

Module II: Relativity and Electrodynamics

Lecture 12: Higher-rank tensors

Amol Dighe
TIFR, Mumbai

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- 1 Second rank 4-tensors: symmetric and antisymmetric
- 2 Higher-rank 4-tensors
- 3 Length, area, 3-volume and 4-volume

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Definition of a 4-tensor

- A second rank 4-tensor B is the object that transforms under frame changes like the direct product of two 4-vectors.
- The contravariant components B^{mn} of B should transform as

$$B'^{mn} = \Lambda^m_k \Lambda^n_\ell B^{k\ell} \quad (1)$$

and its covariant components B_{mn} should transform as

$$B'_{mn} = \bar{\Lambda}^k_m \bar{\Lambda}^\ell_n B_{k\ell} . \quad (2)$$

One can also express B in terms of its “mixed” components B^m_n or B_m^n , which satisfy appropriate transformation conditions.

- B^m_m , the trace of B , is a Lorentz invariant.
- The Λ 's here are composed of proper rotations and boosts. Space reflections are not included. However we know how elements of **direct products of 4-vectors** behave under space reflections, since we know how 4-vectors behave under space reflections. The tensors that behave like **these** are called “proper tensors”, the ones that do not are “pseudotensors”.

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Second-rank tensors: $\Lambda, \bar{\Lambda}$

- Till now, we have treated Λ to be simply a matrix. However note that $\Lambda^m_n = \partial x'^m / \partial x^n$ transforms like mixed components of a 4-tensor, and hence Λ is a 4-tensor.
- The defining equation for a 4-vector, $X' = \Lambda X$ is then not simply a matrix equation, but a tensor equation.
- Similarly, $\bar{\Lambda}^m_n = \partial x^m / \partial x'^n$ transform like mixed components of a 4-tensor, hence $\bar{\Lambda}$ is also a 4-tensor.
- All second rank tensors can be written as a sum of a symmetric and an antisymmetric tensor. The tensor g is always symmetric, while $\Lambda, \bar{\Lambda}$ are symmetric for only boosts, but are not so when rotations are involved.

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Symmetric second-rank tensor: the metric g

- Also, $g_{mn} = \partial x_m / \partial x^n$ and $g^{mn} = \partial x^m / \partial x_n$ transform like covariant and contravariant components of a 4-tensor, g . The mixed components of this tensor are $g^m_n = \partial x^m / \partial x^n = \delta^m_n$.
- Since g is a tensor, the operation of raising and lowering indices in a 4-vector (or, by extension, a 4-tensor) is a tensor operation (“trivial”).

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Antisymmetric second-rank 4-tensor

- An antisymmetric second-rank tensor B may be written in terms of its contravariant components as

$$B^{mn} = \begin{pmatrix} 0 & B^{01} & B^{02} & B^{03} \\ -B^{01} & 0 & B^{12} & B^{13} \\ -B^{02} & -B^{12} & 0 & B^{23} \\ -B^{03} & -B^{13} & -B^{23} & 0 \end{pmatrix}.$$

There are six independent components, which transform as

$$B'^{mn} = \Lambda^m_k \Lambda^n_\ell B^{k\ell}.$$

- If we look at the “mixed” components $B^{0\alpha}$ ($\alpha \in \{1, 2, 3\}$ are the space indices) in primed frame, we get

$$B'^{0\alpha} = \Lambda^0_k \Lambda^\alpha_\ell B^{k\ell}.$$

For space rotations, $\Lambda^0_k = 1$ when $k = 0$ and vanishes otherwise. Then one gets

$$B'^{0\alpha} = \Lambda^\alpha_\mu B^{0\mu}.$$

Thus, the components $B^{0\alpha}$ behave like a 3-vector under space rotation. (Note: B is a proper tensor, not a pseudotensor.)

