Module III: Relativistic ED: applications

Lecture 16: Motion of charges in $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields

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Outline

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Lorentz force and relativistic 3-momentum

- We have seen in the last module (Relativity and Electrodynamics) that the equation of motion for a particle with charge e in an electromagnetic field is given by the Lorentz force law

$$\frac{d\vec{\mathbf{p}}}{dt} = e\,(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \tag{1}$$

- This of course looks the same as the non-relativistic version, however here $\vec{\mathbf{p}}$ is the relativistic 3-momentum (the space-components of the 4-momentum)

$$\vec{\mathbf{p}} = m\,\gamma\,\vec{\mathbf{v}}\,, \tag{2}$$

  where γ is the Lorentz boost.

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Kinematic energy

- Also, the "kinematic energy" $\mathcal{E}_{\rm kin}$
  (where $\mathcal{E}_{\rm kin}/c$ is the time-component of the 4-momentum)
  is $\mathcal{E}_{\rm kin} = \sqrt{m^2c^4 + |\vec{\mathbf{p}}|^2c^2}$.
  We use the notation $\mathcal{E}$ or the energy to distinguish it from the notation E for the electric field components, and the suffix "kin" to distinguish it from
  the total energy $\mathcal{E}$ of the particle, which will also include its electromagnetic potential energy.

- The kinematic energy relates the velocity and momentum of the particle:

$$\vec{\mathbf{v}} = \frac{\vec{\mathbf{p}}c^2}{\mathcal{E}_{\rm kin}} \ . \qquad (3)$$

- Using the Lorentz force, the trajectory of the particle is then

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{\vec{\mathbf{p}}c^2}{\mathcal{E}_{\rm kin}}\right) \ . \qquad (4)$$

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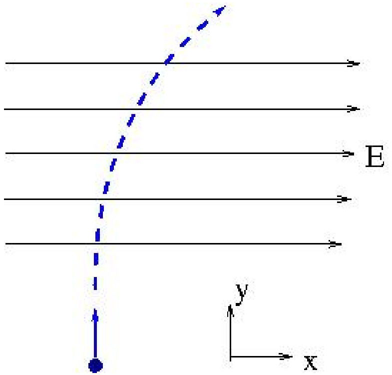
Outline

1. Relativistic equations of motion
2. Particle in a constant uniform electric field
3. Particle in a constant uniform magnetic field
4. Particle in combinations of electric and magnetic fields

Trajectory of particle starting out perpendicular to $\vec{\mathbf{E}}$



Formal statement of the problem

- The electric field is $\vec{\mathbf{E}} = E_x \hat{\mathbf{x}}$, which is a constant.
- The initial momentum is $\vec{\mathbf{p}}(0) = p_0 \hat{\mathbf{y}}$
- Determine the trajectory $x(t)$ and $y(t)$.

Equations of motion

- The Lorentz force equations give

$$\frac{dp_x}{dt} = eE_x \ , \quad \frac{dp_y}{dt} = 0 \ , \tag{5}$$

which lead to

$$p_x = eE_x t \ , \quad p_y = p_0 \ . \tag{6}$$

- The kinematic energy is

$$\mathcal{E}_{\rm kin} = \sqrt{m^2c^4 + |\vec{\mathbf{p}}|^2c^2} = \sqrt{m^2c^4 + p_0^2c^2 + (eE_xt)^2c^2} \ , \tag{7}$$

which may be written for convenience as

$$\mathcal{E}_{\rm kin} = \sqrt{\mathcal{E}_0^2 + (eE_xct)^2} \tag{8}$$

where $\mathcal{E}_0 = \sqrt{m^2c^4 + p_0^2c^2}$ is the initial kinematical (and total) energy of the particle.

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The trajectory

- The relation $\vec{\mathbf{v}} = \vec{\mathbf{p}}c^2/\mathcal{E}_{\text{kin}}$ gives

$$\frac{dx}{dt} = \frac{eE_x t\, c^2}{\sqrt{\mathcal{E}_0^2 + (eE_x ct)^2}}\,, \quad \frac{dy}{dt} = \frac{p_0 c^2}{\sqrt{\mathcal{E}_0^2 + (eE_x ct)^2}}\,. \tag{9}$$

- The solutions for $x(t)$ and $y(t)$ are then

$$x(t) = \frac{\sqrt{\mathcal{E}_0^2 + (eE_x ct)^2}}{eE_x} + x_0\,, \quad y(t) = \frac{p_0 c}{eE_x} \sinh^{-1}\left(\frac{eE_x ct}{\mathcal{E}_0}\right) \tag{10}$$

where we have taken $y(0) = 0$.

- The trajectory may then be described as

$$x = \frac{\mathcal{E}_0}{eE_x} \cosh\left(\frac{eE_x y}{p_0 c}\right) + x_0 \tag{11}$$

which is a catenary (the form taken by a uniform chain whose two end points are fixed at the same height.)

