Electrodynamics II : Autumn 2018

Assignment 4

Given: Tuesday Nov 20, Expected: Monday Dec 10

This assignment has two sections. You need to submit answers to only the questions from Section I. You are strongly recommended to solve the questions from Section II, however you need not submit them and will not be graded on them.

Section I

1. A relativistic particle is losing energy at a constant rate $\mathcal{R} = dE/dt'$ while moving through a material in a straight line. In the process, the speed of the particle decreases from $v = 0.9c$ to $v = 0$.

   (i) Plot the power radiated as a function of $\cos\theta$ (the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$), when the speed of the particle is $v = 0.9c$, $v = 0.5c$ and $v = 0.1c$ (on the same plot, showing the relative magnitudes, in appropriate units).

   (ii) Calculate the total energy radiated by the particle in the form of Bremsstrahlung radiation. You may need to integrate numerically.

2. For a charge in uniform circular motion on a circle of radius a with angular frequency ω,

   (i) Plot the average power radiated by the charge as a function of θ for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on this angular dependence.

   (ii) Numerically perform the angular integration of dU/dt_r (without the approximation of small $\tilde{\theta}$) to calculate the total power radiated for $\gamma = 1, 10, 100$ in appropriate units. Compare the answers with those obtained analytically by using $\cos\tilde{\theta} \approx 1 - \tilde{\theta}^2$.

3. Starting with the 4-vector expression for the radiation reaction force $f^k_{\rm rad}$,

   (i) Determine the 3-vector radiation reaction force $\vec{\mathbf{F}}$ in the small-velocity limit. Keep terms up to and including the order $(v/c)^2$.

   (ii) Determine the 3-vector radiation reaction force $\vec{\mathbf{F}}$ in the large-velocity limit. Keep the two leading powers of γ.

4. EM wave through dilute electron gas:

   (i) Using the expression for the refractive index of a dilute electron gas with number density N, plot the phase velocity and group velocity of an EM wave travelling through the gas as a function of ω. Neglect the damping term and choose an appropriate value for N. Where is the "dilute" nature of the gas relevant ?

   (ii) Show that a dilute gas of free electrons will not allow an EM wave to propagate through it, if the frequency of the wave is less than a cutoff frequency ω_{cutoff}. Determine the cutoff frequency in terms of the number density of electrons and other universal constants. For $\omega > \omega_{\text{cutoff}}$, qualitatively plot the behaviour of the wavenumber k as a function of ω.

5. A charge q is travelling with speed $0.99c$ through a dilute electron gas of number density N. Plot the spectrum of Cherenkov photons emitted per unit length. Use $\Gamma = 0$, and choose appropriate values of parameters that are realistic, at the same time bring out some interesting features.

Section II

1. (i) By an explicit calculation, show that the magnetic field $\vec{\mathbf{B}}$ from an accelerating charge reduces to the form $\vec{\mathbf{B}} = \vec{\mathbf{r}} \times \vec{\mathbf{E}}/(rc)$, where $\vec{\mathbf{r}} = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}'(t_r)$.

   (ii) Calculate the total energy radiated in Bremsstrahlung when the speed of a charge q decreases at a constant rate a, from v_0 to 0.

2. Starting with

$$\frac{4\pi\epsilon c^2}{q}\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \left(\frac{\vec{\mathbf{v}}(t_r)}{s(t_r)} \right)\Bigg|_t , \tag{1}$$

   (i) Calculate $\vec{\mathbf{B}}$ and separate it into a component independent of $\vec{\mathbf{a}}$ and a component linear in $\vec{\mathbf{a}}$.

   (ii) Show that, the non-radiative part is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)_{\vec{\mathbf{a}}=0} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{s^3\gamma^2} \, \vec{\mathbf{v}}(t_r) \times \vec{\mathbf{r}}(t_r) \tag{2}$$

   (iii) Show that, for the radiative component,

$$\vec{\mathbf{B}}_{\text{rad}}(\vec{\mathbf{x}}, t) = \frac{\vec{\mathbf{r}}(t_r) \times \vec{\mathbf{E}}_{\text{rad}}(\vec{\mathbf{x}}, t)}{r(t_r)c} \tag{3}$$

3. For a charge in uniform circular motion on a circle of radius a with angular frequency ω,

   (i) Plot the φ-dependence of dU/dt in the x-y plane, for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on this angular dependence.

   (ii) Show the angular distribution of the radiated power as a SphericalPlot3D (Mathematica notation) for $\gamma = 1, 10, 100$ in appropriate units. Comment on the angular dependence.

   (iii) Plot dU/dt_r as a function of $\tilde{\theta}$, for two values of $\tilde{\phi} : 0, \pi/2$ and three values of $\gamma : 1, 10, 100$. Comment on your results.

4. Find the conditions of validity of the radiation reaction force in the ultra-relativistic limit.

5. Rutherford scattering describes the process when an electron of mass m_e scatters off a nucleus of a much larger mass m_N. In this process, the energy absorbed by the nucleus is very small, and the electron energy is unchanged, though the direction of the electron changes by an angle θ. The cross section for Rutherford scattering is given by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{4|\vec{\mathbf{p}}_e|^2\beta^2\sin^4(\theta/2)} \ .$$

   where $|\vec{\mathbf{p}}_e| = |\vec{\mathbf{p}}_{e1}| = |\vec{\mathbf{p}}_{e2}|$. Here $\vec{\mathbf{p}}_{e1}$ and $\vec{\mathbf{p}}_{e2}$ are the momenta of the electron before and after the scattering, respectively, Z is the atomic number of the nucleus, α is the fine structure constant, and $\beta = |\vec{\mathbf{v}}|/c$, where $\vec{\mathbf{v}}$ is the velocity of the incoming electron. Let $q^i = p^i_{e1} - p^i_{e2}$ be the 4-momentum transferred by the electron to the nucleus.
   (a) Neglecting the electron mass m_e, calculate the Lorentz invariant quantity q^2 in terms of $|\vec{\mathbf{p}}_e|$ and θ.
   (b) Hence, calculate the cross section $d\sigma/dq^2$.

   Now view this process in the reference frame where the electron is originally stationary, and the nucleus scatters on it, transferring an energy $\mathcal{E}$ to it. In this frame,
   (c) Calculate the energy transfer $\mathcal{E}$ in terms of q^2. Do not neglect the electron mass.
   (d) Hence, determine $d\sigma/d\mathcal{E}$.
   (e) Find an upper bound on the energy transfer $\mathcal{E}$.

6. Exercises 2,3,8,9, page 376, Panofsky and Phillips