Anomaly at finite density and chiral fermions on the lattice

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Outline

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3. Anomaly at $\mu \neq 0$: Non-Perturbative
4. Introducing chemical potential for chiral fermions on the lattice
5. Conclusions

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2. Anomaly at non-zero density: Perturbative

3. Anomaly at $\mu \neq 0$: Non-Perturbative

4. Introducing chemical potential for chiral fermions on the lattice

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Introduction

- Anomalies arise when the classical symmetries are broken at the quantum level. One such example is the axial-vector current anomaly in QED.
- $U_A(1)$ in QCD is anomalous. The non zero mass of η' due to this $U(1)$ axial anomaly[Witten(79)].
- In recent years, QCD phase diagram has been an important subject of study. For two light flavours we expect a critical point at T_c and small finite density.
- The size of the anomaly term affects the order of chiral phase transition. For $N_f = 2$ the chiral phase transition is of second order if anomaly term is sizable at T_c [Pisarski & Wilczek(84)] .

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- We calculate the continuum anomaly equation at finite density both perturbatively from the triangle diagram as well as non-perturbatively from Fujikawa's method.
- There has been recent efforts to incorporate chemical potential in chiral fermion operators on the lattice. We discuss some of the properties of such operators like the anomaly and discuss the consequences.

Outline

Triangle diagram at $\mu \neq 0$

- We compute the axial anomaly perturbatively at $\mu \neq 0$ in the Euclidean spacetime$\longrightarrow$ easily extendable to weak coupling lattice QCD.
- The partition function in QCD with fermion fields ψ coupled to SU(3) gauge fields $A_\nu = A_\nu^a T^a$, $\nu = 1 - 4$ is,

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\nu]e^{-\int d^4x \ \bar{\psi}(\not{D}_\nu + m - \mu\gamma_4)\psi + S_{YM}}$$

  with $D_\nu = \partial_\nu - igA_\nu^a T_a$ and S_{YM} is the free Yang-Mills action with appropriate gauge-fixing. The matrices $\gamma_\nu, \gamma_5 \rightarrow$ hermitian.
- Under chiral transformation of the massless fermion fields the action remains invariant with a classically conserved chiral current

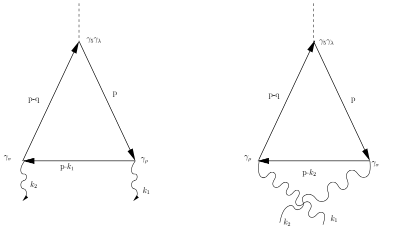
$$\partial_\nu j_5^\nu(x) = 0.$$

Triangle diagram at $\mu \neq 0$

- We calculate the quantum effects to the flavour singlet anomaly by measuring $\langle \partial_\mu j_{\mu 5} \rangle$.

$$\langle \partial_\mu j_{\mu,5} \rangle = \frac{1}{Z} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \; \partial_\mu j_{\mu,5} \; \mathrm{e}^{-\mathrm{S}},$$
$$= -\frac{1}{2} \int d^4x_1 d^4x_2 \partial_\lambda \langle T\{j_{5,\lambda}(x) j_\rho(x_1) j_\sigma(x_2)\}\rangle A^\rho(x_1) A^\sigma(x_2) \; .$$

- Now the expectation value of the divergence amounts to the calculation of the triangle diagram [Adler, Bell, Jackiw(69)].



- We calculate the triangle diagrams in the momentum space, $\langle \partial_\mu j_{\mu,5} \rangle \rightarrow q_\lambda \Delta^{\lambda\rho\sigma}(k_1, k_2, q)$. The propagators at finite density changes from $p_4 \rightarrow p_4 - i\mu$.

Triangle diagram at $\mu \neq 0$

- At non-zero μ the $q_\lambda \Delta^{\lambda\rho\sigma}(k_1, k_2, q)$ has terms like $(p_4 - i\mu)^2 + \vec{p}^2$ in the denominator and terms proportional to μ^0 , μ and μ^2 in the numerator.
- In the numerator, the term $\propto \mu^2$ vanishes because $Tr[\gamma^5\gamma^4\gamma^\sigma\gamma^4\gamma^\rho] \sim \epsilon^{4\sigma 4\rho} = 0$.
- The integrals are linearly divergent and hence must be regulated by introducing a cut-off scale, Λ.
- Gauge invariance has to be maintained by demanding that $k_{1\rho}\Delta^{\lambda\rho\sigma}(k_1, k_2) = k_{2\sigma}\Delta^{\lambda\rho\sigma}(k_1, k_2) = 0 \Rightarrow \Lambda \to \infty$.
- The term $\propto \mu$, goes as $\frac{4i\mu}{\Lambda}\epsilon^{4\sigma\beta\rho}k_{1\mu}k_{2\beta}$ and hence vanishes as $\Lambda \to \infty$.
- The anomaly equation remains same as in the absence of μ as, $q_\lambda \Delta^{\lambda\rho\sigma} = -\mathrm{tr}[T^a T^b]\frac{ig^2}{2\pi^2}\epsilon^{\alpha\beta\sigma\rho}k_{1\alpha}k_{2\beta}$.

