Electrodynamics II: Lecture 8

EM radiation

Amol Dighe

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Outline

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$\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ from $\vec{\mathbf{A}}$ and ϕ

- Now that we know

$$\phi(\vec{\mathbf{x}},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{x}}',t_-)}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \ , \quad \vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}',t_-)}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \tag{1}$$

- We can calculate $\vec{\mathbf{E}} = -\nabla\phi - \partial\vec{\mathbf{A}}/\partial t$ and $\vec{\mathbf{B}} = \nabla\times\vec{\mathbf{A}}$. Note that we'll have to use

$$\nabla\rho(\vec{\mathbf{x}}',t_-) = \hat{\mathbf{r}}\frac{d}{dr}\rho = \hat{\mathbf{r}}\left(\frac{\partial\rho}{\partial r} + \frac{\partial\rho}{\partial t_-}\frac{dt_-}{dr}\right) = \hat{\mathbf{r}}\left(\frac{\partial\rho}{\partial r} + \frac{1}{c}\frac{\partial\rho}{\partial t}\right) \tag{2}$$

  where $\vec{\mathbf{r}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}'$.

- The electric and magnetic fields turn out to be

$$\begin{aligned}
\vec{\mathbf{E}}(\vec{\mathbf{x}},t) &= \frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho(\vec{\mathbf{x}}',t_-)}{r^2}\hat{\mathbf{r}} + \frac{\dot{\rho}(\vec{\mathbf{x}}',t_-)}{cr}\hat{\mathbf{r}} - \frac{\dot{\vec{\mathbf{J}}}(\mathbf{x}',\mathbf{t}_-)}{c^2r}\right)d^3x' \\
\vec{\mathbf{B}}(\vec{\mathbf{x}},t) &= \frac{\mu_0}{4\pi}\int\left(\frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}',t_-)}{r^2}\times\hat{\mathbf{r}} + \frac{\dot{\vec{\mathbf{J}}}(\mathbf{x}',\mathbf{t}_-)}{cr}\times\hat{\mathbf{r}}\right)d^3x'
\end{aligned} \tag{3}$$

$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$: behaviour at large $|\vec{\mathbf{x}}|$

- In both, $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$, there are terms that behave as $1/r^2$ and there are terms that behave as $1/r$. The former are proportional to the sources, the latter are proportional to the rate of change of sources.
- When the sources are confined to a small region $|\vec{\mathbf{x}}'| < d$, then for $|\vec{\mathbf{x}}| >> d$, the $1/r$ terms dominate over the others. These are the "radiative" components of the fields.
- The radiative $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are then

$$\vec{\mathbf{E}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\frac{[\dot{\rho}(\vec{\mathbf{x}}')]}{cr} \hat{\mathbf{r}} - \frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')]}{c^2 r} \right) d^3x' \tag{4}$$

$$\vec{\mathbf{B}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \int \left(\frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}}{cr} \right) d^3x' \tag{5}$$

$\vec{\mathbf{E}}_{\mathrm{rad}}$ and $\vec{\mathbf{B}}_{\mathrm{rad}}$ in terms of $\vec{\mathbf{J}}$ only

- Given the continuity equation

$$\nabla' \cdot J(\vec{\mathbf{x}}', t) + \dot{\rho}(\vec{\mathbf{x}}', t) = 0 \ , \tag{6}$$

  it is clear that if $[\vec{\mathbf{J}}(\vec{\mathbf{x}}')]$ is known everywhere, so is $[\dot{\rho}(\vec{\mathbf{x}}')]$, and the radiative $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields can be written in terms of $\vec{\mathbf{J}}$ only.

- Some algebraic manipulation yields

$$\vec{\mathbf{B}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^3} \int \frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}}{r} d^3x' \tag{7}$$

$$\vec{\mathbf{E}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{([\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{r} d^3x' \tag{8}$$

  Note that $\mu_0/(4\pi) = 1/(4\pi\epsilon c^2)$, one can use any combination.

Frequency components of radiation fields

- Fourier components of $\vec{\mathbf{B}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{E}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t)$ give the radiation fields in terms of their frequency components:

$$B_{\omega}^{rad}(\vec{\mathbf{x}}) = \frac{-i}{4\pi\epsilon c^2}\int\left(\vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}')\times\vec{\mathbf{k}}\right)\frac{e^{ikr}}{r}d^3x' \qquad (9)$$

$$E_{\omega}^{rad}(\vec{\mathbf{x}}) = \frac{i}{4\pi\epsilon_0 c}\int\left(\hat{\mathbf{r}}\times(\vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}')\times\vec{\mathbf{k}})\right)\frac{e^{ikr}}{r}d^3x' \qquad (10)$$

- If the sources are monochromatic, the above directly give the corresponding $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields. We shall use this in calculating the power radiated by periodically time-varying charges and currents.

