Module II: Relativity and Electrodynamics

Lecture 10: Four-vectors, co-variance and contravariance

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Outline

1. Contravariant and covariant 4-vectors
2. Examples of 4-vectors: $x, \partial, p, J, A, u, a$

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1. Contravariant and covariant 4-vectors

2. Examples of 4-vectors: x, ∂, p, J, A, u, a

Qualifications for being termed a 4-vector

- A 4-vector is a 4-component object whose components transform under a change of frame either like the differentials (cdt, dx, dy, dz), or like the derivatives $\left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.
  In the former case, it is a contravariant vector.
  In the latter case, it is a covariant vector.
- We shall later see that a vector itself may be treated as a geometrical object, which may be represented by its covariant or contravariant components. We shall represent the 4-vectors by sans-serif notation X. Their contravariant components will be represented by X^k and the covariant components will be represented by X_k $(k \in \{0, 1, 2, 3\})$.

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Lorentz transformations of dx

- We already know how the coordinates transform under a boost along x-axis:

$$\begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} . \qquad (1)$$

- We write the above equation as

$$\mathrm{dx}' = \Lambda\, \mathrm{dx} \,, \qquad \text{or} \qquad \mathrm{dx}'^k = \sum_m \Lambda^k{}_m\, \mathrm{dx}^m \qquad (2)$$

where $\mathrm{dx}^m = (\mathrm{dx}^0, \mathrm{dx}^1, \mathrm{dx}^2, \mathrm{dx}^3) = (cdt, dx, dy, dz)$.

- The above equation may be looked upon simply as a matrix equation at the moment. (Later, we'll interpret it as a tensor equation, but that is getting ahead of ourselves.)

Contravariant components

- Chain rule for differentials implies

$$dx'^k = \sum \frac{\partial x'^k}{\partial x^m} dx^m \tag{3}$$

Thus, $\Lambda^k{}_m = \partial x'^k / \partial x^m$.

- Any 4-component object X that transforms as $X' = \Lambda X$ is a contravariant vector. Thus, a contravariant vector transforms as

$$X'^k = \frac{\partial x'^k}{\partial x^m} X^m \,, \tag{4}$$

where we have started using the convention where repeated indices are summed over.

- "contravariant vector X^m" (physicists' loose language) $\equiv$ "contravariant components of the vector X"

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A few comments

- The definition of Λ as $\Lambda^k{}_m = \partial x'^k / \partial x^m$ is a general one, valid in all circumstances.
- Here we have considered only Lorentz boosts along x direction, so the 4×4 matrix that appears in Eq. (1) is a special case of Λ.
- However, once we know how a 4-component object behaves under this special Lorentz boost, we can combine this with our prior information on how the “space” components of this object behave under rotations, to figure out how the whole 4-component object behaves under a general Lorentz boost.
- The space components of all the relevant objects we'll consider here are 3-vectors. Hence we know that they behave “as a vector should” under space rotations, and it only remains to check that they transform properly under Lorentz boosts. We shall hence only focus on this last point.

Lorentz transformations of ∂

- Let us represent

$$\partial_m \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \frac{\partial}{\partial x^m} = \left(\frac{\partial}{\partial (ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) .$$

- The chain rule for derivatives gives

$$\partial'_k = \frac{\partial}{\partial x'^k} = \frac{\partial x^m}{\partial x'^k} \frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x'^k} \partial_m = \bar{\Lambda}^m{}_k \partial_m . \tag{5}$$

- For the boost in x direction with speed v, one gets

$$\bar{\Lambda}^m{}_k = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6}$$

Covariant vectors

- The requirement in eq. (5) may be written as a matrix equation

$$\partial' = \partial \bar{\Lambda} \ ,$$

  where ∂ should be written as a row vector.
- Any 4-component object that transforms as

$$Y'_k = \frac{\partial x^m}{\partial x'^k} Y_m = \bar{\Lambda}^m{}_k Y_m \tag{7}$$

  is a covariant vector.
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Relationship between Λ and $\bar{\Lambda}$

- Note that

$$\Lambda^m{}_n \, \bar{\Lambda}^n{}_k = \frac{\partial x'^m}{\partial x^n} \, \frac{\partial x^n}{\partial x'^k} = \delta^m_k \, . \tag{8}$$

Therefore, $\Lambda\bar{\Lambda} = 1$, or $\bar{\Lambda} = \Lambda^{-1}$.

- This also leads to

$$X'^m Y'_m = \Lambda^m_n X^n \, \bar{\Lambda}^k{}_m Y_k = \delta^k{}_n X^n Y_k = X^k Y_k \tag{9}$$

So the Lorentz transformations do not affect the "inner product" $X^k Y_k$ of a contravariant and a covariant vector. Such a quantity is therefore frame-independent.

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The position 4-vector x

- Clearly the position vector

$$x^k = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{\mathbf{x}})$$

  transforms like dx^k, i.e. $x'^k = \Lambda^k{}_m x^m$. It is therefore a contravariant 4-vector.