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- Can be generalized to finite temperature, the finite $T - \mu$ part are modulated by the Fermi-Dirac distribution function, $Lt_{|\vec{p}|\to\infty}\frac{4\pi|\vec{p}|}{(2\pi)^3}\left[(\vec{k}_1 \cdot \vec{p})f(|\vec{p}|)\left(\frac{1}{e^{\beta(|\vec{p}|-\mu)}+1} + \frac{1}{e^{\beta(|\vec{p}|+\mu)}+1}\right) + \{\rho, k_1 \leftrightarrow \sigma, k_2\}\right]$ $\longrightarrow 0$.

Outline

Review of Fujikawa's method at $T = 0, \ \mu = 0$

- Under chiral transformation, the fermion measure changes by a Jacobian factor $\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi Det|\frac{\partial(\bar{\psi}',\psi')}{\partial(\bar{\psi},\psi)}| = \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathrm{e}^{-2\mathrm{i}\int \mathrm{d}^4\mathrm{x}\ \alpha(\mathrm{x})\mathrm{Tr}\gamma_5}$.
- $\not{D}$ is anti-Hermitian in this spacetime. The chiral Jacobian is evaluated in the eigenspace of $\not{D}$ represented by ϕ_n satisfying $\not{D}\phi_n = \lambda_n\phi_n$.
- Since $\{\gamma_5, \not{D}\} = 0$, for every eigen vector ϕ_n with eigenvalue λ_n there is a corresponding $\gamma_5\phi_n$ with e.v. $-\lambda_n$ hence the trace of γ_5 vanishes for non-zero eigenvalues of fermion operator.
- The only finite contribution comes from the zero modes of the Dirac operator. Thus,

$$\mathrm{Tr}\gamma_5 = \sum_n \phi_n^\dagger\gamma_5\phi_n = n_+ - n_-$$

with $n_\pm$ being the no. of left(right-handed) zero modes.

Anomaly at $\mu \neq 0$ in the continuum

- Fujikawa's analysis can be extended to finite density by noting that $S(\mu) \rightarrow S(\mu = 0) - \mu \int d^4x\ \bar{\psi}\gamma_4\psi$.
- The fermion operator in this case $\rightarrow \not{D} - \mu\gamma_4$. It still anti-commutes with $\gamma_5 \Rightarrow$ the action is still invariant.
- The measure has to be evaluated with some care $\rightarrow$ to be calculated in the eigenspace of $\not{D}(\mu)$.
- $\not{D}(\mu)$ is no longer Hermitian yet diagonalizable. Evident in a new set of vectors obtained from the $\mu = 0$ basis vectors by a non-unitary transformation

$$\zeta_m(\mathbf{x}, \tau) = \mathrm{e}^{\mu\tau}\phi_{\mathrm{m}}(\mathbf{x}, \tau) \quad , \quad \upsilon^\dagger_{\mathrm{m}}(\mathbf{x}, \tau) = \phi^\dagger_{\mathrm{m}}(\mathbf{x}, \tau)\mathrm{e}^{-\mu\tau} \ .$$

Anomaly at $\mu \neq 0$ in the continuum

- ζ_m and $\upsilon_m^\dagger$ are the eigenvectors of $\not{D}(\mu)$ and $\not{D}^\dagger(\mu)$ with (purely imaginary) eigenvalues λ_m , λ_m^*,

$$\not{D}(\mu)\zeta_m = \lambda_m \zeta_m \ , \ \upsilon_m^\dagger \not{D}^\dagger(\mu) = -\lambda_m \upsilon_m^\dagger .$$

- The sets of vectors $\{\zeta\}$ and $\{\upsilon\}$ are in one-to-one correspondence with the complete set $\{\phi\}$ as these follow
  a)completeness
  b)normality.
- For non-zero eigenvalues, for each eigenvector ζ_m has corresponding $\gamma_5\zeta_m$, hence $\mathrm{Tr}\gamma_5 = 0$.
- In the space of the zero modes of $\not{D}(\mu)$, given by ζ_n and $\upsilon_n^\dagger$, the chiral Jacobian

$$Tr\gamma_5 = \sum_n \upsilon_n^\dagger \gamma_5 \zeta_n = \sum_n \phi_n^\dagger \mathrm{e}^{-\mu\tau} \gamma_5 \mathrm{e}^{\mu\tau} \phi_\mathrm{n} = \mathrm{n}_+ - \mathrm{n}_- .$$

