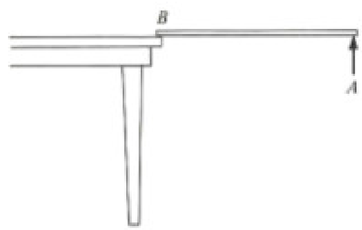
Classical Mechanics (Autumn 2017)

Drop Test, 20/08/2017

1. **Newtonian mechanics:** [6 + 4]

   (a) A uniform stick of mass m and length l is suspended horizontally with the end B at the edge of a table, and the other end A is held by hand. Point A is suddenly released.

   At the instant after release, determine:
   (i) the torque and angular acceleration about the end B
   (ii) the vertical acceleration of the centre-of-mass
   (iii) the vertical and horizontal components of the hinge force by the table on the stick at B.

   (b) A system of two particles m_1 and m_2, with coordinates $\vec{x}_1$ and $\vec{x}_2$ respectively, interact through a mutual force $\vec{F}$. Express the angular momentum and kinetic energy of the system in the centre-of-mass frame in terms of a virtual particle having the coordinate $\vec{x} \equiv \vec{x}_1 - \vec{x}_2$.

2. **Calculus of variation, Lagrangian and Hamiltonian:** [5 + 5]

   (a) A particle of mass m is constrained to move on the surface of a sphere of radius a. There are no other forces on the system (not even gravity). Prove that the particle will move along a great circle. [All arguments need to be clearly and explicitly stated.]

   (b) Prove that for any natural system, if the potential energy does not have any terms higher than linear order in the generalized velocities, it is always possible to go from the Lagrangian to the Hamiltonian formulation. [All arguments need to be clearly and explicitly stated.]

3. **Constructing Lagrangians:** [5 + 5]

(a) Write down the Lagrangian for a particle of mass m, in a frame of reference rotating with an angular velocity Ω, if the particle is subjected to a force of the form $\vec{F} = -\nabla\Phi$ (as measured in the inertial frame).

(b) A particle moves in a plane under the influence of a central force whose magnitude is given by

$$F(r) = \frac{\lambda}{r^2}(\mu - \dot{r}^2 + 2r\ddot{r}) \ .$$

where λ and μ are constants.
Choose appropriate generalized coordinates and find the velocity-dependent potential for this system. Hence find the Lagrangian and set up the equations of motion.

4. **Oscillations:** [5 + 5]

(a) Model the CO_2 molecule, which has a linear structure, as a system of point masses, connected by springs. Choose generalized coordinates for this system, write down the Lagrangian and set up the equations of motion.

(b) Find the possible oscillation frequencies of the above system, and indicate the nature of oscillation corresponding to each of them.

5. **Central forces:** [10]

A particle of mass m moves under the influence of a central force

$$\vec{F} = -\frac{k}{r^2}\left(1 - \frac{\alpha}{r}\right)\hat{r} \ .$$

Show that this particle will move in a precessing ellipse, and find the time period of the precession.

6. **Conservation laws:** [5 + 5]

(a) Consider a two-dimensional system with a Lagrangian

$$L = m(\dot{x}^2 + \dot{y}^2) - k(x^2 + y^2) \ .$$

Find the symmetry of the Lagrangian, and express it as a transformation under which L remains invariant. Find the corresponding Noether invariant.

(b) Consider a particle of mass m moving under the influence of an inverse square field of force

$$\vec{F} \equiv -\frac{mk\vec{r}}{r^3} \ ,$$

where $k > 0$ is a constant. Define the vector

$$\vec{A} = \vec{l} \times \vec{v} + \frac{mk\vec{r}}{r} \ ,$$

where $\vec{l} = m\vec{r}\times\vec{v}$. Evaluate the Poisson Brackets $\{\vec{A}, H\}$, where H is the Hamiltonian. Does this lead to any conservation principle?

7. **Canonical transformations, Hamilton-Jacobi:** [3 + 3 + 4]

A system with one degree of freedom is described by canonical coordinates $\{q, p\}$, and has a Hamiltonian

$$H = \frac{p^2}{2} + \frac{q^2}{2} \ .$$

(a) Make a canonical transformation with the generator

$$F_1(q, Q) = \frac{1}{2} q^2 \cot Q$$

and obtain the new canonical coordinates $\{Q, P\}$.

(b) In terms of the new coordinates, write down the Hamiltonian, the canonical equations of motion, and their solution.

(c) Write down the Hamilton-Jacobi equation for the above system and solve it to obtain the time variation of $q = q(t)$.

8. **Continuous media:** [4 + 2 + 4]

Consider the Lagrangian density

$$\mathcal{L}(x,t) = i\hbar \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \vec{\nabla}\psi^* \cdot \vec{\nabla}\psi - V(\vec{x})\psi^*\psi$$

where $\psi(x,t)$ and $\psi^*(x,t)$ are considered as independent fields and $V(\vec{x})$ is real.

(a) Write down the Euler-Lagrange equations for these fields and show that these are identical up to complex conjugation.

(b) The Lagrangian density above is clearly not symmetric in $\psi(x,t)$ and $\psi^*(x,t)$. How can you motivate the symmetry in the equations of motion ?

(c) The Lagrangian density above is symmetric under transformations of the form

$$\psi \to \psi' = e^{i\theta}\psi \ , \qquad \psi^* \to \psi^{*\prime} = e^{-i\theta}\psi^* \ ,$$

where θ is a real constant. Find the corresponding Noether current.

9. **Rigid body dynamics:** [5 + 3 + 2]

(a) Calculate the moment of inertia tensor $\mathbb{I}$ for a thin uniform circular disk of radius R and mass M, about its centre. Let the z axis be normal to the plane of the disk.

(b) Write down the kinetic energy for a rigid body of mass M, translating with velocity $\vec{V}$ and rotating about its centre of mass with angular velocity $\vec{\omega}$, in terms of $M, \vec{V}, \vec{\omega}, \mathbb{I}$.

(c) Using the above results, calculate the kinetic energy of a disc rolling without slipping, with velocity $\vec{V}$ on a flat surface.

10. **Misc:** [10]

Consider a particle whose equation of motion in one dimension is

$$\ddot{x} + \beta\dot{x} + \omega^2 x = \alpha \cos \Omega t \ ,$$

for $t > 0$. At $t = 0$, the particle is at rest at a distance d from the origin. Qualitatively sketch the trajectory of the particle *in phase space* (no need to show any analytic calculations), and point out its important features, in each of the following cases.

In each case, choose the meanings of "small" and "large" so that important features of the motion are clearly seen. Specify concrete meanings of "small" and "large" used in each case.

(a) $\alpha = 0$, "small" β

(b) $\alpha = 0$, "large" β

(c) "small" α, "small" β

(d) "small" α, "large" β

(e) "large" α

Comment on the Ω-dependence of the last three trajectories.

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