

Module III: Relativistic ED: applications

Lecture 18: EM fields from a uniformly moving charge

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Outline

- 1 \vec{E} and \vec{B} fields from Lienard-Wiechert potentials
- 2 \vec{E} and \vec{B} fields from Lorentz transformations
- 3 Force between two uniformly moving charges

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\vec{E} and \vec{B} fields from potentials

- The potentials from a moving charge with coordinates $\vec{x}'_0(t')$ are:

$$\phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{s(t_r)}, \quad \vec{A}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2} \frac{q\vec{v}(t_r)}{s(t_r)}. \quad (1)$$

- Now we need to calculate the electric and magnetic fields

$$\vec{E}(\vec{x}, t) = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B}(\vec{x}, t) = \nabla \times \vec{A}. \quad (2)$$

- Note that the fields are $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. So we need to calculate the derivatives ∇ and at constant t , and not constant t_r .
- Therefore, we need to write down $\vec{v}(t_r)$ and $s(t_r)$ in terms of the time t of observation.

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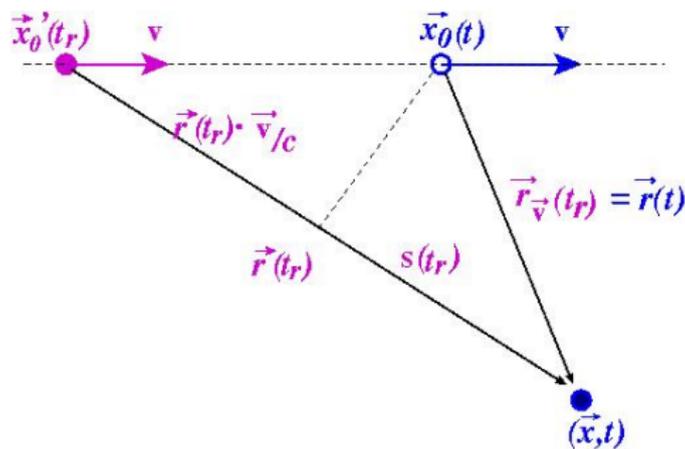
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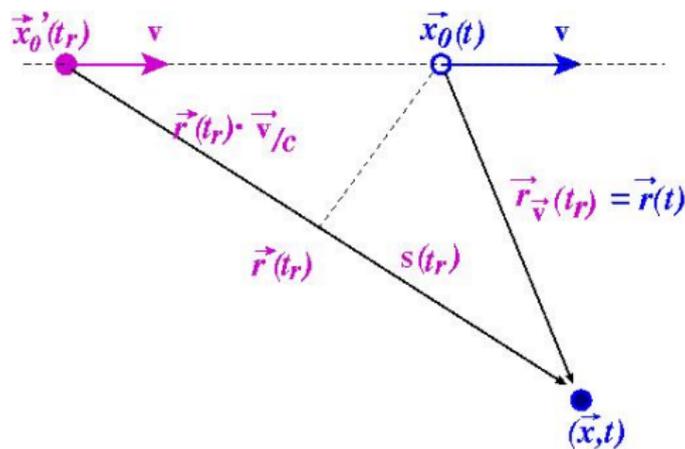
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Charge moving with a constant velocity



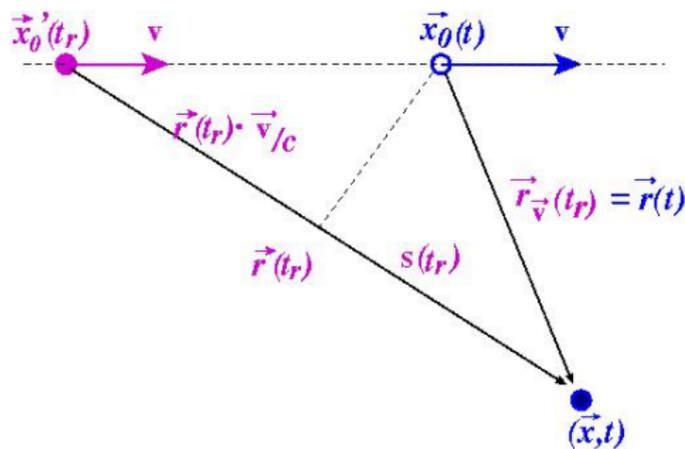
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- For a charge moving with a constant velocity, $\vec{v}(t_r) = \vec{v}(t) = \vec{v}$. Then we can also write $\vec{r}_{\vec{v}}(t_r) = \vec{r}(t) = \vec{x} - \vec{x}_0(t)$. Here, $\vec{x}_0(t)$ is the position the charge would be at time t .