Outline

Calculation of radiated power

- The Poynting vector is $\vec{\mathbf{N}} = \vec{\mathbf{E}}^* \times \vec{\mathbf{H}}$, which gives the power radiated per unit area along it.
- Total energy radiated per unit area normal to $\vec{\mathbf{N}}$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} \vec{\mathbf{N}}(t)dt &= \int_{-\infty}^{\infty} \vec{\mathbf{E}}^*(t) \times \vec{\mathbf{H}}(t)dt && (11)\\
&= \int_{\omega,\omega',t} \vec{\mathbf{E}}_\omega^{\mathrm{rad}*} e^{i\omega t} d\omega \times \vec{\mathbf{H}}_{\omega'} e^{-i\omega' t} d\omega' dt \\
&= 2\pi \int \vec{\mathbf{E}}_\omega^{\mathrm{rad}*} \times \vec{\mathbf{H}}_\omega^{\mathrm{rad}} d\omega && (12)
\end{aligned}$$

- Substituting the expressions for $\vec{\mathbf{E}}_\omega^{\mathrm{rad}}$ and $\vec{\mathbf{H}}_\omega^{\mathrm{rad}} = \vec{\mathbf{B}}_\omega^{\mathrm{rad}}/\mu_0$ obtained earlier, after a bit of algebra, gives

$$\vec{\mathbf{E}}_\omega^{\mathrm{rad}*} \times \vec{\mathbf{H}}_\omega^{\mathrm{rad}} = \frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) \frac{e^{ikr}}{r} d^3x' \right| \qquad (13)$$

Total energy radiated

- Total radiated energy across a surface $d\vec{\mathbf{S}}$ is

$$U = 2\pi \int \vec{\mathbf{E}}^{\mathrm{rad}*}_{\omega} \times \vec{\mathbf{H}}^{\mathrm{rad}}_{\omega} d\omega \cdot d\vec{\mathbf{S}} \quad (14)$$

$$= \frac{1}{8\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \int_{\vec{\mathbf{S}},\omega} \left| \int \left(\vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) \frac{e^{ikr}}{r} d^3x' \right| d\omega \hat{\mathbf{r}} \cdot d\vec{\mathbf{S}} \quad (15)$$

- Taking the surface to be a part of the surface of a sphere of radius r, using $d\vec{\mathbf{S}} = r^2 \partial\Omega \hat{\mathbf{r}}$, one gets

$$\frac{d^2U}{d\omega d\Omega} = \frac{1}{8\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) e^{ikr} d^3x' \right| \quad (16)$$

- For a monochromatic source, one can define average power radiated by averaging over one cycle:

$$\frac{dU}{d\Omega} = \frac{1}{2}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) e^{ikr} d^3x' \right| \quad (17)$$

The factor of $1/2$ comes from averaging.

Long-distance approximation

- When $|\vec{\mathbf{x}}| >> |\vec{\mathbf{x}}'|$, then we have

$$kr = k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| \approx k|\vec{\mathbf{x}}| - \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}' \tag{18}$$

so that

$$\frac{d^2U}{d\omega d\Omega} = \frac{1}{8\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x'\right| \tag{19}$$

This can be used in many instances, for example for radiation from antennas, as will be seen in the next section.

Outline

Sinusoidal driving current

- An antenna may be represented as a monochromatic sinusoidal current with frequency ω:

$$J(x', y', z') = I_0 \delta(y') \delta(z') \frac{\sin[k(L/2 - |x'|)]}{\sin(kL/2)} \tag{20}$$

- An useful result here is

$$\left| \int (\vec{\mathbf{J}} \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'} d^3x' \right| = \frac{2I_0}{\sin\theta \sin(kL/2)} [\cos(kL\cos\theta/2) - \cos(kL/2)] \tag{21}$$

- The radiation from such an antenna can be calculated in closed form in special cases like (i) long wavelength limit: $kL \ll 1$ (ii) half-wave antenna: $kL = \pi/2$, (iii) full-wave antenna: $kL = \pi$.

Recap of topics covered in this lecture

- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields in the presence of moving sources
- Radiative components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$: the $1/r$ behaviour that dominates at large distances
- Poynting vector and power radiated by EM waves
- Long distance approximation for radiated power
- Antennas as sources of EM radiation