Electrodynamics II: Lecture 7

EM wave equation with sources

Amol Dighe

Aug 31, 2011

Outline

1. Wave equation for ϕ and $\vec{A}$ with sources
2. Solving the wave equation with sources

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2. Solving the wave equation with sources

Why the potentials $\vec{\mathbf{A}}$ and ϕ ?

- We already have the wave equation satisfied by $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ in the absence of any charge or current sources:

$$\nabla^2\vec{\mathbf{E}} - \mu\sigma\frac{\partial\vec{\mathbf{E}}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0 \tag{1}$$

  and similarly for $\vec{\mathbf{B}}$.

- However when sources (ρ and $\vec{\mathbf{J}}$) are introduced, they affect $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ in rather complicated ways. Therefore (with hindsight) we formulate our problem in terms of $\vec{\mathbf{A}}$ and ϕ, the vector and scalar potentials, respectively.
- When we come to relativity and covariance of equations, we'll appreciate the importance of $\vec{\mathbf{A}}$ and ϕ even more.

A and ϕ: definitions

- Maxwell's equation $\nabla \cdot \vec{\mathbf{B}} = 0$ alows us to write $\vec{\mathbf{B}}$ as the curl of a vector, we define this vector as $\vec{\mathbf{A}}$:

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \tag{2}$$

  Note that this does not define $\vec{\mathbf{A}}$ completely, since $\nabla \cdot \vec{\mathbf{A}}$ has not yet been defined, so the uniqueness theorem 1 is not satisfied.

- Maxwell's equation $\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$ then implies

$$\nabla \times \vec{\mathbf{E}} = -\nabla \times \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{3}$$

  This allows us to write

$$\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla \phi \tag{4}$$

  where ϕ is a scalar. This is the definition of ϕ.

- Note that ϕ is also not uniquely defined.

Gauge freedom for $\vec{\mathbf{A}}$ and ϕ

- As obserbed earlier, A and ϕ are not uniquely defined.
- Indeed, we can carry out simultaneous gauge trasformations

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} - \nabla\psi \,, \phi' = \phi + \frac{\partial \psi}{\partial t} \tag{5}$$

  with any arbitrary scalar ψ, and these new potentials $\vec{\mathbf{A}}'$ and ϕ' will still give us the same $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.
- Since $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the physically measurable quntities, the potentials $(\vec{\mathbf{A}}, \phi)$ and $(\vec{\mathbf{A}}', \phi')$ are equivalent
- This freedom of choosing any ψ corresponds to the "gauge symmetry". We can choose to do the calculations in any convenient gauge, the final measurable quantities will turn out to be identical / gauge invariant.

Wave equation for $\vec{\mathbf{A}}$

- We have already used two Maxwell's equations while defining $\vec{\mathbf{A}}$ and ϕ: they will be satisfied automatically.
- Using $\nabla \times \vec{\mathbf{B}} = \mu \vec{\mathbf{J}} + \mu\epsilon(\partial \vec{\mathbf{E}}/\partial t)$, where $\vec{\mathbf{J}} = \vec{\mathbf{J}}_{\rm ext} + \sigma \vec{\mathbf{E}}$, we get

$$\nabla \times (\nabla \times \vec{\mathbf{A}}) = \mu \vec{\mathbf{J}}_{\rm ext} + \mu\sigma \left(-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla\phi \right) + \epsilon\mu \left(-\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} \right) \tag{6}$$

- Using $\nabla \times (\nabla \times \vec{\mathbf{A}}) = -\nabla^2 \vec{\mathbf{A}} + \nabla(\nabla \cdot \vec{\mathbf{A}})$, this leads to

$$\nabla^2 \vec{\mathbf{A}} - \mu\sigma \frac{\partial \vec{\mathbf{A}}}{\partial t} - \epsilon\mu \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}}_{\rm ext} + \nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla(\mu\sigma\phi) + \nabla\left(\epsilon\mu \frac{\partial \phi}{\partial t}\right) \tag{7}$$

- If we now use our gauge freedom to make $\nabla \cdot \vec{\mathbf{A}} + \mu\sigma\phi + \epsilon\mu(\partial\phi/\partial t) = 0$, (called as the Lorentz gauge), the we get the wave equation for $\vec{\mathbf{A}}$:

$$\nabla^2 \vec{\mathbf{A}} - \mu\sigma \frac{\partial \vec{\mathbf{A}}}{\partial t} - \epsilon\mu \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}}_{\rm ext} \tag{8}$$

Wave equation for ϕ

- We now see where the remaining Maxwell's equation, $\nabla \cdot \vec{\mathbf{E}} = \rho/\epsilon$ leads us to

$$\nabla \cdot \left(-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla\phi \right) = \frac{\rho}{\epsilon} \Rightarrow -\frac{\partial}{\partial t}(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2\phi = \frac{\rho}{\epsilon} \qquad (9)$$

- Now we use the same Lorentz condition as before to replace $\nabla \cdot \vec{\mathbf{A}}$ by $-\mu\sigma\phi - \epsilon\mu(\partial\phi/\partial t)$, which leads to

$$\nabla^2\phi - \mu\sigma\frac{\partial\phi}{\partial t} - \epsilon\mu\frac{\nabla^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon} \qquad (10)$$

- Note that the form of the equation for ϕ is the same as that for $\vec{\mathbf{A}}$, with the charge a the source, instead of the current.

Outline

The wave equations in vacuum

- In vacuum, the wave equations take the form

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad (11)$$

$$\nabla^2\vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{A}}}{\partial t^2} = -\mu_0\vec{\mathbf{J}} \quad (12)$$

We drop the suffix on $\vec{\mathbf{J}}$ for the sake of brevity.