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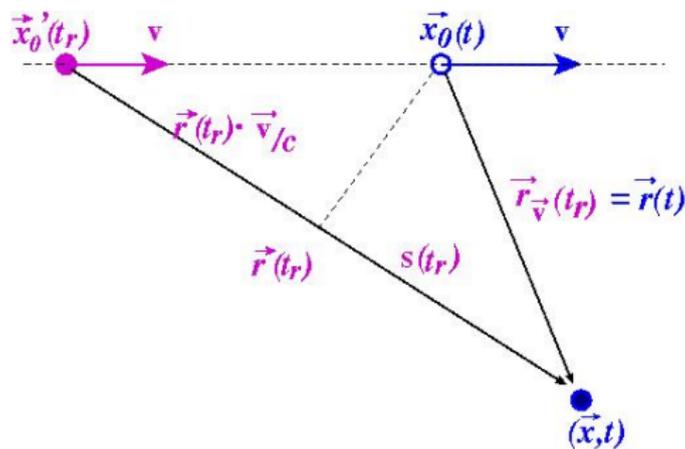
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Determining $[s(t_r)]^2$

- From the figure, it is clear that

$$[s(t_r)]^2 = [r_{\vec{v}}(t_r)]^2 - \left(\frac{\vec{r}(t_r) \times \vec{v}}{c} \right)^2 \quad (3)$$

- Also, since $\vec{r}(t_r) = \vec{r}(t) + \vec{v}(t - t_r)$, we have $\vec{r}(t_r) \times \vec{v} = \vec{r}(t) \times \vec{v}$.
Then

$$[s(t_r)]^2 = r(t)^2 - \left(\frac{\vec{r}(t) \times \vec{v}}{c} \right)^2 \quad (4)$$

We have thus written $s(t_r)$ in terms of $\vec{r}(t) = (\tilde{x}, \tilde{y}, \tilde{z})$.

- Let the motion of the charge be along x-direction, i.e. $\vec{v} = v\hat{x}$.
Explicitly in terms of $(\tilde{x}, \tilde{y}, \tilde{z})$, we get

$$\begin{aligned} [s(t_r)]^2 &= \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 - \frac{v^2}{c^2} (\tilde{y}^2 + \tilde{z}^2) \\ &= \tilde{x}^2 + \frac{1}{\gamma^2} (\tilde{y}^2 + \tilde{z}^2) \end{aligned} \quad (5)$$

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Potentials in terms of quantities at time t

- Now we have the potentials in terms of all quantities at time t :

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{s} = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{\tilde{x}^2 + \frac{1}{\gamma^2}(\tilde{y}^2 + \tilde{z}^2)}} \quad (6)$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^2} \frac{q v \hat{\mathbf{x}}}{s} = \frac{1}{4\pi\epsilon_0 c^2} \frac{q v \hat{\mathbf{x}}}{\sqrt{\tilde{x}^2 + \frac{1}{\gamma^2}(\tilde{y}^2 + \tilde{z}^2)}} \quad (7)$$

- Remember that here

$$\begin{aligned}(\tilde{x}, \tilde{y}, \tilde{z}) &= [x - x_0(t), y - y_0(t), z - z_0(t)] \\ &= [\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)].\end{aligned}$$

- Now the electric and magnetic fields can be calculated in a straightforward manner.

