# ED II: Lecture 4

## Time dependent EM fields: relaxation, propagation

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# Outline

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# Stationary and non-stationary states

- Stationary state, by definition, means that the currents are steady and there is no net charge movement, i.e.

$$\nabla \cdot \vec{\mathbf{J}}_s = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

  These statements are equivalent, due to continuity.
- If the initial distribution of charges and currents does not satisfy the above criteria, they will redistribute themselves so that a stationary state is reached.
- This process of “relaxation” happens over a time scale that is characteristic of the medium, called the relaxation time.

# Relaxation time

- The continuity equation, combining with $\nabla \cdot \vec{\mathbf{D}} = \rho$, gives

$$\nabla \cdot \frac{\partial \vec{\mathbf{D}}}{\partial t} = -\nabla \cdot \vec{\mathbf{J}} \tag{2}$$

- Using $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ and $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$,

$$\nabla \cdot (1 + \frac{\epsilon}{\sigma} \frac{\partial}{\partial t}) \vec{\mathbf{J}} = 0 \tag{3}$$

- The solution to this differential equation is

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_s + (\vec{\mathbf{J}}_0 - \vec{\mathbf{J}}_s) e^{-t/\tau} \tag{4}$$

  where $J_0$ is the initial current distribution
- $\tau = \epsilon/\sigma$ is the relaxation time
- $\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{\mathbf{J}}$, $\vec{\mathbf{E}} = \sigma \vec{\mathbf{J}}$, etc. relax at the same rate.

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# Time-dependent electric field

- No free charges, no external EMF sources. Maxwell $\Rightarrow$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\frac{\partial}{\partial t}(\nabla \times \mu \vec{\mathbf{H}}) \tag{5}$$

$$\nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\mu \frac{\partial}{\partial t}(\vec{\mathbf{J}}_{\mathrm{fr}} + \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}) \tag{6}$$

- This gives the second order partial differential equation

$$\nabla^2 \vec{\mathbf{E}} - \mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \tag{7}$$

- Depending on whether the $(\partial^2 \vec{\mathbf{E}}/\partial t^2)$ term dominates or the $(\partial \vec{\mathbf{E}}/\partial t)$ one, we'll get two different extremes of behaviour. The former will lead to a propagating wave, the latter will lead to a diffusion equation, corresponding to a decaying wave.

# Looking for solution of the form $\vec{\mathbf{E}}(\vec{\mathbf{x}})e^{-i\omega t}$

- The differential equation becomes

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2(1 + \frac{i\sigma}{\epsilon\omega})\vec{\mathbf{E}} = 0 \tag{8}$$

- There are two time scales here: $1/\omega$ and $\tau = \epsilon/\sigma$

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2(1 + \frac{i}{\tau\omega})\vec{\mathbf{E}} = 0 \tag{9}$$

- When $\tau\omega >> 1$,

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2\vec{\mathbf{E}} = 0 \tag{10}$$

  which is a wave propagating with speed $c = 1/\sqrt{\mu\epsilon}$
- When $\tau\omega << 1$,

$$\nabla^2\vec{\mathbf{E}} + \frac{i\omega}{\tau c^2}\vec{\mathbf{E}} = 0 \tag{11}$$

  which is the equation for diffusion. In the context of EM waves, this will lead to a decaying solution.
- we shall explore these behaviours in detail now.

# Outline

# Propagating (plane wave) solution for $\vec{\mathbf{E}}$

- $\omega\tau >> 1 \Rightarrow$ displacement current dominates over conduction current

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2\vec{\mathbf{E}} + i\omega\mu\sigma\vec{\mathbf{E}} = 0 \tag{12}$$

- Plane wave: all fields are functions of the distance $\zeta$ of a plane from the origin. $\hat{\mathbf{n}}$ is the normal to this plane.
- $\nabla \to \hat{\mathbf{n}}(\partial/\partial\zeta)$
- Maxwell's equations in this language:

$$\hat{\mathbf{n}} \cdot \frac{\partial\vec{\mathbf{D}}}{\partial\zeta} = 0\ , \qquad \hat{\mathbf{n}} \times \frac{\partial\vec{\mathbf{E}}}{\partial\zeta} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \tag{13}$$

$$\hat{\mathbf{n}} \cdot \frac{\partial\vec{\mathbf{B}}}{\partial\zeta} = 0\ , \qquad \hat{\mathbf{n}} \times \frac{\partial\vec{\mathbf{H}}}{\partial\zeta} = \frac{\partial\vec{\mathbf{D}}}{\partial t} \tag{14}$$

# Longitudinal components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

## $\vec{\mathbf{E}}_{||}$: longitudinal component of $\vec{\mathbf{E}}$

- $(\partial\vec{\mathbf{D}}/\partial\zeta)$ equation and dot product of $\hat{\mathbf{n}}$ with the $(\partial\vec{\mathbf{H}}/\partial\zeta)$ equation $\Rightarrow$

$$\frac{\partial\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}}{\partial\zeta} = 0 \,, \quad \frac{\partial\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}}{\partial t} = 0 \tag{15}$$

- For non-conducting media (e.g. vacuum), $\vec{\mathbf{E}}_{||}$ is a constant.

