Large Mass and Chemical Potential Model
- a Laboratory for QCD ? -

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Features:

- **A Model based on the Hopping Parameter Expansion**:
  - double limit $\kappa \to 0$ and $\mu \to \infty$ with $\zeta = \kappa e^{\mu}$: fixed ("Quenched" QCD with Non-Zero Baryon Density)
  - corrections to order κ^2 ($1/M^2$ corrections to static charges)
- **"Laboratory" for QCD at Large Matter Density**:
  - as an approximation near the "quenched" limit
  - as a model by itself at any μ, κ
- **Still acknowledges the Sign Problem**:
  - but refined (local) algorithms with reweighting can converge in reasonable computer time
  - allows simulations across large μ "transition" at $T \sim 0$

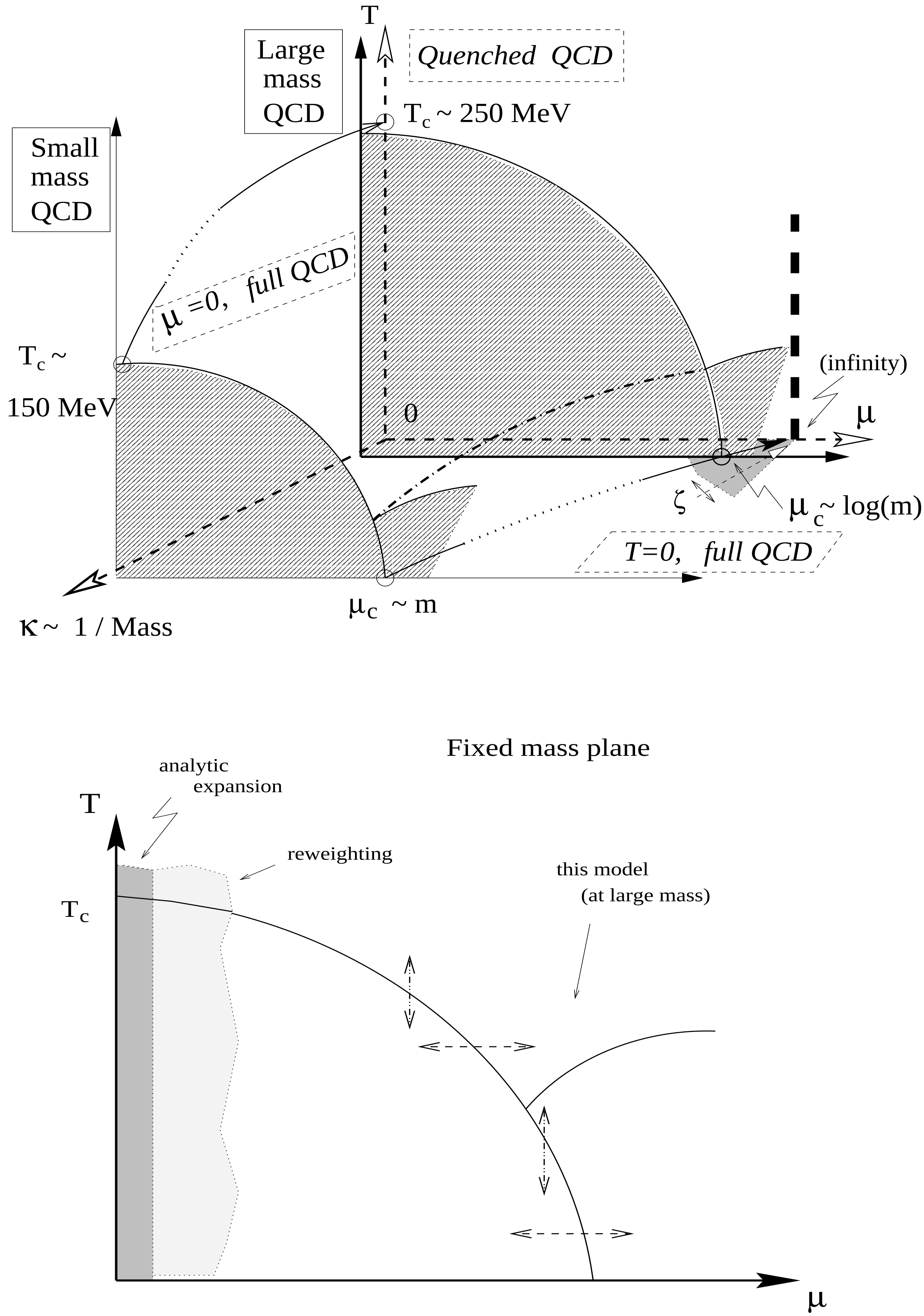


Figure 1: Tentative phase diagram and where we want to look.

QCD grand canonical partition function

(f : flavour index, U : links, T : lattice translations):

$$\mathcal{Z}(\beta,\kappa,\mu) = \int [DU]\,\mathrm{e}^{-S_G(\beta,\{U\})}\mathcal{Z}_F(\kappa,\mu,\{U\}),$$

$$\mathcal{Z}_F(\kappa,\mu,\{U\}) = \mathrm{Det}\,W(\kappa,\mu,\{U\}), \tag{1}$$

$$\begin{aligned} W_{ff'} = & \delta_{ff'}[1-\kappa_f \sum_{i=1}^{3}(\Gamma_{+i}\,U_i\,T_i + \Gamma_{-i}\,T_i^*\,U_i^*) \\ & -\kappa_f\,\left(\mathrm{e}^{\mu_f}\,\Gamma_{+4}\,U_4\,T_4 + \mathrm{e}^{-\mu_f}\,\Gamma_{-4}\,T_4^*\,U_4^*\right)], \end{aligned} \tag{2}$$

$$\Gamma_{\pm\mu} = 1 \pm \gamma_\mu, \quad \kappa = \frac{1}{2(M+3+\cosh\mu)} = \frac{1}{2(M_0+4)}$$