- We already know that for the static situation, i.e. when the $(\partial^2/\partial t^2)$ terms are absent:

$$\nabla^2\phi(\vec{\mathbf{x}}) = -\frac{\rho(\vec{\mathbf{x}})}{\epsilon_0} \quad \Rightarrow \quad \phi(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \quad (13)$$

$$\nabla^2\vec{\mathbf{A}}(\vec{\mathbf{x}}) = -\mu_0\vec{\mathbf{J}}(\vec{\mathbf{x}}) \quad \Rightarrow \quad \vec{\mathbf{A}}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \quad (14)$$

- We expect (hope) that the solution to the time-dependent wave equation may be similar.

Fourier analysis

- Let us try solving a general equation

$$\nabla^2\psi(\vec{\mathbf{x}},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{\mathbf{x}},t) = -g(\vec{\mathbf{x}},t) \tag{15}$$

by using the method of Fourier transform and Green's function.

- Write the solution $\psi(\vec{\mathbf{x}},t)$ and the source $g(\vec{\mathbf{x}},t)$ in terms of their Fourier transforms ψ_ω and g_ω:

$$\psi(\vec{\mathbf{x}},t) = \int_{-\infty}^{\infty}\psi_\omega(\vec{\mathbf{x}})e^{-i\omega t}d\omega\ , \quad g(\vec{\mathbf{x}},t) = \int_{-\infty}^{\infty}g_\omega(\vec{\mathbf{x}})e^{-i\omega t}d\omega\ , \tag{16}$$

where the Fourier transforms are defined as

$$\psi_\omega(\vec{\mathbf{x}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{\mathbf{x}},t))e^{i\omega t}d\omega\ , \quad g_\omega(\vec{\mathbf{x}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}g(\vec{\mathbf{x}},t))e^{i\omega t}d\omega\ , \tag{17}$$

- In terms of the Fourier transforms, the wave equation becomes

$$\nabla^2\psi_\omega(\vec{\mathbf{x}}) + \frac{\omega^2}{c^2}\psi_\omega(\vec{\mathbf{x}}) = -g_\omega(\vec{\mathbf{x}}) \tag{18}$$

which we'll now try solving using the method of Green's function.

The method of Green's function

- The method of Green's functions implies that:
  If $G(\vec{\mathbf{x}}, \vec{\mathbf{x}}')$ is a solution to the Green's equation

$$\nabla^2 G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') + \frac{\omega^2}{c^2} G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \tag{19}$$

  then the solution to $\psi_\omega(\vec{\mathbf{x}})$ is obtained as

$$\psi_\omega(\vec{\mathbf{x}}) = \int g_\omega(\vec{\mathbf{x}}') G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') d^3 x' \tag{20}$$

  This may be checked by explicit substitution.
- The Green's equation is spherically symmetric, so we expect a spherically symmetric solution, i.e. $G(\vec{\mathbf{x}} - \vec{\mathbf{x}}')$ is simply $G(r)$.
- The Green's equation is then

$$\frac{1}{r}\frac{\partial}{\partial r}[rG(r)] + k^2 G(r) = -\delta(r) \tag{21}$$

- This has a solution (that may be checked by substitution):

$$G(r) = \frac{1}{4\pi r} e^{\pm ikr} \Rightarrow G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \frac{1}{4\pi|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{\pm ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} \tag{22}$$

Solution for $\psi_\omega(\vec{\mathbf{x}})$ and $\psi(\vec{\mathbf{x}}, t)$

- The Green's function method has now given us the solution for $\psi_\omega(\vec{\mathbf{x}})$:

$$\psi_\omega(\vec{\mathbf{x}}) = \frac{1}{4\pi} \int \frac{g_\omega(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{\pm ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{23}$$

- Inverse Fourier transform gives us the solution for $\psi(\vec{\mathbf{x}}, t)$:

$$\psi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi} \int_{\vec{\mathbf{x}}'} \int_\omega \frac{g_\omega(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{i(\omega t \pm k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|)} d^3x' \tag{24}$$

- In terms of $t_\pm \equiv t \pm k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$, this becomes

$$\psi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi} \int_{\vec{\mathbf{x}}'} \frac{g_\omega(\vec{\mathbf{x}}', t_\pm)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{25}$$

- This is the solution to our wave equation

Properties of the solution to wave equation

- The solution formally looks the same as the solution for the static case, except the time dependence, which appears through t_+ and t_-: the advanced and retarded times respectively.
- This implies that the potentials at any point depend on the source distribution at some other times: in particular, at times $t_\pm t \pm |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$. This is akin to a signal taking time $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$ to travel from the source at $\vec{\mathbf{x}}'$ to affect the potential at x.
- Thus, the disturbance caused by the sources travels with the speed c. That is, the speed of light is c.
- When we are dealing with the effect of time-varying sources on the potentials, advanced solutions are not physical since they would violate causality. They will need to be considered when, later in the course, we'll be dealing with the back-reaction of the changes in potential on the sources.

The retarded potentials

- The retarded potentials, caused by time-varying sources, are:

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\vec{\mathbf{x}}')]}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{26}$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' = \frac{\mu_0}{4\pi} \int \frac{[\vec{\mathbf{J}}(\vec{\mathbf{x}}')]}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{27}$$

where $[f(\vec{\mathbf{x}})]$ is a convention used to write $f\left(\vec{\mathbf{x}}, t - \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c}\right)$

Recap of topics covered in this lecture

- Definitions of the potentials $\vec{\mathbf{A}}$ and ϕ, gauge freedom
- Lorentz gauge and wave equations for $\vec{\mathbf{A}}$ and ϕ in the presence of sources (charges and currents)
- Solution to the wave equation in vacuum, using Fourier transforms and Green's function
- Advanced and retarded solutions for the potentials $\vec{\mathbf{A}}$ and ϕ