Module II: Relativity and Electrodynamics

Lecture 13: EM field tensor, Maxwell's equations

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Outline

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Motivation for the EM field tensor F

- We have seen that the electromagnetic potential A is a 4-vector, and hence a useful quantity to deal with relativistically. However, it is a gauge-dependent quantity, and hence does not have a unique value. The following changes in A does not change the physics:

$$\left.\begin{array}{l}\phi \to \phi + \partial\psi/\partial t \\ \vec{\mathbf{A}} \to \vec{\mathbf{A}} - \nabla\psi\end{array}\right\} \quad \Rightarrow \quad A^m \to A^m + \partial^m\psi \tag{1}$$

  where ψ is any scalar function.

- Note that $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, the physically measurable quantities, are written in terms of derivatives of ϕ and $\vec{\mathbf{A}}$. We therefore expect that there will be some quantity that is expressed in terms of derivatives of A that will not change under the above gauge transformation.

- The EM field tensor is such a quantity (indeed, the simplest such nontrivial quantity)

$$F^{mn} = \partial^m A^n - \partial^n A^m \tag{2}$$

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F in terms of $\vec{E}$ and $\vec{B}$

- Note that

$$\begin{aligned} F^{0\alpha} &= \partial^0 A^\alpha - \partial^\alpha A^0 \\ \text{i.e.} \quad \vec{V} &= \frac{\partial}{\partial(ct)}\vec{A} + \nabla\frac{\phi}{c} = -\frac{\vec{E}}{c} \, , \end{aligned}$$

Recall: $\vec{V}$ is the vector part of the rank-2 antisymmetric tensor F as discussed earlier. (Note: we shall use $c = 1$ from henceforth in this lecture.)

- For the purely space components of F, one gets

$$F^{12} = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \, , \quad \text{etc.}$$

- The net expression for the contravariant components of F is

$$F^{km} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{3}$$

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Different representations of F

- Since F has a central role in electrodynamics, it is a good idea to be familiar with its different representations. For example, the covariant and mixed components of F are:

$$F_{km} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, F_k{}^m = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

- The dual pesudotensor $\widetilde{F}$ that has the same information as F can be written as

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Lorentz invariants with F and $\widetilde{F}$

- With only F, one can form the Lorentz invariant

$$F^{km}F_{km} = 2(|\vec{B}|^2 - |\vec{E}|^2) \ . \tag{4}$$

This is a scalar quantity, and indicates that there are limits on how much the relative values of $\vec{B}$ and $\vec{E}$ can change. In particular,

  - If $|\vec{B}| = |\vec{E}|$ in one frame, $|\vec{B}| = |\vec{E}|$ in all frames.
  - If $|\vec{B}| > |\vec{E}|$ in one frame, $|\vec{B}| > |\vec{E}|$ in all frames, and vice versa.

- With F and $\widetilde{F}$, one can also form

$$F^{km}\tilde{F}_{km} = 4\vec{E}\cdot\vec{B} \ , \tag{5}$$

which is a pseudoscalar. This also puts severe constraints on $\vec{E}$ and $\vec{B}$ in different frames. In particular,

  - If $\vec{E} \perp \vec{B}$ in one frame, $\vec{E} \perp \vec{B}$ in all frames.
  - If the angle betweem $\vec{E}$ and $\vec{B}$ is acute ($\vec{E}\cdot\vec{B} > 0$) in one frame, it stays acute in all frames. Similarly for obtuse angles ($\vec{E}\cdot\vec{B} < 0$).

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The source-free equation: $\partial_\ell \widetilde{F}^{\ell m} = 0$

- Since $F_{ik} = \partial_i A_k - \partial_k A_i$ is an antisymmetric tensor, it trivially satisfies the identity

$$\partial_\ell F_{ik} + \partial_i F_{k\ell} + \partial_k F_{\ell i} = 0 \ .$$

- Multiplying this equation by $\epsilon^{ik\ell m}$ gives

$$\partial_\ell \widetilde{F}^{\ell m} = 0 \ . \tag{6}$$

- This corresponds to two of the Maxwell's equations:

$$\begin{aligned} \partial_\alpha \widetilde{F}^{\alpha 0} = 0 \quad &\Rightarrow \quad \nabla \cdot \vec{B} = 0 \ , \\ \partial_\alpha \widetilde{F}^{\alpha\beta} = 0 \quad &\Rightarrow \quad \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \ . \end{aligned} \tag{7}$$

- Thus, the source-free equations of Maxwell emerge simply from the definition of F. This is not surprising, since even in the non-relativistic analysis before, these equations are trivially satisfied once $\vec{A}$ and ϕ exist that lead to $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \partial A/\partial t$.