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Comparison with the non-relativistic limit

- In the non-relativistic (NR) limit, the velocity (and hence the momentum) along the y-direction is constant. In the relativistic case, the y-momentum is conserved, but not the y-velocity.
- In the NR case, the motion along x direction corresponds to uniform acceleration, and hence $x(t)$ is quadratic in t at large t. In the relativistic case, $x(t)$ is linear in t at large t.
- In the NR case, the trajectory is a parabola. As we have seen here, the relativistic trajectory is a catenary. Of course in the limit $eE_x y \ll p_0$, the catenary becomes a parabola.

Problem

Generalize this analysis to the case where the particle is moving at an angle θ with the electric field at $t = 0$.

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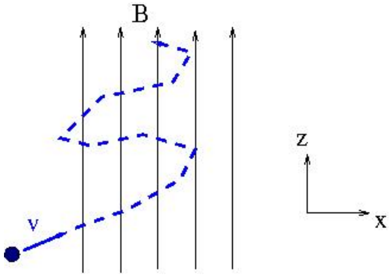
Particle starting out at an angle with $\vec{\mathbf{B}}$



Formal statement of the problem

- The magnetic field is $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$, which stays constant.
- Initial velocity is $\vec{\mathbf{v}}(0) = (v_{x0}, 0, v_{z0})$.
- Find the trajectory $x(t), y(t), z(t)$.

Equations of motion and conserved quantities

- The Lorentz force equations give

$$\frac{dp_x}{dt} = ev_y B \ , \quad \frac{dp_y}{dt} = -ev_x B \ , \quad \frac{dp_z}{dt} = 0 \ . \tag{12}$$

- Note that $(d\vec{\mathbf{p}}/dt) \cdot \vec{\mathbf{v}} = 0$, so no work is done on the particle, and the energy $\mathcal{E}$ is a constant. The identification of this conservation law simplifies our treatment a lot. Also, note that since there is no scalar potential, $\mathcal{E}_{\text{kin}} = \mathcal{E}$.
- Using $\vec{\mathbf{v}} = \vec{\mathbf{p}}c^2/\mathcal{E}_{\text{kin}}$, and denoting $\omega = eBc^2/\mathcal{E}$, we get

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The trajectory

- The motion along the z direction is straightforward:

$$z(t) = z_0 + v_{z0} t \tag{14}$$

- The motion in the transverse plane is simply a circular motion:

$$v_x = v_{x0} \cos(\omega t) , \quad v_y = v_{X0} \sin(\omega t) , \tag{15}$$

- Thus, the particle follows a helical trajectory, with a uniform linear motion along the magnetic field (z-direction) and a uniform circular motion in the transverse plane. The energy of the particle does not change, neither does the angle between the motion of the particle and the magnetic field.

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- Here the motion is very similar to that in the NR limit: the trajectory is always uniform circular in the transverse plane and uniform linear in along the magnetic field.
- The only difference is in the frequency of precession, ω. In the NR limit, $\omega = eB/m$, while in the relativistic case, $\omega = eBc^2/\mathcal{E}$.
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$$m\frac{d\vec{\mathbf{v}}}{dt} = e\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad \text{(NR)} \tag{16}$$

$$\frac{\mathcal{E}}{c^2}\frac{d\vec{\mathbf{v}}}{dt} = e\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad \text{(relativistic)}\ , \tag{17}$$

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Radius of curvature

- NR: $R \propto m$
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- Impact on particle identification

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Outline

Parallel and constant $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

Problem

- Let $\vec{\mathbf{E}} = E_z\hat{\mathbf{z}}$ and $\vec{\mathbf{B}} = B_z\hat{\mathbf{z}}$.
- Initially, the particle has $\vec{\mathbf{p}} = (p_{x0}, p_{y0}, p_{z0})$.
- Show that the solutions for the coordinates $x(t), y(t), z(t)$ are of the form

$$x = \frac{p_T}{eB_z}\sin\phi\,, \quad y = \frac{p_T}{eB_z}\cos\phi\,, \quad z = \frac{\mathcal{E}_0}{eE_z}\cosh\left(\frac{E_z\phi}{cB}\right) \quad (18)$$

  Determine p_T in terms of the initial conditions given above.
- Draw the trajectory, and comment on the differences between the relativistic and non-relativistic case.

Orthogonal and constant $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

Problem

- Take $\vec{\mathbf{E}} = E_x \hat{\mathbf{x}}$ and $\vec{\mathbf{B}} = B_z \hat{\mathbf{z}}$.
- Take the initial velocity of the particle to be $\vec{\mathbf{p}} = (p_{x0}, p_{y0}, p_{z0})$.
- Find the trajectory of the particle.
- This problem involves some rather complicated algebra and different initial conditions may give rise to qualitatively different trajectories. It is advised to solve this problem numerically on a computer for different sets of initial conditions (even if you get an analytical answer) and comment on the results.

Take-home message from this lecture

- Lorentz force law and the relation $\vec{\mathbf{v}} = \vec{\mathbf{p}}c^2/\mathcal{E}_{\text{kin}}$ are the ingredients needed to solve problems of relativistic motion of charges.