(M : "bare mass", M_0 : bare mass at $\mu = 0$. The exponential prescription for μ ensures cancelling of divergences in small a limit *[Hasenfratz and Karsch, Ph.Lett. 125 B (1983) 308].*)

Hopping parameter expansion $\longrightarrow$ expansion in closed loops:

$$\begin{aligned} \mathrm{Det}\,W &= \exp(\mathrm{Tr}\,\ln W) = \exp\left[-\sum_{l=1}^{\infty}\sum_{\{\mathcal{C}_l\}}\sum_{s=1}^{\infty}\frac{\left(\kappa_f^l g_{\mathcal{C}_l}^f\right)^s}{s}\,\mathrm{Tr}_{\,\mathrm{D,C}}\mathcal{L}_{\mathcal{C}_l}^s\right] \\ &= \prod_{l=1}^{\infty}\prod_{\{\mathcal{C}_l\}}\prod_{f}\mathrm{Det}_{\,\mathrm{Dirac,Color}}\left(1\;-\;(\kappa_f)^l g_{\mathcal{C}_l}^f\mathcal{L}_{\mathcal{C}_l}\right) \end{aligned} \tag{3}$$

$\mathcal{C}_l$ are distinguishable, non-exactly-self-repeating closed paths of length l, s is the number of times a loop $\mathcal{L}_{\mathcal{C}_l}$ covers $\mathcal{C}_l$,

$$\begin{aligned} g_{\mathcal{C}_l}^f &= \left(\epsilon\,\mathrm{e}^{\pm N_\tau \mu_f}\right)^r \text{ if } \mathcal{C}_l = \text{“}\textit{Polyakov r–path}\text{”}, \\ &= 1 \text{ otherwise.} \end{aligned} \tag{4}$$

A "*Polyakov r–path*" closes over the lattice in the ± 4 direction with winding number r and periodic(antiperiodic) b.c. [$\epsilon = +1(-1)$].

Quenched limit at $\mu > 0$

$$\kappa \to 0,\ \mu \to \infty, \quad \kappa\, \mathrm{e}^{\mu} \equiv \zeta :\ \text{fixed} \tag{5}$$

$$\mathcal{Z}_F^{[0]}(C, \{U\}) = \exp\left[- \sum_{\{\vec{x}\}} \sum_{s=1}^{\infty} \frac{(\epsilon C)^s}{s}\ \mathrm{Tr}_{\,\mathrm{C}} (\mathcal{P}_{\vec{x}})^s \right]$$

$$= \prod_{\{\vec{x}\}} \mathrm{Det}_{\,\mathrm{C}}\, (1\ -\ \epsilon\, C \mathcal{P}_{\vec{x}})^2, \quad C = (2\,\zeta)^{N_\tau} \tag{6}$$

[Bender et al, Nucl. Phys. B (Proc.Suppl.) 26 (1992) 323; Engels et al, Nucl. Phys. B 558 (1999) 307]

Next order corrections

$$\mathcal{Z}_F^{[2]}(\kappa, \mu, \{U\}) = \exp\Bigg\{ -2 \sum_{\{\vec{x}\}} \sum_{s=1}^{\infty} \frac{(\epsilon\, C)^s}{s} \times$$

$$\mathrm{Tr}_{\,\mathrm{C}} \left[(\mathcal{P}_{\vec{x}})^s + \kappa^2 \sum_{r,q,i,t,t'} (\epsilon\, C)^{s(r-1)} (\mathcal{P}^{r,q}_{\vec{x},i,t,t'})^s \right] \Bigg\}$$

$$= \mathcal{Z}_F^{[0]}(C, \{U\}) \times \prod_{\vec{x},r,q,i,t,t'} \mathrm{Det}_{\,\mathrm{C}} \left(1 - (\epsilon\, C)^r\, \kappa^2\, \mathcal{P}^{r,q}_{\vec{x},i,t,t'} \right)^2 . \tag{7}$$

For easy bookkeeping we use the temporal gauge

$U_{n,4} = 1$, except for $U_{(\vec{x},n_4=N_\tau),4} \equiv V_{\vec{x}}$: free,

then

$$\mathcal{P}^{r,q}_{\vec{x},i,t,t'} = (V_{\vec{x}})^{r-q} U_{(\vec{x},t),i} (V_{\vec{x}+\hat{\imath}})^q U^*_{(\vec{x},t'),i} \tag{8}$$

with $r > q \geq 0$, $i = \pm 1, \pm 2, \pm 3$, $1 \leq t \leq t' \leq N_\tau$ ($t < t'$ for $q = 0$).

[Kaczmarek et al, Nucl. Phys. B (Proc.Suppl.) 26 (2002) 323]