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Maxwell's equations with sources

- The Maxwell's equation we just obtained contained information about the derivative of $\widetilde{F}$. It would be interesting to check what the derivatives of F itself are. Therefore we calculate $\partial_m F^{mn}$.
- We have

$$\begin{aligned} \partial_m F^{m0} &= \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \partial_m F^{m\beta} &= -\frac{\partial E_\alpha}{\partial t} + (\nabla \times \vec{B})_\alpha = \mu_0 J_\alpha \end{aligned} \tag{8}$$

  These two equations may be combined into (using $\epsilon_0 \mu_0 = 1/c^2$)

$$\partial_m F^{mn} = \mu_0 J^n \; . \tag{9}$$

- The above represent the remaining two Maxwell's equations in terms of the EM field tensor F and the sources J.

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Some analogies to familiar theorems in 3-d

Gauss's law

- In 3-d: $\oint \vec{\mathbf{C}} \cdot dS = \int (\nabla \cdot \vec{\mathbf{C}}) dV$, i.e. $\oint d\widetilde{S}_\alpha C^\alpha = \int dV \partial_\beta C^\beta$.
- In 4-d:

$$
\begin{aligned}
\oint d\widetilde{V}_k \, C^k &= \int d\widetilde{\Omega} \, \partial_\ell C^\ell \\
\oint d\widetilde{V}_k \, A^k &= \int d\widetilde{\Omega} \, \partial_\ell A^\ell
\end{aligned}
\qquad (10)
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Stokes' theorem

- In 3-d: $\oint \vec{\mathbf{C}} \cdot d\ell = \int (\nabla \times \vec{\mathbf{C}}) \cdot dS$, i.e. $\oint dx^\alpha C_\alpha = \int d\widetilde{S}_\alpha \epsilon^{\alpha\beta\gamma} \partial_\beta C_\gamma$
- In 4-d:

$$
\begin{aligned}
\oint dx^k C_k &= \int d\widetilde{S}_{mn} \, \epsilon^{mnk\ell} \partial_k C_\ell = 2 \int dS^{k\ell} \, \partial_k C_\ell \\
\oint dx^k A_k &= \int dS^{k\ell} \, F_{k\ell}
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- In 3-d: $\oint \vec{\mathbf{C}} \cdot dS = \int (\nabla \cdot \vec{\mathbf{C}}) dV$, i.e. $\oint d\widetilde{S}_\alpha C^\alpha = \int dV \partial_\beta C^\beta$.
- In 4-d:

$$\begin{aligned} \oint d\widetilde{\mathrm{V}}_k \, \mathrm{C}^k &= \int d\widetilde{\Omega} \, \partial_\ell \mathrm{C}^\ell \\ \oint d\widetilde{\mathrm{V}}_k \, \mathrm{A}^k &= \int d\widetilde{\Omega} \, \partial_\ell \mathrm{A}^\ell \end{aligned} \tag{10}$$

Stokes' theorem

- In 3-d: $\oint \vec{\mathbf{C}} \cdot d\ell = \int (\nabla \times \vec{\mathbf{C}}) \cdot dS$, i.e. $\oint dx^\alpha C_\alpha = \int d\widetilde{S}_\alpha \epsilon^{\alpha\beta\gamma} \partial_\beta C_\gamma$
- In 4-d:

$$\begin{aligned} \oint \mathrm{dx}^k \mathrm{C}_k &= \int \mathrm{d}\widetilde{\mathrm{S}}_{mn} \, \epsilon^{mnk\ell} \partial_k \mathrm{C}_\ell = 2 \int \mathrm{dS}^{k\ell} \, \partial_k \mathrm{C}_\ell \\ \oint \mathrm{dx}^k \mathrm{A}_k &= \int \mathrm{dS}^{k\ell} \, \mathrm{F}_{k\ell} \end{aligned} \tag{11}$$

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Maxwell's equations in integral form

- The third connection: 2-surface enclosing 3-volume
- If C^{mn} is an antisymmetric tensor, then

$$\oint d\widetilde{S}_{mp}\, C^{mn} = \int d\widetilde{V}_m\, \partial_p\, C^{mn}$$

- When C is the EM field tensor F, we get

$$\begin{aligned} \oint d\widetilde{S}_{mn} F^{mn} &= \int d\widetilde{V}_m\, \partial_n\, F^{mn} \\ \oint d\widetilde{S}_{mn}\, F^{mn} &= \mu_0 \int d\widetilde{V}_m\, J^m \end{aligned} \quad (12)$$

This is one way the Maxwell's equations can be written in the integral form.

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Take-home message from this lecture

- The electromagnetic field tensor F is the logical choice for an object with the right transformation properties whose elements are uniquely measurable (components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$).
- Maxwell's four equations can be written in a compact form in terms of F as two equations that give the derivatives of F and its dual, $\widetilde{\mathrm{F}}$.
- Gauss's law and Stokes' theorem in 4-d
- Maxwell's equations in integral form