Calculating electric field

- The Electric field

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = -\nabla\phi(\vec{\mathbf{x}}, t) - \frac{\partial\vec{\mathbf{A}}(\vec{\mathbf{x}}, t)}{\partial t} \quad (8)$$

- Check:

$$\nabla\left(\frac{1}{s}\right) = \frac{(-\gamma^2\tilde{x}, -\tilde{y}, -\tilde{z})}{\gamma^2s^3}, \quad (9)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{s}\right) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \nabla\left(\frac{1}{s}\right) = -\vec{\mathbf{v}} \cdot \nabla\left(\frac{1}{s}\right) \quad (10)$$

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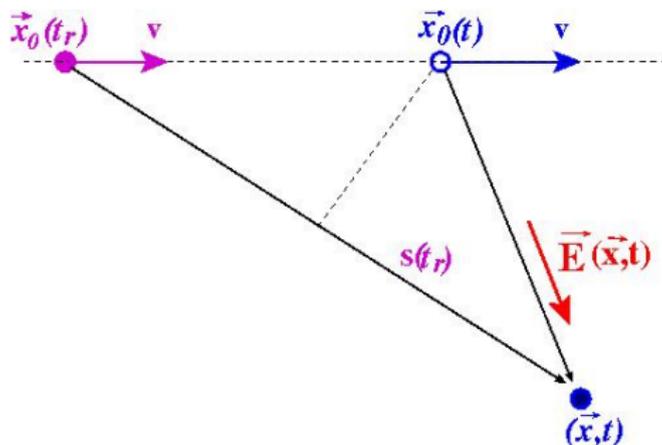
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Comments on the electric field

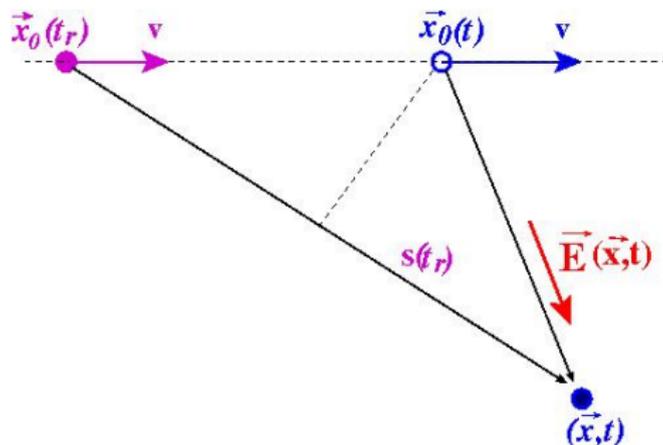


- In a compact form,

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\gamma^2 s^3} (\tilde{x}, \tilde{y}, \tilde{z}) \quad (13)$$

- Thus, \vec{E} is pointed to the observation point, from the position the charge *would have been* at the same time (not the retarded time). Note that this result is true only for a constant velocity of the charge, and hence does not violate the limit on the propagation speed of a signal.

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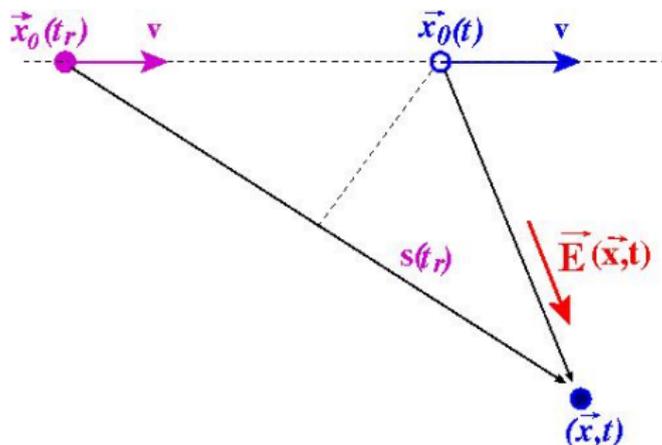


