Module I: Electromagnetic waves

Lecture 1: Maxwell's equations, static solutions

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Outline

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In the language of differential vector calculus

Gauss's law

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{1}$$

Gauss's law for magnetism

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{2}$$

Maxwell-Faraday equation

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{3}$$

Ampere's law, with Maxwell's correction

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) \tag{4}$$

Intuitive interpretations obtained through integral forms $\Rightarrow$

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Intuitive interpretations obtained through integral forms $\Rightarrow$

Gauss's law: enclosed charges

$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$

- Integrate over a closed volume:

$$\int_V (\nabla \cdot \vec{\mathbf{E}}) dV = \int_V \frac{\rho}{\epsilon_0} dV \quad (5)$$

- Use a mathematical identity (Gauss's theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \quad (6)$$

- Relationship between electric field on a closed surface and the charge enclosed inside it
- The part in red: source of the electric field
- Leads to Coulomb's law if Q is a point charge at the centre of a sphere of radius r: $E_r \cdot 4\pi r^2 = Q/\epsilon_0$

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Gauss's law: no magnetic monopoles

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- Relationship between magnetic field on a closed surface and the magnetic charge enclosed inside it
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- Vanishing of the source $\Rightarrow$ no magnetic monopoles

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Maxwell-Faraday equation: flux through a loop

$\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{\mathbf{S}}} (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{S}} = \int_{\vec{\mathbf{S}}} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} \tag{9}$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t}(\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}) \tag{10}$$

(If the loop does not change with time)

- $$\mathcal{E} \equiv \oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{\vec{\mathbf{S}}} (\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}) = -\frac{\partial \Phi}{\partial t} \tag{11}$$

$\Phi = \int_{\vec{\mathbf{S}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$

More comments on the next page ⇒

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More about Maxwell-Faraday equation

- Relationship between electric field along a loop and the rate of change of magnetic flux through an open surface whose boundary is the loop
- No sources needed: it is a relationship between $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$
- The "$\mathcal{E} = -\partial\Phi/\partial t$" equation does not hold for all situations, since it does not take into account the Lorentz force on a moving charge in a magnetic field. For example, see the discussion about Faraday Wheel in Feynman lectures. We'll return to this point later in the course.

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Ampere's law with Maxwell's corrections

$\nabla \times \vec{\mathbf{B}} = \mu_0(\vec{\mathbf{J}} + \epsilon_0 \partial \vec{\mathbf{E}} / \partial t)$

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{\mathbf{S}}} (\nabla \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{S}} = \mu_0 \int_{\vec{\mathbf{S}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}} + \mu_0 \epsilon_0 \int_{\vec{\mathbf{S}}} \frac{\partial \vec{\mathbf{E}}}{\partial t} \cdot d\vec{\mathbf{S}} \tag{12}$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t} (\vec{\mathbf{E}} . d\vec{\mathbf{S}}) \tag{13}$$

(if the loop does not change with time)

- Relationship between $\vec{\mathbf{B}}$ along a loop and the rate of change of electric flux through an open surface whose boundary is the loop
- $I = \oint_{\vec{\mathbf{S}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}$ is the conduction current
- $\epsilon_0 \int_{\vec{\mathbf{S}}} \frac{\partial}{\partial t} (\vec{\mathbf{E}} . d\vec{\mathbf{S}})$ is often called "displacement current", this is the correction by Maxwell to Ampere's law

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Inside a dielectric medium (static case)

- Gauss's law always valid, when ρ is the total charge: $\nabla \cdot \vec{\mathbf{E}} = \rho/\epsilon_0$
- Part of the charge is due to polarization induced in the medium, which gives rise to the "bound charge": $\rho_b = -\nabla \cdot \vec{\mathbf{P}}$, where $\vec{\mathbf{P}}$ is the polarization
- Then $\epsilon_0 \nabla \cdot \vec{\mathbf{E}} = (\rho_b + \rho_{\mathrm{fr}}) = -\nabla \cdot P + \rho_{\mathrm{fr}}$, where ρ_{fr} is the free charge density
- Defining $\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$, we get Gauss's law in terms of the free charge density:

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{\mathrm{fr}} \tag{14}$$

- The relation $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ defines the dielectric permittivity of the medium, ϵ. This is in general not a number but a tensor, and may not be constant. Wherever it is constant, the dielectric is called "linear".

