ED II: Lecture 3

Energy in electric and magnetic fields

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Outline

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Work done in vacuum

- Work done in increasing charge density by $\delta\rho(\vec{\mathbf{x}})$:

$$\delta W = \int \phi(\vec{\mathbf{x}})\ \delta\rho(\vec{\mathbf{x}}) dV = \epsilon_0 \int \phi(\vec{\mathbf{x}}) \delta(\nabla \cdot \vec{\mathbf{E}}) dV \qquad (1)$$

- Integrate by parts:

$$\delta W = \epsilon_0 \int \nabla \cdot (\phi \delta \vec{\mathbf{E}}) dV - \epsilon_0 \int (\nabla \phi) \cdot (\delta \vec{\mathbf{E}}) dV$$

$$\delta W = \epsilon_0 \int \vec{\mathbf{E}} \cdot \delta \vec{\mathbf{E}} dV \qquad (2)$$

Total work done

$$W = \frac{\epsilon_0}{2} \int \vec{\mathbf{E}}^2 dV \qquad (3)$$

Work done in a dielectric

- Work done in increasing charge density by $\delta\rho(\vec{\mathbf{x}})$:

$$\delta W = \int \phi(\vec{\mathbf{x}})\ \delta\rho(\vec{\mathbf{x}}) dV = \int \phi(\vec{\mathbf{x}})\delta(\nabla \cdot \vec{\mathbf{D}}) dV \qquad (4)$$

- Integrate by parts:

$$\begin{aligned} \delta W &= \int \nabla \cdot (\phi\delta\vec{\mathbf{D}}) dV - \int (\nabla\phi) \cdot (\delta\vec{\mathbf{D}}) dV \\ \delta W &= \int \vec{\mathbf{E}} \cdot \delta\vec{\mathbf{D}} dV \end{aligned} \qquad (5)$$

Only for a linear dielectric ($\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$)

$$W = \frac{1}{2} \int \vec{\mathbf{E}} \cdot \vec{\mathbf{D}} dV \qquad (6)$$

Matching with energy of collection of discrete charges

- For N charges q_i, we expect that the energy is

$$U = \frac{\epsilon_0}{2} \sum_n q_i \phi_i \tag{7}$$

where ϕ_i is the potential at the i^{th} charge q_i due to the others. Let $\vec{\mathbf{E}}_i$: electric field due to charge i

- Using $U = (\epsilon_0/2) \int \vec{\mathbf{E}}^2 dV$, we get

$$U = \frac{\epsilon_0}{2} \int (\Sigma_i \vec{\mathbf{E}}_i^2 + \sum_{i \neq j} \vec{\mathbf{E}}_i \cdot \vec{\mathbf{E}}_j) dV \tag{8}$$

$$= U_0 + \frac{\epsilon_0}{2} \sum_i \int \vec{\mathbf{E}}_i \cdot (-\nabla \phi_i) dV \tag{9}$$

- The first term, U_0, does not depend on the distribution of charges, it is the “self-energy”.
- The second term matches with what we expect.

Outline

Differences compared to the electric field

- A steady current has to be maintained, for which energy should continue to be supplied by an external EMF, $\vec{\mathcal{E}}$.
- This energy will partly be stored in the magnetic field (caused by the current) and the rest of it will be dissipated as heat in the conductor.
- Total current $\vec{\mathbf{J}} = \sigma(\vec{\mathbf{E}} + \vec{\mathcal{E}}) \Rightarrow \vec{\mathbf{J}}^2 = \sigma(\vec{\mathcal{E}} \cdot \vec{\mathbf{J}} + \vec{\mathbf{E}} \cdot \vec{\mathbf{J}})$
- Rate of energy input:

$$\frac{dU_{\text{in}}}{dt} = \vec{\mathcal{E}} \cdot \vec{\mathbf{J}} dV = \int \frac{\vec{\mathbf{J}}^2}{\sigma} dV - \int \vec{\mathbf{E}} \cdot \vec{\mathbf{J}} dV \tag{10}$$

- The first term is rate of energy dissipation as heat, the second term is rate of storage of energy in the magnetic field

Energy stored in static magnetic field

- Rate of energy storage in magnetic field:

$$\frac{dU_m}{dt} = -\int \vec{\mathbf{E}}\cdot\vec{\mathbf{J}}dV = -\int \vec{\mathbf{E}}\cdot(\nabla\times\vec{\mathbf{H}})dV \quad (11)$$

$$= -\int \vec{\mathbf{H}}\cdot(\nabla\times\vec{\mathbf{E}})dV + \int \nabla\cdot(\vec{\mathbf{E}}\times\vec{\mathbf{H}})dV$$

$$= \int \vec{\mathbf{H}}\cdot\frac{\partial\vec{\mathbf{B}}}{\partial t}dV \quad (12)$$

- Incremental energy stored: $\boxed{\delta U_m = \int \vec{\mathbf{H}}\cdot\delta\vec{\mathbf{B}}dV}$

Only if $\vec{\mathbf{H}}$ is linear in $\vec{\mathbf{B}}$

$$U_m = \frac{1}{2}\int \vec{\mathbf{H}}\cdot\vec{\mathbf{B}}dV = \frac{1}{2}\int \vec{\mathbf{H}}\cdot(\nabla\times\vec{\mathbf{A}})dV \quad (13)$$

$$= \frac{1}{2}[\int \vec{\mathbf{A}}\cdot(\nabla\times\vec{\mathbf{H}})dV - \int \nabla\cdot(\vec{\mathbf{H}}\times\vec{\mathbf{A}})dV]$$

$$= \frac{1}{2}\int \vec{\mathbf{A}}\cdot\vec{\mathbf{J}}dV \quad (14)$$

A note about $\vec{\mathbf{E}} \times \vec{\mathbf{H}}$

- We neglected the term $\int \nabla \cdot (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) dV = \oint (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) \cdot d\vec{\mathbf{S}}$
- This is indeed valid for static electromagnetic fields
- But for time-dependent fields, we shall see later that the leading terms in $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ go as $1/r$, so over the surface of a sphere with large radius r, the integral actually will have a constant nonzero vale.
- This will be the energy radiated away at infinity due to the changing currents. $\vec{\mathbf{N}} \equiv \vec{\mathbf{E}} \times \vec{\mathbf{H}}$ will be defined as the Poynting vector, which gives the rate of loss of energy through radiation.

A recap of topics covered in the lecture

- Energy stored in static electric fields: in vacuum and in a dielectric
- Energy stored in a static magnetic field
- Note: some of the results are only valid when the material media have linear properties.