Electrodynamics II: Lecture 6

EM waves in confined spaces

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Aug 19, 2011

Outline

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Travelling waves with the same (x, y) profile

- We are looking for waves travelling in z direction, while keeping the same (x, y) profile. I.e. the form

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}^0(x, y)e^{i(k_z z - \omega t)} \;, \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}^0(x, y)e^{i(k_z z - \omega t)} \tag{1}$$

- Maxwell's $(\nabla \times \vec{\mathbf{E}})$ and $(\nabla \times \vec{\mathbf{B}})$ equations then become

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \;, \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \tag{2}$$

$$\frac{\partial E_z}{\partial y} - ik_z E_y = i\omega B_x \;, \qquad \frac{\partial B_z}{\partial y} - -ik_z B_y = -\frac{i\omega}{c^2}E_x \tag{3}$$

$$ik_z E_x - \frac{\partial E_z}{\partial x} = i\omega B_y \;, \qquad ik_z B_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \tag{4}$$

- Note that one can factor out the $e^{i(k_z z - \omega t)}$ dependence of E_x, E_y, E_z and B_x, B_y, B_z, so now onwards they have no z- or t-dependence in this lecture.
- Using the last two lines (4 equations), one can write E_x, E_y, B_x, B_y in terms of the other two quantities, E_z and B_z

All components in terms of E_z and B_z

$$E_x = \frac{1}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y}\right) \quad (5)$$

$$E_y = \frac{1}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x}\right) \quad (6)$$

$$B_x = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y}\right) \quad (7)$$

$$B_y = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x}\right) \quad (8)$$

- Note that if E_z and B_z both vanish (or are constants), no other components of $\vec{\mathbf{E}}$ or $\vec{\mathbf{B}}$ can survive (unless $k_z = 0$, which case needs to be treated separately.)
- However E_z and B_z are not free parameters; the above equations just give four constraints on $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, two more constraints from the last page are still remaining.

Constraining E_z, B_z themselves

E_z, B_z themselves must satisfy consistency conditions

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \quad (9)$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z \quad (10)$$

These correspond to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_z^2 + \frac{\omega^2}{c^2}\right) E_z = 0 \quad (11)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_z^2 + \frac{\omega^2}{c^2}\right) B_z = 0 \quad (12)$$

If there were no boundary conditions in the $x - y$ plane, this would have a plane wave solution – a flat $x - y$ profile . But conducting boundaries imply that these fields must have a non-trivial $x - y$ profile.

EM wave propagation in waveguides

- Let us consider rectangular / circular hollow conducting cylinders, through which an EM wave will be "guided" by bending the boundaries of the cylinders.
- A simple solution would have been a plane wave travelling along z direction, such that $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are transverse, $E_z = B_z = 0$. Such a solution is called as TEM (transverse electric and magnetic) mode.
- Such a mode is not possible in a hollow cylinder, proof given on the next page
- However E_z and B_z can individually vanish, such modes are termed TE (Transverse electric: $E_z = 0$) and TM (Transverse magnetic: $B_z = 0$).

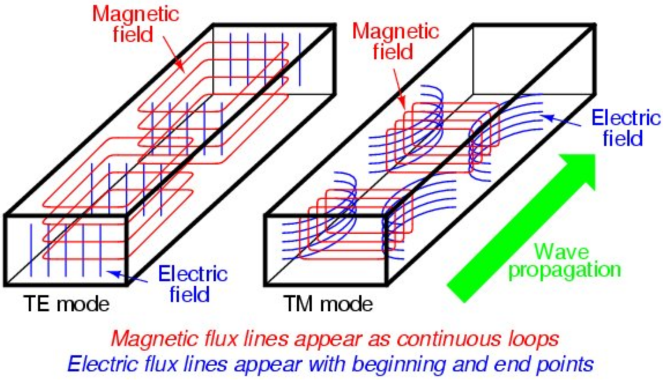
Hollow cylinder cannot have both $E_z = 0$ and $B_z = 0$

- Since $B_z = 0$, we have $(\nabla \times \vec{\mathbf{E}})_z = -\partial B_z/\partial t = 0$. Then