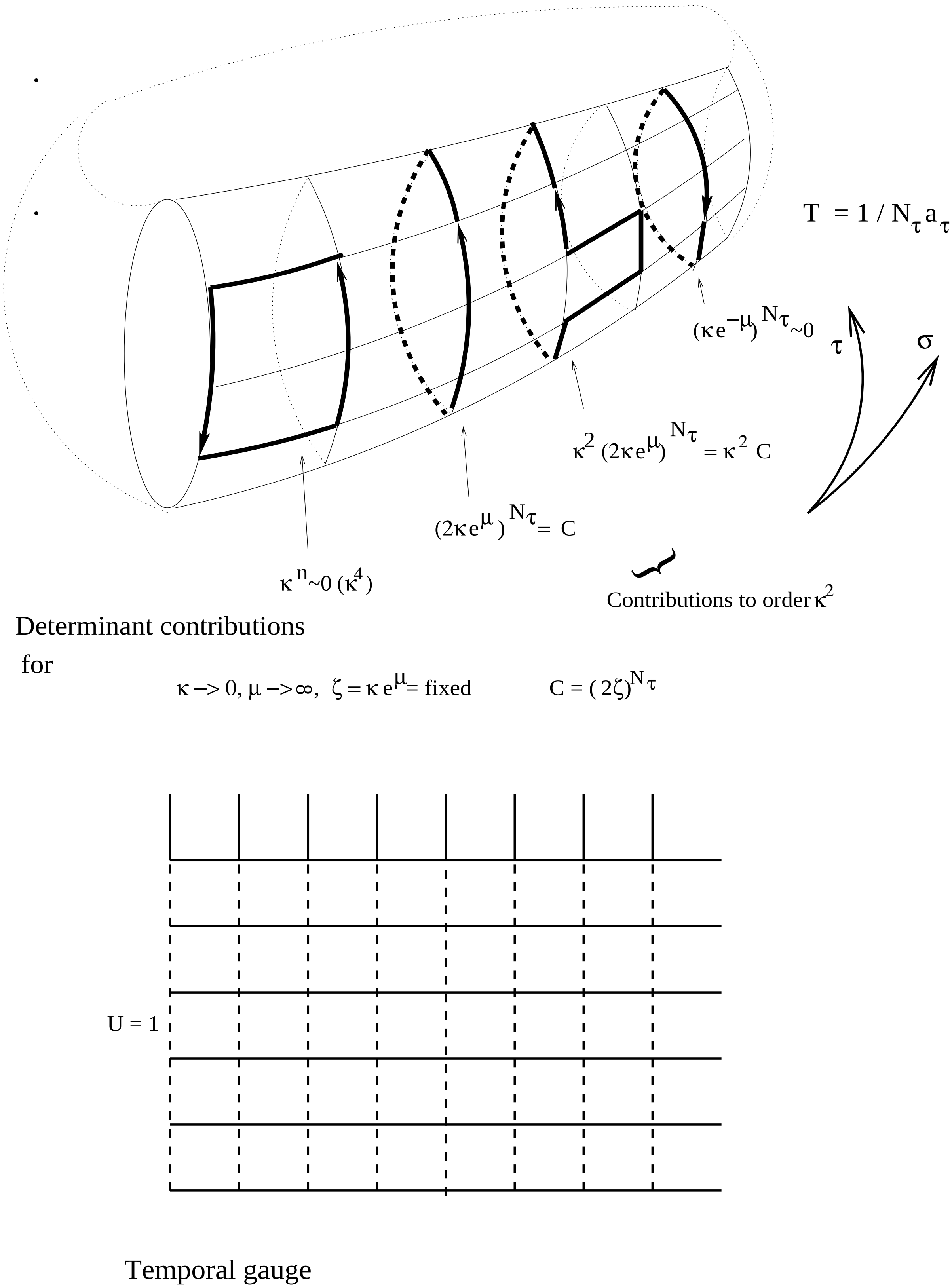


Figure 2: Periodic lattice, loops, temporal gauge.

Measurements:

Bulk quantities and correlators, under variation of μ, κ, T, to check the properties of the different phases for small T, large μ

- baryon number density n_B, "mobility" (measures the κ^2 corrections), Polyakov loop $P = \frac{1}{3}\mathrm{Tr}\langle \frac{1}{N_\sigma^3}\sum_{\vec{x}} TrP_{\vec{x}}\rangle$:

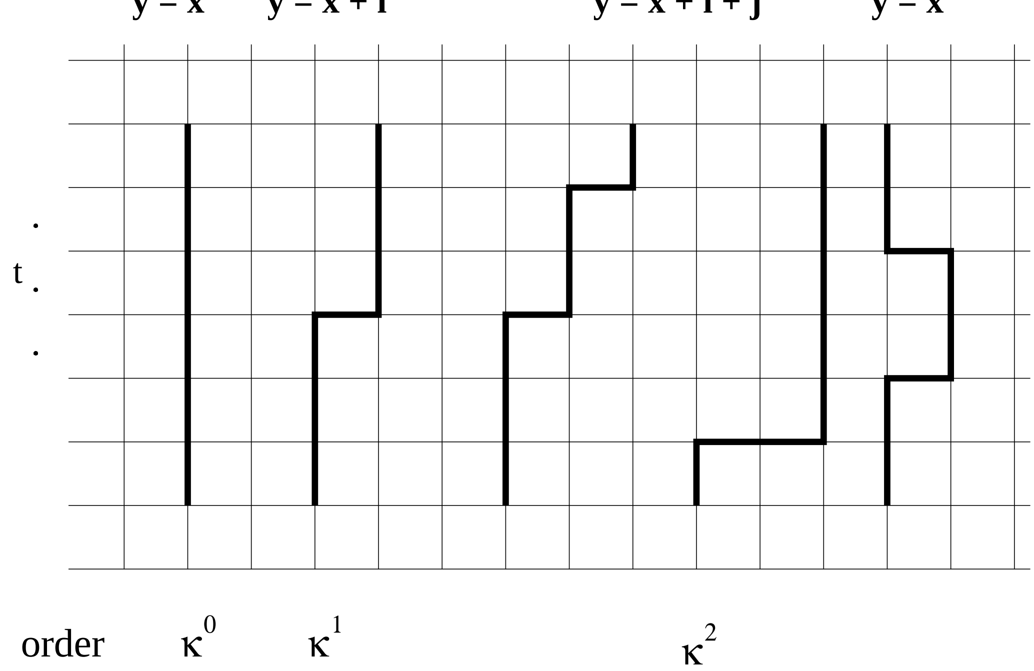
$$\begin{aligned}
\frac{n_B}{T^3} &= \frac{N_\tau^3}{3N_\sigma^3}\hat{n}, \quad \hat{n} = \hat{n}_0 + \hat{n}_1, \quad mob = \frac{\hat{n}_1}{\hat{n}_0 + \hat{n}_1} \qquad (9)\\
\hat{n}_0 &= \langle \frac{\partial}{\partial\mu}\mathcal{Z}_F^{[0]}\rangle \simeq 2C\langle\sum_{\vec{x}} TrP_{\vec{x}}\rangle,\\
\hat{n}_1 &= \langle \frac{\partial}{\partial\mu}\left(\frac{\mathcal{Z}_F^{[2]}}{\mathcal{Z}_F^{[0]}}\right)\rangle \simeq 2C\kappa^2\langle \sum_{\vec{x};\pm i,\,(t,t')} \mathcal{P}_{\vec{x},\,i,\,(t,t')}\rangle
\end{aligned}$$