Anomaly at $\mu \neq 0$ in the continuum

- Most general method of introducing μ, by the (unphysical) non-unitary transformation of the fermion fields in the QCD action:

$$\psi^{'}(\mathbf{x},\tau) = \mathrm{e}^{-\mu\tau}\psi(\mathbf{x},\tau) \quad , \quad \bar{\psi}^{'}(\mathbf{x},\tau) = \bar{\psi}(\mathbf{x},\tau)\mathrm{e}^{\mu\tau} \ .$$

- Such transformation changes the $\mathcal{L}(\mu = 0) \rightarrow \mathcal{L}(\mu)$.
- Commutes with flavour and chiral transformations, leaving the spectrum unchanged $\rightarrow$ Preserves the anomaly relation too, as derived in the previous slide.

Outline

Chiral fermions on the lattice

- Having exact chiral symmetry on lattice is desirable for the study of chiral symmetry restoration in QCD.
- The Domain wall fermions [Kaplan(92)] with $N_5 = \infty$ and the Overlap fermions[Narayanan & Neuberger(95), Neuberger(98), Narayanan(98)] have exact chiral symmetry on the lattice. Example: The Overlap operator for $N_f = 1$ is,

$$aD_{ov} = 1 + \gamma_5 \epsilon(\gamma_5 D_W(-M)) \;,\; \epsilon(\gamma_5 D_W) = \frac{D_W}{\sqrt{D_W D_W^\dagger}} \;, 0 < M < 2.$$

- These operators satisfy the Ginsparg-Wilson(GW) relation $\{\gamma_5, D\} = aD\gamma_5 D$.
- These actions are invariant under infinitesimal chiral transformations [Luscher(98)] on the lattice:

$$\delta\psi = \alpha\gamma_5(1 - \frac{a}{2}D(\mu = 0))\psi \;\text{ and }\; \delta\bar{\psi} = \alpha\bar{\psi}(1 - \frac{a}{2}D(\mu = 0))\gamma_5 \;,$$

- Such transformations also give the correct index on the lattice[Hasenfratz, Laliena and Neidermeyer(98), Luscher(98)].

Forms of $D_W(\mu)$

- Naively adding a μN term to lattice fermion operator like Wilson-Dirac operator leads to μ^2 divergence in the energy density in the continuum limit.
- Adding $\mathrm{e}^{\pm\mu a_4}$ with U_4 , $U_4^\dagger$ respectively in $D_W(0)$ leads to a $D_W(\mu)$ with cancellation of such divergences [Hasenfratz-Karsch method(83)].
  In general functions, $f(\mu a_4),\ g(\mu a_4)$ satisfying $f.g = 1$, $f - g \approx \mu a_4$ lead to cancellation of the divergences [Gavai(85)].
- Interestingly, non-unitary rotation of the fermion fields gives $D_W(\mu)$ as prescribed by the H-K method $\longrightarrow$ true for any local fermion operator.

$$S = \sum_x \bar{\psi}_x D_W(0)_{xy} \psi_y \longrightarrow \sum_x \bar{\psi}'_x D_W(\mu)_{xy} \psi'_y$$

with

$$\psi'(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\psi(\mathbf{x},\tau) \quad , \quad \bar{\psi}'(\mathbf{x},\tau) = \bar{\psi}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau} \ ,$$

Forms of $D_{ov}(\mu)$ and anomaly on the lattice

- Recently a $D_{ov}(\mu)$ was suggested by Bloch & Wettig (06) as, $aD_{ov}(\mu) = 1 + \gamma_5 \epsilon(\gamma_5 D_W(\mu))$.
- Satisfies the relation: $\{\gamma_5, D_{ov}(\mu)\} = aD_{ov}(\mu)\gamma_5 D_{ov}(\mu)$.
- Thermodynamic quantities has no potentially divergent μ^2/a^2 terms. [Gattringer, Liptak(07);Banerjee,Gavai,Sharma (08)]
- The action is not invariant under chiral transformations as

$$\begin{aligned} \delta S &= \alpha \sum_{x,y} \bar{\psi}_x [\gamma_5 D_{ov}(\mu) + D_{ov}(\mu)\gamma_5 \\ &- \frac{a}{2} D_{ov}(0)\gamma_5 D_{ov}(\mu) - \frac{a}{2} D_{ov}(\mu)\gamma_5 D_{ov}(0)]_{x,y} \psi_y \neq 0 \end{aligned}$$

- Incorporating chemical potential to chiral fermion operators like the Overlap and Domain wall operators necessarily leads to chiral symmetry breaking.

