ED II: Lecture 2

EM waves: propagation, decay, energy content

Amol Dighe

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Outline

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- Stationary state, by definition, means that the currents are steady and there is no net charge movement, i.e.

$$\nabla \cdot \vec{\mathbf{J}}_s = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

These statements are equivalent, due to continuity.

- If the initial distribution of charges and currents does not satisfy the above criteria, they will redistribute themselves so that a stationary state is reached.

- This process of "relaxation" happens over a time scale that is characteristic of the medium, called the relaxation time.

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Stationary and non-stationary states

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Relaxation time

- The continuity equation, combining with $\nabla \cdot \vec{\mathbf{D}} = \rho$, gives

$$\nabla \cdot \frac{\partial \vec{\mathbf{D}}}{\partial t} = -\nabla \cdot \vec{\mathbf{J}} \tag{2}$$

- Using $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ and $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$,

$$\nabla \cdot \left(1 + \frac{\epsilon}{\sigma} \frac{\partial}{\partial t} \right) \vec{\mathbf{J}} = 0 \tag{3}$$

- The solution to this differential equation is

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_s + (\vec{\mathbf{J}}_0 - \vec{\mathbf{J}}_s) e^{-t/\tau} \tag{4}$$

where J_0 is the initial current distribution

- $\tau = \epsilon/\sigma$ is the relaxation time

- $\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{\mathbf{J}}$, $\vec{\mathbf{E}} = \vec{\mathbf{J}}/\sigma$, etc. relax at the same rate.

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- No free charges, no external EMF sources. Maxwell $\Rightarrow$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\frac{\partial}{\partial t}(\nabla \times \mu \vec{\mathbf{H}}) \tag{5}$$

$$\nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\mu \frac{\partial}{\partial t}(\vec{\mathbf{J}}_{\text{fr}} + \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}) \tag{6}$$

- This gives the second order partial differential equation

$$\nabla^2 \vec{\mathbf{E}} - \mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \tag{7}$$

- Depending on whether the $(\partial^2 \vec{\mathbf{E}}/\partial t^2)$ term dominates or the $(\partial \vec{\mathbf{E}}/\partial t)$ one, we'll get two different extremes of behaviour. The former will lead to a propagating wave, the latter will lead to a diffusion equation, corresponding to a decaying wave.

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Time-dependent electric field

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- The differential equation becomes

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2(1 + \frac{i\sigma}{\epsilon\omega})\vec{\mathbf{E}} = 0 \tag{8}$$

- There are two time scales here: $1/\omega$ and $\tau = \epsilon/\sigma$

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2(1 + \frac{i}{\tau\omega})\vec{\mathbf{E}} = 0 \tag{9}$$

- When $\tau\omega >> 1$,

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2\vec{\mathbf{E}} = 0 \tag{10}$$

which is a wave propagating with speed $c = 1/\sqrt{\mu\epsilon}$

- When $\tau\omega << 1$,

$$\nabla^2\vec{\mathbf{E}} + \frac{i\omega}{\tau c^2}\vec{\mathbf{E}} = 0 \tag{11}$$

which is the equation for diffusion. In the context of EM waves, this will lead to a decaying solution.

- we shall explore these behaviours in detail now.

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- $\omega\tau >> 1 \Rightarrow$ displacement current dominates over conduction current

$$\nabla^2\vec{\mathbf{E}} + \mu\epsilon\omega^2\vec{\mathbf{E}} + i\omega\mu\sigma\vec{\mathbf{E}} = 0 \qquad (12)$$

- Plane wave: all fields are functions of the distance ζ of a plane from the origin. $\hat{\mathbf{n}}$ is the normal to this plane.