- Spatial/temporal plaquettes $P_{\sigma\sigma}$, $P_{\sigma\tau}$, topol. susceptibility χ^2

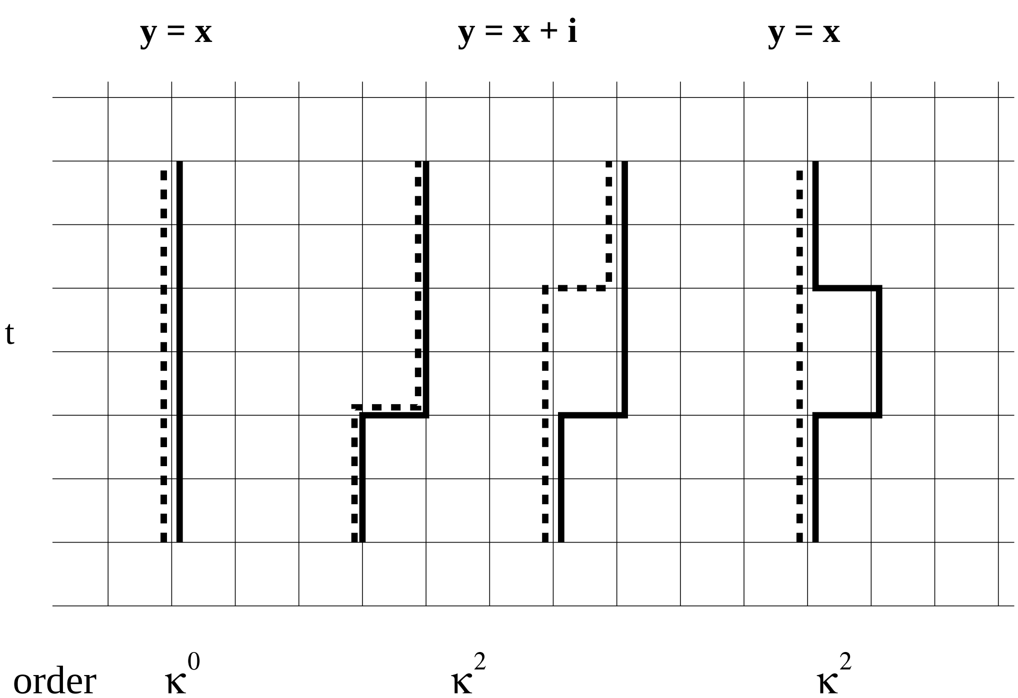
- quark and di-quark correlators (fixed gauge, maximal), e.g.:

$$\begin{aligned}
C_{(qq)}(\tau) &= (\delta_i^a\,\delta_j^b + \xi\,\delta_j^a\,\delta_i^b)\,(\delta_k^c\,\delta_l^d + \xi\,\delta_l^c\,\delta_k^d)\\
&\times \sum_{\vec{x},\vec{y},t}\langle[\psi_i^a\,\mathcal{C}\,\psi_j^b\,(\vec{x},t)]\,[\psi_l^c\,\mathcal{C}\,\psi_k^d\,(\vec{y},t+\tau)]^*\rangle\\
&= (\delta_i^a\,\delta_j^b + \xi\,\delta_j^a\,\delta_i^b)\,(\delta_k^c\,\delta_l^d + \xi\,\delta_l^c\,\delta_k^d)\\
&\times \sum_{\vec{x},\vec{y},t}\Big\{W^{-1}_{ik;ac}(\vec{x},t;\vec{y},t+\tau)\,\mathcal{C}\,W^{-1,T}_{jl;bd}(\vec{x},t;\vec{y},t+\tau)\,\mathcal{C}^T\\
&- W^{-1}_{il;ad}(\vec{x},t;\vec{y},t+\tau)\,\mathcal{C}^T\,W^{-1,T}_{jk;bc}(\vec{x},t;\vec{y},t+\tau)\,\mathcal{C}\Big\} \qquad (10)
\end{aligned}$$

Here W^{-1} is the quark propagator, $\mathcal{C}$ the charge conjugation matrix $\{a,\ldots;i,\ldots\}$ the colour and flavour indices, respectively, and we have dropped the (summed over) Dirac indices. ξ is a parameter allowing various combinations of colour-flavour "locking" (see, e.g. *[Alford, Rajagopal and Wilczek, Nucl. Phys. B537 (1999) 443]*).



Contributions to the quark propagator to order $\kappa^2 \zeta^t$



Contributions to the di-quark propagator to order $\kappa^2 \zeta^{2t}$

Figure 3: Paths contributing to quark and diquark "propagators".