- It was shown by Bloch & Wettig (06) that $D_{ov}(\mu)$ satisfies the index theorem and has exact chiral symmetry on the lattice.
- To keep the action invariant, one has to modify the chiral transformations to $\delta\psi = \alpha\gamma_5(1 - \frac{a}{2}D_{ov}(\mu))\psi$ [BGS (08)].
- The anomaly due to change of measure in this case, $Tr\ [2\gamma_5(1 - a/2D_{ov}(\mu))]$ is μ-dependent on the lattice $\longrightarrow$ depend on the zero modes of $D_{ov}(\mu)$.
  Contradicts our finding that the measure is indep. of μ.

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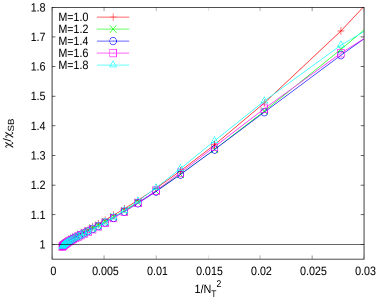
  Contradicts our finding that the measure is indep. of μ.
- Making the symmetry transformations dependent on the intensive thermodynamic variable μ is unphysical[BGS (08)] as there is no unique order parameter for chiral symmetry breaking.
- It is always desirable to keep the symmetries on the lattice as close to the continuum as possible with a unique order parameter.

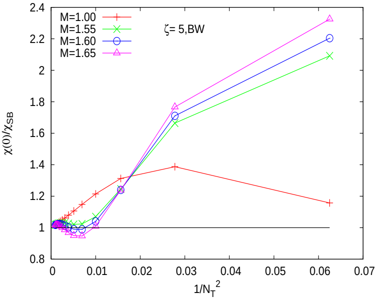
A simple method for introducing μ

- Introduce μ as a Lagrange multiplier corresponding to a "number density" defined in a point-split form to $D_{ov}(0)$.

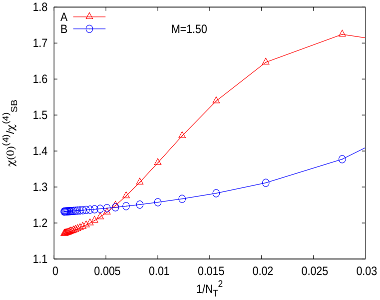
$$D_{ov}(\hat{\mu})_{xy} = (D_{ov})_{xy} - \frac{\hat{\mu}}{2a_4\ M}\left[(\gamma_4 + 1)U_4^\dagger(y)\delta_{x,y+\hat{4}} - (1-\gamma_4)U_4(x)\delta_{x,y-\hat{4}}\right] .$$

- This operator breaks chiral symmetry explicitly on the lattice with the same $\mathcal{O}(a)$ corrections as the Bloch-Wettig operator.
- Potentially divergent $1/a^2$ terms present in the lattice expression of energy density, quark no. susceptibility$\longrightarrow$ have to do a zero-temperature subtraction.





- The convergence towards continuum is linear
- no oscillations for odd and even values of N_T as seen in the B-W operator
- M-dependence is less pronounced
- Divergences are completely cancelled $\longrightarrow$ verified by calculating $\chi^{(4)}$



Advantages

- The proposed $D(\mu)$ has exact chiral symmetry at $\mu = 0$ so best for critical point search by Taylor expansion about $\mu = 0$ rather than the oft-used staggered fermions.
- This method of introducing μ is useful for computing higher order quark no. susceptibilities(QNS) in QCD even for other fermion operators like the staggered operator since

$$D' = \sum_{x,y} N(x,y), \qquad \text{and} \qquad D'' = D''' = D''''... = 0 \ ,$$

  incontrast to the to the popular exp($\pm\mu$)-prescription, where,

$$D' = D'''... = \sum_{x,y} N(x,y) \qquad \text{and} \qquad D'' = D'''' = D''''''... \neq 0 \ .$$

- Computing higher order QNS is important for locating critical end point of QCD[Gavai& Gupta('03)].

Outline

Conclusions

- The anomaly is unaffected at finite density in the continuum as observed
  a)from the perturbative calculation of ABJ triangle diagram.
  b)Non-perturbatively from Fujikawa's method.
- At finite density the eigenvectors of $\not{D}(\mu)$ are related to that of $\not{D}(0)$ by non-unitary rotation.
- The chiral determinant is still independent of μ so its relation to the zero modes is independent of finite density effects.
- Introduction of chemical potential in chiral fermion operators necessarily leads to chiral symmetry breaking on the lattice.

Conclusions

- On the lattice however if we try to keep the action invariant at finite μ by making μ-dep. chiral transformations then the anomaly has finite density corrections. This has been argued to be inconsistent physically.
- Finally we propose a new method of incorporating μ in lattice Dirac operators which still breaks chiral invariance but useful in computing higher order QNS in QCD.