Electrodynamics II : Autumn 2011
Assignment 3
Given: Sunday Nov 13, Expected: Monday Nov 28

This assignment has two sections. You need to submit answers to only the questions from Section I. The questions from Section II are strongly recommended for practice / understanding of concepts, however you need not submit them and will not be graded on them.

Section I

1. Given the action $S = -mc \int_a^b ds - (e/c) \int_a^b A_i dx^i$,
   (i) Determine the canonical 4-momentum P_i, and hence the Lorentz invariant quantity $P_i P^i$. Show how the invariance of $P_i P^i$ leads to the classical Hamilton-Jacobi equation in the limit $e \to 0$ and $c \to \infty$.
   (ii) Write down the Hamiltonian (in the 3-vector language), and the equations of motion that follow from that.

2. For a Lagrangian density $\mathcal{L}(q, \partial_i q)$, the energy-momentum tensor T_i^k is given by $T_i^k = \partial_i q (\partial \mathcal{L} / \partial_k q) - \delta_i^k \mathcal{L}$. For the electromagnetic field in the absence of charges, $\mathcal{L} = -(\epsilon/4c) F_{kl} F^{kl}$.
   (i) Taking q as the 4-potential A_m, determine T_i^k in terms of the components of the electromagnetic field tensor F.
   (ii) Calculate the components of T_i^k in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

3. A particle with mass m and charge q is moving in a region with constant electric field $\vec{\mathbf{E}}$ and constant magnetic field $\vec{\mathbf{B}}$, both along the z axis. The initial momentum of the particle is (p_{x0}, p_{y0}, p_{z0}).
   (i) Determine the trajectory of the particle, i.e. determine $x(t), y(t), z(t)$, You may find it convenient to use the variables $p_\perp$ and ϕ instead of p_x and p_y, such that $p_\perp e^{i\phi} = p_x + i p_y$.
   (ii) Plot v_z and $v_\perp = \sqrt{v_x^2 + v_y^2}$ as a function of t (on the same plot) for some appropriate values of parameters that will bring out the main features.

4. A relativistic particle is losing energy at a constant rate $\mathcal{R} = dE/dt'$ while moving through a material in a straight line. In the process, the speed of the particle decreases from $v = 0.9c$ to $v = 0$.
   (i) Plot the power radiated as a function of $\cos\theta$ (the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$), when the speed of the particle is $v = 0.9c$, $v = 0.5c$ and $v = 0.1c$ (on the same plot, showing the relative magnitudes, in appropriate units).
   (ii) Calculate the total energy radiated by the particle in the form of Bremsstrahlung radiation. You may need to integrate numerically.

5. The position vector for a charge q moving with a constant angular velocity ω_0 is $x'(t') = A(\cos\phi(t'), \sin\phi(t'), 0)$.
   (i) Calculate the Poynting vector $\vec{\mathbf{N}}$ at a point $\vec{\mathbf{r}} = r(0, \sin\alpha, \cos\alpha)$.
   (ii) When $\phi = 0$, show the power radiated as a function of α for $v = 0.5c$ and $v = 0.9c$ on the same plot in appropriate units.
   (iii) Integrating over α, calculate the total power radiated in terms of the acceleration $|\vec{a}|$ and the boost γ.

6. In an electron gas with number density N of electro,
   (i) Calculate the total absorption cross section, $\int \sigma(\omega) d\omega = \Delta U / S_{\omega_0}$, where ΔU is the average rate of absorption of energy integrated over all frequencies, and S_{ω_0} is the magnitude of the Poynting vector at the resonant frequency.
   (ii) Using the expression for the refractive index of such a dilute gas, plot the phase velocity and group velocity of an EM wave as a function of ω. Neglect the damping term and choose an appropriate value for N. Where is the "dilute" nature of the gas relevant ?

Section II

1. Add the term $\eta \int \tilde{F}_{ik} F^{ik} d\Omega$ to the action of the electromagnetic field. Using the variation of the 4-potential A, determine the equations of motion. Comparing these with Maxwell's equations, determine the value of η.

2. Angular momentum tensor in 4-d:
   (i) An infinitesimal rotation in 4-d is defined as $x'^i - x^i = \delta x^i = x_k \delta\Omega^{ik}$. Show that $\delta\Omega_{ik}$ is an antisymmetric tensor.
   (ii) For a collection of free particles, the action is $S = -\sum mc \int_a^b ds$. Show that $\delta S = \delta\Omega_{ik} M^{ik}$ where $M^{ik} = (1/2)\sum(p^i x^k - p^k x^i)$. Hence argue that M^{ik} is conserved.
   (iii) M^{ik} is the angular momentum 4-tensor. Calculate the components of this tensor in terms of $\vec{\mathbf{r}}, \vec{\mathbf{p}}$ and $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ of the individual particles.

3. Determine the relativistic trajectory of a charge q in uniform electric field $\vec{\mathbf{E}} = A_0 \hat{y}$ and uniform magnetic field $\vec{\mathbf{B}} = A_0 \hat{z}$.

4. Accelerating charge:
   (i) By an explicit calculation, show that the magnetic field $\vec{\mathbf{B}}$ from an accerating charge reduces to the form $\vec{\mathbf{B}} = \vec{\mathbf{r}} \times \vec{\mathbf{E}}/(rc)$ where $\vec{\mathbf{r}} = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}'(t')$.
   (ii) Calculate the total energy radiated in Bremsstrahlung when the speed of a charge q decreases at a constant rate a, from v_0 to 0.

5. Exercises 2,3,8,9, page 376, Panofsky and Phillips

6. Radiation reaction force:
   (i) Low velocities: using the condition that the radiation reaction force should be much smaller than the external electromagnetic force, find an upper limit on the magnitude of the magnetic field $\vec{\mathbf{B}}$.
   (ii) High velocities: calculate the leading term (in powers of the boost γ) for the radiation reaction force on an ultrarelativistic particle travelling through electric field $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$ (in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$).