ED II: Lecture 2

Solving static boundary value problems

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Outline

1. Uniqueness theorems
2. Separation of variables for $\nabla^2\Phi = 0$

Outline

Unique vector, given divergence and curl

Uniqueness theorem 1

Given $\nabla \cdot \vec{\mathbf{V}} = s(\vec{\mathbf{x}})$
and $\nabla \times \vec{\mathbf{V}} = \vec{\mathbf{c}}(\vec{\mathbf{x}})$ (with $\nabla \cdot \vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$, of course),
if $\vec{\mathbf{V}}$ goes to zero at infinity (fast enough),
then $\vec{\mathbf{V}}$ can be uniquely written in terms of $s(\vec{\mathbf{x}})$ and $\vec{\mathbf{c}}(\vec{\mathbf{x}})$.

Indeed the solution can be given:

$$V = -\nabla\phi(\vec{\mathbf{x}}) + \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}) \tag{1}$$

with

$$\phi(\vec{\mathbf{x}}) = \frac{1}{4\pi}\int \frac{s(\vec{\mathbf{x}})}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}' \tag{2}$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}) = \frac{1}{4\pi}\int \frac{\vec{\mathbf{c}}(\vec{\mathbf{x}})}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}' \tag{3}$$

Proof of uniqueness theorem 1

Steps involved

- Show $\nabla \cdot \vec{\mathbf{V}} = s(\vec{\mathbf{x}})$, using $\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(r)$
- Show $\nabla \times \vec{\mathbf{V}} = \vec{\mathbf{c}}(\vec{\mathbf{x}})$ using integration by parts.
  You'll have to use the conditions
  $\nabla \cdot \vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$ everywhere, and
  $\vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$ at large distances (or goes to zero fast enough)

Unique scalar, given $\nabla^2\phi$ and boundary conditions

Uniqueness theorem 2

For a scalar $\phi(\vec{\mathbf{x}})$,
given $\nabla^2\phi$ everywhere,
and given $\phi(\vec{\mathbf{x}})$ or $\nabla\phi \cdot \hat{\mathbf{n}}$ on a closed surface
a unique solution for $\phi(\vec{\mathbf{x}})$ exists.

Steps for proving uniqueness theorem 2

- Consider two solutions ϕ_1 and ϕ_2, and define $\psi = \phi_1 - \phi_2$
- Using $\oint(\psi\nabla\psi)\cdot d\vec{\mathbf{S}} = \int(\nabla\psi)\cdot(del\psi)dV + \int\psi\nabla^2\psi dV$,
  Show that $|\nabla\psi| = 0$ everywhere in the enclosed volume
  (Use $\psi = 0$ or $\nabla\psi \cdot \hat{\mathbf{n}} = 0$ at the boundary)
- Note: the boundary conditions may be of the form $\psi = 0$ on some part of the boundary and $\nabla\psi = 0$ on the remaining part.

Unique vector, given $\nabla \times (\nabla \times \vec{\mathbf{A}})$

Uniqueness theorem 3 [for a vector $\vec{\mathbf{A}}(\vec{\mathbf{x}})$]

Given $\nabla \times (\nabla \times \vec{\mathbf{A}})$ everywhere,
and given $\vec{\mathbf{A}} \times \hat{\mathbf{n}}$ or $(\nabla \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}$ on a closed surface
a unique solution for $\vec{\mathbf{A}}(\vec{\mathbf{x}})$ exists.

Steps for proving uniqueness theorem 3

- Consider two solutions $\vec{\mathbf{A}}_1$ and $\vec{\mathbf{A}}_2$, and define $\vec{\mathbf{a}} = \vec{\mathbf{A}}_1 - \vec{\mathbf{A}}_2$
- Using
  $\oint [\vec{\mathbf{a}} \times (\nabla \times \vec{\mathbf{a}})] \cdot d\vec{\mathbf{S}} = \int (\nabla \times \vec{\mathbf{a}}) \cdot (\nabla \times \vec{\mathbf{a}}) dV - \int \vec{\mathbf{a}} \cdot [\nabla \times (\nabla \times \vec{\mathbf{a}})] dV$,
  Show that $|\nabla \times \vec{\mathbf{a}}| = 0$ everywhere in the enclosed volume
  (Use $\vec{\mathbf{a}} \times \hat{\mathbf{n}} = 0$ or $(\nabla \times \vec{\mathbf{a}}) \times \hat{\mathbf{n}} = 0$ at the boundary)
- One may have $\vec{\mathbf{a}} \times \hat{\mathbf{n}} = 0$ on some part of the boundary and $(\nabla \times \vec{\mathbf{a}}) \times \hat{\mathbf{n}} = 0$ on the remainder.
- $\vec{\mathbf{a}} \times \hat{\mathbf{n}}$: tangential component of $\vec{\mathbf{a}}$ to the surface

A caution about $\nabla^2\vec{\mathbf{A}}$

- $\nabla^2\vec{\mathbf{A}} \neq (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$
- In fact $\nabla^2\vec{\mathbf{A}}$ is defined through

$$\nabla^2\vec{\mathbf{A}} = -\nabla \times (\nabla \times \vec{\mathbf{A}}) + \nabla(\nabla \cdot \vec{\mathbf{A}}) \tag{4}$$

Uniqueness theorems: applications

- If a solution is found by hook by or crook, we can be sure that this is the only solution
- A search for simple solutions, with certain symmetry properties, if successful, can solve the problem completely.
- Tricks like the method of images work.