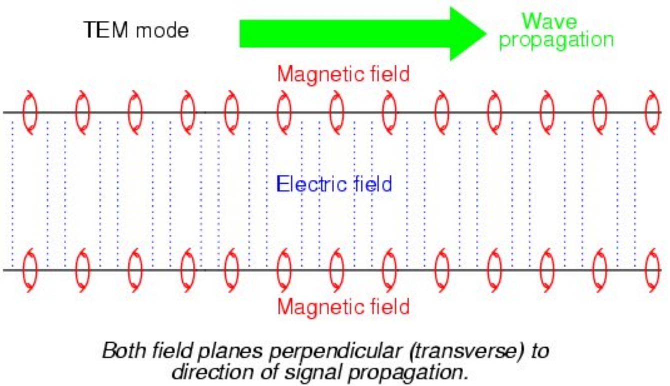
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad (13)$$

- Since E_x and E_y are independent of z, and $E_z = 0$, we get $\nabla \times \vec{\mathbf{E}} = 0$, i.e. $\vec{\mathbf{E}}$ can be written as $\vec{\mathbf{E}} = \nabla\Phi$.
- In addition, no charges inside the cylinder, so $\nabla \cdot E = 0$. That is, $\nabla^2\Phi = 0$.
- Now we have a boundary value problem, with $\nabla^2\Phi = 0$ inside the boundary and $\Phi =$constant on the complete boundary (the hollow conductor).
- This boundary value problem has a solution, $\Phi =$constant everywhere, and the uniqueness theorem states that this is the only solution.
- Thus, there can be no electric / magnetic fields inside the waveguide.

TE and TM modes



TEM mode


TEM mode
Wave propagation
Magnetic field
Electric field
Magnetic field
Both field planes perpendicular (transverse) to direction of signal propagation.

Outline

TE modes ($E_z = 0, B_z \neq 0$) in a rectangular waveguide

- Let the walls of the waveguide be at $y = 0, b$ and $x = 0, a$.
  The boundary conditions are then
  $E_x = 0$ at $y = 0, b$ and $E_y = 0$ at $x = 0, a$
- The equations that give $E_{x,y}$ in terms of B_z then imply
  $\frac{\partial B_z}{\partial y} = 0$ at $y = 0, b$ and $\frac{\partial B_z}{\partial y} = 0$ at $x = 0, a$
- The solution to the differential equation for B_z, with these boundary conditions, is

  $$B_z = A\cos(k_x x)\cos(k_y y) \tag{14}$$

  where $k_x = (m\pi/a)$ and $k_y = (n\pi/b)$.
- Such a mode is called TE_{mn} mode. Note that at least one of m or n has to be nonzero, else all fields will vanish.

Cutoff frequencies for TE modes

- The TE_{mn} solution, when substituted in the differential equation for B_z, gives

$$-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2 + (\omega/c)^2 = 0 \qquad (15)$$

- For consistency with the physical situation, k_z must be real; i.e. $k_z^2 > 0$. This gives the condition

$$\omega > c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \equiv \omega_{mn} \qquad (16)$$

- Thus, for a TE mode TE_{mn} to propagate, it must have a minimum frequency ω_{mn}. A waveguide thus acts like a high-pass filter.

TM modes

- A similar analysis is possible for TM modes, but this will not be done here.
- Note that the cutoff frequencies ω_{mn} for the TM modes are the same as those for TE modes.

Phase velocity and group velocity

- Phase velocity: simply the speed at which the crest of the wavefront travels in a given direction.
- For a plane wave $Ae^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}-\omega t)}$, the phase velocity along the direction $\hat{\mathbf{r}}$ is

$$v_{ph} = \left.\frac{dr}{dt}\right|_{\text{constant phase}} = \frac{\omega}{|\vec{\mathbf{k}}.\hat{\mathbf{r}}|} \tag{17}$$

- If $\vec{\mathbf{k}}$ is not along $\hat{\mathbf{r}}$, typically $v_{ph} > c$. This does not mean that any signal is travelling faster than light, though.
- Group velocity measures the speed at which a signal is transported. The signal is embedded in the distribution of frequencies, and group velocity measures how fast the peak of this distribution shifts. Details on the next page.

