Electrodynamics II : Autumn 2018

Assignment 3

Given: Friday Oct 12, Expected: Tuesday Oct 30

This assignment has two sections. You need to submit answers to only the questions from Section I. The questions from Section II are strongly recommended for practice / understanding of concepts, however you need not submit them and will not be graded on them.

Section I

1. For a Lagrangian density $\mathcal{L}(q, \partial_i q)$, [Note: q here is the generalized coordinate, not the charge] the energy-momentum tensor T_i^k is given by $T_i^k = \partial_i q[\partial\mathcal{L}/\partial(\partial_k q)] - \delta_i^k \mathcal{L}$. For the electromagnetic field in the absence of charges, $\mathcal{L} = -(\epsilon_0 c/4)F_{kl}F^{kl}$.
   (a) Taking q as the 4-potential A_m, determine T_i^k in terms of the components of the electromagnetic field tensor F.
   (b) Calculate the components of T_i^k in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

2. Angular momentum tensor in 4-d:
   (a) An infinitesimal rotation in 4-d is defined as $x'^i - x^i = \delta x^i = x_k \delta\omega^{ik}$. Show that $\delta\omega_{ik}$ is an antisymmetric tensor.
   (b) For a collection of free particles, the action is $S = -\sum mc \int_a^b ds$. Show that $\delta S = \delta\omega_{ik} M^{ik}$ where $M^{ik} = (1/2)\sum(p^i x^k - p^k x^i)$. Hence argue that M^{ik} is conserved.
   (c) M^{ik} is the angular momentum 4-tensor. Calculate the components of this tensor in terms of $\vec{\mathbf{r}}, \vec{\mathbf{p}}$ and $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ of the individual particles.

3. A particle with mass m and charge q is moving in a region with constant electric field $\vec{\mathbf{E}}$ and constant magnetic field $\vec{\mathbf{B}}$, both along the z axis. The initial momentum of the particle is (p_{x0}, p_{y0}, p_{z0}).
   (a) Determine the trajectory of the particle, i.e. determine $x(t), y(t), z(t)$. You may find it convenient to use the variables $p_\perp$ and ϕ instead of p_x and p_y, such that $p_\perp e^{i\phi} = p_x + ip_y$.
   (b) Plot v_z and $v_\perp = \sqrt{v_x^2 + v_y^2}$ as a function of t (on the same plot) for some appropriate values of parameters that will bring out the main features.

4. A particle with mass m and charge q is moving in a region with constant electric field $\vec{\mathbf{E}}$ and constant magnetic field $\vec{\mathbf{B}}$, such that $\vec{\mathbf{E}} = E_x \hat{x}$ and $\vec{\mathbf{B}} = B_z \hat{z}$. The initial momentum of the particle is $\vec{\mathbf{p}} = (p_{x0}, p_{y0}, p_{z0})$. Plot the trajectories for different sets of initial conditions and point out different possible behaviours. (Analytical answers not expected, but they may help you identifying which initial conditions to look for.)

5. Two charges q_1 and q_2 are moving with uniform velocities along the x and y axis respectively. Their position vectors are given as

$$\vec{\mathbf{x}}_1 = v_1 t \; \hat{\mathbf{x}} \; , \quad \vec{\mathbf{x}}_2 = v_2 t \; \hat{\mathbf{y}} \; .$$

(a) Calculate the potentials $\phi(x, y, z, t)$ and $\vec{\mathbf{A}}(x, y, z, t)$ due to the charge q_1.
(b) Hence calculate the fields $\vec{\mathbf{E}}(x, y, z, t)$ and $\vec{\mathbf{B}}(x, y, z, t)$ due to the charge q_1.
(c) Draw a diagram showing the positions of q_1 and q_2 at an arbitrary time t. Qualitatively show the directions of $\vec{\mathbf{E}}(\vec{\mathbf{x}}_2, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}_2, t)$. Point out the important features.
(d) Calculate the force $\vec{\mathbf{F}}_{12}$ on the charge q_2 due to the charge q_1.
(e) Calculate, and show in the figure, the force $\vec{\mathbf{F}}_{21}$ on the charge q_1 due to the charge q_2.
Comment on the relative directions of $\vec{\mathbf{F}}_{12}$ and $\vec{\mathbf{F}}_{21}$. Your answers should be in terms of x, y, z, t, v_1, v_2 and other universal constants, but no other variables. There is no need to combine terms to simplify.

Section II

1. Given the action $S = -mc \int_a^b ds - q \int_a^b A_i dx^i$, Write down the Hamiltonian (in the 3-vector language), and the equations of motion that follow from that.

2. Add the term $\eta \int \tilde{F}_{ik} F^{ik} d\Omega$ to the action of the electromagnetic field. Using the variation of the 4-potential A, determine the equations of motion. Is it possible to compare these with Maxwell's equations to determine the value of η ?

3. Find the trajectory of a charge which enters a constant uniform electric field $\vec{\mathbf{E}}$, making an angle θ with it. Which are the different qualitative behaviours ?