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Inside a magnetic medium (static case)

- Maxwell-Faraday equation always valid, when $\vec{\mathbf{J}}$ is the total current: $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$
- Part of the current is due to magnetization induced in the medium, which gives rise to the "surface current": $\vec{\mathbf{J}}_{\text{surface}} = \nabla \times \vec{\mathbf{M}}$, where $\vec{\mathbf{M}}$ is the magnetization
- Then $\nabla \times \vec{\mathbf{B}} = \mu_0(\vec{\mathbf{J}}_{\text{surface}} + \vec{\mathbf{J}}_{\text{fr}}) = \mu_0 \nabla \times \vec{\mathbf{M}} + \mu_0 \vec{\mathbf{J}}_{\text{fr}}$, where $\vec{\mathbf{J}}_{\text{fr}}$ is the free current density
- Defining $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$, we get Ampere's law in terms of the free charge density:

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- Maxwell-Faraday equation always valid, when $\vec{\mathbf{J}}$ is the total current: $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$
- Part of the current is due to magnetization induced in the medium, which gives rise to the "surface current": $\vec{\mathbf{J}}_{\text{surface}} = \nabla \times \vec{\mathbf{M}}$, where $\vec{\mathbf{M}}$ is the magnetization
- Then $\nabla \times \vec{\mathbf{B}} = \mu_0(\vec{\mathbf{J}}_{\text{surface}} + \vec{\mathbf{J}}_{\text{fr}}) = \mu_0 \nabla \times \vec{\mathbf{M}} + \mu_0 \vec{\mathbf{J}}_{\text{fr}}$, where $\vec{\mathbf{J}}_{\text{fr}}$ is the free current density
- Defining $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$, we get Ampere's law in terms of the free charge density:

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{\text{fr}} \tag{15}$$

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Maxwell's equations: "macroscopic" form

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{\text{fr}} \tag{16}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{17}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{18}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{\text{fr}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{19}$$

These are equivalent to the equations (1)–(4), with the substitutions

$$\rho = \rho_{\text{fr}} + \rho_b \,, \qquad \vec{\mathbf{J}} = \vec{\mathbf{J}}_{\text{fr}} + \vec{\mathbf{J}}_{\text{surface}} \tag{20}$$

$$\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}} \,, \qquad \vec{\mathbf{B}} = \mu_0 (\vec{\mathbf{H}} + \vec{\mathbf{M}}) \tag{21}$$

$$\rho_b = -\nabla \cdot \vec{\mathbf{P}} \,, \qquad \vec{\mathbf{J}}_{\text{surface}} = \nabla \times \vec{\mathbf{M}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \,. \tag{22}$$

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Recap of topics covered so far

- Maxwell's equations: in differential and integral form
- Maxwell's equations in the presence of dielectrics and magnetic media

Outline

Unique vector, given divergence and curl

Uniqueness theorem 1

Given $\nabla \cdot \vec{\mathbf{V}} = s(\vec{\mathbf{x}})$
and $\nabla \times \vec{\mathbf{V}} = \vec{\mathbf{c}}(\vec{\mathbf{x}})$ (with $\nabla \cdot \vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$, of course),
if $\vec{\mathbf{V}}$ goes to zero at infinity (fast enough),
then $\vec{\mathbf{V}}(\vec{\mathbf{x}})$ can be uniquely written in terms of $s(\vec{\mathbf{x}})$ and $\vec{\mathbf{c}}(\vec{\mathbf{x}})$.

Indeed the solution can be given:

$$\vec{\mathbf{V}}(\vec{\mathbf{x}}) = -\nabla\phi(\vec{\mathbf{x}}) + \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}) \tag{23}$$

with

$$\phi(\vec{\mathbf{x}}) = \frac{1}{4\pi}\int \frac{s(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}' \tag{24}$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}) = \frac{1}{4\pi}\int \frac{\vec{\mathbf{c}}(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}' \tag{25}$$

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Proof of uniqueness theorem 1

Steps involved

- Show $\nabla \cdot \vec{\mathbf{V}} = s(\vec{\mathbf{x}})$, using $\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(r)$
- Show $\nabla \times \vec{\mathbf{V}} = \vec{\mathbf{c}}(\vec{\mathbf{x}})$ using integration by parts.
  You'll have to use the conditions
  $\nabla \cdot \vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$ everywhere, and
  $\vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$ at large distances (or goes to zero fast enough)

