Module II: Relativity and Electrodynamics

Lecture 14: Lagrangian formulation of relativistic mechanics

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Outline

1. Lagrangian, Hamiltonian, energy, EoMs
2. Non-relativistic particle in a potential
3. Relativistic free particle
4. Relativistic particle in EM fields

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What these two lectures (and the next) will do

- The principle of least action in classical mechanics (CM) suggests that, while going from a state to another, the system travels along such a path in coordinate space so as to minimize a quantity called the action S. This principle gives rise to the equations of motion (EoMs).
- The equations of motion may be obtained using the Lagrangian formulation or the Hamiltonian formulation.
- Specifying the Lagrangian (whose time integral is the action) is then equivalent to specifying the system completely.
- We want to reach a stage whexre, using principles learnt in last few lectures, like Lorentz invariance and Gauge invariance, we should be able to derive the equations of motion satisfied by the EM fields and by particles travelling in them.
- We'll start in small steps, first with non-relativistic CM, leading to relativistic CM and then finally to relativistic electrodynamics (in the next lecture). We expect to lead ourselves to Maxwell's equations and the Lorentz force law.

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The Lagrangian formulation

- The action S is given by

$$S = \int_{t_1}^{t_2} L(\vec{\mathbf{x}}, \vec{\mathbf{v}}) dt \tag{1}$$

- The canonical momentum $\vec{\mathbf{P}}$ is

$$P_\alpha = \frac{\partial L}{\partial v^\alpha} \tag{2}$$

- The principle of least action leads to the equations of motion (see Goldstein)

$$\frac{dP_\alpha}{dt} = \frac{\partial L}{\partial x^\alpha} \tag{3}$$

- The definition of canonical momentum and the EoM above specify the motion of the system in the Lagrangian formulation.

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The Hamiltonian formulation

- The Hamiltonian is defined as

$$H = P_\alpha v^\alpha - L \, . \tag{4}$$

This is the energy of the system.

- Equations of motion in terms of the Hamiltonian are

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial P_\alpha} \, , \qquad \frac{dP_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha} \, . \tag{5}$$

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The 4-d covariant formulation

- From the action S, using the principle of least action, one can get EoMs in a covariant form (i.e. a form that will stay valid in all reference frames.)
- The EoM is to be obtained by extremizing the action, and will be obtained by an explicit procedure in all the examples we shall consider here.

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Lagrangian formulation: potential $V(x)$

As a warm-up, let us take a non-relativistic particle of mass m, inside a potential $V(\vec{\mathbf{x}})$.

- Lagrangian:

$$L = \frac{1}{2}mv^2 - V(x)$$

- Momentum:

$$P_\alpha = mv_\alpha$$

- EoM:

$$\frac{d}{dt}(mv_\alpha) = -\frac{\partial V}{\partial x^\alpha}$$

This is just Newton's second law for a force that arises from a potential.

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Hamiltonian formulation: potential $V(x)$

- Hamiltonian:

$$H = mv^2 - L = \frac{1}{2}mv^2 + V(x) = \frac{p^2}{2m} + V(x)$$

  Thus, the total energy is the sum of kinetic energy and potential energy.

- EoM1:

$$\frac{dx^\alpha}{dt} = \frac{P^\alpha}{m}, \quad \text{i.e.} \quad P^\alpha = m\frac{dx^\alpha}{dt}$$

  This is the momentum in terms of velocity.

