Module II: Relativity and Electrodynamics

Lecture 15: Lagrangian formulation of relativistic ED

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Outline

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From a single charge to volume distribution

- We have seen that the Lagrangian for a charge q in EM fields is given by

$$S = -mc \int ds - q \int A_k dx^k \; . \qquad (1)$$

The first term ($S_{\rm matter}$) describes the kinematics of a free particle, while the second one ($S_{\rm int}$) gives the interaction between a charge and the EM field.

- For many particles with masses m_i and charges q_i, the charge-field interaction term $S_{\rm int}$ gives

$$S_{\rm int} = -\sum_i q_i \int_{x_1}^{x_2} A_k dx^k \; . \qquad (2)$$

- Going to the continuum, $\sum q_i \to \int \rho dV$ where ρ is the charge density. This quantity may be written in a Lorentz-invariant form as

$$\frac{1}{c}\int J^\ell d\widetilde{V}_\ell = -\frac{1}{6c}\int J^\ell \epsilon_{\ell mnp} dV^{mnp} \; .$$

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Alternative way of writing the interaction term

- The interaction term can then be written as

$$S_{\text{int}} = \frac{1}{6c} \int \mathrm{J}^{\ell} A_k \epsilon_{\ell mnp} d\mathrm{V}^{mnp} d\mathrm{x}^k \ .$$

- Given

$$\epsilon_{\ell mnp} d\mathrm{V}^{mnp} d\mathrm{x}^k = -6\delta^k_\ell d\tilde{\Omega} \tag{3}$$

(the negative sign comes from the sign of ϵ_{0123}), one gets

$$S_{\text{int}} = -\frac{1}{c} \int d\tilde{\Omega}\, \mathrm{J}^k \mathrm{A}_k \ . \tag{4}$$

- Thus, the interaction term in the Lagrangian can be written in terms of the current and the EM potential.

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Gauge invariance and charge conservation

- We have already seen that the gauge freedom of $\int A_k dx^k$ does not affect the EOMs. Let us see that the case is similar with "$A \cdot J$":

$$\begin{aligned}\int d\tilde{\Omega} A'_k J^k &= \int d\tilde{\Omega} A_k J^k + \int d\tilde{\Omega} (\partial_k \psi) J^k \\ &= \int d\tilde{\Omega} A_k J^k + \int d\tilde{\Omega} \partial_k (\psi J^k) - \int d\tilde{\Omega} \psi (\partial_k J^k) \,.\end{aligned}$$

  The second term is the 4-volume integral of a total derivative, while the last term vanishes by the continuity condition. Thus, the Gauge transformation leaves the action effectively invariant.

- Note that the Gauge invariance is thus equivalent to the continuity relation $\partial_k J^k = 0$, and hence the conservation of electric charge.

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A term quadratic in EM potential A

- So far, we have seen the charge-field interaction term (or equivalently, the current-field interaction term) in the action. This term, when analyzed through the principle of least action under variation of the coordinates x^k, leads to the EoM

$$m\,c\,\frac{du_k}{ds} = q\,F_{km}\,u^m \tag{5}$$

  which yields the Lorentz force law.
- There are no other linear terms in A that will satisfy both, Lorentz invriance and Gauge invariance. Therefore, let us go to the quadratic terms.
- The only non-trivial terms that satisfy the above properties have the form $F_{km}F^{km}$ and $\tilde{F}_{km}F^{km}$. The latter violates parity, so it should not be a part of electromagnetism.
- The terms in the action that involve the EM fields are then

$$S = -\frac{1}{c}\int d\tilde{\Omega} A_k J^k - \eta_{FF}\int d\tilde{\Omega} F_{km}F^{km}\,. \tag{6}$$

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Variation of action w.r.t. A_k

- Now we shall use the principle of least action where the initial and final states stand for the initial and final configurations of the EM potential A. over the complete Cauchy surface. The action will be extremized with respect to the variation of the potential A.
- The variation of the action is

$$\begin{aligned}\delta S &= -\frac{1}{c}\int d\tilde{\Omega} J^k \delta A_k - 2\eta_{FF}\int d\tilde{\Omega} F^{km}\delta F_{km} \\ &= -\int d\tilde{\Omega}\left[\frac{1}{c}J^k\delta A_k + 2\eta_{FF}(F^{km}\partial_k\delta A_m - F^{km}\partial_m\delta A_k)\right] \\ &= -\int d\tilde{\Omega}\left[\frac{1}{c}J^m\delta A_m + 4\eta_{FF}F^{km}\partial_k\delta A_m\right] \qquad (7)\end{aligned}$$

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Maxwell's equations !

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$$\int d\tilde{\Omega} F^{km} \partial_k \delta A_m = \int d\tilde{\Omega} \partial_k (F^{km} \delta A_m) - \int d\tilde{\Omega} (\partial_k F^{km}) \delta A_m \ ,$$

and getting rid of the total derivative term, one gets

$$\delta S = -\int d\tilde{\Omega} \left[\frac{1}{c} J^m \delta A_m - 4\eta_{FF} (\partial_k F^{km}) \delta A_m \right] .$$

- The minimization of action, $\delta S = 0$, gives

$$4\eta_{FF} \partial_k F^{km} = \frac{1}{c} J^m \ . \tag{8}$$

This becomes the Maxwell's equation

$$\partial_k F^{km} = \mu_0 J^k \ , \tag{9}$$

with the identification

$$\eta_{FF} = \frac{1}{4\mu_0 c} = \frac{\epsilon_0 c}{4} \tag{10}$$

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Consolidation / take-home message

- Thus, the action in relativistic electrodynamics is

$$S = -\sum_i m_i c \int ds - \frac{1}{c} \int d\tilde{\Omega} J^k A_k - \frac{\epsilon_0 c}{4} \int d\tilde{\Omega} F_{km} F^{km} \qquad (11)$$

- The first term is the matter term S_{matter} that is present even in the absence of any EM fields.
- The second term is the interaction term S_{int} that represents the interaction term between currents and EM fields.
- The third term, S_{FF}, is the term that represents the interaction of the field with itself. It may be interpreted as the kinetic energy term of EM fields, and is present even when there are no source currents.
- Extremizing the action with respect to the variation in the coordinates x gives the Lorentz force law $m du^k/ds = qF^{km}u_m$, as seen is the last lecture.
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