Module III: Relativistic ED: applications

Lecture 17: EM potentials from a moving charge (Lienard-Wiechert)

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Outline

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Retarded potentials

- We have seen in the first module (Electromagnetic waves) that a charge distribution $\rho(\vec{\mathbf{x}})$ and a current distribution $\vec{\mathbf{J}}(\vec{\mathbf{x}})$ give rise to the retarded potentials

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} , \quad (1)$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} . \quad (2)$$

- For future convenience, we write them in a form that looks symmetric in $(\vec{\mathbf{x}}, t)$ and $(\vec{\mathbf{x}}', t')$:

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x'\, dt'\, \delta\left(t' - t + \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c}\right) \frac{\rho(\vec{\mathbf{x}}', t')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} , \quad (3)$$

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Note that we always use $(\vec{\mathbf{x}}, t)$ for the coordinates where we measure the fields, whereas $(\vec{\mathbf{x}}', t')$ is the source position.

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Potentials due to a moving charge

- Let a charge q be moving along a trajectory $\boxed{\vec{\mathbf{x}}'_0(t')}$. Then

$$\rho(\vec{\mathbf{x}}', t') = q\, \delta^3[\vec{\mathbf{x}}' - \vec{\mathbf{x}}'_0(t')] \tag{5}$$

$$\vec{\mathbf{J}}(\vec{\mathbf{x}}', t') = q\, \vec{\mathbf{v}}(t')\, \delta^3[\vec{\mathbf{x}}' - \vec{\mathbf{x}}'_0(t')]\,. \tag{6}$$

- Then for the scalar potential, after integrating over $\vec{\mathbf{x}}'$ using the δ^3 function, we get

$$\phi(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0} \int dt'\, \delta\left(t' - t + \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'_0(t')|}{c}\right) \frac{1}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'_0(t')|}\,. \tag{7}$$

- Similarly, for the vector potential,

$$\vec{\mathbf{J}}(\vec{\mathbf{x}}, t) = \frac{\mu_0 q}{4\pi} \int dt'\, \delta\left(t' - t + \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'_0(t')|}{c}\right) \frac{\vec{\mathbf{v}}(t')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'_0(t')|}\,. \tag{8}$$

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Integration over t'

- Now we need to integrate over t' using the δ-function. Note that

$$\int dy\ \delta[f(y)] = \frac{1}{|\partial f(y)/\partial y|_{y_0}} \tag{9}$$

where y_0 is the point such that $f(y_0) = 0$.

- Let us denote $\vec{\mathbf{r}}(t') \equiv \vec{\mathbf{x}} - \vec{\mathbf{x}}'_0(t')$. Then $f(t') = t' - t + \frac{r(t')}{c}$, leading to

$$\frac{\partial f}{\partial t'} = 1 - \frac{\vec{\mathbf{v}}(t') \cdot \hat{\mathbf{r}}(t')}{c} \tag{10}$$

where $\hat{\mathbf{r}}(t')$ is a unit vector along the direction of $\vec{\mathbf{r}}(t')$. Note that the dependence of $\vec{\mathbf{r}}$ and r on $\vec{\mathbf{x}}$ and t is implicit.

- The quantity $\partial f/\partial t'$ needs to be calculated at t' that satisfies $f(t') = 0$, i.e. at the time $t' = t_r$, where t_r satisfies the implicit equation $t_r = t - r(t_r)/c$.

Problem

Obtain the above expression for $\partial f/\partial t'$

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Lienard-Wiechert potential

- Thus, we finally have

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{r(t_r)}\frac{q}{\left[1 - \frac{\vec{\mathbf{v}}(t_r)\cdot\hat{\mathbf{r}}(t_r)}{c}\right]} \quad (11)$$

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\left[r - \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{r}}}{c}\right]_{t_r}} . \quad (12)$$

This is the Lienard-Wiechert scalar potential.

- The Lienard-Wiechert vector potential is, similarly,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi}\frac{q\,\vec{\mathbf{v}}(t_r)}{\left[r - \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{r}}}{c}\right]_{t_r}} . \quad (13)$$

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Show that, given $\vec{\mathbf{x}}$, t and $\vec{\mathbf{x}}_0'(t')$, the implicit equation for t_r can have at most one solution. What happens when there is no solution ?

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Lienard-Wiechert potentials: simple form

Scalar potential

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{s(t_r)} \tag{14}$$

Vector potential

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \frac{q\vec{\mathbf{v}}(t_r)}{s(t_r)} \tag{15}$$

where $s(t_r) \equiv r(t_r) - \frac{\vec{\mathbf{v}}(t_r) \cdot \vec{\mathbf{r}}(t_r)}{c}$

Note that t_r is the solution of the implicit equation $t_r = t - \frac{r(t_r)}{c}$. We shall denote quantities to be calculated at t_r in magenta.