- EoM2:

$$\frac{dP_\alpha}{dt} = -\frac{\partial V}{\partial x^\alpha}$$

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Defining Lorentz-invariant Lagrangian

- For the non-relativistic case, we knew that the Lagrangian has the form $L = T - V$, where T is the kinetic energy.
- We do not know a priori what the Lagragian becomes with special relativity. However there is a strong principle guiding us: the action has to be a Lorentz invariant, and has to reduce to the non-relativistic limit (moduli total derivatives) in the limit of small velocities.
- For a free particle, the only relevant Lorentz invariant linear in dt is the proper distance $\int ds$. The action therefore has to be

$$S = \alpha \int_{x_1}^{x_2} ds \tag{6}$$

whete x_1 and x_2 are the initial and final coordinates.
- Since we want to get to the Lagrangian, which appears in the action in the form $S = \int L dt$, we use $ds = c dt/\gamma$ to write

$$S = \alpha c \int_{t_1}^{t_2} \frac{dt}{\gamma} \quad \Rightarrow \quad L = \frac{\alpha c}{\gamma} .$$

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Lagrangian and EoMs: free particle

- Now at small velocities,

$$L = \alpha c/\gamma = \alpha c\sqrt{1 - v^2/c^2} \approx \alpha c\left[1 - v^2/(2c^2)\right] \tag{7}$$

- This should correspond to the non-relativistic limit for the Lagrangian of a free particle, $L = \frac{1}{2}mv^2$, up to a derivative. This gives $\alpha = -mc$, and one gets

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}\ . \tag{8}$$

- With this Lagrangian, the momentum is

$$P_\alpha = \frac{\partial L}{\partial v^\alpha} = m\gamma v_\alpha \tag{9}$$

Thus, the expression for relativistic momentum has emerged naturally with our formalism.

- The EoM is $dP_\alpha/dt = 0$, or $P_\alpha = \text{constant}$, not surprising since this is a free particle, so momentum has to be conserved.

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Hamiltonian and EoMs

- Hamiltonian:

$$H = P_\alpha v^\alpha - L = mv^2\gamma - \frac{mc^2}{\gamma} = mc^2\gamma = \sqrt{P^2c^2 + m^2c^4} \quad (10)$$

This is exactly the relativistic energy E of a particle.

- EoM1:

$$\frac{dx^\alpha}{dt} = \frac{P^\alpha c^2}{\sqrt{P^2c^2 + m^2c^4}} = \frac{P^\alpha c^2}{E} \quad (11)$$

This gives $\vec{\mathbf{P}} = E\vec{\mathbf{v}}/c^2$.

- EoM2:

$$\frac{dP_\alpha}{dt} = 0\,, \quad (12)$$

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4-d covariant treatment

- Let us see the consequences of the minimization of action,

$$S = -mc \int_{x_1}^{x_2} ds$$

in the 4-d formalism. We denote by δX the deviation in a quantity X along a path that deviates slightly from the actual one.

- Using $(ds)^2 = dx_k dx^k$, we get

$$S = -mc \int_{x_1}^{x_2} ds = -mc \int_{x_1}^{x_2} \frac{dx_k dx^k}{ds} = -mc \int_{x_1}^{x_2} u_k dx^k \ .$$

- The variation of the above equation gives

$$\begin{aligned} \delta S &= -mc \int_{x_1}^{x_2} (\delta u_k) dx^k - mc \int_{x_1}^{x_2} u_k d(\delta x^k) \\ &= 0 - mc \left[u_k \left(\delta x^k\right)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{du_k}{ds} \delta x^k ds \right] \end{aligned}$$

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Equations of motion in covariant notation

- The first term inside the square bracket in eq. (13) vanishes since $\delta x^k = 0$ at x_1 and x_2.
- The vanishing of δS then corresponds to

$$\frac{du_k}{ds} = 0 \ . \tag{13}$$

This is the EoM. Note that this leads to the conservation of 4-velocity.

Outline

The action and the Lagrangian

- With the addition of the electromagnetic field, we now have an additional quantity, the EM potential 4-vector A_k
- Then in addition to ds, we can make one more Lorentz invariant quantity: $A_k dx^k$.The most general expression for the Lagrangian is then

$$S = \int_{x_1}^{x_2} \left(-m\, c\, ds - \eta\, A_k\, dx^k\right) \tag{14}$$

where η is a constant that we'll have to determine after matching the EoMs with experiment.