## Longitudinal component of $\vec{\mathbf{B}}$

- $(\partial\vec{\mathbf{B}}/\partial\zeta)$ equation and dot product of $\hat{\mathbf{n}}$ with the $(\partial\vec{\mathbf{E}}/\partial\zeta)$ equation $\Rightarrow$

$$\frac{\partial\vec{\mathbf{B}}\cdot\hat{\mathbf{n}}}{\partial\zeta} = 0 \,, \quad \frac{\partial\vec{\mathbf{B}}\cdot\hat{\mathbf{n}}}{\partial t} = 0 \tag{16}$$

- Only stationary longitudinal component of $\vec{\mathbf{B}}$ is possible, i.e. $\vec{\mathbf{B}}_{||}$ is constant (note: we have taken $\mu = \mu_0$)

# Transverse components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

- Combining the two $\hat{\mathbf{n}}\times$ equations:

$$\hat{\mathbf{n}} \times \left(\frac{\partial^2 \vec{\mathbf{E}}}{\partial \zeta^2} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}\right) = 0 \qquad (17)$$

- Differential equation for $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}} \times \hat{\mathbf{n}}$
- General solution: $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0}[f(\zeta - ut) + g(\zeta + ut)]$
- If $\vec{\mathbf{E}}_\perp$ is sinusoidal:

$$\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0} e^{-i(\omega t \pm k\zeta)} \qquad (18)$$

- Direction of propagation $\vec{\mathbf{k}} \Rightarrow$

$$\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)} \qquad (19)$$

- Using $\hat{\mathbf{n}} \times (\partial \vec{\mathbf{E}}_\perp / \partial \zeta) = -\partial \vec{\mathbf{B}}_\perp / \partial t$,

$$i\vec{\mathbf{k}} \times \vec{\mathbf{E}}_\perp = i\omega \vec{\mathbf{B}}_\perp \Rightarrow \vec{\mathbf{B}}_\perp = \frac{\vec{\mathbf{k}}}{\omega} \times \vec{\mathbf{E}} \qquad (20)$$

# Propagating wave in short

- $\vec{\mathbf{E}}_{||}$ and $\vec{\mathbf{B}}_{||}$ are constants in space and time, hence not interesting for wave propagation
- $\vec{\mathbf{E}}_{\perp}$ and $\vec{\mathbf{B}}_{\perp}$ can have $e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$ dependence, with $\vec{\mathbf{B}}_{\perp} = (\vec{\mathbf{k}}/\omega) \times \vec{\mathbf{E}}$
- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are transverse to the direction of motion, and also orthogonal to each other.

# Outline

# Decaying plane wave

- When $\omega\tau << 1$, conduction current dominates over displacement current

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2\vec{\mathbf{E}} + i\omega\mu\sigma\vec{\mathbf{E}} = 0 \qquad (21)$$

- The solution of the form $\vec{\mathbf{E}}_0 e^{\pm i(kx-\omega t)}$ implies

$$k^2 = -\frac{i\omega}{c^2\tau} = \frac{\omega}{c^2\tau}e^{-i\pi/2} \qquad (22)$$

$$\Rightarrow k = \sqrt{\frac{\omega}{c^2\tau}}e^{-i\pi/4} = \sqrt{\frac{\omega}{c^2\tau}}\left(\frac{1-i}{\sqrt{2}}\right) \qquad (23)$$

- This gives

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{\pm i(\mathrm{Re}(k)x-\omega t)} e^{-\mathrm{Im}(k)x} \qquad (24)$$

- The wave then decays with a $e^{-\mathrm{Im}(k)x}$ dependence inside the conducting medium.

# Skin depth in metals

- For metals, $\tau \sim 10^{-14}$ sec. So for $\omega < 10^{14}$, conduction current dominates.
- A wave incident on a metallic surface will decay as

$$|\vec{\mathbf{E}}| = |\vec{\mathbf{E}}_0| e^{-r/\delta} \tag{25}$$

where, from the last page, (check factor of 2)

$$\delta = \sqrt{\frac{2c^2\tau}{\omega}} = \sqrt{\frac{2}{\sigma\omega}} \tag{26}$$

- Within a distance $\delta$ from the surface of the metal, the wave would decrease in magnitude by a factor $1/e$. This $\delta$ is the "skin depth" of the metal. The surface currents will flow within this width.
- "Ideal" conductor $\Rightarrow \delta \to 0$.