Calculations:

- algorithm: Wilson Plaquette action, reweighting procedure

  updating with Boltzmann factor (local, vectorizable)

$$\prod_{Plaq} \mathrm{e}^{\frac{\beta}{3}\,\mathrm{Tr}\,Plaq} \prod_{\vec{x}} \exp\left\{ 2\,C\,\mathcal{R}e\mathrm{Tr}_{\,\mathrm{C}} \left[\mathcal{P}_{\vec{x}} + \kappa^2 \sum_{r,q,i,t,t'} \mathcal{P}^{r,q}_{\vec{x},i,t,t'} \right] \right\} \quad (11)$$

  reweighting (global, vectorizable) with

$$\prod_{\vec{x}} \exp\left\{ -\,2\,C\,\mathcal{R}e\mathrm{Tr}_{\,\mathrm{C}} \left[\mathcal{P}_{\vec{x}} + \kappa^2 \sum_{r,q,i,t,t'} \mathcal{P}^{r,q}_{\vec{x},i,t,t'} \right] \right\} \mathcal{Z}_F^{[2]}(\{U\}) \quad (12)$$

  note:
  - the updating already takes into account $\mu \neq 0$ effetcs
  - further tuning possibilities: modify the shifted factor, consider local reweighting (in agreement with detailed balance etc), use center transformations in confining region

- simulation:

  "low" temperature lattices 6^4, $6^3 \times 8$ at $\beta = 5.55$ and 5.6

  large μ, rather "small" bare mass $M_0 = 0.167$ ($\kappa = 0.12$)

  check behaviour of bulk properties around the prospective "transition" line

  calculate correlators (*in work*).

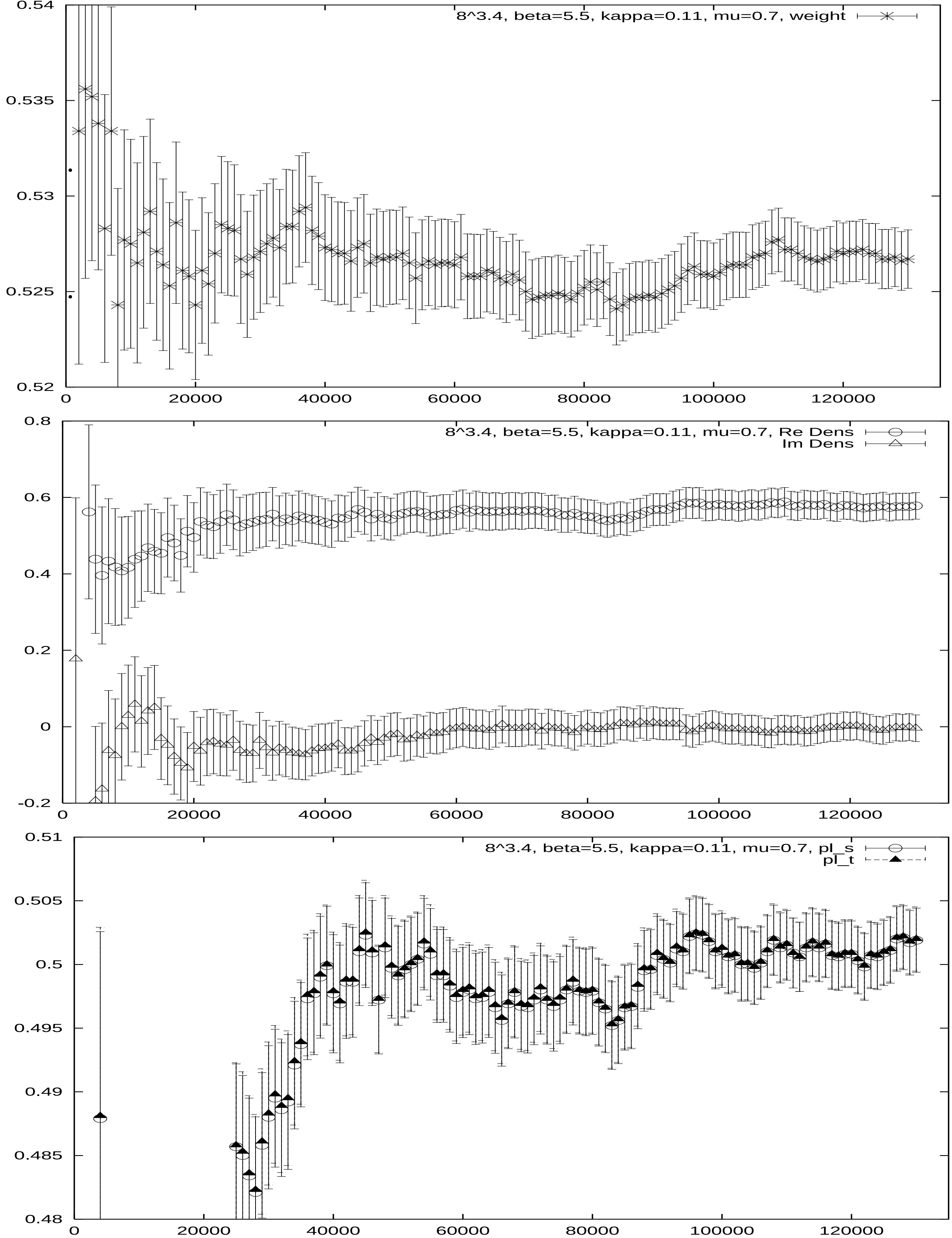
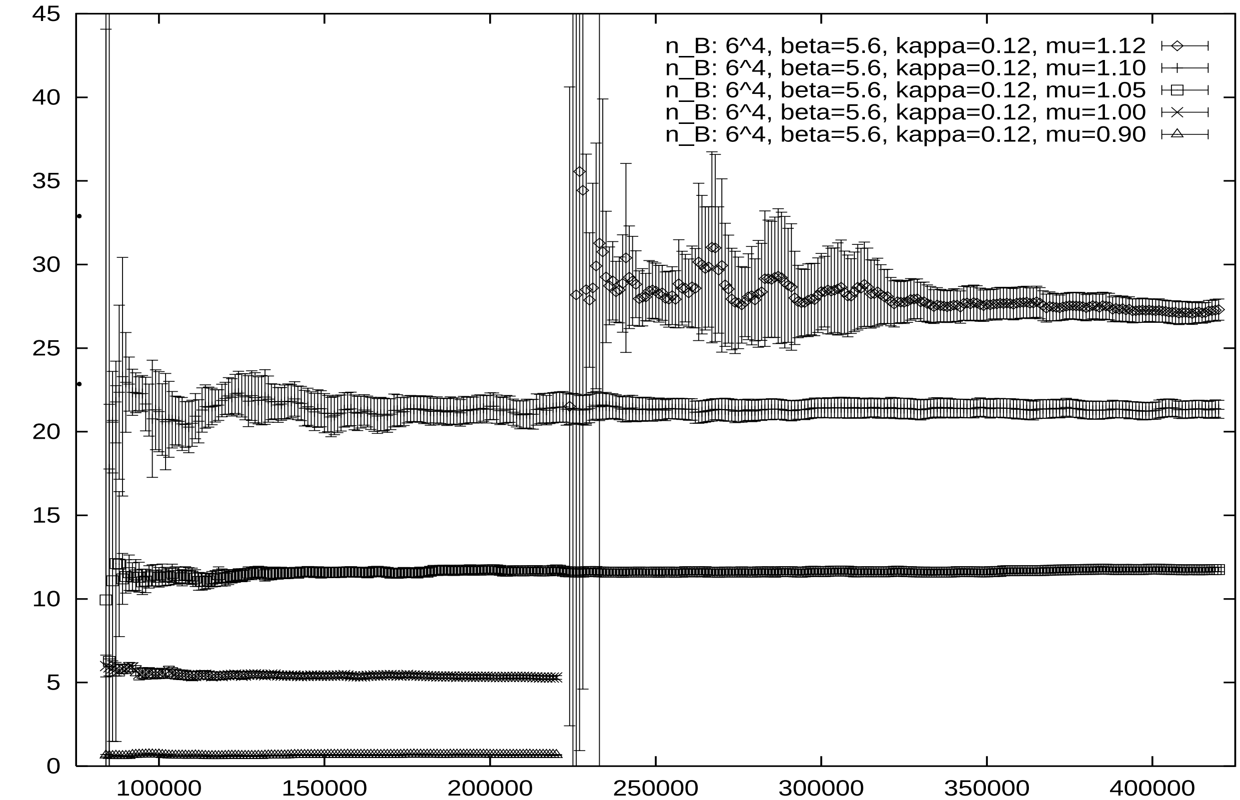


