# Module II: Relativity and Electrodynamics

## Lecture 11: Metric, Lorentz invariants, kinematics

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# Outline

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# Relating covariant and contravariant components

- As observed in all examples, the change of signs of the space components of a covariant vector converts it to a contravariant one, and vice versa. The covariant and contravariant components thus contain the same information, and describe the same 4-vector.
- If a 4-vector X has covariant components $X_m$ and contravariant components $X^k$, they can be related through

$$X_m = \frac{\partial x_m}{\partial x^k} X^k \ , \quad X^k = \frac{\partial x^k}{\partial x_m} X_m \ . \tag{1}$$

- These matrices, which "lower" or "raise" the indices of the 4-vectors are the "metrics":

$$g_{km} = \frac{\partial x_k}{\partial x^m} \ , \quad g^{km} = \frac{\partial x^k}{\partial x_m} \ . \tag{2}$$

Here in Special Relativity, $g_{km} = g^{km} = \mathrm{Diag}(1, -1, -1, -1)$. At the moment, the metrics may be just considered as matrices. Later we shall discuss their "tensorial" nature,

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# Scalar products in terms of the metric

- We saw earlier that the inner product (product with all indices summed over) of a covariant and a contravariant vector is invariant under Lorentz transformation: $X'^m Y'_m = X^k Y_k$.
- Using $Y_m = g_{mk} Y^k$, one may write this as

$$X \cdot Y = g_{mk} X^m Y^k \tag{3}$$

  Thus one can talk about a "scalar product" $X \cdot Y$ of two vectors X and Y, without referring to their components explicitly.
- The scalar product of two 4-vectors is invariant under Lorentz transformations, and hence is a frame-independent physical quantity.

# Proper distance in special relativity

$$s^2 = \mathrm{x}^m \mathrm{x}_m = g_{mk} \mathrm{x}^m \mathrm{x}^k \tag{4}$$

is the square of the "length" of the 4-vector x. The "proper" distance between two "events" $\mathrm{x}_1$ and $\mathrm{x}_2$ is $\Delta s$, with

$$(\Delta s)^2 = (\Delta \mathrm{x})^i (\Delta \mathrm{x})_i = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{5}$$

where $\Delta \mathrm{x} = \mathrm{x}_2 - \mathrm{x}_1$.

- When $(\Delta s)^2 > 0$, the two events are said to have a "timelike" separation: they are causally connected, i.e. one can influence the other, and the time ordering of these events is the same in all frames.
- When $(\Delta s)^2 < 0$, the two events are said to have a "spacelike" separation: they are not causally connected, and one cannot influence the other. The time ordering of these events is frame-depedent.
- $(\Delta s)^2 = 0$ is the "null cone", or the light cone. Electromagnetic waves in vacuum travel on this cone.

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# More Lorentz-invariant scalar products

Many Lorentz-invariant scalar products of 4-vectors will play an important role in our discussions.

- $\partial^m \partial_m = \Box$ is the D'Alembertian. We have earlier explicitly verified its frame-dependence.
- $\mathrm{p}^m \mathrm{p}_m = (E/c)^2 - |\vec{\mathbf{p}}|^2 = m^2 c^2$, clearly invariant since it is the mass of the particle.
- $\mathrm{p}_m \mathrm{x}^m = Et - \vec{\mathbf{p}} \cdot \vec{\mathbf{x}} = \hbar \phi$, the phase of a plane wave. We used this quantity while determining the aberration and Doppler shift.
- $\partial_m J^m = \frac{\partial(\rho c)}{\partial(ct)} + \nabla \cdot \vec{\mathbf{J}} = 0$ is the continuity condition / conservation of charge.
- $\partial_m A^m = \frac{\partial(\phi/c)}{\partial(ct)} + \nabla \cdot \vec{\mathbf{A}} = 0$ is the Lorentz gauge condition, which was also a frame-independent statement, though we did not bother about it then.

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# Analyzing situations involving relativistic kinematics

- To become familiarized with handling relativistic kinematics, let us start with the example of Compton scattering, which will represent the general two-body scattering AB$\rightarrow$CD.
- We'll first analyze the kinematics using the 3-vector notation, where the conservations of energy and momentum are treated separately.
- Then we'll analyze the same situation from another point of view: We'll consider all the Lorentz invariant quantities and study their interrelations, with an aim to understand what is needed to specify the final state completely.