- However also note that

$$x_k = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) = (ct, -\vec{\mathbf{x}})$$

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- From the above, x^k and x_k may be interpreted as the contravariant and covariant components of the same object, a 4-vector x.

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The differential operator ∂ as a 4-vector

- The differential operator

$$\partial_m = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial(ct)}, \nabla\right)$$

  is a covariant 4-vector by definition.

- Also note that

$$\partial^m = (\partial^0, \partial^1, \partial^2, \partial^3) = \left(\frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial(ct)}, -\nabla\right)$$

  transforms like $\partial'^m = \Lambda^m_{\ k}\partial^k$. It is therefore a contravariant vector.

- Thus, ∂^m and ∂_m may be interpreted as the contravariant and covariant components of the same object, a 4-vector ∂.

The momentum 4-vector p

- We have seen that in relativity, the momentum of a particle travelling with velocity $\vec{\mathbf{u}}$ is $\vec{\mathbf{p}} = m\gamma\vec{\mathbf{u}}$, while its energy is $E = m\gamma c^2$.
- Consider the 4-component quantity $(E/c, \vec{\mathbf{p}})$. In the rest frame of the particle, it equals $(mc, \vec{0})$. Let this be the frame S.
- In the frame S' (moving with a speed v along x direction, as seen from S), the above quantity should be $(m\gamma c, -m\gamma\vec{\mathbf{v}})$. This is obtained by a trasformation through the matrix Λ, as can be checked explicitly. Therefore,

$$\mathrm{p}^m = (\mathrm{p}^0, \mathrm{p}^1, \mathrm{p}^2, \mathrm{p}^3) = (E/c, \vec{\mathbf{p}})$$

  is a contravariant 4-vector.
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The current 4-vector J

- Consider a wire of constant cross section, and uniform charge density ρ in frame S, in which no current is flowing, i.e. $\vec{\mathbf{J}} = 0$. Let the wire be along x direction.
- In a frame S' moving with a speed v along x direction, one sees a charge density $\rho' = \gamma\rho$ due to Lorentz contraction, and a current density $\vec{\mathbf{J}}' = -\gamma\rho\vec{\mathbf{v}}$.
- Thus, the transformation of the object $(\rho c, \vec{\mathbf{J}})$ to a moving frame can be obtained through the matrix Λ.
- Therefore,

$$J^m = (J^0, J^1, J^2, J^3) = (\rho c, \vec{\mathbf{J}})$$

is a contravariant 4-vector, while

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Electromagnetic potential 4-vector A

- In the Lorentz gauge

$$\nabla \cdot \vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \ , \tag{10}$$

the vector and scalar potentials $\vec{\mathbf{A}}$ and ϕ satisfy the wave equations

$$\frac{\partial^2 \vec{\mathbf{A}}}{\partial (ct)^2} - \nabla^2 \vec{\mathbf{A}} = \mu_0 \vec{\mathbf{J}} \ , \quad \frac{\partial^2 \phi}{\partial (ct)^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0} \ . \tag{11}$$

- Since the wave equations do not change under Lorentz transformations as seen earlier, clearly $(\phi/c, \vec{\mathbf{A}})$ should have the same transformation properties like $(\rho c, \vec{\mathbf{J}})$.
- Therefore,

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4-velocity u

- While defining new 4-vectors in terms of the known ones, one should ensure that they obey the required transformations. The covariant / contravariant indexing (subscripts / superscripts) and the summation convention are useful tools to take care of this.
- The 4-velocity is defined as

$$\mathrm{u}^m = c\frac{\mathrm{dx}^m}{ds} \tag{12}$$

which is clearly a 4-vector, since dx^m is a 4-vector and $ds = \sqrt{(c\ dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$ is a Lorentz-invariant scalar.

- Since $ds = dt/\gamma$, the 4-velocity may be written as

$$\mathrm{u}^m = (\gamma c, \gamma\vec{\mathbf{v}}) \tag{13}$$

- Note that

$$\mathrm{u}^m \mathrm{u}_m = c^2 \frac{\mathrm{dx}^m \mathrm{dx}_m}{(ds)^2} = c^2 \tag{14}$$

Thus, u is a “unit” 4-vector (in the units $c = 1$).

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4-acceleration a

- The acceletration 4-vector is naturally defined as

$$a^m = c^2 \frac{d^2 x^m}{ds^2} = c \frac{du^m}{ds} \tag{15}$$

- Given that $u^m u_m = c^2$, a constant, one gets $a^m u_m = 0$. Thus, 4-velocity and 4-acceleration are orthogonal to each other (in the 4-dimensional sense): $u \cdot a = 0$.

Take-home message from this lecture

- 4-vectors in Special Relativity are useful quantities because of their transformation properties. They may be represented in terms of their contravariant or covariant components.
- 4-position x, 4-derivative ∂, 4-momentum p, 4-current J and 4-potential A: all are 4-vectors. The 4-vectors for velocity and acceleration can be appropriately defined.