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Calculating magnetic field

- Using

$$\vec{B}(\vec{x}, t) = \nabla \times \vec{A}(\vec{x}, t) = \frac{q}{4\pi\epsilon_0 c^2} \nabla \times \left(\frac{\vec{v}}{s} \right), \quad (14)$$

we get

$$B_x = 0, \quad (15)$$

$$B_y = -\frac{qv}{4\pi\epsilon_0 c^2} \frac{\tilde{z}}{\gamma^2 s^3}, \quad (16)$$

$$B_z = \frac{qv}{4\pi\epsilon_0 c^2} \frac{\tilde{y}}{\gamma^2 s^3}. \quad (17)$$

Problem

Check the above answer by explicit calculation.

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Comments on the magnetic field

- We have, in compact form,

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^2} \frac{qv}{\gamma^2 s^3} (0, -\tilde{z}, \tilde{y}) \quad (18)$$

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Outline

- 1 \vec{E} and \vec{B} fields from Lienard-Wiechert potentials
- 2 \vec{E} and \vec{B} fields from Lorentz transformations
- 3 Force between two uniformly moving charges

From a coulomb potential

- The fields produced by a charge at rest are

$$\begin{aligned}\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) &= \frac{q\vec{\mathbf{r}}(t)}{4\pi\epsilon_0[r(t)]^3} = \frac{q}{4\pi\epsilon_0(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{3/2}}(\tilde{x}, \tilde{y}, \tilde{z}), \\ \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) &= \vec{\mathbf{0}}.\end{aligned}\tag{19}$$

- Here we want to go to a frame moving with a velocity $-\vec{\mathbf{v}}$, so that the charge is seen to be moving with a velocity $\vec{\mathbf{v}}$. In this frame,

$$\tilde{x}' = \tilde{x}/\gamma, \quad \tilde{y}' = \tilde{y}, \quad \tilde{z}' = \tilde{z}.\tag{20}$$

- We already know how $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ transform under these Lorentz transformations:

$$\begin{aligned}E'_x &= E_x, & E'_y &= \gamma(E_y + vB_z), & E'_z &= \gamma(E_z - vB_y), \\ B'_x &= B_x, & B'_y &= \gamma(B_y - \frac{v}{c^2}E_z), & B'_z &= \gamma(B_z + \frac{v}{c^2}E_y).\end{aligned}$$

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Fields in the primed frame, with primed coordinates

- Example of an $\vec{\mathbf{E}}$ component:

$$\begin{aligned} E'_x &= \frac{q}{4\pi\epsilon_0} \frac{\tilde{x}}{(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{3/2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{\gamma\tilde{x}'}{(\gamma^2\tilde{x}'^2 + \tilde{y}'^2 + \tilde{z}'^2)^{3/2}} = \frac{q}{4\pi\epsilon_0} \frac{\tilde{x}'}{\gamma^2 s^3} \end{aligned} \quad (21)$$

- Example of a $\vec{\mathbf{B}}$ component:

$$\begin{aligned} B'_y &= -\frac{\gamma v}{c^2} E_z = -\frac{q}{4\pi\epsilon_0} \frac{\gamma v \tilde{z}}{(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{3/2}} \\ &= -\frac{q}{4\pi\epsilon_0} \frac{\gamma v \tilde{z}'}{(\gamma^2\tilde{x}'^2 + \tilde{y}'^2 + \tilde{z}'^2)^{3/2}} = -\frac{q}{4\pi\epsilon_0} \frac{v \tilde{z}'}{\gamma^2 s^3} \end{aligned} \quad (22)$$

Problem

Explicitly calculate all the components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields and check that they match the results from the Lienard-Wiechert route.