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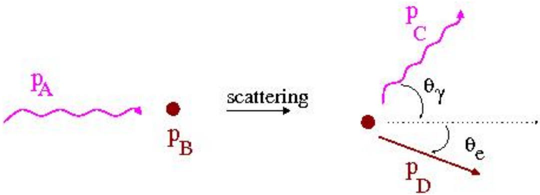
# Compton scattering




- The momenta:

$$\vec{\mathbf{p}}_i = \left(\frac{h\nu}{c}, 0, 0\right) + (0, 0, 0) ,$$
$$\vec{\mathbf{p}}_f = \left(\frac{h\nu'}{c}\cos\theta_\gamma, \frac{h\nu'}{c}\sin\theta_\gamma, 0\right) + (m_e v_e \gamma_e \cos\theta_e, -m_e v_e \gamma_e \sin\theta_e, 0)$$

- The energies:

$$E_i = h\nu + m_e c^2 \ , \quad E_f = h\nu' + m_e c^2 \gamma_e \tag{6}$$

- Three non-trivial constraints from conservation of momentum (two) and conservation of energy (one). Four uknowns: $\nu', \theta_\gamma, \theta_e, v_e$. Given any one of these four (usually $\theta_\gamma$), the rest can be determined.

# Scattering in 4-vector notation

- Now we shall have another look at the same process, but using the language of 4-vectors and Lorentz invariant scalar products. Our results here can be generalized to any scattering process AB$\rightarrow$CD, and we do not need to be in any particular frame.
- Let the 4-momenta of the four particles involved be $\mathrm{p}_A, \mathrm{p}_B, \mathrm{p}_C, \mathrm{p}_D$. The invariants that can be formed from these four vectors are:

$$\mathrm{p}_A \cdot \mathrm{p}_A = m_A^2 c^2 \ , \ \mathrm{p}_B \cdot \mathrm{p}_B = m_B^2 c^2 \ , \ \mathrm{p}_C \cdot \mathrm{p}_C = m_C^2 c^2 \ , \ \mathrm{p}_D \cdot \mathrm{p}_D = m_D^2 c^2$$

  that correspond to the four masses, and the six scalar products

$$\mathrm{p}_A \cdot \mathrm{p}_B \ , \quad \mathrm{p}_A \cdot \mathrm{p}_C \ , \quad \mathrm{p}_A \cdot \mathrm{p}_D \ , \quad \mathrm{p}_B \cdot \mathrm{p}_C \ , \quad \mathrm{p}_B \cdot \mathrm{p}_D \ , \quad \mathrm{p}_C \cdot \mathrm{p}_D \ .$$

- Not all the above invariants are independent. Indeed, conservation of momentum leads to relations among them.

# Momentum conservation in the scattering AB→CD

- Conservation of 4-momentum gives

$$\mathrm{p}_A + \mathrm{p}_B = \mathrm{p}_C + \mathrm{p}_D \, . \tag{7}$$

- Equating magnitudes of both sides gives

$$\begin{aligned} (\mathrm{p}_A + \mathrm{p}_B)\cdot(\mathrm{p}_A + \mathrm{p}_B) &= (\mathrm{p}_C + \mathrm{p}_D)\cdot(\mathrm{p}_C + \mathrm{p}_D) \\ m_A^2c^2 + m_B^2c^2 + 2\mathrm{p}_A\cdot\mathrm{p}_B &= m_C^2c^2 + m_D^2c^2 + 2\mathrm{p}_C\cdot\mathrm{p}_D \end{aligned}$$

  Thus, given the masses of particles, the two scalar products $\mathrm{p}_A\cdot\mathrm{p}_B$ and $\mathrm{p}_C\cdot\mathrm{p}_D$ are related.