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Antisymmetric second-rank tensor: general properties

- Now let us look at the other components, of the form $B^{\alpha\beta}$. With the advantage of hindsight, consider the quantity $W_\mu = \epsilon_{\alpha\beta\mu} B^{\alpha\beta}$ where ϵ is the completely antisymmetric, Levi-Civita 3-tensor.
- The transformation properties of W_μ under space rotations can be obtained as:

$$\begin{aligned}W'_\mu &= \epsilon'_{\alpha\beta\mu} B'^{\alpha\beta} = \bar{\Lambda}^\nu_\alpha \bar{\Lambda}^\rho_\beta \bar{\Lambda}^\sigma_\mu \epsilon_{\nu\rho\sigma} \Lambda^\alpha_\lambda \Lambda^\beta_\delta B^{\lambda\delta} \\ &= \delta^\nu_\lambda \delta^\rho_\delta \bar{\Lambda}^\sigma_\mu \epsilon_{\nu\rho\sigma} B^{\lambda\delta} = \bar{\Lambda}^\sigma_\mu \epsilon_{\lambda\delta\sigma} B^{\lambda\delta} \\ &= \bar{\Lambda}^\sigma_\mu W_\sigma .\end{aligned}$$

Thus, W acts like a vector under space rotations. Moreover, since B is a proper tensor and ϵ is a pseudotensor, W is a **pseudovector / axial vector**.

- Thus, the six independent components of the antisymmetric second-rank tensor can be separated into a 3-vector $V^\alpha = B^{0\alpha} = (V_x, V_y, V_z)$, and a 3-axial vector $W_\mu = \epsilon_{\alpha\beta\mu} B^{\alpha\beta} = (W_x, W_y, W_z)$.

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Vector \oplus axial vector \Rightarrow antisymmetric tensor

- In terms of the vector V and the axial vector W , one can write the components of B^{ij} as

$$B^{mn} = \begin{pmatrix} 0 & V_x & V_y & V_z \\ -V_x & 0 & W_z & -W_y \\ -V_y & -W_z & 0 & W_x \\ -V_z & W_y & -W_x & 0 \end{pmatrix}.$$

- This should remind us of the electromagnetic field tensor F , whose components are written in terms of \vec{E} , a vector, and \vec{B} , an axial vector:

$$F^{mn} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (3)$$

We shall later devote a complete lecture to the exploration of F .

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Outline

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Rank- n tensors and completely antisymmetric ϵ

- Along the same lines as before, an n^{th} rank tensor is an object whose components transform as the direct product of n 4-vectors. The tensors may be represented in terms of their contravariant, covariant, or mixed components.
- The 4-d analogue of the Levi-Civita tensor is ϵ , whose components are defined through $\epsilon^{0123} = +1$, and $\epsilon^{k\ell mn} = \pm 1$ depending on whether (k, ℓ, m, n) is an even or odd permutation of $(0, 1, 2, 3)$. In all the other cases (i.e. when any two or more of k, ℓ, m, n are equal), $\epsilon^{k\ell mn} = 0$.
- Components of ϵ do not change with proper Lorentz transformations. Moreover, they do not change sign even under space reflections, which is what a proper tensor would normally be expected to do. Therefore, ϵ is a pseudotensor.
- Quantities that are made up from a product of tensors and a single ϵ are pseudotensors. We shall come across a few of them later in this course.

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Dual tensors with ϵ

Because of its completely antisymmetric nature, ϵ plays an important role in creating “dual” tensors. For example,

- If X is a vector, \tilde{X} defined through $\tilde{X}^{k\ell m} = \frac{1}{2}\epsilon^{k\ell mn}X_n$ is a rank-3 pseudotensor, which is completely antisymmetric in its three indices.
- If X is an antisymmetric rank-2 tensor, \tilde{X} defined through $\tilde{X}^{k\ell} = \frac{1}{2}\epsilon^{k\ell mn}X_{mn}$ is a rank-2 antisymmetric pseudotensor.
- If X is a completely antisymmetric rank-3 tensor, \tilde{X} defined through $\tilde{X}^k = \frac{1}{2}\epsilon^{k\ell mn}X_{\ell mn}$ is an axial vector.
- The operation of forming a dual projects out the antisymmetric parts of the original tensors, and convert rank- n tensors to rank- $(4 - n)$ pseudotensors and vice versa. If the original tensor is completely antisymmetric, the dual contains the same information as the original tensor, but written in a form that transforms differently (tensor vs. pseudotensor, change of rank).

