Electrodynamics II: Lecture 5

EM waves with boundaries

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Aug 17, 2011

Outline

1. EM waves at dielectric boundaries: reflection, transmission
   - $\vec{\mathbf{E}}$ in the plane of incidence
   - $\vec{\mathbf{E}}$ normal to the plane of incidence
2. EM waves in conductors: inside and at the boundary

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Reflection and refraction

An EM wave is incident from one medium ($\epsilon_1, \mu_1, n_1, c_1$) to another medium ($\epsilon_1, \mu_1, n_1, c_1$), at an angle θ_I with the normal to the boundary.

The figure:

Incident, reflected and refracted waves

Incident wave

$$\vec{\mathbf{E}}_I = \vec{\mathbf{E}}_{I0} e^{i(\vec{\mathbf{k}}_I \cdot \vec{\mathbf{r}} - \omega t)} \quad (1)$$

$$\vec{\mathbf{B}}_I = \frac{\vec{\mathbf{k}}_I}{\omega} \times \vec{\mathbf{E}}_I = \frac{1}{c_1}(\hat{\mathbf{k}}_I \times \vec{\mathbf{E}}_I) \quad (2)$$

Reflected wave

$$\vec{\mathbf{E}}_R = \vec{\mathbf{E}}_{R0} e^{i(\vec{\mathbf{k}}_R \cdot \vec{\mathbf{r}} - \omega t)} \quad (3)$$

$$\vec{\mathbf{B}}_R = \frac{\vec{\mathbf{k}}_R}{\omega} \times \vec{\mathbf{E}}_R = \frac{1}{c_1}(\hat{\mathbf{k}}_R \times \vec{\mathbf{E}}_R) \quad (4)$$

Transmitted wave

$$\vec{\mathbf{E}}_T = \vec{\mathbf{E}}_{T0} e^{i(\vec{\mathbf{k}}_T \cdot \vec{\mathbf{r}} - \omega t)} \quad (5)$$

$$\vec{\mathbf{B}}_T = \frac{\vec{\mathbf{k}}_T}{\omega} \times \vec{\mathbf{E}}_T = \frac{1}{c_2}(\hat{\mathbf{k}}_T \times \vec{\mathbf{E}}_T) \quad (6)$$

Boundary conditions on phases

$\vec{\mathbf{D}}_\perp$ is continuous across the boundary

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp} + \epsilon_1 \vec{\mathbf{E}}_{R\perp} = \epsilon_2 \vec{\mathbf{E}}_{T\perp} \tag{7}$$

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp 0} e^{i(\vec{\mathbf{k}}_I \cdot \vec{\mathbf{r}} - \omega t)} + \epsilon_1 \vec{\mathbf{E}}_{R\perp 0} e^{i(\vec{\mathbf{k}}_R \cdot \vec{\mathbf{r}} - \omega t)} = \epsilon_2 \vec{\mathbf{E}}_{T\perp 0} e^{i(\vec{\mathbf{k}}_T \cdot \vec{\mathbf{r}} - \omega t)} \tag{8}$$

- The equatity should be valid at all $\vec{\mathbf{r}}$ on the boundary

$$\vec{\mathbf{k}}_I \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_R \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_T \cdot \vec{\mathbf{r}} \tag{9}$$

- With origin at the point of incidence:

$$|\vec{\mathbf{k}}_I| r \sin\theta_I = |\vec{\mathbf{k}}_R| r \sin\theta_R = |\vec{\mathbf{k}}_T| r \sin\theta_T \tag{10}$$

- Using $|k_I| = |k_R|$ and $|k_T|/|k_I| = n_2/n_1$,

$$\sin\theta_I = \sin\theta_R \,, \quad \frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \tag{11}$$

The first is the law of reflection the second is the Snell's law

Boundary conditions on amplitudes

The discussion on the previous page would have worked for any of the boundary conditions, we just took $\vec{\mathbf{D}}_\perp$ as an example. Now we need not worry about the phases, since the laws of reflection and refraction derived there guarantee that the phase conditions will be satisfied.

Boundary conditions

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp 0} + \epsilon_1 \vec{\mathbf{E}}_{R\perp 0} = \epsilon_2 \vec{\mathbf{E}}_{T\perp 0} \tag{12}$$

$$\vec{\mathbf{B}}_{I\perp 0} + \vec{\mathbf{B}}_{R\perp 0} = \vec{\mathbf{B}}_{T\perp 0} \tag{13}$$

$$\vec{\mathbf{E}}_{I\|0} + \vec{\mathbf{E}}_{R\|0} = \vec{\mathbf{E}}_{T\|0} \tag{14}$$

$$\frac{1}{\mu_1}\vec{\mathbf{B}}_{I\|0} + \frac{1}{\mu_1}\vec{\mathbf{B}}_{R\|0} = \frac{1}{\mu_2}\vec{\mathbf{B}}_{T\|0} \tag{15}$$

For convenience we'll divide the incident electric field into a component in the plane of incidence (the plane that contains $\vec{\mathbf{k}}_I, \vec{\mathbf{k}}_R, \vec{\mathbf{k}}_T$) and a component normal to the plane of incidence. These two clearly won't interfere, and they can be added together at any time, using the principle of superposition, to get the net electric field.