Figure 4: Convergence of weight, n_B and τ/σ plaq. at small μ.



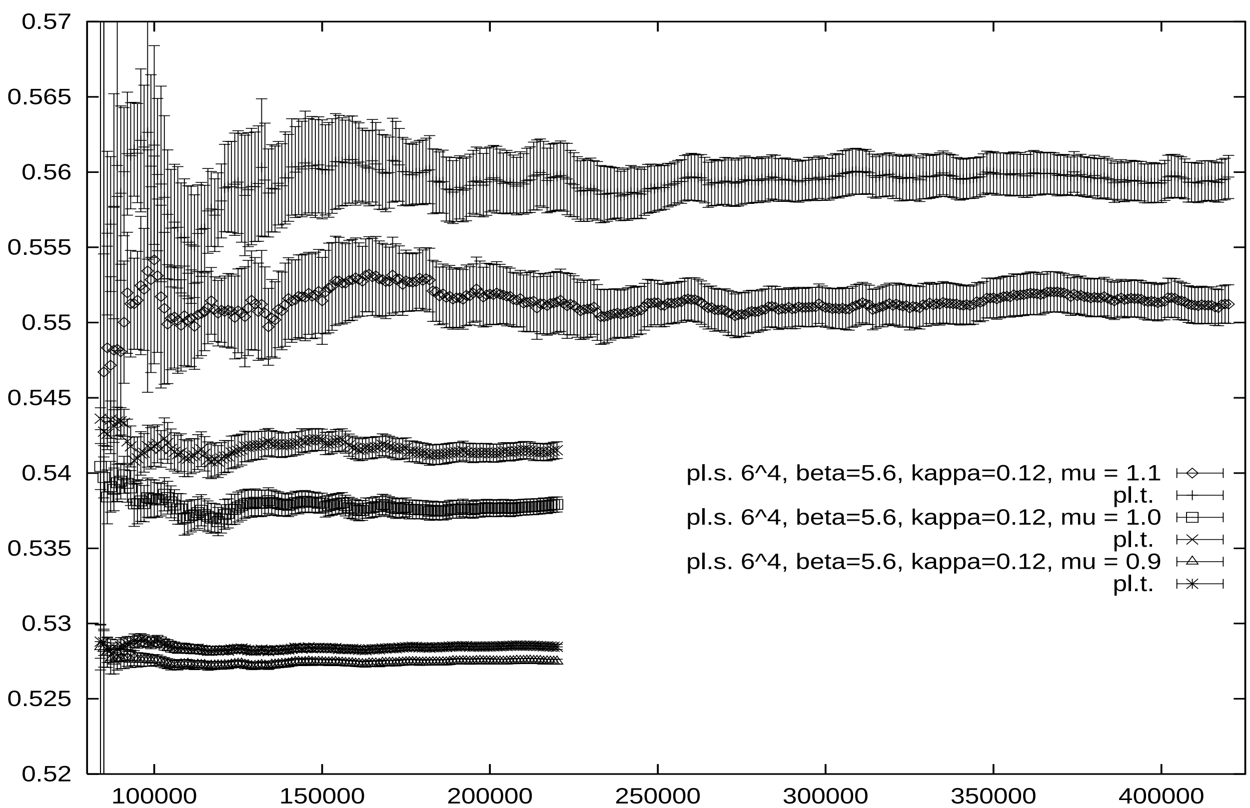


Figure 5: Convergence of n_B and τ/σ plaquettes at large μ.