- $\nabla \to \hat{\mathbf{n}}(\partial/\partial\zeta)$

- Maxwell's equations in this language: (neglect $\vec{\mathbf{J}}_{\mathrm{fr}} = \sigma\vec{\mathbf{E}}$)

$$\hat{\mathbf{n}} \cdot \frac{\partial\vec{\mathbf{D}}}{\partial\zeta} = 0 \ , \qquad \hat{\mathbf{n}} \times \frac{\partial\vec{\mathbf{E}}}{\partial\zeta} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad (13)$$

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Propagating (plane wave) solution for $\vec{\mathbf{E}}$

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$\vec{\mathbf{E}}_{\parallel}$: longitudinal component of $\vec{\mathbf{E}}$

- $(\partial\vec{\mathbf{D}}/\partial\zeta)$ equation and dot product of $\hat{\mathbf{n}}$ with the $(\partial\vec{\mathbf{H}}/\partial\zeta)$ equation $\Rightarrow$

$$\frac{\partial\vec{\mathbf{D}}\cdot\hat{\mathbf{n}}}{\partial\zeta} = 0 \;, \quad \frac{\partial\vec{\mathbf{D}}\cdot\hat{\mathbf{n}}}{\partial t} = 0 \tag{15}$$

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Longitudinal component of $\vec{\mathbf{B}}$

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- Combining the two $\hat{\mathbf{n}}\times$ equations:

$$\hat{\mathbf{n}} \times (\frac{\partial^2 \vec{\mathbf{E}}}{\partial \zeta^2} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}) = 0 \qquad (17)$$

- Differential equation for $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}} \times \hat{\mathbf{n}}$
- General solution: $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0}[f(\zeta - ut) + g(\zeta + ut)]$
- If $\vec{\mathbf{E}}_\perp$ is sinusoidal:

$$\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0}e^{-i(\omega t \pm k\zeta)} \qquad (18)$$

- Direction of propagation $\vec{\mathbf{k}} \Rightarrow$

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- Using $\hat{\mathbf{n}} \times (\partial \vec{\mathbf{E}}_\perp / \partial \zeta) = -\partial \vec{\mathbf{B}}_\perp / \partial t$,

$$i\vec{\mathbf{k}} \times \vec{\mathbf{E}}_\perp = i\omega\vec{\mathbf{B}}_\perp \Rightarrow \vec{\mathbf{B}}_\perp = \frac{\vec{\mathbf{k}}}{\omega} \times \vec{\mathbf{E}} \tag{20}$$

Transverse components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

- Combining the two $\hat{\mathbf{n}} \times$ equations:

$$\hat{\mathbf{n}} \times \left(\frac{\partial^2 \vec{\mathbf{E}}}{\partial \zeta^2} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}\right) = 0 \tag{17}$$

- Differential equation for $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}} \times \hat{\mathbf{n}}$
- General solution: $\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0}[f(\zeta - ut) + g(\zeta + ut)]$
- If $\vec{\mathbf{E}}_\perp$ is sinusoidal:

$$\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_{\perp,0} e^{-i(\omega t \pm k\zeta)} \tag{18}$$

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Propagating wave in short

- $\vec{\mathbf{E}}_\parallel$ and $\vec{\mathbf{B}}_\parallel$ are constants in space and time, hence not interesting for wave propagation

- $\vec{\mathbf{E}}_\perp$ and $\vec{\mathbf{B}}_\perp$ can have $e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$ dependence, with $\vec{\mathbf{B}}_\perp = (\vec{\mathbf{k}}/\omega) \times \vec{\mathbf{E}}$

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Outline

- When $\omega\tau << 1$, conduction current dominates over displacement current

$$\nabla^2\vec{E} + \mu\epsilon\omega^2\vec{E} + i\omega\mu\sigma\vec{E} = 0 \tag{21}$$

- The solution of the form $\vec{E}_0 e^{\pm i(kx-\omega t)}$ implies

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\begin{aligned}
k^2 &= i\omega\mu\sigma = \omega\mu\sigma e^{i\pi/2} \tag{22} \\
\Rightarrow k &= \sqrt{\omega\mu\sigma} e^{i\pi/4} = \sqrt{\omega\mu\sigma}\left(\frac{1+i}{\sqrt{2}}\right) \tag{23}
\end{aligned}
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- This gives

$$\vec{E} = \vec{E}_0 e^{\pm i(\mathrm{Re}(k)x-\omega t)} e^{-\mathrm{Im}(k)x} \tag{24}$$

- The wave then decays with a $e^{-\mathrm{Im}(k)x}$ dependence inside the conducting medium.

