from the sigma model.

\[ F(x, p') = P_2(x) p'^2 + P_1(x) p' + P_0(x). \]

Constraints:

- Sum of solutions vanishes \( p'_1 + p'_2 = 0 \).
- Asymptotics \( p'(x) \sim 1/x^2 \) at \( x = \infty \) and \( p'(x) \sim 1 \) at \( x = 0 \).
- Double poles in \( p'(x) \) at \( x = \pm 1 \). No residues.
- Physical branch points \( p'(x) \sim 1/\sqrt{x - x^*} \).
- No unphysical branch points \( p'(x) \sim \sqrt{x - x^\times} \).
- \( p(x) = \int_\infty^x p'(x') \, dx' \) must be single-valued (modulo \( 2\pi \)).

Count moduli of admissible curves.
Coefficients

Ansatz for derivative of quasi-momentum

\[ p'_{1,2}(x) = \pm \frac{c_{A+2} x^{A+2} + \ldots + c_0 x^0}{(x^2 - 1)^2 \sqrt{(x - x_1^*) \cdots (x - x_{2A}^*)}}. \]

The ansatz satisfies already:

- **3A + 3** free coefficients, \( c_0, \ldots c_{A+2}, x_1^*, \ldots, x_{2A}^* \).
- Asymptotics \( p'(x) \sim 1/x^2 \) at \( x = \infty \) and \( p'(x) \sim 1 \) at \( x = 0 \).
- Physical branch points at \( x = x_a^* \).
- No unphysical branch cuts: No square root in numerator.

**Further constraints:**

- Double poles at \( x = \pm 1 \) with equal coefficient: 1 constraint.
- No residue at \( x = \pm 1 \): 2 constraints.

Remaining coefficients: 3A.
Cycles

Single-valuedness: All closed cycles must be integer

\[ \oint dp \in 2\pi \mathbb{Z} \quad \text{as well as} \quad \int_{0}^{\infty} dp \in 2\pi \mathbb{Z} \quad \text{and} \quad \int_{\infty}^{\infty} dp \in 2\pi \mathbb{Z}. \]

Can arrange cuts \( C_a \) such that

\[ \oint A_a \ dp = 0, \quad \int_{B_a} \ dp = 2\pi n_a \]

with \( A, B \)-cycles defined as in

From \( A, B \)-cycles: \( A - 1 \) and \( A + 1 \) constraints. Remaining: \( A \).
String Moduli

Precisely \( A \) moduli remain. **Conclusions:**

One “mode number” \( n_a \in \mathbb{Z} \) and one “amplitude” \( K_a \in \mathbb{R} \) for each cut

\[
n_a = \frac{1}{2\pi} \int_{B_a} dp, \quad K_a = -\frac{1}{2\pi i} \oint_{A_a} \left( 1 - \frac{1}{x^2} \right) p(x) \, dx.
\]

One constraint among \( \{(n_a, K_a)\} \): “level matching”.

One additional parameter \( L = J_{L,3} \): “length”.

Solutions classified by \( \{(n_a, K_a)\} \) and \( L \) or if \( n_a \neq n_b \)

**Solutions classified by \( \{K_n\} \) and \( L \).**

Similar to gauge theory:

- Need to rescale \( u \sim x/L \) to move poles at \( x = \pm 1 \) to \( u = 0 \).

Agreement up to two loops. Clearly different at three loops. [Kazakov, Marshakov, Minahan, Zarembo]
Order of Limits

\[ E(\lambda', J) = \left(1 + \frac{c}{\lambda' J^2}\right)^{-J} \rightarrow E(\lambda') = 1 \]

\[ D(\lambda, J) = \frac{\lambda^J}{(\lambda + c)^J} \]

\[ D(\lambda, J) \xrightarrow{J \rightarrow \infty} E(\lambda') \]

\[ D(\ell,J) \xrightarrow{J \rightarrow \infty} D_\ell \neq E_\ell \]

\[ D_{\ell < J}(J) = 0 \rightarrow D_\ell = 0 \]

- Cannot compare in perturbation theory.
- Spinning strings and near-BMN proposals do not quite work.
- Integrability might lead to the exact solution of either theory.

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Wrapping Interactions

At higher loop orders there is an additional type of planar interaction

- Starts contributing at $O(\lambda^L)$.
  - May nevertheless repair discrepancy (see example).
- Asymptotic Bethe ansatz does not incorporate wrappings.
- Algebraic construction apparently not useful here.
Conclusions

⋆ Strings on $AdS_5 \times S^5$
- Classical Spinning Strings Solutions.
- Integrability, Lax connection, Monodromy.
- Construction of an algebraic curve and classification.

⋆ $\mathcal{N} = 4$ Gauge Theory
- Construction of dilatation operator.
- Coherent states in the thermodynamic limit.
- One-loop/higher-loop integrability, Bethe Ansätze.
- Algebraic curve in the thermodynamic limit.

⋆ Comparison for AdS/CFT, BMN & Spinning Strings
- Direct, Hamiltonian, Integrable Structures.
- Agreement up to two loops.
- Problems at three loops. BMN & Spinning Strings does not work.
Open Questions

• Will this talk ever end?

★ Near-BMN & Spinning Strings
• Understand the problems at three loops. Wrappings?
• Better question: Why did it work up to two loops in the first place?

★ Various Directions
• Theories with less supersymmetry. Phenomenology?
• Can we use integrability for non-planar/stringy interactions?

★ Exact Bethe ansätze
• Gauge theory at higher loops: $\lambda$-dependence.
  Inspiration from classical string theory.
• Quantum strings: Discretised curve, Bethe roots.
  Inspiration from one-loop gauge theory.
• Compare & see whether AdS/CFT really works.