Group velocity

- The Fourier transform of a wave gives the frequencies the wave consists of. Consider the situation where the spread in frequencies is small, which is the only one where we can define a group velocity easily. Let the frequencies be confined to the range $\omega = \omega_0 \pm \Delta\omega$. The corresponding wave vectors are confined to $\vec{\mathbf{k}} = \vec{\mathbf{k}}_0 \pm \Delta\vec{\mathbf{k}}$.
- The wave is

$$\psi(\vec{\mathbf{x}}, t) = \int a(\vec{\mathbf{k}}) e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t)} d^3k \ , \tag{18}$$

which may be written as

$$\psi(\vec{\mathbf{x}}, t) = A(\vec{\mathbf{x}}, t) e^{i(\vec{\mathbf{k}}_0\cdot\vec{\mathbf{x}} - \omega_0 t)} \ , \tag{19}$$

where

$$A(\vec{\mathbf{x}}, t) = \int a(\vec{\mathbf{k}}) e^{i(\Delta\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \Delta\omega t)} d^3k \tag{20}$$

- The frequency distribution shifts as a wavepacket, the velocity of the peak of the distribution is the approximate velocity of the wavepacket.

Group velocity: continued

- Let us consider a one-dimensional case of a wave travelling along z-axis. At the peak,

$$0 = \frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial z}\frac{dz}{dt} \tag{21}$$

- The group velocity is then

$$v_g = \frac{dz}{dt} = -\frac{\partial A/\partial t}{\partial A/\partial z} = \frac{\Delta\omega}{\Delta k} = \left.\frac{d\omega}{dk}\right|_{\omega_0} \tag{22}$$

Velocities along z axis for the waveguide

- $\omega = \sqrt{\omega_{mn}^2 + k_z^2}$
- Phase velocity $v_{ph} = \frac{\omega}{k_z} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$
- Group velocity $v_g = \frac{d\omega}{dk_z} = c\sqrt{1-(\omega_{mn}/\omega)^2}$
- Waveguide transports different frequencies at different speeds: dispersion

Outline

Circular cylindrical waveguides

No TEM mode, as per the earlier arguments

For TM mode

$$E_z = AJ_m(k_\ell r)e^{im\phi}e^{i(k_z z - \omega t)} \tag{23}$$

If the cylinder has radius r_0,
then the boundary condition is $J_m(k_\ell r_0) = 0$, gives $k_\ell(m)$

For TE mode

$$B_z = AJ_m(k_\ell r)e^{im\phi}e^{i(k_z z - \omega t)} \tag{24}$$

If the cylinder has radius r_0,
then the boundary condition is $J'_m(k_\ell r_0) = 0$, gives $k_\ell(m)$

- $k_z^2 = (\omega/c)^2 - k_\ell^2 \Rightarrow$ cutoff frequency $\omega_{m,\ell} = ck_\ell(m)$
- TM and TE modes have different cutoff frequencies, unlike rectangular waveguides !

Power transmitted by a waveguide

- Consider TE mode. i.e. $E_z = 0$.
- The equations for $\vec{\mathbf{E}}_\perp = (E_x, E_y)$ and $\vec{\mathbf{B}}_\perp = (B_x, B_y)$ become

$$\left.\begin{aligned} B_x &= \frac{ik_z}{k_\perp^2}\frac{\partial B_z}{\partial x} \\ B_y &= \frac{ik_z}{k_\perp^2}\frac{\partial B_z}{\partial y} \end{aligned}\right\} \Rightarrow \quad \vec{\mathbf{B}}_\perp = \frac{ik_z}{k_\perp^2}\nabla_\perp B_z \tag{25}$$

$$\left.\begin{aligned} E_x &= \frac{ck}{k_\perp^2}\frac{\partial B_z}{\partial y} \\ E_y &= -\frac{ck}{k_\perp^2}\frac{\partial B_z}{\partial x} \end{aligned}\right\} \Rightarrow \quad \vec{\mathbf{E}}_\perp = \frac{ick}{k_z}\vec{\mathbf{B}}_\perp \times \vec{\mathbf{z}} \tag{26}$$