Outline

When to use

This technique works when there is some symmetry in the boundary conditions of the problem, which suggests the use of certain coordinates.

If the boundary conditions are of the form

- $\Phi(x = a) = \phi_0$, for all (y, z) $\Rightarrow$ cartesian coordinates
- $\Phi(r = a) = \phi_0$, for all (θ, ϕ) $\Rightarrow$ spherical polar coordinates
- $\Phi(r = a) = \phi_0$, for all (z, ϕ) $\Rightarrow$ cylindrical coordinates

Cartesian coordinates

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{5}$$

- Form of the solution:

$$X(x) = \begin{cases} Ae^{ik_x x} + Be^{-ik_x x} \\ Ae^{\kappa_x x} + Be^{-\kappa_x x} \end{cases} \tag{6}$$

  Similarly for $Y(y)$ and $Z(z)$.
- The solutions along x, y, z direction can be individually oscillatory ($e^{\pm ikx}$) or hyperbolic ($e^{\pm \kappa x}$).
- All three solutions cannot be propagating simultaneously, neither can all be hyperbolic at the same time.

Spherical polar coordinates

$$\Phi(r,\theta,\phi) = R_\ell(r)\Theta_\ell^m(\theta)\Phi_m(\phi) \tag{7}$$

Form of the solution

- $R_\ell = A\ell r^\ell + B_\ell r^{-\ell-1}$
  - $A_\ell = 0$ if the solution is to be finite at infinity,
  - $B_\ell = 0$ if it is to be finite at the origin
- $\Theta_\ell^m(\theta) = C_\ell P_\ell^m(\cos\theta) + D_\ell Q_\ell^m(\cos\theta)$
  - P_ℓ^m, Q_ℓ^m: associated Legendre polynomials
  - $P_\ell^m = |Y_\ell^m|$, magnitudes of spherical harmonics
  - $D_\ell = 0$ if the solution is finite along z axis, since Q_ℓ^m blows up there
- $\Phi(\phi) = \begin{cases} Ee^{im\phi} + Fe^{-im\phi} & (m \neq 0) \\ E\phi + F & (m = 0) \end{cases}$
  - Azimuthal symmetry $\Rightarrow m = 0$
  - $\oplus$ single-valued solution $\Rightarrow E = 0$
  - $P_\ell^0(\cos\theta) = P_\ell(\cos\theta)$, Legendre polynomials

Cylindrical coordinates

$$\Phi(r, \phi, z) = R_n(r)\mathit{\Phi}_n(\phi)Z(z) \tag{8}$$

Form of the solution

- $R_\ell = \begin{cases} A_n J_n(kr) + B_n N_n(kr) & (k \neq 0) \\ A_n r^n + B_n r^{-n} & (k = 0, n \neq 0) \\ A \ln r + B & (k = n = 0) \end{cases}$
  - J_n: Bessel functions, K_n: associated Bessel functions
  - $B_n = 0$ if Φ is to be finite at the origin
- $\mathit{\Phi}_n(\phi) = \begin{cases} C_n e^{in\phi} + D_n e^{-in\phi} & (n \neq 0) \\ C\phi + D & (n = 0) \end{cases}$
  - Azimuthal symmetry $\Rightarrow n = 0$
  - $\oplus$ single-valued solution $\Rightarrow C = 0$
- $Z(z) = \begin{cases} Ee^{kz} + Fe^{-kz} & (k \neq 0) \\ Ez + F & (k = 0) \end{cases}$
  - $Z(z)$ can be oscillatory, in which case $R(r)$ involves modified Bessel functions

A recap of topics covered in this lecture

- Uniqueness theorem for $\vec{\mathbf{V}}$, given its divergence and curl both of which fall sufficiently fast at infinity
- Uniqueness theorem for Φ, given $\nabla^2\Phi$ everywhere and Φ or $(\nabla\Phi \cdot \hat{\mathbf{n}})$ on a closed boundary.
- Uniqueness theorem for $\vec{\mathbf{A}}$, given $\nabla \times (\nabla \times \vec{\mathbf{A}})$ everywhere and the components of $\vec{\mathbf{A}}$ or $(\nabla \times \vec{\mathbf{A}})$ tangential to a closed boundary
- Solutions with separation of variables in the cartesian, spherical and cylindrical coordinates