Decaying plane wave

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Skin depth in metals

- For metals, $\tau \sim 10^{-14}$ sec. So for $\omega \lesssim 10^{14}$/s, conduction current dominates.

- A wave incident on a metallic surface will decay as

$$|\vec{\mathbf{E}}| = |\vec{\mathbf{E}}_0| e^{-r/\delta} \tag{25}$$

where, from the last page,

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}} = \sqrt{\frac{2c^2 \tau}{\omega}} \tag{26}$$

- Within a distance δ from the surface of the metal, the wave would decrease in magnitude by a factor $1/e$. This δ is the "skin depth" of the metal. The surface currents will flow within this width.

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Work done for creating a charge distribution

- Work done in increasing charge density by $\delta\rho(\vec{\mathbf{x}})$:

$$\delta W = \int \phi(\vec{\mathbf{x}}) \, \delta\rho(\vec{\mathbf{x}}) dV = \epsilon_0 \int \phi(\vec{\mathbf{x}})\delta(\nabla \cdot \vec{\mathbf{E}}) dV \qquad (27)$$

- Integrate by parts:

$$\delta W = \epsilon_0 \int \nabla \cdot (\phi\delta\vec{\mathbf{E}}) dV - \epsilon_0 \int (\nabla\phi) \cdot (\delta\vec{\mathbf{E}}) dV$$

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Total energy stored in electric field

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- Just like a small incremental charge gives rise to $\delta q \longrightarrow \delta \vec{\mathbf{E}}$ against which δW needs to be done to bring incremental charge (that is stored in the resultant $\vec{\mathbf{E}}$),

- A small incremental current gives rise to
  $\delta_t \vec{\mathbf{J}} \longrightarrow \delta_t(\nabla \times \vec{\mathbf{H}}) \longrightarrow \delta_t \vec{\mathbf{H}} \longrightarrow \delta_t(\nabla \times \vec{\mathbf{E}}) \longrightarrow \delta_t \vec{\mathbf{E}}$
  $(\delta_t \Rightarrow \text{small change with time})$

  δW needs to be done against this "back emf" to produce incremental current (that will be stored in the resultant $\vec{\mathbf{B}}$).

- The energy stored in the magnetic field during time δt :
  $\delta U_m = -\delta_t \vec{\mathbf{E}} \cdot \vec{\mathbf{J}} \, \delta t \, dV$

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&= \int \vec{\mathbf{H}} \cdot \frac{\partial(\delta_t \vec{\mathbf{B}})}{\partial t} dV \delta t \quad (31)
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- Incremental energy stored: $\boxed{\delta U_m = \int \vec{\mathbf{H}} \cdot \delta \vec{\mathbf{B}} dV}$

Total energy stored in magnetic field (μ constant)

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W = \frac{1}{2\mu} \int \vec{\mathbf{B}}^2 dV \quad (32)
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- This is indeed valid for static electromagnetic fields

- But for time-dependent fields, we shall see later that the leading terms in $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ go as $1/r$, so over the surface of a sphere with large radius r, the integral actually will have a constant nonzero vale.

- This will be the energy radiated away at infinity due to the changing currents. $\vec{\mathbf{N}} \equiv \vec{\mathbf{E}} \times \vec{\mathbf{H}}$ will be defined as the Poynting vector, which gives the rate of loss of energy through radiation.