Unique scalar, given $\nabla^2\phi$ and boundary conditions

Uniqueness theorem 2

For a scalar $\phi(\vec{\mathbf{x}})$,
given $\nabla^2\phi$ everywhere,
and given $\phi(\vec{\mathbf{x}})$ or $\nabla\phi \cdot \hat{\mathbf{n}}$ on a closed surface
a unique solution for $\phi(\vec{\mathbf{x}})$ exists. (*caveat for* $\nabla\phi \cdot \hat{\mathbf{n}}$ *case ?*)

Steps for proving uniqueness theorem 2

- Consider two solutions ϕ_1 and ϕ_2, and define $\psi = \phi_1 - \phi_2$
- Using $\oint(\psi\nabla\psi)\cdot d\vec{\mathbf{S}} = \int(\nabla\psi)\cdot(\nabla\psi)dV + \int\psi\nabla^2\psi dV$,
  Show that $|\nabla\psi| = 0$ everywhere in the enclosed volume
  (Use $\psi = 0$ or $\nabla\psi\cdot\hat{\mathbf{n}} = 0$ at the boundary)
- Note: the boundary conditions may be of the form $\psi = 0$ on some part of the boundary and $\nabla\psi = 0$ on the remaining part.

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Unique vector, given $\nabla \times (\nabla \times \vec{\mathbf{A}})$

Uniqueness theorem 3 [for a vector $\vec{\mathbf{A}}(\vec{\mathbf{x}})$]

Given $\nabla \times (\nabla \times \vec{\mathbf{A}})$ everywhere,
and given $\vec{\mathbf{A}} \times \hat{\mathbf{n}}$ or $(\nabla \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}$ on a closed surface
a unique solution for $\vec{\mathbf{A}}(\vec{\mathbf{x}})$ exists. (*caveat for* $(\nabla \times A) \times \hat{\mathbf{n}}$?)

Steps for proving uniqueness theorem 3

- Consider two solutions $\vec{\mathbf{A}}_1$ and $\vec{\mathbf{A}}_2$, and define $\vec{\mathbf{a}} = \vec{\mathbf{A}}_1 - \vec{\mathbf{A}}_2$
- Using $\oint [\vec{\mathbf{a}} \times (\nabla \times \vec{\mathbf{a}})] \cdot d\vec{\mathbf{S}} = \int (\nabla \times \vec{\mathbf{a}}) \cdot (\nabla \times \vec{\mathbf{a}}) dV - \int \vec{\mathbf{a}} \cdot [\nabla \times (\nabla \times \vec{\mathbf{a}})] dV$, Show that $|\nabla \times \vec{\mathbf{a}}| = 0$ everywhere in the enclosed volume (Use $\vec{\mathbf{a}} \times \hat{\mathbf{n}} = 0$ or $(\nabla \times \vec{\mathbf{a}}) \times \hat{\mathbf{n}} = 0$ at the boundary)
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A caution about $\nabla^2 \vec{\mathbf{A}}$

- In a general coordinate system with coordinates (s, t, u),
$$\nabla^2 \vec{\mathbf{A}} \neq (\nabla^2 A_s)\hat{s} + (\nabla^2 A_t)\hat{t} + (\nabla^2 A_u)\hat{u}$$

- For the special case of cartesian coordinates (x, y, z),
$$\nabla^2 \vec{\mathbf{A}} = (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$$

- In general, $\nabla^2 \vec{\mathbf{A}}$ is defined through

$$\nabla^2 \vec{\mathbf{A}} = -\nabla \times (\nabla \times \vec{\mathbf{A}}) + \nabla(\nabla \cdot \vec{\mathbf{A}}) \tag{26}$$

Uniqueness theorems: applications

- If a solution is found by hook by or crook, we can be sure that this is the only solution
- A search for simple solutions, with certain symmetry properties, if successful, can solve the problem completely.
- Tricks like the method of images work.

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Outline

When to use

This technique works when there is some symmetry in the boundary conditions of the problem, which suggests the use of certain coordinates.