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Dual tensors with ϵ

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Outline

- 1 Second rank 4-tensors: symmetric and antisymmetric
- 2 Higher-rank 4-tensors
- 3 Length, area, 3-volume and 4-volume

Line element

- We shall develop the calculus of 4-tensors in short, since this will be useful later in developing the Lagrangian formulation of electrodynamics.
- We shall proceed by using the analogy with quantities in 3-d.

Line element

- In 3-d: line element dx^α is a 3-vector, (dx, dy, dz)
- In 4-d: line element dx^k is a 4-vector, (cdt, dx, dy, dz) .

Area element

- The area element in 3-d is normally written as $d\vec{x}_1 \times d\vec{x}_2$, which, in the language of 3-d tensors, can be written as

$$d\tilde{a}_\gamma = \frac{1}{2} \epsilon_{\alpha\beta\gamma} dx_1^\alpha dx_2^\beta.$$

This is an axial vector in 3-d (hence the tilde).

- A formal way to look at the above definition is to go in two steps: First define the area as a “two-form”

$$da = d\vec{x}_1 \wedge d\vec{x}_2 \Rightarrow da^{\alpha\beta} = dx_1^\alpha dx_2^\beta - dx_1^\beta dx_2^\alpha.$$

This is a rank-2 tensor in 3 dimensions.

Then define the dual vector through

$$d\tilde{a}_\gamma = \frac{1}{2} \epsilon_{\alpha\beta\gamma} a^{\alpha\beta}.$$

- By analogy, the area tensor in 4-d is defined through the two-form

$$df = dx_1 \wedge dx_2 \Rightarrow df^{km} = dx_1^k dx_2^m - dx_1^m dx_2^k,$$

whose dual is the area tensor (analogous to the $d\tilde{a}$ above)

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3-volume element

- In 3-d, the 3-volume element is $d\vec{x}_1 \cdot (d\vec{x}_2 \times d\vec{x}_3)$, which can be written in the language of 3-vectors as a pseudoscalar

$$d\tilde{V} = \frac{1}{6} \epsilon_{\alpha\beta\gamma} dx_1^\alpha dx_2^\beta dx_3^\gamma.$$

- In 4-d, the 3-volume element is a rank-3 tensor

$$dV^{k\ell m} = \begin{vmatrix} dx_1^k & dx_2^k & dx_3^k \\ dx_1^\ell & dx_2^\ell & dx_3^\ell \\ dx_1^m & dx_2^m & dx_3^m \end{vmatrix} \quad (4)$$

- This is also equivalent to a 4-axial vector

$$d\tilde{V}_k = -\frac{1}{6} \epsilon_{k\ell mn} dV^{\ell mn} \quad (5)$$

which is normal to the hypersurface spanned by dx_1, dx_2, dx_3 .
(Note: $\epsilon^{0123} = +1$, so $\epsilon_{0123} = -1$. That corresponds to the negative sign.)

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4-volume element

- This is very similar to the definition of 3-volume in 3-d. The 4-volume element is the rank-4 tensor

$$d\Omega^{k\ell mn} = \begin{vmatrix} dx_1^k & dx_2^k & dx_3^k & dx_4^k \\ dx_1^\ell & dx_2^\ell & dx_3^\ell & dx_4^\ell \\ dx_1^m & dx_2^m & dx_3^m & dx_4^m \\ dx_1^n & dx_2^n & dx_3^n & dx_4^n \end{vmatrix} \quad (6)$$

- This may be written in terms of the pseudoscalar

$$d\tilde{\Omega} = -\frac{1}{24} \epsilon_{k\ell mn} d\Omega^{k\ell mn} . \quad (7)$$

Recap of topics in this lecture

- Symmetric second-rank tensors: the Lorentz-transformation matrices Λ and $\bar{\Lambda}$, the index-raising/lowering matrix g
- Antisymmetric second rank tensor = vector \oplus axial vector
- Totally antisymmetric rank-4 pseudotensor ϵ , dual tensors
- Line element, area element, 3-volume element, 4-volume element