Discussion, Outlook:

General behaviour:

- algorithm works reasonably well over a large range of parameters (from $n_B < 0.01\, n_B(sat)$ to $n_B > 0.3\, n_B(sat)$), even at small T

- can be tuned to work also beyond this interval

- the model permits to vary μ, κ, T as independent parameters

- it is reasonably cheap to measure various correlations

First results:

- strong variation of baryon density around $\mu \sim 1.15$ on the 6^4 lattice, the more pronounced the smaller the temperature (smaller β), compare also with the higher T lattice $6^3 \times 4$ at the same β

- accompanying signal in Pol. loop, temporal - spatial plaquette and χ^2

$\longrightarrow$ cross over (or transition) to a high density phase above $\sim 20\%\, n_B(sat)$

- effect of κ^2 correction ("mobility") increases with decreasing T (from about 0.25 at $N_\tau = 6$ to about 0.13 at $N_\tau = 4$)

Significance:

- incorporates a number of features of QCD at large μ, large M

- can be used as a model for itself, or as an approximation to QCD
- in the latter case the approximation concerns the neighborhood of

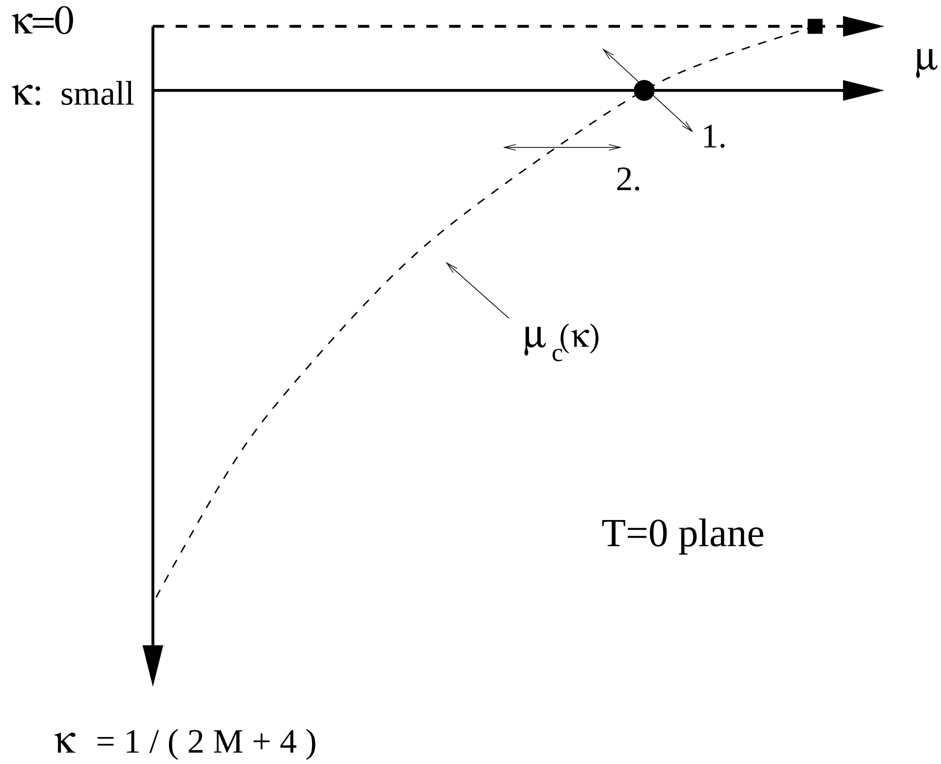
a non-physical point ($M \to \infty$), therefore it is difficult to introduce physical units (but see below).

Developments:

- precise calculations on 6^4 and $6^3 \times 8$ lattices in the interesting region appear straightforward (for bulk observables, densities, correlators, condensates)

- it seems difficult to go to larger lattices, however improving may help.
- one could use this model to provide a heavy, dense, charged background for light quarks correlators and calculate light hadron spectra, etc. This would also fix a corresponding scale.

The calculations are done on the VPP5000 computer at University of Karlsruhe and the Forschungszentrum Karlsruhe.