- The Lagrangian is thus

$$L = -\frac{mc^2}{\gamma} - \eta\phi + \eta\vec{\mathbf{A}} \cdot \vec{\mathbf{v}} \tag{15}$$

since $A_k = (\phi/c, -\vec{\mathbf{A}})$, $dx^0 = c\,dt$, and $d\vec{\mathbf{x}} = \vec{\mathbf{v}}dt$.

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- The Lagrangian is thus

$$L = -\frac{mc^2}{\gamma} - \eta\phi + \eta\vec{\mathbf{A}}\cdot\vec{\mathbf{v}} \tag{15}$$

  since $A_k = (\phi/c, -\vec{\mathbf{A}})$, $dx^0 = c\,dt$, and $d\vec{\mathbf{x}} = \vec{\mathbf{v}}dt$.

What about gauge invariance ?

- The Lorentz invariance of the action has been guarateed by construction. However we have not yet checked Gauge invariance explicitly. Let us do it for the interaction term now.
- The Gauge transformation $A'_k = A_k + \partial_k \psi$ gives

$$\int A'_k dx^k = \int (A_k + \partial_k \psi) dx^k = \int A_k dx^k + \int \partial_k \psi dx^k$$
$$= \int A_k dx^k + \psi(x_2) - \psi(x_1) \ .$$

Thus, the action changes by a constant, and hence the EoMs do not change. The Gauge transformation thus leaves the action effectively invariant.

- The term $\int A_k dx^k$ is thus Lorentz invariant as well as Gauge invariant. These are two conditions that each term in the Lagrangian should necessarily satisfy.

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EoMs from the Lagrangian

- The canonical momentum $\vec{\mathbf{P}}$ is

$$P_\alpha = \frac{\partial L}{\partial v^\alpha} = m\gamma v_\alpha + \eta A_\alpha = p_\alpha + \eta A_\alpha \ , \tag{16}$$

where $\vec{\mathbf{p}} = m\gamma\vec{\mathbf{v}}$ is the kinematic momentum.

- The EoM is

$$\frac{d}{dt}(p_\alpha + \eta A_\alpha) = -\eta\frac{\partial \phi}{\partial x^\alpha} + \eta\frac{\partial}{\partial x^\alpha}(A_\rho v^\rho) \tag{17}$$

or

$$\frac{d}{dt}(\vec{\mathbf{p}} + \eta\vec{\mathbf{A}}) = -\eta\nabla\phi + \eta\nabla(\vec{\mathbf{A}}\cdot\vec{\mathbf{v}}) \tag{18}$$

- To simplify this, use

$$\begin{aligned}\frac{d\vec{\mathbf{A}}}{dt} &= \frac{\partial\vec{\mathbf{A}}}{\partial t} + (\vec{\mathbf{v}}\cdot\nabla)\vec{\mathbf{A}} \ , \\ \nabla(\vec{\mathbf{A}}\cdot\vec{\mathbf{v}}) &= (\vec{\mathbf{v}}\cdot\nabla)\vec{\mathbf{A}} + (\vec{\mathbf{A}}\cdot\nabla)\vec{\mathbf{v}} + \vec{\mathbf{v}}\times(\nabla\times\vec{\mathbf{A}}) + \vec{\mathbf{A}}\times(\nabla\times\vec{\mathbf{v}})\end{aligned}$$

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The Lorentz force

- Since $L(\vec{\mathbf{x}}, \vec{\mathbf{v}})$, the partial derivatives of $\vec{\mathbf{v}}$ with respect to $\vec{\mathbf{x}}$ vanish. As a result, $\nabla \times \vec{\mathbf{v}} = 0$. Adding this simplification, one gets

$$\begin{aligned} \frac{d}{dt}\vec{\mathbf{p}} &= \eta\left(-\nabla\phi - \frac{\partial\vec{\mathbf{A}}}{\partial t} + \vec{\mathbf{v}} \times (\nabla \times \vec{\mathbf{A}})\right) \\ &= \eta(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \end{aligned} \tag{19}$$

- With the knowledge of the Lorentz force law, we can identify $\eta = q$, in order to get

$$\frac{d}{dt}\vec{\mathbf{p}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})\ . \tag{20}$$

  Thus, we have derived the Lorentz force law, starting from the action that we obtained simply by using the Lorentz invariance of action.