Comments on the Lienard-Wiechert potential

- The scalar potential due to a moving charge is not simply equal to $\frac{q}{4\pi\epsilon_0 r(t_r)}$ as one may naively guess. This is a purely geometric effect, that arises from the Jacobian between dt' and dt_r.
- Many different derivations of Lienard-Wiechert potential may be found in literature. The one in Griffiths gives a nice intuitive understanding.
- It is extremely crucial to keep track the times at which various quantities are calculated: t, t' or t_r. We shall try to be explicit about it throughout the rest of the course. (These notes use magenta for the quantities to be calculated at the retarded time t_r.)
- Note that $\vec{\mathbf{r}}(\vec{\mathbf{x}}, t, \vec{\mathbf{x}}_0(t'))$, and hence r is also, in principle, a function of $\vec{\mathbf{x}}$, t and $\vec{\mathbf{x}}_0(t')$. The $\vec{\mathbf{x}}$ and t dependence will be kept implicit unless specifically needed.

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Lorentz invariants and Lorentz transformations

- Relativity provides us with a powerful tool: if we know the 4-potential in a particular frame, we can use Lorentz transformations to convert it to any other frame.
- And of course we know the 4-potential for a point charge q in its rest frame. Since $\phi = q/(4\pi\epsilon_0|\vec{\mathbf{r}}|)$ the 4-potential is simply

$$A^k = \left(\frac{q}{4\pi\epsilon_0|\vec{\mathbf{r}}|c}, \vec{0} \right) \quad (16)$$

- Applying Lorentz transformations directly to the above A^k is not practical, since $|\vec{\mathbf{r}}|$ itself should change under these transformations. To avoid this problem, we should write a Lorentz-invariant quantity that equals r in the rest frame of the particle.
- We shall use brown to denote the 4-vectors and 4-tensors.

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Lienard-Wiechert 4-potential

- Such a quantity is $\mathrm{r}\cdot\mathrm{u}/c = \mathrm{r}_k\mathrm{u}^k/c$, where $\mathrm{r}_k \equiv (r, -\vec{\mathbf{r}})$ is the position 4-vector from the source point $(ct_r, \vec{\mathbf{x}}')$ to the observation point $(ct, \vec{\mathbf{x}})$, and $\mathrm{u}^k = (\gamma c, \gamma\vec{\mathbf{v}})$ is the 4-velocity of the charge.
- In the rest frame of the particle, $\mathrm{r}_k^{(0)} = (r, -\vec{\mathbf{r}})$ and $\mathrm{u}^{k(0)} = (c, \vec{0})$, so that $\mathrm{r}_k^{(0)}\mathrm{u}^{k(0)}/c = r$. Thus, $\mathrm{r}\cdot\mathrm{u}/c$ equals r in the rest frame of the charge.
- The right form of the 4-potential in the rest frame of the charge is then

$$\mathrm{A}^{k(0)} = \left(\frac{q}{4\pi\epsilon_0(\mathrm{r}\cdot\mathrm{u})}, \vec{0}\right) \tag{17}$$

- In the frame where the charge is moving with a velocity $\vec{\mathbf{v}}$, Lorentz transformations give

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Components of the Lienard-Wiechert 4-potential

- We have

$$\mathrm{r}\cdot\mathrm{u} = r\gamma c - \gamma\vec{\mathbf{v}}\cdot\vec{\mathbf{r}} = \gamma c\left(r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c}\right) = \gamma\, c\, s\,. \tag{19}$$

- The components of the 4-potential A^k are then

$$\phi = c\mathrm{A}_0 = \frac{q\gamma c}{4\pi\epsilon_0\gamma c s} = \frac{q}{4\pi\epsilon_0 s} \tag{20}$$

$$\vec{\mathbf{A}}_\mu = \mathrm{A}_\mu = \frac{q\gamma\vec{\mathbf{v}}}{4\pi\epsilon_0\gamma c^2 s} = \frac{\mu_0 q\vec{\mathbf{v}}}{4\pi s} \tag{21}$$

the same as the potentials we had obtained without using relativity explicitly.

- Note that the form of s appeared automatically from the Lorentz invariant quantity.

Components of the Lienard-Wiechert 4-potential

- We have

$$\mathrm{r}\cdot\mathrm{u} = r\gamma c - \gamma\vec{\mathbf{v}}\cdot\vec{\mathbf{r}} = \gamma c\left(r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c}\right) = \gamma\, c\, s\,. \qquad (19)$$

- The components of the 4-potential A^k are then

$$\phi = c\mathrm{A}_0 = \frac{q\gamma c}{4\pi\epsilon_0\gamma c s} = \frac{q}{4\pi\epsilon_0 s} \qquad (20)$$

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Take-home message from this lecture

- The scalar and vector potential at a point due to a moving charge is due to the location and velocity of the charge at the retarded time t_r, with an additional geometric factor of $1 - \frac{v(t_r) \cdot \hat{\mathbf{r}}(t_r)}{c}$.
- The geometric factor naturally emerges if we use covariant notation.