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Force on q_2 due to the fields produced by q_1

- The force on a charge q_2 at the location $(\vec{\mathbf{x}}_2, t)$ is

$$\vec{\mathbf{F}}(\vec{\mathbf{x}}_2, t) = q_2 \left[\vec{\mathbf{E}}(\vec{\mathbf{x}}_2, t) + \vec{\mathbf{v}}_2 \times \vec{\mathbf{B}}(\vec{\mathbf{x}}_2, t) \right] \quad (23)$$

where $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are generated due to the motion of the charge q_1 with velocity $\vec{\mathbf{v}}_1$.

- Using the results obtained earlier in this lecture,

$$\vec{\mathbf{F}}(\vec{\mathbf{x}}_2, t) = \frac{q_1 q_2}{4\pi\epsilon_0} \left[-\nabla \left(\frac{1}{s} \right) + (\vec{\mathbf{v}}_1 \cdot \nabla) \frac{\vec{\mathbf{v}}_1}{c^2 s} + \frac{\vec{\mathbf{v}}_2}{c^2} \times \left(\nabla \times \frac{\vec{\mathbf{v}}_1}{s} \right) \right] \quad (24)$$

∇ is with respect to the coordinates $\vec{\mathbf{x}}_2$.

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Charges moving with the same velocities

- When $\vec{v}_1 = \vec{v}_2$, we can use the identity

$$\frac{\vec{v}}{c^2} \times \left(\nabla \times \frac{\vec{v}}{s} \right) = \nabla \left(\frac{v^2}{c^2 s} \right) - (\vec{v} \cdot \nabla) \frac{\vec{v}}{c^2 s} \quad (25)$$

to get

$$\vec{F}(\vec{x}_2, t) = \frac{q_1 q_2}{4\pi\epsilon_0} \left[-\nabla \left(\frac{1 - v^2/c^2}{s} \right) \right]. \quad (26)$$

- This may be written as

$$\vec{F}(\vec{x}_2, t) = -\nabla\psi, \quad \text{with} \quad \psi = \frac{q_1 q_2}{4\pi\epsilon_0 \gamma^2 s}. \quad (27)$$

The force can thus be derived from a scalar potential, called the “convection potential”.

- It is not surprising, since in the frame that moves along with both the charges, the two charges are stationary and experience Coulomb potential of each other.

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Charges moving in orthogonal directions

Problem

Two charges q_1 and q_2 are moving with uniform velocities along the x and y axis respectively. Their position vectors are given as

$$\vec{x}_1 = v_1 t \hat{x}, \quad \vec{x}_2 = v_2 t \hat{y}.$$

- Calculate the potentials $\phi(x, y, z, t)$ and $\vec{A}(x, y, z, t)$ due to the charge q_1 .
- Hence calculate the fields $\vec{E}(x, y, z, t)$ and $\vec{B}(x, y, z, t)$ due to the charge q_1 .
- Draw a diagram showing the positions of q_1 and q_2 at an arbitrary time t . Qualitatively show the directions of $\vec{E}(\vec{x}_2, t)$ and $\vec{B}(\vec{x}_2, t)$. Point out the important features.
- Calculate the force \vec{F}_{12} on the charge q_2 due to the charge q_1 .
- Calculate, and show in the figure, the force \vec{F}_{21} on the charge q_1 due to the charge q_2 .

Comment on the relative directions of \vec{F}_{12} and \vec{F}_{21} . Your answers should be in terms of x, y, z, t, v_1, v_2 and other universal constants, but no other variables. There is no need to combine terms to simplify.

Take-home message from this lecture

- The \vec{E} field due to a uniformly moving charge seems to point to where the charge currently is, rather than where it was at the retarded time.
- The expressions for fields obtained from Lienard-Wiechert potentials and from a Lorentz boost of the Coulomb potential match, as they have to.
- Even if the charge is not moving uniformly, the results may be generalized to “where the charge would have been currently, if it were travelling uniformly”.
- For two charges moving with equal constant velocities, the mutual force may be written in terms of a scalar potential, which is just the Coulomb potential with appropriate Lorentz and geometric factors.