- The Lagrangian is thus

$$L = -\frac{mc^2}{\gamma} - q\phi + q(\vec{\mathbf{A}} \cdot \vec{\mathbf{v}})\ . \tag{21}$$

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Analysis through Hamiltonian formalism

- The Hamiltonian is

$$
\begin{aligned}
H &= P_\alpha v^\alpha - L = (m\gamma v^2 + qA_\alpha v^\alpha) - (-\frac{mc^2}{\gamma} - q\phi + qA_\alpha v^\alpha) \\
&= m\gamma c^2 + q\phi = \sqrt{p^2c^2 + m^2c^4} + q\phi \qquad (22) \\
&= \sqrt{(\vec{\mathbf{P}} - q\vec{\mathbf{A}})^2c^2 + m^2c^4} + q\phi \qquad (23)
\end{aligned}
$$

  which shows that the energy is equal to the total relativistic kinematic energy ($= m\gamma c^2$) plus an additional energy due to the scalar potential ϕ. The vector potential $\vec{\mathbf{A}}$ plays no role here.

- In the non-relativistic limit,

$$H \approx mc^2 + \frac{p^2}{2m} + q\phi \qquad (24)$$

  which matches the results we already know.

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which matches the results we already know.

EoMs from the Hamiltonian formulation

Problem

Using the Hamiltonian

$$H = \sqrt{(\vec{\mathbf{P}} - e\vec{\mathbf{A}})^2 c^2 + m^2 c^4} + q\phi \ ,$$

determine the equations of motion.

The 4-d formalism

- From $S = -m \int u_k dx^k - q \int A_k dx^k$, we get
- the variation of S as

$$\begin{aligned} \delta S &= -m \int (\delta u_k) dx^k - m \int u_k d(\delta x^k) - q \int A_k d(\delta x^k) - q \int (\delta A_k) dx^k \\ &= 0 + m \int du_k \delta x^k + q \int dA_k \delta x^k - q \int (\delta A_k) dx^k \end{aligned} \quad (25)$$

- Further using

$$dA_k = \frac{\partial A_k}{\partial x^m} dx^m \,, \quad \delta A_k = \frac{\partial A_k}{\partial x^m} \delta x^m \,,$$

one obtains

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EoM from 4-d formalism

- The condition $\delta S = 0$ gives

$$\left[\int m\frac{du_k}{ds}\delta x^k + q\int \frac{\partial A_k}{\partial x^m}\frac{dx^m}{ds}\delta x^k - q\int \frac{\partial A_k}{\partial x^m}\frac{dx^k}{ds}\delta x^m\right] ds = 0 \qquad (27)$$

- This reduces to

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That is,

$$mc\frac{du_k}{ds} = q(\partial_k A_m - \partial_m A_k)u^k = q F_{km} u^m \qquad (29)$$

- The last equation, when written explicitly in terms of $\vec{E}$ and $\vec{B}$, gives the relativistic Lorentz force equation.

Problem

Show that the above equation corresponds to the Lorentz force law.

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Take-home message from this lecture

- The principle of least action, combined with the Lorentz invariance of action, allows us to obtain the expressions for relativistic energy, momentum, and the 4-momentum conservation equation through the Lagrangian, Hamiltonian, or the 4-d covariant formalism.
- When the interaction with an EM potential is included, modified expressions for energies and momenta are obtained, and the Lorentz force law emerges.