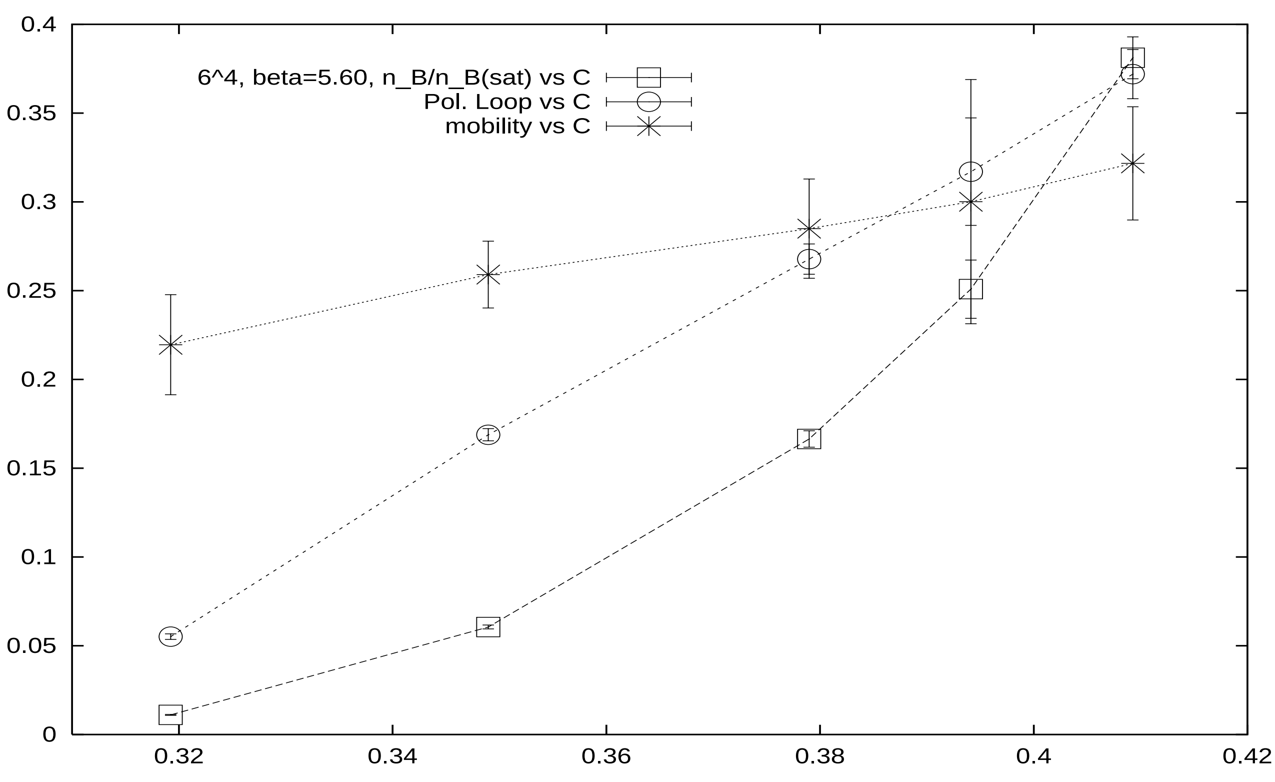


Figure 6: Crossing the "transition" at $T \sim 0$. $n_B/n_B(sat)$, Pol. loop and mobility vs ζ, path 1 above ($n_B(sat)/T^3 = 216$).

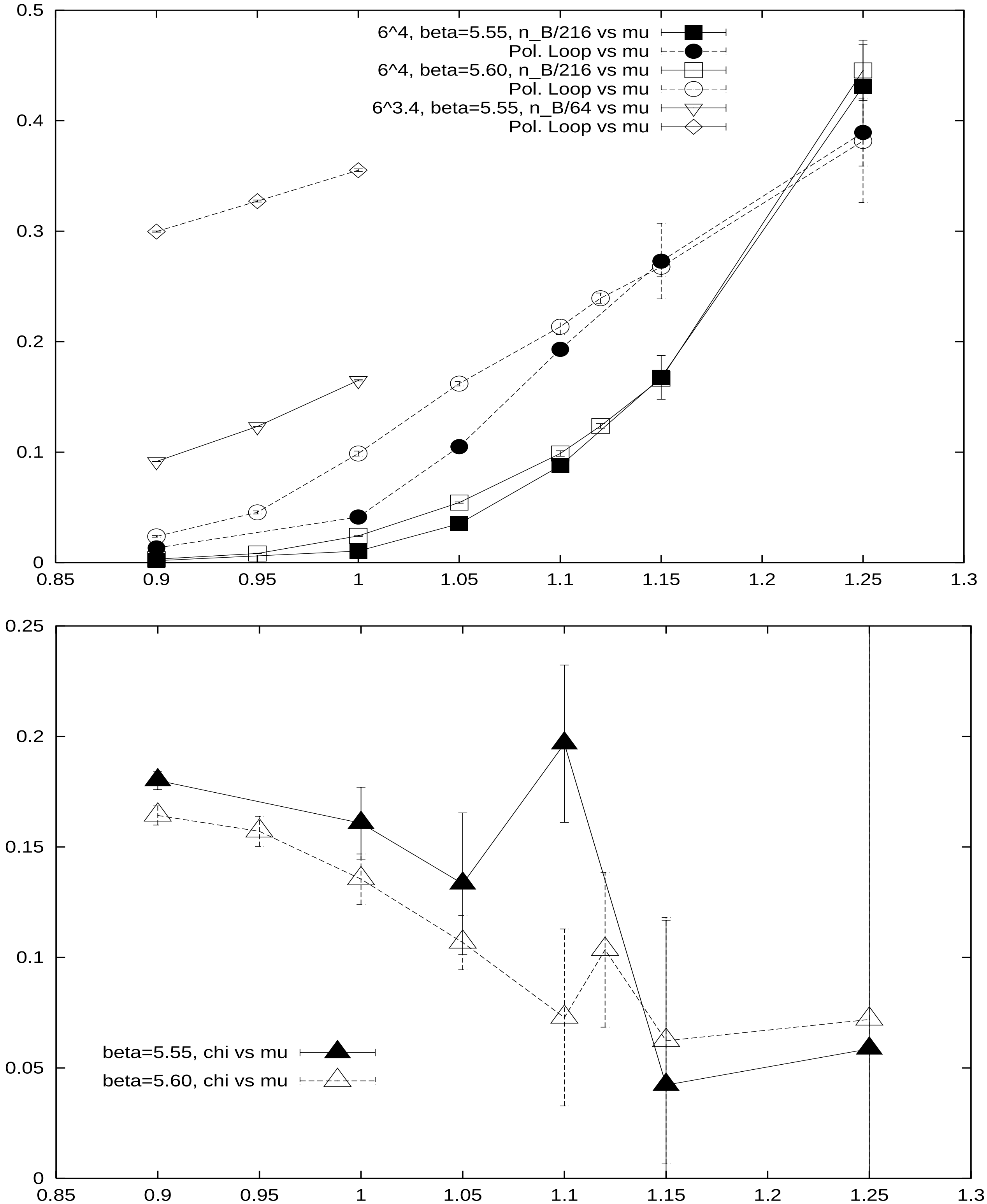


Figure 7: Density $n_B/n_B(sat)$ ($n_B(sat)/T^3 = N_\tau^3$) and Pol. loop (upper plot), and χ^2 (lower plot) vs μ at $\kappa = 0.12$ (path 2).