# Outline

# Quadratic quantities and factors of 2

- In Electrodynamics, for convenience, we often use notation involving complex numbers (mainly exponentials), e.g.

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{i(kx-\omega t)} \;, \quad \vec{\mathbf{B}} = -i\vec{\mathbf{B}}_0 e^{i(kx-\omega t)} \tag{27}$$

  when we actually want to represent

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cos(kx-\omega t) = \mathrm{Re}(\vec{\mathbf{E}}_0 e^{i(kx-\omega t)}) \tag{28}$$

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \sin(kx-\omega t) = \mathrm{Re}(i\vec{\mathbf{B}}_0 e^{i(kx-\omega t)}) \tag{29}$$

- While performing calculations in complex notation and taking the real part of the final answer works as long as we are dealing with quantities linear in $\vec{\mathbf{E}}$ or $\vec{\mathbf{B}}$, one has to be careful while dealing with quadratic (or higher order) quantities.
- For example, in the complex notation above,

$$\langle|\vec{\mathbf{E}}|^2\rangle = \langle|\vec{\mathbf{E}}^* \cdot \vec{\mathbf{E}}|\rangle = |\vec{\mathbf{E}}_0|^2 \tag{30}$$

  while the actual answer should be (using real notation)

$$\langle|\vec{\mathbf{E}}|^2\rangle = |\vec{\mathbf{E}}_0|^2\langle\cos^2(kx-\omega t)\rangle = \frac{1}{2}|\vec{\mathbf{E}}_0|^2 \tag{31}$$

# Energy density stored in EM fields

- We have already seen that the energy stored in electric field is $U_e = (1/2)\epsilon_0|\vec{\mathbf{E}}|^2$ (we showed this result for a static field). When the electric field represents a propagating wave, then taking into account the "factor of 2" for averaged quadratic quantities, we get

$$\langle U_e \rangle = \frac{1}{4}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{32}$$

- The energy stored in magnetic field is $U_m = (1/2)|\vec{\mathbf{B}}|^2/\mu_0$ (we showed it for a static magnetic field). For a propagating wave, $|\vec{\mathbf{B}}| = |\vec{\mathbf{k}}/\omega||\vec{\mathbf{E}}|$. Including the "factor of 2", we get

$$\langle U_m \rangle = \frac{1}{4}\frac{|\vec{\mathbf{B}}_0|^2}{\mu_0} = \frac{1}{4}\frac{|\vec{\mathbf{k}}|^2}{\omega^2\mu_0}|\vec{\mathbf{E}}_0|^2 = \frac{1}{4}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{33}$$

- For a plane EM wave, energy stored in electric and magnetic field is equal. The total energy of an EM wave is

$$\langle U \rangle = \langle U_e \rangle + \langle U_m \rangle = \frac{1}{2}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{34}$$

# Energy transported by the EM fields

- The rate of energy transport is given by the Poynting vector,

$$\vec{\mathbf{N}} = \vec{\mathbf{E}}^* \times \vec{\mathbf{H}} \tag{35}$$

- The time-averaged value of this quantity is

$$\langle|\vec{\mathbf{N}}|\rangle = \frac{1}{2}|\vec{\mathbf{E}}_0^*|.|\vec{\mathbf{H}}_0| = \frac{1}{2}|\vec{\mathbf{E}}_0^*|\frac{|\vec{\mathbf{k}}|}{\omega\mu}|\vec{\mathbf{E}}_0| = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{\mathbf{E}}_0|^2 \tag{36}$$

- Compared with the rate of energy consumption in a conductor, $(1/2)\sigma|\vec{\mathbf{E}}_0|^2$, the quantity $\sqrt{\epsilon_0/\mu_0}$ is termed the conductance of vacuum
- Similarly, $\sqrt{\epsilon/\mu}$ is the conductance of a medium through which an EM wave propagates
- Note that $\langle|\vec{\mathbf{N}}|\rangle = c\langle U\rangle$, since the wave transports energy at the speed $c$.

# Recap of topics covered in this lecture

- Relaxation to stationary state, relaxation time
- Electromagnetic wave: displacement current and conduction current
- Transverse electromagnetic field solutions for a propagating wave
- Decay of EM waves in a conductor, skin depth
- Energy stored and transported by an EM wave