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Outline

Quadratic quantities and factors of 2

- In Electrodynamics, for convenience, we often use notation involving complex numbers (mainly exponentials), e.g.

$$\vec{E} = \vec{E}_0 e^{i(kx - \omega t)} \ , \quad \vec{B} = -i\vec{B}_0 e^{i(kx - \omega t)} \tag{33}$$

when we actually want to represent

$$\vec{E} = \vec{E}_0 \cos(kx - \omega t) = \mathrm{Re}(\vec{E}_0 e^{i(kx - \omega t)}) \tag{34}$$

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- While performing calculations in complex notation and taking the real part of the final answer works as long as we are dealing with quantities linear in $\vec{E}$ or $\vec{B}$, one has to be careful while dealing with quadratic (or higher order) quantities.

- For example, in the complex notation above,

$$\langle |\vec{E}|^2 \rangle = \langle |\vec{E}^* \cdot \vec{E}| \rangle = |\vec{E}_0|^2 \tag{36}$$

while the actual answer should be (using real notation)

$$\langle |\vec{E}|^2 \rangle = |\vec{E}_0|^2 \langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} |\vec{E}_0|^2 \tag{37}$$

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- We have already seen that the energy stored in electric field is $U_e = (1/2)\epsilon_0|\vec{\mathbf{E}}|^2$ (we showed this result for a static field). When the electric field represents a propagating wave, then taking into account the "factor of 2" for averaged quadratic quantities, we get

$$\langle U_e \rangle = \frac{1}{4}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{38}$$

- The energy stored in magnetic field is $U_m = (1/2)|\vec{\mathbf{B}}|^2/\mu_0$ (we showed it for a static magnetic field). For a propagating wave, $|\vec{\mathbf{B}}| = |\vec{\mathbf{k}}/\omega||\vec{\mathbf{E}}|$. Including the "factor of 2", we get

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$$\langle U_m \rangle = \frac{1}{4}\frac{|\vec{\mathbf{B}}_0|^2}{\mu_0} = \frac{1}{4}\frac{|\vec{\mathbf{k}}|^2}{\omega^2\mu_0}|\vec{\mathbf{E}}_0|^2 = \frac{1}{4}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{39}$$

- For a plane EM wave, energy stored in electric and magnetic field is equal. The total energy of an EM wave is

$$\langle U \rangle = \langle U_e \rangle + \langle U_m \rangle = \frac{1}{2}\epsilon_0|\vec{\mathbf{E}}_0|^2 \tag{40}$$

- The rate of energy transport is given by the Poynting vector,

$$\vec{\mathbf{N}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}} \tag{41}$$

- The time-averaged value of this quantity is

$$\langle |\vec{\mathbf{N}}| \rangle = \frac{1}{2} |\vec{\mathbf{E}}_0| . |\vec{\mathbf{H}}_0| = \frac{1}{2} |\vec{\mathbf{E}}_0| \frac{|\vec{\mathbf{k}}|}{\omega\mu} |\vec{\mathbf{E}}_0| = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |\vec{\mathbf{E}}_0|^2 \tag{42}$$

- Compared with the rate of energy consumption in a conductor, $(1/2)\sigma|\vec{\mathbf{E}}_0|^2$, the quantity $\sqrt{\epsilon_0/\mu_0}$ is termed the conductance of vacuum

- Similarly, $\sqrt{\epsilon/\mu}$ is the conductance of a medium through which an EM wave propagates

- Note that $\langle |\vec{\mathbf{N}}| \rangle = c\langle U \rangle$, since the wave transports energy at the speed c.

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Energy transported by the EM fields

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Recap of topics covered in this lecture

- Relaxation to stationary state, relaxation time
- Electromagnetic wave: displacement and conduction current
- Transverse EM field solutions for a propagating wave
- Decay of EM waves in a condunctor, skin depth
- Energy stored in EM fields
- Energy transported by an EM wave