If the boundary conditions are of the form

- $\Phi(x = a) = \phi_0$, for all (y, z) $\Rightarrow$ cartesian coordinates
- $\Phi(r = a) = \phi_0$, for all (θ, ϕ) $\Rightarrow$ spherical polar coordinates
- $\Phi(r = a) = \phi_0$, for all (z, ϕ) $\Rightarrow$ cylindrical coordinates

Cartesian coordinates

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{27}$$

- Form of the solution:

$$X(x) = \begin{cases} Ae^{ik_x x} + Be^{-ik_x x} \\ Ae^{\kappa_x x} + Be^{-\kappa_x x} \end{cases} \tag{28}$$

  Similarly for $Y(y)$ and $Z(z)$.
- The solutions along x, y, z direction can be individually oscillatory ($e^{\pm ikx}$) or hyperbolic ($e^{\pm \kappa x}$).
- All three solutions cannot be propagating simultaneously, neither can all be hyperbolic at the same time.

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Spherical polar coordinates

$$\Phi(r,\theta,\phi) = R_\ell(r)\Theta_\ell^m(\theta)\Phi_m(\phi) \tag{29}$$

Form of the solution

- $R_\ell = A_\ell r^\ell + B_\ell r^{-\ell-1}$
  - $A_\ell = 0$ if the solution is to be finite at infinity,
  - $B_\ell = 0$ if it is to be finite at the origin
- $\Theta_\ell^m(\theta) = C_\ell P_\ell^m(\cos\theta) + D_\ell Q_\ell^m(\cos\theta)$
  - P_ℓ^m, Q_ℓ^m: associated Legendre polynomials
  - $P_\ell^m = |Y_\ell^m|$, magnitudes of spherical harmonics
  - $D_\ell = 0$ if the solution is finite along z axis, since Q_ℓ^m blows up there
- $\Phi(\phi) = \begin{cases} Ee^{im\phi} + Fe^{-im\phi} & (m \neq 0) \\ E\phi + F & (m = 0) \end{cases}$
  - Azimuthal symmetry $\Rightarrow m = 0$
  - $\oplus$ single-valued solution $\Rightarrow E = 0$
  - $P_\ell^0(\cos\theta) = P_\ell(\cos\theta)$, Legendre polynomials

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- $\Phi(\phi) = \begin{cases} Ee^{im\phi} + Fe^{-im\phi} & (m \neq 0) \\ E\phi + F & (m = 0) \end{cases}$
  - Azimuthal symmetry $\Rightarrow m = 0$
  - $\oplus$ single-valued solution $\Rightarrow E = 0$
  - $P_\ell^0(\cos\theta) = P_\ell(\cos\theta)$, Legendre polynomials

Cylindrical coordinates

$$\Phi(r,\phi,z) = R_n(r)\mathit{\Phi}_n(\phi)Z(z) \tag{30}$$

Form of the solution

- $R_\ell = \begin{cases} A_n J_n(kr) + B_n N_n(kr) & (k \neq 0) \\ A_n r^n + B_n r^{-n} & (k = 0, n \neq 0) \\ A \ln r + B & (k = n = 0) \end{cases}$
  - J_n: Bessel functions, N_n: associated Bessel functions
  - $B_n = 0$ if Φ is to be finite at the origin
- $\mathit{\Phi}_n(\phi) = \begin{cases} C_n e^{in\phi} + D_n e^{-in\phi} & (n \neq 0) \\ C\phi + D & (n = 0) \end{cases}$
  - Azimuthal symmetry $\Rightarrow n = 0$
  - $\oplus$ single-valued solution $\Rightarrow C = 0$
- $Z(z) = \begin{cases} Ee^{kz} + Fe^{-kz} & (k \neq 0) \\ Ez + F & (k = 0) \end{cases}$
  - $Z(z)$ can be oscillatory, in which case $R(r)$ involves modified Bessel functions

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A recap of topics covered in this lecture

- Maxwell's equations: in differential and integral form
- Maxwell's equations in the presence of dielectrics and magnetic media
- Uniqueness theorem for $\vec{\mathbf{V}}$, given its divergence and curl both of which fall sufficiently fast at infinity
- Uniqueness theorem for Φ, given $\nabla^2\Phi$ everywhere and Φ or $(\nabla\Phi \cdot \hat{\mathbf{n}})$ on a closed boundary.
- Uniqueness theorem for $\vec{\mathbf{A}}$, given $\nabla \times (\nabla \times \vec{\mathbf{A}})$ everywhere and the components of $\vec{\mathbf{A}}$ or $(\nabla \times \vec{\mathbf{A}})$ tangential to a closed boundary
- Solutions with separation of variables in the cartesian, spherical and cylindrical coordinates