ED II: Lecture 1

Maxwell's equations: a review

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Outline

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In the language of differential vector calculus

Gauss's law

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{1}$$

Gauss's law for magnetism

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{2}$$

Maxwell-Faraday equation

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{3}$$

Ampere's law, with Maxwell's correction

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) \tag{4}$$

Intuitive interpretations obtained through integral forms $\Rightarrow$

Gauss's law: enclosed charges

$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$

- Integrate over a closed volume:

$$\int_V (\nabla \cdot \vec{\mathbf{E}}) dV = \int_V \frac{\rho}{\epsilon_0} dV \tag{5}$$

- Use a mathematical identity (Gauss's theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \tag{6}$$

- Relationship between electric field on a closed surface and the charge enclosed inside it
- The part in red: source of the electric field
- Leads to Coulomb's law if Q is a point charge at the centre of $\vec{\mathbf{S}}$, a sphere of radius r: $\boxed{E_r \cdot 4\pi r^2 = Q/\epsilon_0}$

Gauss's law: no magnetic monopoles

$\nabla \cdot \vec{\mathbf{B}} = 0$

- Integrate over a closed volume:

$$\int_V (\nabla \cdot \vec{\mathbf{B}}) dV = 0 \tag{7}$$

- Use a mathematical identity (Gauss's theorem)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0 \tag{8}$$

- Relationship between magnetic field on a closed surface and the magnetic charge enclosed inside it
- The part in red: source of the magnetic field.
- Vanishing of the source $\Rightarrow$ no magnetic monopoles

Maxwell-Faraday equation: flux through a loop

$\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}} / \partial t$

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{\mathbf{S}}} (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{S}} = \int_{\vec{\mathbf{S}}} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} \tag{9}$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = \int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t} (\vec{\mathbf{B}} . d\vec{\mathbf{S}}) \tag{10}$$

- If the loop does not change with time

$$\mathcal{E} \equiv \oint \vec{\mathbf{E}} \cdot d\vec{\ell} = \frac{\partial}{\partial t} \int_{\vec{\mathbf{S}}} (\vec{\mathbf{B}} . d\vec{\mathbf{S}}) = -\frac{\partial \Phi}{\partial t} \tag{11}$$

$\boxed{\Phi = \int_{\vec{\mathbf{S}}} \vec{\mathbf{B}} . d\vec{\mathbf{S}}}$

More comments on the next page $\Rightarrow$

More about Maxwell-Faraday equation

- Relationship between electric field along a loop and the rate of change of magnetic flux through an open surface whose boundary is the loop
- No sources needed: it is a relationship between $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$
- The "$\mathcal{E} = -\partial\Phi/\partial t$" equation does not hold for all situations, since it does not take into account the Lorentz force on a moving charge in a magnetic field. For example, see the discussion about Faraday Wheel in Feynman lectures. We'll return to this point later in the course.

Ampere's law with Maxwell's corrections

$\nabla \times \vec{\mathbf{B}} = \mu_0(\vec{\mathbf{J}} + \epsilon_0 \partial \vec{\mathbf{E}}/\partial t)$

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{\mathbf{S}}} (\nabla \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{S}} = \mu_0 \int_{\vec{\mathbf{S}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}} + \mu_0 \epsilon_0 \int_{\vec{\mathbf{S}}} \frac{\partial \vec{\mathbf{E}}}{\partial t} \cdot d\vec{\mathbf{S}} \qquad (12)$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t} (\vec{\mathbf{E}}.d\vec{\mathbf{S}}) \qquad (13)$$

- Relationship between magnetic field along a loop and the rate of change of magnetic flux through an open surface whose boundary is the loop
- $I = \oint_{\vec{\mathbf{S}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}$ is the conduction current
- $\mu_0 \int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t} (\vec{\mathbf{E}}.d\vec{\mathbf{S}})$ is often called "displacement current", this is the correction by Maxwell to Ampere's law

Outline

Inside a dielectric medium (static case)

- Gauss's law always valid, when ρ is the total charge: $\nabla \cdot \vec{\mathbf{E}} = \rho/\epsilon_0$
- Part of the charge is due to polarization induced in the medium, which gives rise to the "bound charge":
  $\rho_b = \nabla \cdot \vec{\mathbf{P}}$, where $\vec{\mathbf{P}}$ is the polarization
- Then $\epsilon_0 \nabla \cdot \vec{\mathbf{E}} = (\rho_b + \rho_{\mathrm{fr}}) = \nabla \cdot P + \rho_{\mathrm{fr}}$, where ρ_{fr} is the free charge density
- Defining $\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$, we get Gauss's law in terms of the free charge density:

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{\mathrm{fr}} \tag{14}$$

- The relation $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ defines the dielectric permittivity of the medium, ϵ. This is in general not a number but a tensor, and may not be constant. Wherever it is constant, the dielectric is called "linear".

Inside a magnetic medium (static case)

- Maxwell-Faraday equation always valid, when $\vec{\mathbf{J}}$ is the total current: $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$
- Part of the current is due to magnetization induced in the medium, which gives rise to the “surface current”: $\vec{\mathbf{J}}_{\text{surface}} = \nabla \times \vec{\mathbf{M}}$, where $\vec{\mathbf{M}}$ is the magnetization
- Then $\nabla \times \vec{\mathbf{B}} = (\vec{\mathbf{J}}_{\text{surface}} + \vec{\mathbf{J}}_{\text{fr}}) = \mu_0 \nabla \times M + \mu_0 \vec{\mathbf{J}}_{\text{fr}}$, where $\vec{\mathbf{J}}_{\text{fr}}$ is the free current density
- Defining $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$, we get Ampere’s law in terms of the free charge density:

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{\text{fr}} \tag{15}$$

- The relation $\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$ defines the magnetic permeability of the medium, μ. This is in general not a number but a tensor, and may not be constant. Wherever it is constant, the magnetic medium is called “linear”.

Maxwell's equations: "macroscopic" form

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{\mathrm{fr}} \tag{16}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{17}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{18}$$

$$\nabla \times \vec{\mathbf{B}} = \vec{\mathbf{J}}_{\mathrm{fr}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{19}$$

These are equivalent to the equations (1)–(4), with the substitutions

$$\rho = \rho_{\mathrm{fr}} + \rho_b \,, \qquad \vec{\mathbf{J}} = \vec{\mathbf{J}}_{\mathrm{fr}} + \vec{\mathbf{J}}_{\mathrm{surface}} \tag{20}$$

$$\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}} \,, \qquad \vec{\mathbf{B}} = \mu_0 (\vec{\mathbf{H}} + \vec{\mathbf{M}}) \tag{21}$$

$$\rho_b = -\nabla \cdot \vec{\mathbf{P}} \,, \qquad \vec{\mathbf{J}}_{\mathrm{surface}} = \nabla \times \vec{\mathbf{M}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \,. \tag{22}$$

A recap of topics covered in this lecture

- Maxwell's equations: in differential and integral form
- Maxwell's equations in the presence of dielectrics and magnetic media