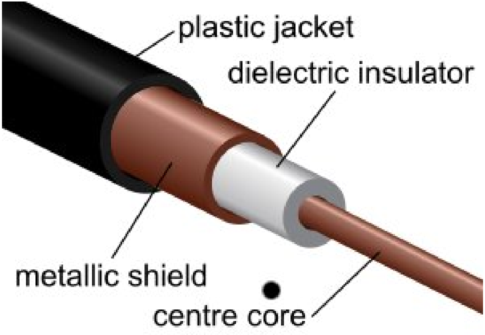
- The magnitude of Poynting vector (power transmitted per unit area) is then

$$|\vec{\mathbf{N}}| = \frac{|\vec{\mathbf{E}}_{\perp 0}^* \times \vec{\mathbf{H}}_{\perp 0}|^2}{2} = \frac{|\vec{\mathbf{E}}_{\perp 0}|^2}{2}\vec{\mathbf{k}}_z ck\mu_0 = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{k_z}{k}|\vec{\mathbf{E}}_0|^2 \tag{27}$$

- Comparing with $|\vec{\mathbf{N}}| = (1/2)\sigma|E_0|^2$, this enables us to define the conductance of the waveguide as $\sigma = \sqrt{\epsilon_0}\mu_0(k_z/k)$. This may be compared with the conductance of free space, $\sqrt{\epsilon_0}\mu_0$.

Outline

Coaxial cable



Propagation through a coaxial cable

- TEM Mode is supported (now there are two disjoint boundaries, so the argument for hollow waveguides does not work.)
- TE and TM modes also propagate, but have a threshold frequency

The TEM mode

- Electric and magnetic fields:

$$\vec{\mathbf{E}} = \frac{E_0 \hat{\mathbf{r}}}{r} e^{i(k_z z - \omega t)} \ , \quad \vec{\mathbf{B}} = \frac{E_0 \hat{\phi}}{cr} e^{i(k_z z - \omega t)} \tag{28}$$

- Group velocity $v_g = c$

Outline

Rectangular cavity

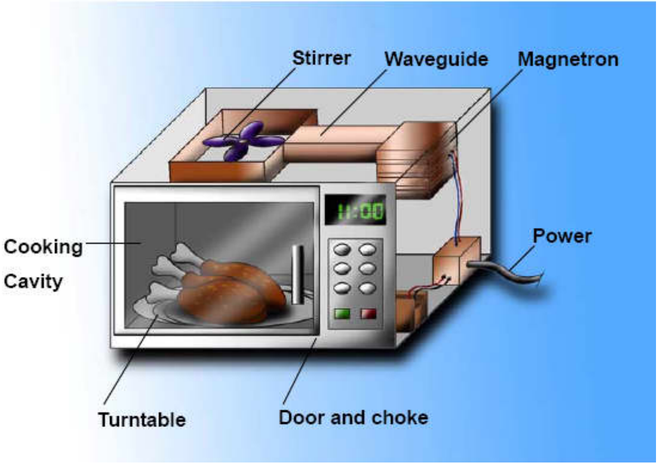
- Conducting walls at $x = 0, a$; at $y = 0, b$ and at $z = 0, c$.
- Potential inside the cavity:

$$\Phi_{mnp} = \sin(k_x x)\sin(k_y y)\sin(k_z z)e^{-i\omega t} \tag{29}$$

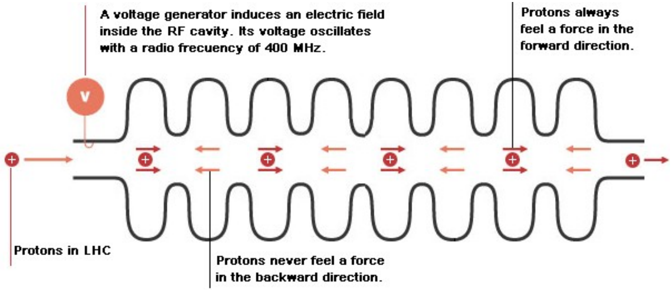
  where $k_x = (m\pi/a), k_y = (n\pi/b), k_z = (p\pi/c)$

- This can be used to obtain $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ inside the cavity.
- A rectangular cavity supports discrete modes.

Microwave: waveguide and cavity



LHC accelerator: cavity principle



LHC accelerator: bunching cavities

Recap of topics civered in this lecture

- Propagation in waveguides in terms of E_z and B_z
- TEM, TE and TM modes from Maxwell's equations
- No TEM modes for hollow waveguides
- Waveguides as high-pass filters, as dispersive media
- Phase velocity and group velocity
- Power transmitted through waveguide
- Coaxial cable: TEM propagation, in addition to TE and TM
- Cavities for bunching protons together at accelerators