Classical Mechanics, Autumn 2017

Assignment #2, Due 03/10/2017 in class

1. Consider a double pendulum constructed with equal masses m and massless rods of equal lengths ℓ. Take any appropriate values of m and ℓ in arbitrary units. Set up and solve the Hamilton's equations of motions (numerically is OK) with initial conditions of your choice. Plot the trajectories of the two masses in (θ_i, p_{θ_i}) space. (Separate diagrams for the two masses). Comment on the features of these solutions. Can you choose initial conditions such that the trajectories look simple ?

2. Two points (θ_1, ϕ_1) and (θ_2, ϕ_2) on the surface of a sphere are to be connected by the curve $\theta(\phi)$ of minimum length along the sphere.

   (a) Using the variational principle, show that this curve is a great circle, and find its length.

   (b) An ant at (θ_1, ϕ_1) wants to start to walk along this shortest curve. Find the direction it should start walking in, in terms of the basis vectors $\hat{\theta}_1, \hat{\phi}_1$ and the parameters $\theta_1, \phi_1, \theta_2, \phi_2$.

3. Consider the motion of two wheels (each with mass m, radius b) connected with a massless axle of length a, rolling without slipping on an inclined plane with slope α. Employ the same coordinates defined in the class: x horizontal along the inclined plane, y along the slope of the plane, z normal to the plane, ϕ_1, ϕ_2 the angles of rotation of the wheel, θ the angle made by the wheel with the x axis. [Some sign conventions are arbitrary here, feel free to use your own.]

   (a) Write down the holonomic and non-holonomic equations of constraint.

   (b) Write down the equations of motion using the method of Lagrange multipliers.

   (c) Solve the EoMs numerically, for any non-trivial initial conditions of your choice, plot x, y, θ as functions of time.

   (d) Use the substitution $\dot{s}\ \hat{s} \equiv \dot{x}\ \hat{x} + \dot{y}\ \hat{y}$ and get an analytic solution of the problem. Explain the features of your numerical solution using this.

4. Consider a relativistic particle of rest mass m under the influence of the electromagnetic 4-potential $A_k = (\phi/c, -\vec{A})$. The "action" to be minimized can only be a linear combination of the two Lorentz-invariant quantities $ds = \sqrt{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$ and $A_k\ dx^k$, where $dx^k = (dt, dx, dy, dz)$. Take the action to be

$$S = \int_1^2 (\alpha\ ds + \eta\ A_k\ dx^k)$$

   (a) Find L in terms of the $\alpha, \eta, \phi, \vec{A}$, and the velocity $\vec{v}$ of the particle.

   (b) Find the value of α using the condition that, in the absence of any electromagnetic field, L should "effectively" reduce to the corresponding non-relativistic Lagrangian.

   (c) Find the relativistic generalized momenta corresponding to the coordinates (t, x, y, z), and write down the corresponding equations of motions using Euler-Lagrange equation.

   (d) Matching the EoMs to the Lorentz force law, determine η.

   (e) Using the values of α and η obtained above, calculate the Hamiltonian, and write down Hamilton's equations of motion.

5. (a) Give an example of a Lagrangian whose Hamiltonian does not exist.

   (b) Draw an electrical circuit that has the same Lagrangian as a particle in a simple harmonic potential that is being acted upon by a periodic force, and a damping term proportional to its speed. State the one-to-one mapping between the terms in both cases. In both cases, interpret the phenomenon when the frequency of applied force equals the characteristic frequency of the harmonic potential. Show an appropriate plot to demonstrate this behaviour.

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