- Writing the 4-momentum conservation equation in the form

$$\mathrm{p}_A - \mathrm{p}_C = \mathrm{p}_D - \mathrm{p}_B$$

  gives a similar relation between $\mathrm{p}_A\cdot\mathrm{p}_C$ and $\mathrm{p}_B\cdot\mathrm{p}_D$.
  Also, the equality

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# Counting invariants in the scattering AB→CD

- The form $p_A = p_C + p_D - p_B$ of the momentum conservation gives a linear relation among $p_B \cdot p_C,\ p_B \cdot p_D$ and $p_C \cdot p_D$. Equivalently, this is a linear relation among $p_A \cdot p_D,\ p_A \cdot p_C$, and $p_A \cdot p_B$.
- Thus, Out of the six scalar products, only two are independent, accounting for two independent observables that have to come from measurements, say $p_A \cdot p_B$ and $p_A \cdot p_C$.
- The initial conditions of the problem specify $p_A$ and $p_B$, and hence $p_A \cdot p_B$. Thus, there is one undetermined quantity left whose knowledge is essential for knowing the final state. This is often the scattering angle.
- Thus in a process AB→CD, the knowledge of scattering angle is necessary to specify the final state.

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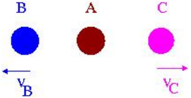
# Two-body decay

- Consider the decay A→BC, when A is at rest.




- The momenta along the direction of motion of decay products:

$$\vec{\mathbf{p}}_i = 0 \,, \qquad \vec{\mathbf{p}}_f = m_B v_B \gamma_B - m_C v_C \gamma_C \,. \tag{8}$$

- The energies:

$$E_i = m_A c^2 \,, \qquad E_f = m_B c^2 \gamma_B + m_C c^2 \gamma_C \,. \tag{9}$$

- Two non-trivial constraints from conservation of momentum and energy, for two unknowns $v_B$ and $v_C$. The values of all the final state observables can therefore be completely predicted, given the masses of the particles A, B and C.

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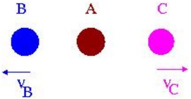


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# Two-body decay in 4-vector notation

- Let us now analyze this decay in the language of 4-vectors. There are three 4-momenta: $\mathrm{p}_A, \mathrm{p}_B, \mathrm{p}_C$.
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  gives, when magnitude squared of both sides is taken,

$$m_A^2 c^2 = m_B^2 c^2 + m_C^2 c^2 + 2\mathrm{p}_B \cdot \mathrm{p}_C \qquad (12)$$

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# Applications of the 4-vector decay kinematics

- Since all the physical quantities can be written in terms of the masses, when A decays at rest, the energies of B and C are fixed. This was expected to happen in beta decays of nuclei, and when the energy of the beta particle was not found to be unique, neutrinos were postulated.
- Just like the energies of all particles in a two-body decay can be determined in terms of their masses, conversely the mass of any of the particles involved can be measured. This has been useful in putting an upper bound on the mass of $\nu_\mu$ through the process $\pi^+ \to \mu^+ \nu_\mu$. (This measurement is obsolete now.)
- Even for the decay to multiple particles, if we know the energies and momenta of all the decay particles, we can determine the mass of the decaying particle through

$$m_A^2 c^2 = \mathsf{p}_A \cdot \mathsf{p}_A = \left(\sum \mathsf{p}_m\right) \cdot \left(\sum \mathsf{p}^m\right) . \qquad (13)$$

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# Applications of the 4-vector decay kinematics

- Since all the physical quantities can be written in terms of the masses, when A decays at rest, the energies of B and C are fixed. This was expected to happen in beta decays of nuclei, and when the energy of the beta particle was not found to be unique, neutrinos were postulated.
- Just like the energies of all particles in a two-body decay can be determined in terms of their masses, conversely the mass of any of the particles involved can be measured. This has been useful in putting an upper bound on the mass of $\nu_\mu$ through the process $\pi^+ \to \mu^+ \nu_\mu$. (This measurement is obsolete now.)
- Even for the decay to multiple particles, if we know the energies and momenta of all the decay particles, we can determine the mass of the decaying particle through

$$m_A^2 c^2 = \mathrm{p}_A \cdot \mathrm{p}_A = \left(\sum \mathrm{p}_m\right) \cdot \left(\sum \mathrm{p}^m\right) . \tag{13}$$

Indeed, this is the way the masses of unknown particles are often determined in High Energy Physics.

# Take-home message from this lecture

- Covariant and contravariant components of a 4-vector may be transformed into each other by the metric.
- Scalar products of 4-vectors are invariant under Lorentz transformations, and hence are frame-independent quantities.
- Relativistic kinematics can be handled purely in terms of the scalar invariants.