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The figure

Applying boundary conditions

Boundary conditions involving $\vec{\mathbf{E}}$

$$-\epsilon_1 E_{I0} \sin\theta_I + \epsilon_1 E_{R0} \sin\theta_R = -\epsilon_2 E_{T0} \sin\theta_T \quad (16)$$

$$-E_{I0} \cos\theta_I + E_{R0} \cos\theta_R = E_{T0} \cos\theta_T \quad (17)$$

- Solution:

$$E_{R0} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{I0}\ , \quad E_{T0} = \left(\frac{2}{\alpha + \beta}\right) E_{I0} \quad (18)$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}\ , \quad \beta = \frac{\mu_1}{\mu_2}\frac{c_1}{c_2} \quad (19)$$

Boundary conditions involving $\vec{\mathbf{B}}$ give exactly the same conditions.

Reflection and transmission coefficient

- Rate of energy transported by incoming wave normal to the boundary: (Correct this, right language, factors of c, connect with N)

$$\text{Incident wave}: \quad I_I = \frac{1}{2}\epsilon_1 c_1 |\vec{\mathbf{E}}_{I0}|^2 \cos\theta_I \tag{20}$$

$$\text{Reflected wave}: \quad I_R = \frac{1}{2}\epsilon_1 c_1 |\vec{\mathbf{E}}_{R0}|^2 \cos\theta_R \tag{21}$$

$$\text{Transmitted wave}: \quad I_T = \frac{1}{2}\epsilon_2 c_2 |\vec{\mathbf{E}}_{T0}|^2 \cos\theta_T \tag{22}$$

- Reflection coefficient

$$R = \frac{I_R}{I_I} = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|^2 \tag{23}$$

- Transmission coefficient

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 c_2}{\epsilon_1 c_1}\frac{\cos\theta_T}{\cos\theta_I} = \frac{4Re(\alpha^*\beta)}{|\alpha + \beta|^2} \tag{24}$$

Comments on reflection and transmission coefficients

- $R + T = 1$, as expected
- $R = 1, T = 0$ possible if α is purely imaginary.

$$\alpha = \frac{\sqrt{1 - \sin^2 \theta_T}}{\cos \theta_I} = \frac{\sqrt{1 - (n_2/n_1)^2 \sin^2 \theta_T}}{\cos \theta_I} , \tag{25}$$

so if $\sin \theta_I > (n_1/n_2)$, there is no transmission.
This is the condition for Total Internal reflection.

- $R = 0, T = 1$ possible if $\alpha = \beta$. This condition takes a simple form if $\mu_1 = \mu_2$, since then

$$\frac{\cos \theta_T}{\cos \theta_I} = \frac{c_1}{c_2} = \frac{\sin \theta_I}{\sin \theta_T} = \frac{c_1}{c_2} \tag{26}$$

This leads to $\sin 2\theta_I = \sin 2\theta_T$, that is $\theta_I + \theta_T = \pi/2$.
In such a case, θ_I is called the Brewster's angle.

Outline

The figure

Comments on this scenario

- The values for R and T will in general be different. In particular, $R = 0$ is not possible here.
- If an unpolarized wave is incident on a dielectric surface, the reflected and transmitted waves will therefore, in general, be polarized.

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Reflection from a conducting surface

- No wave is transmitted inside the conductor; i.e. fields inside the conductor are zero.
- For a normal incidence, $\vec{\mathbf{E}}_I = -\vec{\mathbf{E}}_R$
  I.e. there is a phase-shift by π.
- For incidence at an angle, the components of $\vec{\mathbf{E}}_I$ and $\vec{\mathbf{E}}_R$ parallel to the boundary cancel, i.e. $\vec{\mathbf{E}}_{I\|} = -\vec{\mathbf{E}}_{R\|}$
- There will be charge oscillations at the metal surface corresponding to $\epsilon_1(\vec{\mathbf{E}}_{I\perp} + \vec{\mathbf{E}}_{I\perp}) = \sigma_s$, where σ_s is the surface charge density
- The movements of these charges along the surface correspond to surface currents, which account for finite values of $\vec{\mathbf{H}}_{I\|} + \vec{\mathbf{H}}_{R\|}$ at the boundary.
- The net $\vec{\mathbf{B}}$ normal to the surface vanishes, i.e. $\vec{\mathbf{B}}_{I\perp} + \vec{\mathbf{B}}_{R\perp} = 0$.
  This follows automatically from the $\vec{\mathbf{E}}_{\|}$ conditions above.

Recap of topics covered in this lecture

- Reflection and transmission at the surface of a dielectric
- Boundary conditions at a condusting surface